SCIENCE CHINA
Information Sciences

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Robust adaptive distributed optimization for heterogeneous unknown second-order nonlinear multiagent systems

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Appendix A Literature review

The distributed optimization problem has drawn significant attention recently since it has a variety of applications in economic dispatch of smart grids [1, 2], regression of distributed data [3], optimal coordination of mobile robot systems [4] and unmanned surface vessel (USV) systems [5], and optimal power flow [6], and so on. A class of typical distributed optimization problems in multiagent systems (MASs) is to design a proper optimization algorithm by local information exchange such that all agents' states achieve consensus at the minimizer of a global cost function, where the global cost function is the sum of local cost functions and each local cost function is only known to the agent itself. Different from the traditional distributed consensus problems, such as leaderless consensus [7, 8] and consensus tracking [9], the typical distributed optimization problems are not only required to achieve consensus but also to cooperatively minimize the global cost function.

A large number of results use the damped step size to design discrete-time algorithms for solving distributed optimization problems in various scenarios [10–12]. As a counterpart, many practical physical systems usually described by continuous-time dynamics (e.g., USV systems and mobile robot systems), various continuous-time MASs are extensively investigated in the distributed optimization problem [13–17], to name a few. For example, the continuous-time distributed optimization problem of a single-integrator MAS is addressed in [13] with a connected undirected graph and in [14] with a weight-balanced directed graph. The authors of [15] propose some edge-based fixed-time consensus protocols for a single-integrator MAS to solve the optimization problem. In addition, some fully distributed adaptive optimization algorithms are proposed in [16] for a continuous-time single-integrator MAS with weight-balanced and unbalanced directed graphs. Besides, a primal-dual control approach is employed in [17] to deal with the distributed constrained optimization problem of a double-integrator MAS. The distributed optimization problem of a double-integrator MAS is studied in [18, 19] via designing various discontinuous algorithms. Different from the simple integrator-type MAS, the distributed optimization problem for linear MASs is studied in [20], where both edge- and node-based fully distributed adaptive controllers are designed. By employing the internal model principle approach, the authors in [21, 22] design some distributed optimization controllers for a first-order MAS with exogenous disturbances to achieve an asymptotic optimal consensus convergence. Moreover, the distributed optimization problem is considered in [23] for a higher-order MAS with known smooth nonlinear functions.

Existing works on distributed optimization problems primarily assume all agents have identical linear dynamics, such as integrator-type dynamics [13, 19] and general linear dynamics [20]. However, many physical systems have heterogeneous unknown nonlinear dynamics, such as mobile robot systems, USV systems, pendulum systems, and Euler-Lagrange systems usually described by the second-order systems with heterogeneous unknown nonlinear dynamics, disturbances, and unavailable parameters. The asymptotic distributed optimization problems of MASs with heterogeneous nonlinear dynamics and with heterogeneous disturbances are considered in [23, 24] and [21, 22, 25], respectively, while the system parameters are available. For the case when the upper bound parameters involving nonlinear functions and disturbances are unavailable (unknown), the robust adaptive distributed optimization problem of second-order MAS with heterogeneous nonlinear dynamics and disturbances has not been reported at present. Investigating the distributed optimization problem of second-order MAS with heterogeneous unknown nonlinear functions and disturbances is therefore crucial and significant, especially when the upper bound parameters involving nonlinear functions and disturbances are *unavailable*.

As stated above, this work addresses the distributed optimization problem of a second-order MAS via robust adaptive control, where the heterogeneous unknown nonlinear functions and disturbances are considered. The main contributions of this work can be summarized as the following two aspects. First, the asymptotic distributed optimization problem of a second-order MAS with heterogeneous unknown nonlinear dynamics and disturbances is studied, where the upper bound parameters involving nonlinear functions and disturbances are *unavailable*. Some works focus on the asymptotic distributed

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optimization problem of MAS with integrator-type dynamics [13, 19], linear dynamics [20], Lipschitz-type nonlinear dynamics [24], and known higher-order nonlinear dynamics [23]. Second, a modified command filter approach that obviates the need to compute an analytic derivative of the virtual velocity is applied to design a discontinuous robust adaptive distributed optimization algorithm. To avoid the chattering phenomenon caused by the proposed discontinuous algorithm, a continuous robust adaptive distributed optimization algorithm is further designed. Both of the designed discontinuous and continuous algorithms ensure all agents achieve asymptotic optimal consensus convergence in a fully distributed fashion while guaranteeing strong system robustness.

Appendix B Notations and remarks

Let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$, and let $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ be the sets of n-dimensional real column vectors and $n \times m$ real matrices, respectively. Symbols $\otimes$ and I_n represent the Kronecker product and $n \times n$ identity matrix, respectively. Let $\mathbf{1}_N$ ($\mathbf{0}_N$) be the column vector of N ones (zeros). The signum function of $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ is denoted as $\text{sign}(x) = [\text{sign}(x_1), \ldots, \text{sign}(x_n)]^T$. The gradient and Hessian of a twice differentiable function $f(x) : \mathbb{R}^n \to \mathbb{R}$ are represented as $\nabla f(x) \in \mathbb{R}^n$ and $\nabla^2 f(x) \in \mathbb{R}^{n\times n}$, respectively. For a symmetric matrix A, its biggest eigenvalue is given by $\lambda_{\max}(A)$. For a vector function $g(t) = [g_1(t), \ldots, g_n(t)]^T$, it is said that, $g(t) \in \mathbb{L}_\infty$ if $\sup_t \|g(t)\|_1 < \infty$, $g(t) \in \mathbb{L}_p$ if $\int_0^\infty \|g(s)\|^p ds < +\infty$, where $p > 0$ is a real number, $\|g(t)\|_1 = \sum_{i=1}^n |g_i(t)|$ and $\|g(t)\| = \sqrt{g^T(t)g(t)}$.

Remark 1. Assumption 2 is mild, where φ_i is an available (a computable) scalar function that has been used in [28, 29] for the consensus algorithm design of the uncertain nonidentical MASs. The assumption about the function $g_i(x_i)$ in Assumption 3 is weaker than the one in [19] that $g_i(x_i)$ is required to satisfy $\|g_i - g_j\| \leqslant \bar{g}$ with $\bar{g} > 0$ being a constant, $\forall i, j \in \mathbb{N}$, and the one in [15] that $\phi_i(x_i)$ is assumed to be bounded. The invertibility of the Hessian matrix $\nabla^2 f_i(x_i)$ is a common assumption in the time-varying optimization problems [15, 18, 19, 23] and the designed zero-gradient-sum protocols for distributed optimization [30–32]. The parameters ρ_i, ϱ_i, ς_i, ζ and ϑ_i in Assumptions 2 and 3 are assumed to exist for stability analysis, however, they are unknown and not required in the adaptive optimization controller design. Here, only the available functions φ_i and ϕ_i are known and required in the adaptive optimization controller design.

Remark 2. Some types of local cost functions satisfy Assumption 3 with bounded functions ϕ_i, such as the local cost function $f_i(x_i) = x_i^T A_i x_i + \kappa_{i1}(\mu_i + \mathbf{1}_n^T)x_i + \kappa_{i2}$ used in economic dispatch (where $A_i \in \mathbb{R}^{n\times n}$ is an invertible diagonal matrix, $\mu_i \in \mathbb{R}^{1\times n}$ and $\kappa_{i1}, \kappa_{i2} \in \mathbb{R}$), by some calculations, $\nabla^2 f_i(x_i) = 2A_i$ is invertible and $g_i(x_i)$ satisfies Assumption 3 with $\zeta = 1$, $\phi_i(x_i) = (2A_i)^{-1}(\mu_i^T + \mathbf{1}_n)$ and $\vartheta_i = \kappa_{i1}$.

Remark 3. In the command filter (3), v_i^* and its derivative $\dot{v}_i^*$ are generated by using the pseudocontrol signal input v_i^p, where v_i^* is the command filtered version of v_i^p and $v_i^* - v_i^p$ represents the unachieved portion of v_i^p (called filtered error). As given in [33], by increasing the parameter π_i, the command filter (3) can guarantee its output v_i^* arbitrarily approximates the pseudocontrol signal v_i^p. The command filter control approach is employed in the controller (5) without the need to compute the derivative of virtual velocity, the singularity phenomenon, as a result, is avoided.

Remark 4. The virtual distributed protocol u_i^* in (6) consists of a stabilizing state feedback term $-l_i\tilde{v}_i$ and a robust adaptive compensation term $-\alpha_i\psi_i\text{sign}(\tilde{v}_i)$. The robust adaptive compensation term contains an adaptively adjusted gain is used to accommodate the nonlinearities and disturbances to guarantee global asymptotic convergence of $\tilde{v}_i$. The pseudocontrol signal v_i^p in (10) consists of a robust adaptive compensation term $-\beta_i\chi_i\text{sign}(e_{z_i})$ and an optimization function term $-g_i(x_i)$, where the former is used to compensate for the optimization function to guarantee global consensus of the agents' states z_i, while the latter plays a key role in minimizing $f_i(x_i)$. Neither the designed virtual distributed protocol u_i^* nor the pseudocontrol signal v_i^p used the relative velocity information among neighbors. Similar to [15, 19], we do not use Filippov solution [35] for the nonsmooth analysis (since the designed virtual distributed protocol u_i^* in (6) is discontinuous) to avoid symbol redundancy throughout this work.

Appendix C Preliminaries

Appendix C.1 Basic graph theory

Let a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be an undirected graph with the sets of node $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_N\}$ and edge $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and an adjacency matrix $\mathcal{A} = [a_{ij}]_{N\times N}$ with weights $a_{ij} \geqslant 0$. An edge $(\mathcal{V}_j, \mathcal{V}_i) \in \mathcal{E}$ means that the information of node (agent) j is available to node (agent) i, then $j \in \mathcal{N}_i = \{j \in \mathcal{V}/\{\mathcal{V}_i\} | (\mathcal{V}_j, \mathcal{V}_i) \in \mathcal{E}\}$, where $\mathcal{N}_i$ denotes the neighbors set of agent i. The weights $a_{ij} > 0$ if $j \in \mathcal{N}_i$ and $a_{ij} = 0$ otherwise. The Laplacian matrix associated with the adjacency matrix $\mathcal{A}$ is defined as $\mathcal{L} = [l_{ij}]_{N\times N}$, where $l_{ij} = -a_{ij}$ if $i \neq j$ and $l_{ii} = \sum_{j=1, j\neq i}^N a_{ij}$. An undirected graph is connected if at least one path exists between any pair of distinct nodes.

Appendix C.2 Some useful lemmas

Lemma 1. [26] If $f(x) : \mathbb{R}^n \to \mathbb{R}$ is a convex and continuous differentiable function, $x_* \in \mathbb{R}^n$ is the minimizer of $f(x)$ is equivalent to $\nabla f(x_*) = \mathbf{0}_n$.

The following lemma will be reused in our subsequent asymptotic stability analysis, which is actually a corollary of Barbalat's Lemma.

Lemma 2. [27] For $g(t) \in \mathbb{R}^n$, if $g(t) \in \mathbb{L}_\infty$, $\dot{g}(t) \in \mathbb{L}_\infty$ and $g(t) \in \mathbb{L}_r$ with $r \in [1, \infty)$, then $\lim_{t\to\infty} g(t) = \mathbf{0}_n$.

Lemma 3. The adaptive gains $\alpha_i(t) > 0$ in (7) and (8), and $\beta_i(t) > 0$ in (11) and (12), $\forall t \geqslant 0$, $i \in \mathbb{N}$.

Proof. Denote $\breve{\alpha}_i(t) = \alpha_i(t) - \tilde{\alpha}_i(t)$. From (7) and (8), it holds that $\dot{\breve{\alpha}}_i(t) \geqslant -(\gamma_{i1}\sigma_{i1} + \delta_{i1})\breve{\alpha}_i(t)$, that is $\breve{\alpha}_i(t) \geqslant \breve{\alpha}_i(0)e^{-(\gamma_{i1}\sigma_{i1}+\delta_{i1})t} \geqslant 0$, where $\breve{\alpha}_i(0) \geqslant 0$ due to $\alpha_i(0) \geqslant \tilde{\alpha}_i(0)$. This together with (8) ensures that $\dot{\tilde{\alpha}}_i(t) = \delta_{i1}\breve{\alpha}_i(t) \geqslant 0$, thus $\alpha_i(t) \geqslant \tilde{\alpha}_i(t) \geqslant \tilde{\alpha}_i(0) > 0$, for each $i \in \mathbb{N}$. Similarly, one has $\beta_i(t) \geqslant \tilde{\beta}_i(t) \geqslant \tilde{\beta}_i(0) > 0$, for each $i \in \mathbb{N}$.

Lemma 4. If $e_{z_i} = \sum_{j\in\mathcal{N}_i} a_{ij}(z_i - z_j) \in \mathbb{L}_\infty$ and $e_{z_i} \neq \mathbf{0}_n$, then $z_i \in \mathbb{L}_\infty$, $\forall t \geqslant 0$, each $i \in \mathbb{N}$.

Proof. Let $z = [z_1^T, \dots, z_N^T]^T$ and $e_z = [e_{z_1}^T, \dots, e_{z_N}^T]^T$. Note that $e_z^T e_z = z^T(\mathcal{L}^T\mathcal{L}\otimes I_n)z$, and the N eigenvalues of $\mathcal{L}^T\mathcal{L}$ satisfy $0 = \lambda_1(\mathcal{L}^T\mathcal{L}) < \lambda_2(\mathcal{L}^T\mathcal{L}) \leqslant \cdots \leqslant \lambda_N(\mathcal{L}^T\mathcal{L})$ (see Lemma 5.1 in [36]). This together with Lemma 5 in [28] implies that there exists a constant $k > 0$ such that $\forall e_z \neq \mathbf{0}_{nN}$, $e_z^T e_z = z^T(\mathcal{L}^T\mathcal{L}\otimes I_n)z > kz^Tz$, where c is a constant. Thus, $\forall e_z \neq \mathbf{0}_{nN}$, $\|z\| < \|e_z\|/\sqrt{k} < +\infty$ if $e_z < +\infty$, which means that, $z_i < +\infty$ if $e_{z_i} < +\infty$ and $e_{z_i} \neq \mathbf{0}_n$ for each $i \in \mathbb{N}$.

Lemma 5. The adaptive gains $\alpha_i(t) > 0$ in (14) and (15), and $\beta_i(t) > 0$ in (19) and (20), $\forall t \geqslant 0$, $i \in \mathbb{N}$.

Proof. From (14) and (15), it holds that $\dot{\alpha}_i \geqslant -\gamma_{i1}\sigma_{i1}p_i = \frac{\gamma_{i1}\sigma_{i1}}{\delta_{i1}(\sigma_{i1}+1)}\dot{p}_i$, by integrating from 0 to t derives $\alpha_i(t) - \alpha_i(0) \geqslant \frac{\gamma_{i1}\sigma_{i1}}{\delta_{i1}(\sigma_{i1}+1)}(p_i(t) - p_i(0))$, where $p_i(t) = p_i(0)e^{-\delta_{i1}(\sigma_{i1}+1)t}$ by light of (15). Thus $\alpha_i(t) \geqslant \frac{\gamma_{i1}\sigma_{i1}}{\delta_{i1}(\sigma_{i1}+1)}p_i(t) = \frac{\gamma_{i1}\sigma_{i1}}{\delta_{i1}(\sigma_{i1}+1)}p_i(0)e^{-\delta_{i1}(\sigma_{i1}+1)t} > 0$ if $\alpha_i(0) \geqslant \frac{\gamma_{i1}\sigma_{i1}}{\delta_{i1}(\sigma_{i1}+1)}p_i(0) > 0$, for each $i \in \mathbb{N}$. Similarly, for each $i \in \mathbb{N}$, one has $\beta_i(t) \geqslant \frac{\gamma_{i2}\sigma_{i2}}{\delta_{i2}(\sigma_{i2}+1)}q_i(t) = \frac{\gamma_{i2}\sigma_{i2}}{\delta_{i2}(\sigma_{i2}+1)}q_i(0)e^{-\delta_{i2}(\sigma_{i2}+1)t} > 0$.

Appendix D Proof of Theorem 1

Proof. By using the discontinuous virtual input (6), one gets

$$\dot{\tilde{v}}_i = h_i + d_i + u_i^* = h_i + d_i - l_i\tilde{v}_i - \alpha_i\psi_i\mathrm{sign}(\tilde{v}_i), \quad i \in \mathbb{N}. \tag{D1}$$

To analyze the stability of (D1), a Lyapunov function is introduced as

$$V_{\tilde{v}}(t) = \frac{1}{2}\sum_{i=1}^N\left[\tilde{v}_i^T\tilde{v}_i + \frac{1}{\gamma_{i1}}(\alpha_i - \bar{\alpha}_i)^2 + \frac{\sigma_{i1}}{\delta_{i1}}(\tilde{\alpha}_i - \bar{\alpha}_i)^2\right], \tag{D2}$$

where $\bar{\alpha}_i > 0$ is a constant to be selected later. Using Assumption 2, (7) and (8), $\dot{V}_1$ along (D1) satisfies

$$\begin{aligned}\dot{V}_{\tilde{v}}(t) =& \sum_{i=1}^N\left[\tilde{v}_i^T\left(h_i + d_i - l_i\tilde{v}_i - \alpha_i\psi_i\mathrm{sign}(\tilde{v}_i)\right) + (\alpha_i - \bar{\alpha}_i)\left(\psi_i\|\tilde{v}_i\|_1 - \sigma_{i1}(\alpha_i - \tilde{\alpha}_i)\right) + \sigma_{i1}(\tilde{\alpha}_i - \bar{\alpha}_i)(\alpha_i - \tilde{\alpha}_i)\right]\\ \leqslant& \sum_{i=1}^N\left[(\rho_i + \varrho_i\|\varphi_i\| + \varsigma_i)\|\tilde{v}_i\| - l_i\|\tilde{v}_i\|^2 - \alpha_i\psi_i\|\tilde{v}_i\|_1 + (\alpha_i - \bar{\alpha}_i)\psi_i\|\tilde{v}_i\|_1 - \sigma_{i1}(\alpha_i - \tilde{\alpha}_i)^2\right]\\ \leqslant& \sum_{i=1}^N\left[(c_i - \bar{\alpha}_i)\psi_i\|\tilde{v}_i\|_1 - l_i\|\tilde{v}_i\|^2\right],\end{aligned} \tag{D3}$$

where the last inequality holds owing to $\rho_i + \varrho_i\|\varphi_i\| + \varsigma_i \leqslant c_i(1 + \|\varphi_i\|) = c_i\psi_i$ with $c_i = \max\{\rho_i + \varsigma_i, \varrho_i\}$ and $\|\tilde{v}_i\| \leqslant \|\tilde{v}_i\|_1$. By letting $\bar{\alpha}_i = c_i$, it can be concluded that

$$\dot{V}_{\tilde{v}}(t) \leqslant -\sum_{i=1}^N l_i\|\tilde{v}_i\|^2 \leqslant -\underline{l}\sum_{i=1}^N\|\tilde{v}_i\|^2, \tag{D4}$$

where $\underline{l} = \min_{i\in\mathbb{N}}\{l_i\} > 0$. Then, $V_{\tilde{v}}(t) \in \mathbb{L}_\infty$ since $V_{\tilde{v}}(t) \leqslant V_{\tilde{v}}(0) \in \mathbb{L}_\infty$, this together with (D2) implies that $\tilde{v}_i \in \mathbb{L}_\infty$, $\alpha_i \in \mathbb{L}_\infty$ and $\tilde{\alpha}_i \in \mathbb{L}_\infty$ for all $i \in \mathbb{N}$. Integrating both sides of (D4) from 0 to t derives $\int_0^\infty\sum_{i=1}^N\tilde{v}_i^T(s)\tilde{v}_i(s)ds \leqslant \underline{l}^{-1}(V_{\tilde{v}}(0) - V_{\tilde{v}}(\infty)) < \infty$, thus $\tilde{v}_i \in \mathbb{L}_2$. In the following, if we show that $\tilde{v}_i$ is uniformly continuous, i.e., $\dot{\tilde{v}}_i \in \mathbb{L}_\infty$, then by using the Barbalat lemma (see Lemma 2), we can conclude that the asymptotic convergence of $\tilde{v}_i$.

According to (1) and (4), we have $\dot{z}_i = \dot{x}_i - \dot{\mu}_i = \tilde{v}_i + \bar{v}_i + v_i^p - \dot{\mu}_i$ for each $i \in \mathbb{N}$. This together with (9) and (10) implies that the following relationship holds (each $i \in \mathbb{N}$):

$$\dot{z}_i = \tilde{v}_i + \|\bar{v}_i\|\mathrm{sign}(\mu_i) - \gamma_i e_{z_i} - \beta_i\chi_i\mathrm{sign}(e_{z_i}) - g_i(x_i). \tag{D5}$$

Define $V_\mu = \frac{1}{2}\sum_{i=1}^N\mu_i^T\mu_i$. Its time derivative along (9) gives $\dot{V}_\mu \leqslant -\sum_{i=1}^N k_i\mu_i^T\mu_i \leqslant -2\underline{k}V_\mu$ with $\underline{k} = \min_{i\in\mathbb{N}}\{k_i\} > 0$. So $\mu_i \in \mathbb{L}_\infty$ and $\lim_{t\to\infty}\mu_i = \mathbf{0}_n$, it then follows from (9) that $\bar{v}_i \in \mathbb{L}_\infty$ and $\lim_{t\to\infty}\bar{v}_i = \mathbf{0}_n$, i.e., $v_i^* = v_i^p$ as $t \to \infty$ for all $i \in \mathbb{N}$. We choose a Lyapunov function as

$$V_0 = V_z + \frac{1}{2}\sum_{i=1}^N\left[\frac{1}{\gamma_{i2}}(\beta_i - \bar{\beta}_i)^2 + \frac{\sigma_{i2}}{\delta_{i2}}(\tilde{\beta}_i - \bar{\beta}_i)^2\right], \tag{D6}$$

where $V_z = \frac{1}{2}z^T(\mathcal{L}\otimes I_n)z$, $z = [z_1^T, \dots, z_N^T]^T$ and $\bar{\beta}_i > 0$ is a constant to be selected later. The symmetry of $\mathcal{L}$ implies that $\dot{V}_z = z^T(\mathcal{L}\otimes I_n)\dot{z} = \sum_{i=1}^N e_{z_i}^T\dot{z}_i$. Recall that $\tilde{v}_i \in \mathbb{L}_\infty$, $\mu_i \in \mathbb{L}_\infty$ and $\bar{v}_i \in \mathbb{L}_\infty$, there exist positive constants η_{i1}, η_{i2}, and

η_{i3} such that $\|\tilde{v}_i\| \leqslant \eta_{i1}$, $\|\mu_i\| \leqslant \eta_{i2}$ and $\|\bar{v}_i\| \leqslant \eta_{i3}$ for each $i \in \mathbb{N}$. Time differentiation of V_0, utilization of Assumption 3, (11), (12) and (D5) gives

$$\begin{aligned}\dot{V}_0 =& \sum_{i=1}^{N}\Big[e_{z_i}^T(\tilde{v}_i + \|\bar{v}_i\|\text{sign}(\mu_i) - \gamma_i e_{z_i} - \beta_i\chi_i\text{sign}(e_{z_i}) - \vartheta_i\phi_i \\ &- \zeta(z_i+\mu_i)) + (\beta_i - \bar{\beta}_i)\Big(\chi_i\|e_{z_i}\|_1 - \sigma_{i2}(\beta_i - \tilde{\beta}_i)\Big) + \sigma_{i2}(\tilde{\beta}_i - \bar{\beta}_i)(\beta_i - \tilde{\beta}_i)\Big] \\ \leqslant& \sum_{i=1}^{N}\Big[\left(\eta_{i1}+\eta_{i3}+\vartheta_i\|\phi_i\|+\zeta\eta_{i2}\right)\|e_{z_i}\| - \gamma_i e_{z_i}^T e_{z_i} - \beta_i\chi_i\|e_{z_i}\|_1 - \zeta e_{z_i}^T z_i + (\beta_i - \bar{\beta}_i)\chi_i\|e_{z_i}\|_1 - \sigma_{i2}(\beta_i - \tilde{\beta}_i)^2\Big] \\ \leqslant& \sum_{i=1}^{N}\Big[(m_i - \bar{\beta}_i)\chi_i\|e_{z_i}\|_1 - \gamma_i e_{z_i}^T e_{z_i} - \zeta e_{z_i}^T z_i\Big],\end{aligned} \tag{D7}$$

where the last inequality holds due to $\eta_{i1}+\eta_{i3}+\vartheta_i\|\phi_i\|+\zeta\eta_{i2} \leqslant m_i(1+\|\phi_i\|) = m_i\chi_i$ with $m_i = \max\{\eta_{i1}+\eta_{i3}+\zeta\eta_{i2}, \vartheta_i\}$ and $\|e_{z_i}\| \leqslant \|e_{z_i}\|_1$. By letting $\bar{\beta}_i = m_i$, one has

$$\dot{V}_0 \leqslant -\sum_{i=1}^{N}\left(\gamma_i e_{z_i}^T e_{z_i} + \zeta e_{z_i}^T z_i\right) \leqslant -\zeta\sum_{i=1}^{N} e_{z_i}^T z_i = -2\zeta V_z \leqslant -\frac{\zeta}{\lambda_{\max}(\mathcal{L})}\sum_{i=1}^{N} e_{z_i}^T e_{z_i}, \tag{D8}$$

where $V_z = \frac{1}{2}z^T(\mathcal{L}\otimes I_n)z = \frac{1}{2}((\mathcal{L}\otimes I_n)z)^T z = \frac{1}{2}\sum_{i=1}^N e_{z_i}^T z_i$ has been used in the equality, and $\sum_{i=1}^N e_{z_i}^T e_{z_i} = \|(\mathcal{L}\otimes I_n)z\|^2 = ((\mathcal{L}^{1/2}\otimes I_n)z)^T(\mathcal{L}\otimes I_n)(\mathcal{L}^{1/2}\otimes I_n)z \leqslant \lambda_{\max}(\mathcal{L})z^T(\mathcal{L}\otimes I_n)z = 2\lambda_{\max}(\mathcal{L})V_z$ has been used in the last inequality. Thus $V_0(t) \in \mathbb{L}_\infty$, this together with (D6) implies that $V_z \in \mathbb{L}_\infty$, $\beta_i \in \mathbb{L}_\infty$ and $\tilde{\beta}_i \in \mathbb{L}_\infty$. Recall that $\sum_{i=1}^N e_{z_i}^T e_{z_i} \leqslant 2\lambda_{\max}(\mathcal{L})V_z$, which implies that $e_{z_i} \in \mathbb{L}_\infty$, where $e_{z_i} \in \mathbb{L}_\infty$ further implies that $z_i \in \mathbb{L}_\infty$ if $e_{z_i} \neq \mathbf{0}_n$ by Lemma 4, and thus $x_i = z_i + \mu_i \in \mathbb{L}_\infty$, then $\phi_i \in \mathbb{L}_\infty$ by Assumption 3. It follows from (D5) that $\dot{e}_{z_i} = \sum_{i=1}^N l_{ij}\dot{z}_j \in \mathbb{L}_\infty$ since $\tilde{v}_i \in \mathbb{L}_\infty$, $\bar{v}_i \in \mathbb{L}_\infty$, $\beta_i \in \mathbb{L}_\infty$, $\chi_i = 1+\|\phi_i\| \in \mathbb{L}_\infty$ and $g_i(x_i) = \zeta x_i + \vartheta_i\phi_i \in \mathbb{L}_\infty$. By integrating both sides of (D8), we have $e_{z_i} \in \mathbb{L}_2$. Now, applying Lemma 2 yields $\lim_{t\to\infty} e_{z_i} = \mathbf{0}_n$, i.e., $z_i = z_j$ for all $i, j \in \mathbb{N}$, as $t \to \infty$. This together with $x_i = z_i + \mu_i$ and the fact that $\lim_{t\to\infty}\mu_i = \mathbf{0}_n$ imply that $x_i = x_j$ for all $i, j \in \mathbb{N}$, as $t \to \infty$.

Next, we will show that $\tilde{v}_i$ asymptotically converges to zero vector. According to the command filter (3), $z_{i,1}, z_{i,2} < +\infty$ as $t \in [0, +\infty)$ can be inferred, which ensures that $\dot{x}_i = v_i = \tilde{v}_i + z_{i,1} < +\infty$, then $x_i < +\infty$, $\varphi_i < +\infty$, $h_i < +\infty$ and $d_i < +\infty$ by Assumption 2. Note that $\psi_i = 1 + \|\varphi_i\|$, it follows from that $\psi_i < +\infty$. In light of (D1), it further follows that $\dot{\tilde{v}}_i \in \mathbb{L}_\infty$. Recall that $\tilde{v}_i \in \mathbb{L}_\infty$ and $\tilde{v}_i \in \mathbb{L}_2$. Consequently, using Lemma 2 derives $\lim_{t\to\infty}\tilde{v}_i = \mathbf{0}_n$ for each $i \in \mathbb{N}$.

Note that $u_i^* \in \mathbb{L}_\infty$ in light of (6), $z_{i,2} \in \mathbb{L}_\infty$ can be inferred according to the command filter (3) and $\bar{v}_i = z_{i,1} - v_i^p \in \mathbb{L}_\infty$, it then follows from (5) that $u_i \in \mathbb{L}_\infty$. Since $\lim_{t\to\infty}\tilde{v}_i = \mathbf{0}_n$ for each $i \in \mathbb{N}$, it follow from (7) and (8) that $\dot{\check{\alpha}}_i(t) = -(\gamma_{i1}\sigma_{i1} + \delta_{i1})\check{\alpha}_i(t)$ as $t \to \infty$, where $\check{\alpha}_i(t) = \alpha_i(t) - \tilde{\alpha}_i(t)$, i.e., $\lim_{t\to\infty}\check{\alpha}_i(t) = \lim_{t\to\infty}\check{\alpha}_i(0)e^{-(\gamma_{i1}\sigma_{i1}+\delta_{i1})t} = 0$, so $\lim_{t\to\infty}\alpha_i(t) = \lim_{t\to\infty}\tilde{\alpha}_i(t)$. This together with (7) and (8) ensures that $\lim_{t\to\infty}\dot{\alpha}_i(t) = \lim_{t\to\infty}\dot{\tilde{\alpha}}_i(t) = 0$. Note that $\alpha_i \in \mathbb{L}_\infty$ and $\tilde{\alpha}_i \in \mathbb{L}_\infty$, it follows from that $\lim_{t\to\infty}\alpha_i(t) = \lim_{t\to\infty}\tilde{\alpha}_i(t) = \alpha_i^*$ with α_i^* being a positive constant, for each $i \in \mathbb{N}$. Similarly, one can get that $\lim_{t\to\infty}\beta_i = \lim_{t\to\infty}\tilde{\beta}_i = \beta_i^*$, where β_i^* is a positive constant for each $i \in \mathbb{N}$.

Finally, we will show that the global asymptotic convergence of $x_i - x_*$, $\forall i \in \mathbb{N}$. Consider a Lyapunov function as $V_f(t) = \frac{1}{2}(\sum_{i=1}^N \nabla f_i(x_i))^T\sum_{i=1}^N \nabla f_i(x_i)$. It thus follows from the fact $\dot{x}_i = \tilde{v}_i + \bar{v}_i + v_i^p = \tilde{v}_i + \bar{v}_i - k_i\mu_i - \gamma_i e_{z_i} - \beta_i\chi_i\text{sign}(e_{z_i}) - [\nabla^2 f_i(x_i)]^{-1}\nabla f_i(x_i)$ that

$$\begin{aligned}\dot{V}_f(t) =& \left(\sum_{i=1}^N \nabla f_i(x_i)\right)^T\left(\sum_{i=1}^N \nabla^2 f_i(x_i)\left(\tilde{v}_i + \bar{v}_i - k_i\mu_i - \gamma_i e_{z_i} - \beta_i\chi_i\text{sign}(e_{z_i})\right) - \sum_{i=1}^N \nabla f_i(x_i)\right) \\ =& \left(\sum_{i=1}^N \nabla f_i(x_i)\right)^T\sum_{i=1}^N \nabla^2 f_i(x_i)\left(\tilde{v}_i + \bar{v}_i - k_i\mu_i - \gamma_i e_{z_i} - \beta_i\chi_i\text{sign}(e_{z_i})\right) - 2V_f(t) \\ \leqslant& \frac{1}{2}\left\|\sum_{i=1}^N \nabla^2 f_i(x_i)\left(\tilde{v}_i + \bar{v}_i - k_i\mu_i - \gamma_i e_{z_i} - \beta_i\chi_i\text{sign}(e_{z_i})\right)\right\|^2 - V_f(t),\end{aligned} \tag{D9}$$

where the Young inequality, i.e., $2X^TY \leqslant \|X\|^2 + \|Y\|^2$ for $X, Y \in \mathbb{R}^n$, has been applied. After a simple calculation, it is easily inferred that $V_f(t)$ satisfies the following inequality:

$$V_f(t) \leqslant \left\|\sum_{i=1}^N \nabla^2 f_i(x_i)\left(\tilde{v}_i + \bar{v}_i - k_i\mu_i - \gamma_i e_{z_i} - \beta_i\chi_i\text{sign}(e_{z_i})\right)\right\|^2\frac{1-e^{-t}}{2} + V_f(0)e^{-t}. \tag{D10}$$

Note that $\lim_{t\to\infty}\left(\tilde{v}_i + \bar{v}_i - k_i\mu_i - \gamma_i e_{z_i} - \beta_i\chi_i\text{sign}(e_{z_i})\right) = \mathbf{0}_n$ and $x_i \in \mathbb{L}_\infty$, where $x_i \in \mathbb{L}_\infty$ further implies that $\nabla^2 f_i(x_i) \in \mathbb{L}_\infty$ by Assumption 3. So

$$\lim_{t\to\infty}\sum_{i=1}^N \nabla^2 f_i(x_i)\left(\tilde{v}_i + \bar{v}_i - k_i\mu_i - \beta_i\chi_i\text{sign}(e_{z_i})\right) = \mathbf{0}_n. \tag{D11}$$

It then follows from (D10) that $\lim_{t\to\infty}V_f(t) = 0$, i.e., $\lim_{t\to\infty}\sum_{i=1}^N \nabla f_i(x_i) = \mathbf{0}_n$. According to Lemma 1, we can get that $\lim_{t\to\infty}x_i = x_*$, where x_* is the unique minimizer of problem (2). This theorem is thus established.

Remark 5. To prove $\lim_{t\to\infty} x_i = x_*$ in Theorem 1, we successively proved that $\lim_{t\to\infty} \bar{v}_i = \lim_{t\to\infty} \mu_i = \mathbf{0}_n$, $\lim_{t\to\infty} e_{z_i} = \mathbf{0}_n$, $\lim_{t\to\infty} \tilde{v}_i = \mathbf{0}_n$ and $\beta_i, \chi_i, \nabla^2 f_i(x_i) \in \mathbb{L}_\infty$, $\forall i \in \mathbb{N}$, where $\lim_{t\to\infty} \mu_i = \mathbf{0}_n$ and $\lim_{t\to\infty} e_{z_i} = \mathbf{0}_n$ ensure that $\lim_{t\to\infty} x_i = x$ since $x_i = z_i + \mu_i$. It thus follows from (D10) that $\lim_{t\to\infty} x_i = x_*$, $\forall i \in \mathbb{N}$.

Remark 6. By using the σ-modification method [36], a modification term $-\gamma_{i1}\sigma_{i1}(\alpha_i - \tilde{\alpha}_i)$ and $-\gamma_{i2}\sigma_{i2}(\beta_i - \tilde{\beta}_i)$ is added into (7) and (11), respectively, to render smaller control gains α_i and β_i, thus requiring smaller control input and virtual velocity while guaranteeing the global asymptotic convergence of $\tilde{v}_i$ and consensus of z_i. Therefore, by integrating the adaptively adjusted gains and the σ-modification method, a discontinuous fully distributed robust adaptive optimization algorithm is developed in this work.

Appendix E Proof of Theorem 2

Proof. By using the continuous virtual input (13), one yields

$$\dot{\tilde{v}}_i = h_i + d_i - l_i\tilde{v}_i - \frac{\alpha_i\psi_i^2\tilde{v}_i}{\psi_i\|\tilde{v}_i\| + p_i}, \quad i \in \mathbb{N}. \tag{E1}$$

For the stability analysis of (E1), we consider the following Lyapunov function to replace (D2):

$$V_{\tilde{v}}(t) = \frac{1}{2}\sum_{i=1}^{N}\left[\tilde{v}_i^T\tilde{v}_i + \frac{1}{\gamma_{i1}}(\alpha_i - \bar{\alpha}_i)^2 + \frac{2\bar{\alpha}_i}{\delta_{i1}}p_i\right], \tag{E2}$$

where $\bar{\alpha}_i > 0$ is a constant to be given later. According to Assumption 2, (14) and (15), $\dot{V}_{\tilde{v}}$ along (E1) satisfies

$$\begin{aligned}
\dot{V}_{\tilde{v}} &= \sum_{i=1}^{N}\left[\tilde{v}_i^T\left(h_i + d_i - l_i\tilde{v}_i - \frac{\alpha_i\psi_i^2\tilde{v}_i}{\psi_i\|\tilde{v}_i\| + p_i}\right) + (\alpha_i - \bar{\alpha}_i)\left(\frac{\psi_i^2\|\tilde{v}_i\|^2}{\psi_i\|\tilde{v}_i\| + p_i} - \sigma_{i1}p_i\right) - \bar{\alpha}_i(\sigma_{i1}+1)p_i\right] \\
&\leqslant \sum_{i=1}^{N}\left[(\rho_i + \varrho_i\|\varphi_i\| + \varsigma_i)\|\tilde{v}_i\| - l_i\|\tilde{v}_i\|^2 - \frac{\alpha_i\psi_i^2\|\tilde{v}_i\|^2}{\psi_i\|\tilde{v}_i\| + p_i} + (\alpha_i - \bar{\alpha}_i)\left(\frac{\psi_i^2\|\tilde{v}_i\|^2}{\psi_i\|\tilde{v}_i\| + p_i} - \sigma_{i1}p_i\right) - \bar{\alpha}_i(\sigma_{i1}+1)p_i\right] \\
&\leqslant \sum_{i=1}^{N}\left[c_i\psi_i\|\tilde{v}_i\| - l_i\|\tilde{v}_i\|^2 - \frac{\bar{\alpha}_i\psi_i^2\|\tilde{v}_i\|^2}{\psi_i\|\tilde{v}_i\| + p_i} - \bar{\alpha}_i p_i\right],
\end{aligned} \tag{E3}$$

where the last inequality holds owing to $\rho_i + \varrho_i\|\varphi_i\| + \varsigma_i \leqslant c_i(1 + \|\varphi_i\|) = c_i\psi_i$ with $c_i = \max\{\rho_i + \varsigma_i, \varrho_i\}$ and $-\alpha_i\sigma_{i1}p_i < 0$ from Lemma 5. By letting $\bar{\alpha}_i = c_i$, it can be concluded that

$$\begin{aligned}
\sum_{i=1}^{N}\left(c_i\psi_i\|\tilde{v}_i\| - \frac{\bar{\alpha}_i\psi_i^2\|\tilde{v}_i\|^2}{\psi_i\|\tilde{v}_i\| + p_i} - \bar{\alpha}_i p_i\right) &= \sum_{i=1}^{N}\frac{(c_i - \bar{\alpha}_i)\psi_i^2\|\tilde{v}_i\|^2 + (c_i - \bar{\alpha}_i)p_i\psi_i\|\tilde{v}_i\| - \bar{\alpha}_i p_i^2}{\psi_i\|\tilde{v}_i\| + p_i} \\
&= -\sum_{i=1}^{N}\frac{\bar{\alpha}_i p_i^2}{\psi_i\|\tilde{v}_i\| + p_i} \leqslant 0.
\end{aligned} \tag{E4}$$

Substituting (E4) into (E3) derives

$$\dot{V}_{\tilde{v}} \leqslant -\sum_{i=1}^{N} l_i\|\tilde{v}_i\|^2 \leqslant -\underline{l}\sum_{i=1}^{N}\|\tilde{v}_i\|^2, \tag{E5}$$

where $\underline{l} = \min_{i\in\mathbb{N}}\{l_i\} > 0$. The subsequent proof of this theorem can be completed by using a analysis akin to that of Theorem 1; we omit it here for conciseness.

Remark 7. Same to the designed discontinuous optimization algorithm in Theorem 1, the designed continuous optimization algorithm in Theorem 2 not only implements asymptotic optimization control in a fully distributed fashion but also has strong robustness. Both of the proposed algorithms in Theorems 1 and 2 exhibit the singularity-free feature. Furthermore, under the continuous algorithm in Theorem 2, both unexpected chattering and singularity phenomenons are avoided.

Remark 8. To overcome the problem of "explosion of complexity" for differentiating virtual signals repeatedly, the finite-time command filter is considered in [38], which takes the following form:

$$\begin{aligned}
\dot{z}_{i,1} &= \iota_i = -\omega_{i1}\mathrm{sig}^{\frac{1}{2}}(z_{i,1} - v_i^p) + z_{i,2}, \\
\dot{z}_{i,2} &= -\omega_{i2}\mathrm{sign}(z_{i,2} - \iota_i), \quad \text{each } i \in \mathbb{N},
\end{aligned} \tag{E6}$$

where $v_i^* = z_{i,1}$ and $\dot{v}_i^* = \iota_i$ are the outputs with the virtual controller v_i^p as the input, $\omega_{i1} > 0$ and $\omega_{i2} > 0$ are parameters. Compared with the command filter (3), the command filter (E6) can guarantee its output v_i^* faster approximate the virtual signal v_i^p. From [39], the command filter (E6) is finite-time stable if proper parameters ω_{i1} and ω_{i2} are chosen, that is $z_{i,1}$ and ι_i converge to v_i^p and $\dot{v}_i^p$ in finite time, respectively. Although the convergent speed of the command filter (E6) is faster than the one of the command filter (3), it follows from (E6) that ι_i contains a term $z_{i,2}$ and the derivative of $z_{i,2}$ is discontinuous. Additionaly, to achieve finite-time stable for (E6), the parameters ω_{i1} and ω_{i2} should be chosen large sufficiently. The large parameters may aggravate the unexpected chattering and also increase the control effort. Hence, to circumvent the problem above, the continuous command filter (3) is used in this work.

Appendix F One example

To illustrate the validity of the proposed algorithms, consider an unmanned surface vessel (USV) system with six vessels and the dynamics of each vessel is [5, 40]

$$J_i\ddot{\Theta}_i + \dot{\Theta}_i + \omega_i\dot{\Theta}_i^3 = G_i(\tau_i + \delta_i), \quad \text{each } i \in \mathbb{N}, \tag{F1}$$

where Θ_i and τ_i are respectively the actual course angle and ruder angle, J_i, ω_i, G_i, and δ_i are the time constant, Norrbin coefficient, gain constant and environmental disturbances, respectively. Similar to [5], the asymptotic distributed optimization problem (2) is to design τ_i such that all vessels' course angles Θ_i asymptotically converge to Θ_*, where Θ_* is the minimizer of the following optimization problem:

$$\min_{\text{over all } \Theta_i = \Theta} \sum_{i=1}^{6} \left(\Theta_i - \Theta_i^d\right)^2, \tag{F2}$$

where $\Theta_i^d = \frac{i+1}{18}$. Let $x_i = \Theta_i$ and $v_i = \dot{\Theta}_i$, then system (F1) is equivalent to the MAS (1) with $u_i = \tau_i$, $\theta_i = \frac{G_i}{J_i}$, $h_i = -\frac{1}{J_i}(v_i + \omega_i v_i^3)$ and $d_i = \frac{G_i}{J_i}\delta_i$. The unique minimizer of (F2) is $\Theta_* = x_* = \frac{1}{4}$. Assume that the topology among the six vessels is pictured in Fig. F1

In this numerical example, we consider the following two protocols: (**P1**) the singularity-free asymptotic distributed optimization protocol in Theorem 1; and (**P2**) the singularity-free asymptotic distributed optimization protocol in Theorem 2. For the protocol (**P1**), we choose $G_i = \frac{15}{i}$, $J_i = \frac{75}{i}$, $\omega_i = \frac{2}{3}$, $\delta_i = \frac{8-i}{5}\sin(t)$, $\pi_i = 250$, $\zeta_i = \frac{1}{2}$, $l_i = \frac{1}{100}$, $k_i = 7$, $\gamma_i = 1$, $\gamma_{i1} = i * 10^{-5}$, $\gamma_{i2} = 1$, $\sigma_{i1} = 20$, $\sigma_{i2} = 80$, $\delta_{i1} = 0.5$, $\delta_{i2} = 0.1$, and the initial values are given as $[x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0)] = [4, 2, 1, -4, -1, -2]$, $v_i(0) = 3 - i$, $z_{i,1}(0) = v_i^*(0) = \frac{i-6}{4}$, $z_{i,2}(0) = \mu_i(0) = \frac{i}{2}$, $\alpha_i(0) = \tilde{\alpha}_i(0) + \frac{1}{10} = \frac{i}{5}$ and $\beta_i(0) = \tilde{\beta}_i(0) + \frac{1}{5} = \frac{i}{3}$, for each $i \in \mathbb{N}$. Note that Assumption 2 is satisfied with $\varphi_i = v_i + v_i^3$, $\rho_i = 0$, $\varrho_i = \frac{i}{75}$ and $\varsigma_i = \frac{8-i}{25}$, and Assumption 3 is satisfied with $\zeta = 1$, $\vartheta_i = \frac{i+1}{18}$ and $\phi_i = -1$. For the protocol (**P2**), we assume that $\pi_i = 31$, $\gamma_{i1} = 10^{-4}$, $\gamma_{i2} = 1$, $\delta_{i1} = 2$, $\delta_{i2} = 0.1$, $\sigma_{i1} = \sigma_{i2} = 6$, $p_i(0) = 2\alpha_i(0)$, $q_i(0) = o_i(0) = \frac{1}{2}$, $\delta_{i3} = 1$, for each $i \in \mathbb{N}$, and other parameters are the same as above. The trajectories of $x_i, u_i, \alpha_i, \tilde{\alpha}_i, \beta_i, \tilde{\beta}_i$ using the protocol (**P1**) are given in Fig. F2, where the consensus is evidently reached at the optimal trajectory $x_* = \frac{1}{4}$, each control signal u_i is bounded, $\lim_{t\to\infty} \alpha_i(t) = \lim_{t\to\infty} \tilde{\alpha}_i(t) = \alpha_i^*$ and $\lim_{t\to\infty} \beta_i = \lim_{t\to\infty} \tilde{\beta}_i = \beta_i^*$. The trajectories of $x_i, u_i, \alpha_i, \beta_i$ using the protocol (**P2**) are given in Fig. F3, where the consensus is evidently reached at the optimal trajectory $x_* = \frac{1}{4}$, each control signal u_i is bounded and continuous everywhere, both α_i and β_i converge to some finite steady-state values asymptotically, for each $i \in \mathbb{N}$. Therefore, these simulation results verify the validity of the designed algorithms (**P1**) and (**P2**), and validate the correctness of the theoretical results in Theorems 1 and 2.

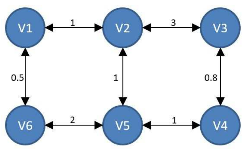

Figure F1 Communication topology $\mathcal{G}$ among six vessels.

References

1 Yu W, Li C, Yu X, et al. Economic power dispatch in smart grids: a framework for distributed optimization and consensus dynamics. Sci China Inf Sci, 2018, 61: 1–16
2 Mao S, Dong Z, Schultz P, et al. A finite-time distributed optimization algorithm for economic dispatch in smart grids. IEEE Trans Syst Man Cybern Syst, 2021, 51: 2068–2079
3 Ram S S, Nedic A, Veeravalli V V. A new class of distributed optimization algorithms: Application to regression of distributed data. Optimiz Methods Software, 2012, 27: 71–88
4 Tang Y, Deng Z, Hong Y. Optimal output consensus of high-order multiagent systems with embedded technique. IEEE Trans Cybern, 2019, 49: 1768–1779
5 Gong P, Han Q L. Distributed fixed-time optimization for second-order nonlinear multiagent systems: State and output feedback designs. IEEE Trans Autom Control, 2024, 69: 3198–3205
6 Bose S, Low S H, Chandy M, et al. Equivalent relaxations of optimal power flow. IEEE Trans Autom Control, 2015, 60: 729–742
7 Olfati-Saber R, Murray R M. Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans Autom Control, 2004, 49: 1520–1533
8 Li Y, Wang X, Sun J, et al. Data-driven consensus control of fully distributed event-triggered multi-agent systems. Sci China Inf Sci, 2023, 66: 152202
9 Gong P, Han Q L. Fixed-time bipartite consensus tracking of fractional-order multi-agent systems with a dynamic leader. IEEE Trans Circuits Syst II Exp Briefs, 2020, 67: 2054–2058
10 Duchi J C, Agarwal A, Wainwright M J. Dual averaging for distributed optimization: Convergence analysis and network scaling. IEEE Trans Autom Control, 2012, 57: 592–606
11 Liu S, Qiu Z, Xie L. Convergence rate analysis of distributed optimization with projected subgradient algorithm. Automatica, 2017, 83: 162–169
12 Alaviani S S, Elia N. Distributed multi-agent convex optimizationover random digraphs. IEEE Trans Autom Control, 2020, 65: 986–998
13 Droge G, Kawashima H, Egerstedt M B. Continuous-time proportional-integral distributed optimisation for networked systems. J Control Decis, 2014, 1: 191–213

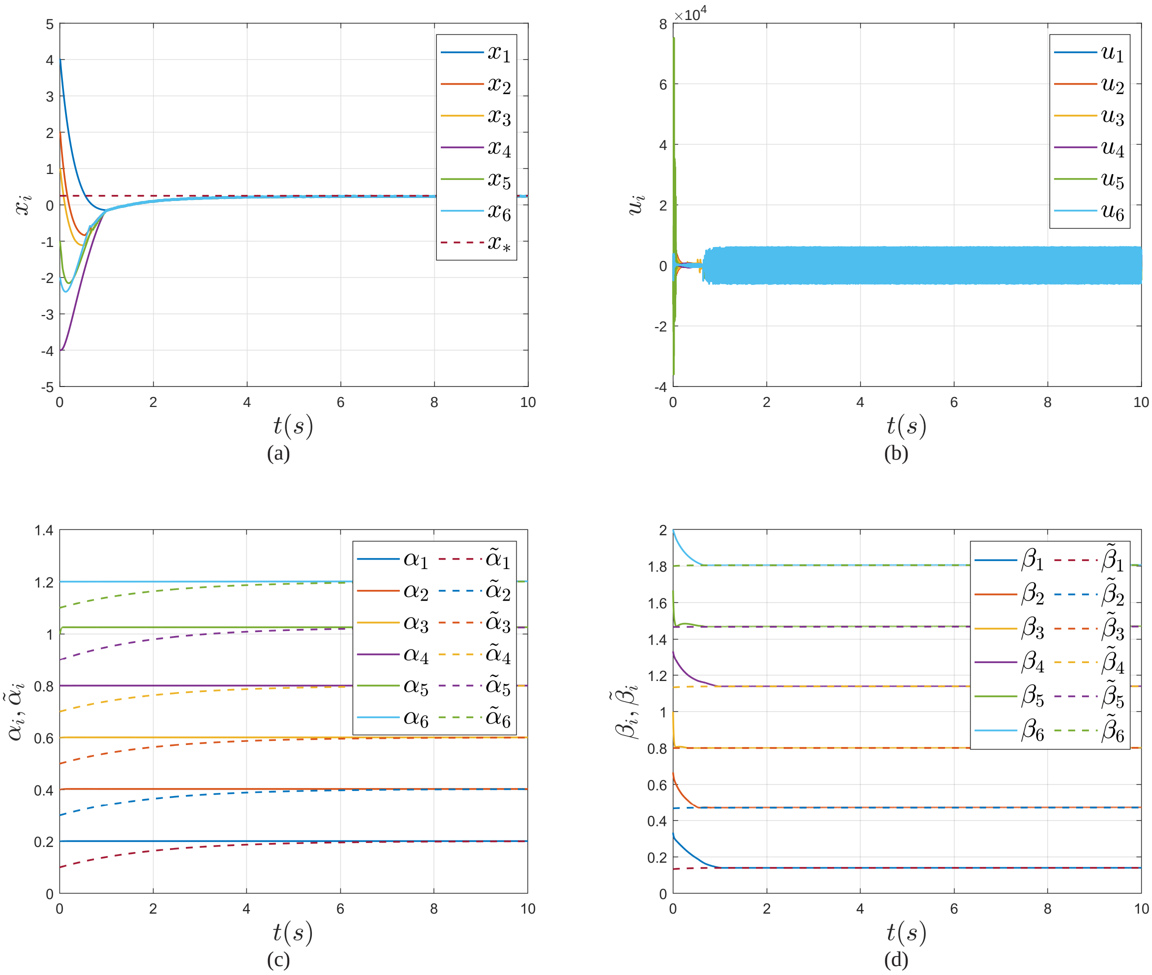

Figure F2 Trajectories of $x_i, u_i, \alpha_i, \tilde{\alpha}_i, \beta_i, \tilde{\beta}_i$ using (**P1**). (a) x_i. (b) u_i. (c) α_i, $\tilde{\alpha}_i$. (d) β_i, $\tilde{\beta}_i$.

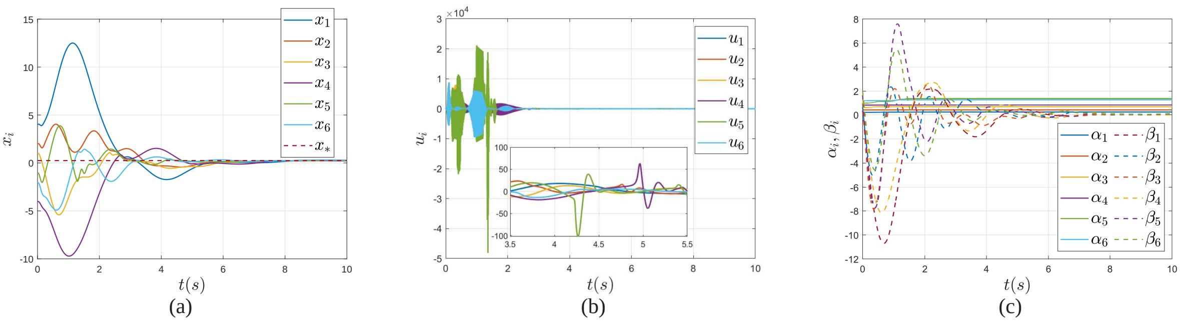

Figure F3 Trajectories of $x_i, u_i, \alpha_i, \beta_i$ using (**P2**). (a) x_i. (b) u_i. (c) α_i, β_i.

14 Gharesifard B, Cortés J. Distributed continuous-time convex optimization on weight-balanced digraphs. IEEE Trans Autom Control, 2014, 59: 781–786

15 Ning B, Han Q L, Zuo Z. Distributed optimization for multiagent systems: An edge-based fixed-time consensus approach. IEEE Trans Cybern, 2019, 49: 122–132

16 Li Z, Ding Z, Sun J, et al. Distributed adaptive convex optimizationon directed graphs via continuous-time algorithms. IEEE Trans Autom Control, 2018, 63: 1434–1441

17 Liu Q, Wang J. A second-order multi-agent network for bound-constrained distributed optimization. IEEE Trans Autom Control, 2015, 60: 3310–3315

18 Rahili S, Ren W, Lin P. Distributed convex optimization of time-varying cost functions for double-integrator systems using nonsmooth algorithms. In: Proceedings of American Control Conference, Chicago, 2015. 68–73

19 Rahili S, Ren W. Distributed continuous-time convex optimization with time-varying cost functions. IEEE Trans Autom Control, 2017, 62: 1590–1605

20 Zhao Y, Liu Y, Wen G, et al. Distributed optimization of linear multiagent systems: Edge- and node-based adaptive designs. IEEE Trans Autom Control, 2017, 62: 3602–3609
21 Wang X, Hong Y, Yi P, et al. Distributed optimization design of continuous-time multiagent systems with unknown-frequency disturbances. IEEE Trans Cybern, 2017, 47: 2058–2066
22 Tran N T, Xiao J, Wang Y, et al. Distributed optimisation problem with communication delay and external disturbance. Int J Syst Sci, 2017, 48: 3530–3541
23 Huang B, Zou Y, Meng Z, et al. Distributed time-varying convex optimization for a class of nonlinear multi-agent systems. IEEE Trans Autom Control, 2020, 65: 801–808
24 Li S L, Nian X H, Deng Z H. Distributed optimization of second-order nonlinear multiagent systems with event-triggered communication. IEEE Trans Control Netw Syst, 2021, 8: 1954–1963
25 Wang X, Zheng W X, Wang G. Distributed finite-time optimization of second-order multiagent systems with unknown velocities and disturbances. IEEE Trans Neural Netw Learn Syst, 2023, 34: 6042–6054
26 Boyd S, Vandenberghe L. Convex Optimization. Cambridge: Cambridge University Press, 2004
27 Sastry S. Nonlinear systems analysis, stability, and control. New York: Springer-Verlag, 1999
28 Wang Y, Song Y, Krstic M, et al. Fault-tolerant finite time consensus for multiple uncertain nonlinear mechanical systems under single-way directed communication interactions and actuation failures. Automatica, 2016, 63: 374–383
29 Gong P, Lan W, Han Q L. Robust adaptive fault-tolerant consensus control for uncertain nonlinear fractional-order multi-agent systems with directed topologies. Automatica, 2020, 117: 109011
30 Lu J, Tang C Y. Zero-gradient-sum algorithms for distributed convex optimization: The continuous-time case. IEEE Trans Autom Control, 2012, 57: 2348–2354
31 Wu Z, Li Z, Yu J. Designing zero-gradient-sum protocols for finite-time distributed optimization problem. IEEE Trans Syst Man Cybern Syst, 2022, 52: 4569–4577
32 Guo G, Zhang R, Zhou Z D. A local-minimization-free zero-gradient-sum algorithm for distributed optimization. Automatica, 2023, 157: 111247
33 Farrell J A, Polycarpou M, Sharma M, et al. Command filtered backstepping. IEEE Trans Autom Control, 2009, 54: 1391–1395
34 Dong W, Farrell J A, Polycarpou M M, et al. Command filtered adaptive backstepping. IEEE Trans Control Syst Technol, 2012, 20: 566–580
35 Filippov A. Differential equations with discontinuous right-hand side. Norwell: Kluwer, 1988
36 Mei J, Ren W, Chen J. Distributed consensus of second-order multi-agent systems with heterogeneous unknown inertias and control gains under a directed graph. IEEE Trans Autom Control, 2016, 61: 2019–2034
37 Young K D, Utkin V I, Ozguner U. A control engineer's guide to sliding mode control. IEEE Trans Control Syst Technol, 1999, 7: 328–342
38 Zhao L, Yu J, Lin C, et al. Adaptive neural consensus tracking for nonlinear multiagent systems using finite-time command filtered backstepping. IEEE Trans Syst Man Cybern Syst, 2018, 48: 2003–2012
39 Levant A. Higher-order sliding modes, differentiation and output-feedback control. Int J Control, 2003, 76: 924–941
40 Hu X, Wei X, Han J, et al. Adaptive disturbance rejection for course tracking of marine vessels under actuator constraint. ISA Trans, 2020, 100: 82–91