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Data-driven output regulation control for constrained linear systems

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Abstract This study introduces a data-driven approach for state and output feedback control addressing the constrained output regulation problem in unknown linear discrete-time systems. Our method ensures effective tracking performance while satisfying the state and input constraints, even when system matrices are not available. We first establish a sufficient condition necessary for the existence of a solution pair to the regulator equation and propose a data-based approach to obtain the feedforward and feedback control gains for state feedback control using linear programming. Furthermore, we design a refined Luenberger observer to accurately estimate the system state, while keeping the estimation error within a predefined set. By combining output regulation theory, we develop an output feedback control strategy. The stability of the closed-loop system is rigorously proved to be asymptotically stable by further leveraging the concept of λ-contractive sets.

Keywords output regulation, constrained system, data-driven

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1 Introduction

The output regulation problem integrates dynamic trajectory tracking, disturbance rejection, and feedback control within into a unified framework. The objective is to devise a feedback control to achieve asymptotic tracking, mitigate disturbances, and ensure the stability of the closed-loop system. The issue has been extensively investigated in the literature in continuous-time and discrete-time systems, as documented in [1, 2].

In practical applications, owing to the structural limitations and the intricacies of the environment, the output regulation problem must also consider state and input constraints, leading to what is known as the constrained output regulation problem, initially proposed in [3]. The mainstream methods used to address this encompass model predictive control, barrier Lyapunov function-based approaches, and controlled-invariant set methods. Ref. [4] presented a model predictive control to address the nonlinear output regulation problem. This approach considered general nonlinear constraints on both the plant state and control inputs. Building on this framework, Ref. [5] investigated the output regulation problem in the presence of state and input constraints by proposing a measurement output feedback predictive algorithm. A robust controller using barrier Lyapunov functions and integral sliding-mode control, has been developed by [6] to solve the output regulation problem for uncertain linear systems with input saturation and state constraints. Furthermore, Ref. [7] has also utilized controlled-invariant sets to design dynamic output feedback controllers for constrained linear discrete-time systems.

The aforementioned approaches rely on models, necessitating an accurate model of system dynamics, including the explicit solution to the output regulation equation. This requires a comprehensive knowledge of system dynamics. In scenarios where systems are unknown, model-free approaches have been

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proposed, ranging from reinforcement learning [8–10] to adaptive dynamic programming [11–13]. Ref. [9] explored the optimal output regulation problem in unknown linear discrete-time systems by combining reinforcement learning with robust output regulation theory. Ref. [10] studied the optimal output synchronization control problem in heterogeneous multi-agent systems with nonidentical communication delays using a reinforcement learning method. Through adaptive dynamic programming, Ref. [11] proposed a non-model-based scheme to solve adaptive optimal output regulation problem in linear systems with unmeasurable disturbance. Furthermore, Ref. [13] combined the internal model principle and adaptive dynamic programming to handle cooperative optimal output regulation in linear multi-agent systems.

Unlike the iterative nature of reinforcement learning and adaptive dynamic programming, data-driven techniques offer a way to derive control directly from the input and state/output data, which can significantly reduce the computational burden. Data-driven control methods have also been applied to various contexts, including tracking problems in [14, 15], state-constrained problems in [16], and safe control problems in [17–19]. Ref. [14] designed a regulator using data informativity, to enable the output of the unknown system to track a reference signal. Ref. [15] proposed both state feedback and dynamic feedback control schemes using state and input data for output regulation problems. A linear state feedback control was designed from noisy input-state data in [16] to ensure robust invariance of a predefined polyhedral state set. Ref. [17] proposed data-based safe controllers for stochastic uncertain linear discrete-time systems with aleatory uncertainties, ensuring a polyhedral set remained λ-contractive probabilistically. Ref. [18] tackled polynomial systems by formulating a data-driven optimization program to maintain the robust invariance of the safe set. It remains theoretically intriguing and practically crucial to continue developing data-driven approaches for output regulation problems that involve both state and input constraints.

This paper introduces a novel approach to state and output feedback control for the constrained linear output regulation problem by combining the principles of λ-contractivity, data-driven control technique, and linear programming. The main contributions of our design lie in three aspects. We first provide a sufficient condition for the existence of solution pairs in the data-driven regulator equation, even when system matrices are unknown, and propose a data-based approach to obtain this solution. This approach fundamentally differs from traditional output regulation problems [20, 21]. Secondly, we utilize linear programming to develop a state feedback control that simultaneously tackles the challenges of state and input constraints along with unknown system dynamics. This is a noteworthy advancement, as existing methods for constrained control [5, 6], model-free output regulation [14, 15, 22], and safe control problems [17–19] do not apply here. Finally, unlike prior state feedback controllers in [17–19] for stabilization with state constraints, we provide a data-driven output feedback control using a refined observer. This observer is required not only to accurately estimate the system state but also to keep the estimation error within a predefined set, which is more demanding than previous methods introduced in [5, 23, 24].

This paper is organized as follows. Section 2 outlines the problem formulation. Sections 3 and 4 detail the state and output feedback control laws, respectively. Section 5 provides an example, and the paper concludes in Section 6.

Notations. Given a set Ω and a scalar $\lambda \geqslant 0$, $\lambda\Omega = \{\lambda x : x \in \Omega\}$. $\mathbb{N}$ denotes the set of natural numbers. For a vector $\zeta = [\zeta_1 \ \zeta_2 \ \cdots \ \zeta_n]^{\mathrm{T}}$ with $\zeta_i \in \mathbb{R}$, $i = 1, \ldots, n$, $|\zeta| = [|\zeta_1| \ |\zeta_2| \ \cdots \ |\zeta_n|]^{\mathrm{T}}$. An identity matrix of appropriate dimension is denoted by I and a matrix of all zeros of appropriate dimensions is denoted by $\mathbf{0}$. For a matrix $A \in \mathbb{R}^{n\times m}$, $\mathrm{vec}(A) = [a_1^{\mathrm{T}} \ a_2^{\mathrm{T}} \ \cdots \ a_m^{\mathrm{T}}]^{\mathrm{T}}$ with $a_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, being the i-column of matrix A. Denote A_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, m$, as the element of the i-th row and j-th column of matrix A and let B_{ij} be the element of the i-th row and j-th column of matrix $B \in \mathbb{R}^{n\times m}$. $A < (\leqslant, >, \geqslant)B$ has to be intended componentwise $A_{ij} < (\leqslant, >, \geqslant)B_{ij}$ for all i and j [25].

2 Problem formulation

We consider the following linear discrete-time system:

$$x(k+1) = Ax(k) + Bu(k) + Ev(k), \tag{1a}$$

$$e(k) = Cx(k) + Du(k) + Fv(k), \tag{1b}$$

where $k \in \mathbb{N}$, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ denotes the input, $e \in \mathbb{R}^p$ represents the regulated output, and $v \in \mathbb{R}^q$ indicates the reference input and/or the external disturbances, generated by a so-called exosystem [1, 2],

$$v(k+1) = Sv(k), \tag{2}$$

with $S \in \mathbb{R}^{q \times q}$ being a constant matrix.

In practical applications, the velocity workspace of multi-robot cooperative system and the acceleration limits are both constrained by a polytope set. Considering the physical constraints on the state and input of robots, unmanned vehicles, flight vehicles, and similar systems, we examine system (1) under these state and input constraints, which are defined by following polytope sets:

$$\begin{aligned} \Omega_1 &= \{x \in \mathbb{R}^n : Hx \leqslant \zeta_1\}, \\ \Omega_2 &= \{u \in \mathbb{R}^m : Gu \leqslant \zeta_2\}, \end{aligned} \tag{3}$$

where $H \in \mathbb{R}^{n_x \times n}$, $G \in \mathbb{R}^{n_u \times m}$, $\zeta_1 \in \mathbb{R}^{n_x}$, and $\zeta_2 \in \mathbb{R}^{n_u}$. It is worth noting that sets Ω_1 and Ω_2 are polytope sets [25, Definition 3.21], meaning that they are compact [5]. The polytope set (3) can accommodate symmetric and asymmetric constraints. According to [26], the shapes from (3) can be more flexible than ellipsoids, offering improved dynamics in the domains of attraction for dynamic systems.

There are some standard assumptions in the output regulation theory [1] as referenced in [22, 23, 27].

Assumption 1. The exosystem (2) is marginally stable.

Assumption 2. The pair (A, B) is controllable, and (C, A) is observable.

Assumption 3. The following linear matrix equations:

$$\begin{aligned} MS &= AM + BN + E, \\ \mathbf{0} &= CM + DN + F \end{aligned} \tag{4}$$

admit a solution pair (M, N).

Remark 1. Assumption 1 pertains to the exosystem (2) and requires that the reference signal remains bounded, as in [15, 23]. This assumption allows the exosystem to generate a broad range of reference trajectories, including step functions, sinusoidal functions with various magnitudes, frequencies, and initial phases, as well as their finite combinations. Assumption 2 is a common prerequisite in controlling linear systems. Assumption 3 is necessary for designing both state feedback and output feedback controllers, as outlined in output regulation theory in [1, 22, 23].

According to [1], given Assumptions 1–3 and when system matrices are known, the unconstrained output regulation problem without constraints (3) can be solved using the state feedback law detailed below:

$$u(k) = Kx(k) + Lv(k), \tag{5}$$

where K is chosen such that $A + BK$ is Schur and $L = N - KM$. The output feedback control law takes the following form:

$$u(k) = K_1\xi(k) + K_2v(k), \tag{6a}$$

$$\xi(k+1) = A\xi(k) + Bu(k) + Ev(k) + L_\xi\big(C\xi(k) + Du(k) + Fv(k) - e(k)\big), \tag{6b}$$

where K_1 and L_ξ are designed such that $A + BK_1$ and $A + L_\xi C$ are Schur, and $K_2 = N - K_1M$.

In this study, system matrices A, B, C, and D are unknown, and there are state and input constraints (3). Therefore, designing the control gains K and L in (5), and K_1, K_2, L_ξ in (6) using traditional output regulation theory is not feasible. Our goal is to introduce a data-based approach to address the linear constrained output regulation problem, and such a problem is described as follows.

Problem 1. Given the linear discrete-time system (1) with unknown system matrices and subject to the state and input constraints (3), design two classes of control laws in the forms of (5) and (6) based on a finite amount of available data such that, for all $(x(0), u(0)) \in \Omega_1 \times \Omega_2$, the trajectory of the closed-loop system satisfies

$$x(k) \in \Omega_1,\ u(k) \in \Omega_2,\ \forall k \in \mathbb{N}, \tag{7a}$$

$$\lim_{k \to +\infty} e(k) = 0. \tag{7b}$$

Remark 2. Problem 1 differs from the output regulation problems in [23, 24], because it deals with unknown system matrices and incorporates state and input constraints (3), known as polytope boundaries. These constraints naturally represent physical constraints found in various practical scenarios. State-of-the-art control methods for tackling constrained output regulation problems predominantly rely on model prediction algorithms [5] or barrier functions [6]. However, these approaches heavily depend on comprehensive knowledge of system matrices, rendering them unsuitable for model-free scenarios. We provide a data-based control framework that addresses these challenges by solving the regulator equation without prior knowledge of system dynamics, designing control laws to satisfy both state and input constraints, and stabilizing the closed-loop system by defining and analyzing appropriate contractive sets.

For the solvability of our problem, we also need the following assumption for the exosystem (2).

Assumption 4. The state $v(k)$ of the exosystem (2) satisfies $(Mv(k), Nv(k)) \in \bar{\Omega}_1 \times \bar{\Omega}_2$ where

$$\bar{\Omega}_1 = \{x \in \mathbb{R}^n : Hx \leqslant \zeta_1 - \Theta_1\}, \tag{8a}$$

$$\bar{\Omega}_2 = \{u \in \mathbb{R}^m : Gu \leqslant \zeta_2 - \Theta_2\}, \tag{8b}$$

with $\Theta_1 \geqslant |H(x(0) - Mv(0))|$ and $\Theta_2 \geqslant |GNv(0)|$.

Remark 3. Physically speaking, Assumption 4 requires that the trajectories of $Mv(k)$ and $Nv(k)$ are distant from the boundaries of Ω_1 and Ω_2. To achieve the tracking objective in Problem 1, the tracking error, containing $v(k)$ in (1b), is required to converge to zero. If Assumption 4 is violated, there could be a conflict between the goal of tracking and the state constraints specified in (3). In such a case, it is unlikely to achieve both objectives outlined in (7a) and (7b). Similar assumptions are necessary for constrained output regulation problems [5, 28].

3 State feedback control design

In this section, we propose a state feedback control law to solve Problem 1 despite the unknown system matrices and the presence of state and input constraints (3). This solution combines a data-driven technique with output regulation theory.

3.1 System data collection

Owing to the lack of a precise system description, we propose an approach for designing the control (5) using the available data.

To implement this, we perform an open-loop experiment. First, we apply T input data points $u(0)$, $u(1)$, …, $u(T-1)$ and exosystem state data points $v(0)$, $v(1)$, …, $v(T-1)$ into the system (1). Second, we collect regulated output data $e(0)$, $e(1)$, …, $e(T-1)$ and state data $x(0)$, $x(1)$, …, $x(T)$. The collected system data is stored in the following matrices:

$$U = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}, \tag{9a}$$

$$X = \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix}, \tag{9b}$$

$$V = \begin{bmatrix} v(0) & v(1) & \cdots & v(T-1) \end{bmatrix}, \tag{9c}$$

$$\mathcal{E} = \begin{bmatrix} e(0) & e(1) & \cdots & e(T-1) \end{bmatrix}, \tag{9d}$$

and

$$\tilde{U} = \begin{bmatrix} \tilde{u}(0) & \tilde{u}(1) & \cdots & \tilde{u}(T-1) \end{bmatrix}, \tag{10a}$$

$$X_1 = \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}, \tag{10b}$$

$$X_2 = \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}, \tag{10c}$$

with $\tilde{u} = [u^{\mathrm{T}}\ v^{\mathrm{T}}]^{\mathrm{T}}$. Since the data in (9) and (10) is generated from (1), the following equations:

$$X_2 = AX_1 + BU + EV, \tag{11a}$$

$$\mathcal{E} = CX_1 + DU + FV \tag{11b}$$

hold. We also need to make the following assumption for the collected data (9) as in [27, 29].

Assumption 5. $\tilde{U}$ is persistently exciting of order $n+1$.

Remark 4. Assumption 5 is the necessary condition for the collected data (9). Under Assumption 5, the matrix $[\,X_1^{\mathrm{T}}\ U^{\mathrm{T}}\ V^{\mathrm{T}}\,]^{\mathrm{T}}$ has full row rank. Thus, the right inverse of matrix $\bar{X}_1$, defined in (22b) in what follows, exists, which is essential for designing control parameter K in (5). It is also necessary to ensure the existence of the solution pair (M, N) in (12) for the data-based regulator equation (13) as outlined in Lemma 1. Assumption 5 is consistently valid if system states (1) and (2) are accessible.

3.2 Data-based solution to the regulator equation

In the framework of output regulation theory [1, 2], a necessary step in designing (5) involves solving the regulator equation (4). However, without knowledge of system matrices (A, B, C, D), the solution pair (M, N) cannot be directly calculated from (4). As a result, we provide a data-driven approach for solving (4), as detailed in Lemma 1.

Lemma 1. Under Assumptions 1–3 and 5, the solution pair (M, N) can be obtained from

$$M = X_1 Y, \quad N = UY, \tag{12}$$

where X_1 and U are given by (10) and (9), respectively, and $Y \in \mathbb{R}^{T\times q}$ satisfies the data-based regulator equation

$$\begin{aligned} X_1 Y S &= (X_2 - EV)Y + E, \\ \mathbf{0} &= (\mathcal{E} - FV)Y + F. \end{aligned} \tag{13}$$

Proof. First, we show that there exists a solution to (13). It should be noted that Eq. (13) is equivalent to

$$\begin{bmatrix} X_1 \\ \mathbf{0} \end{bmatrix} YS - \begin{bmatrix} X_2 - EV \\ \mathcal{E} - FV \end{bmatrix} Y = \begin{bmatrix} E \\ F \end{bmatrix}. \tag{14}$$

By vectorizing both sides of the equation and leveraging the characteristics of the Kronecker product, Eq. (14) can be converted into the following conventional algebraic form:

$$\mathcal{Q}\mathcal{X} = b, \tag{15}$$

where

$$\mathcal{Q} = S^T \otimes \begin{bmatrix} X_1 \\ \mathbf{0} \end{bmatrix} - I \otimes \begin{bmatrix} X_2 - EV \\ \mathcal{E} - FV \end{bmatrix}, \ \mathcal{X} = \mathrm{vec}(Y), \quad b = \mathrm{vec}\left(\begin{bmatrix} E \\ F \end{bmatrix}\right). \tag{16}$$

Then, Eq. (14) admits a solution if and only if $\mathcal{Q}$ has a full row rank. It should be noted that matrix S can be diagonalized into $\mathrm{diag}\{J_1, \ldots, J_l\}$ for some integer $l \geqslant 1$, and $J_i \in \mathbb{R}^{n_i \times n_i}$ with $n_1 + n_2 + \cdots + n_l = q$ taking the following form:

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}, \ i = 1, \ldots, l$$

for λ_i being the eigenvalue of S. Then from (16), $\mathcal{Q}$ is a lower triangular block matrix with l blocks, and its i-th diagonal block has the following structure:

$$\begin{bmatrix} \lambda_i\mathcal{L} - \bar{\mathcal{A}} & \mathbf{0} & \mathbf{0} \cdots & \mathbf{0} & \mathbf{0} \\ \mathcal{L} & \lambda_i\mathcal{L} - \bar{\mathcal{A}} & \mathbf{0} \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots\ \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \cdots & \lambda_i\mathcal{L} - \bar{\mathcal{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \cdots & \mathcal{L} & \lambda_i\mathcal{L} - \bar{\mathcal{A}} \end{bmatrix},$$

where

$$\mathcal{L} = \begin{bmatrix} X_1 \\ \mathbf{0} \end{bmatrix}, \quad \bar{\mathcal{A}} = \begin{bmatrix} X_2 - EV \\ \mathcal{E} - FV \end{bmatrix}.$$

Therefore, $\mathcal{Q}$ has full row rank if and only if $\lambda_i \mathcal{L} - \bar{\mathcal{A}}$ has full row rank. From (11), we obtain

$$-(\lambda_i \mathcal{L} - \bar{\mathcal{A}}) = \begin{bmatrix} X_2 - EV - \lambda_i X_1 \\ \mathcal{E} - FV \end{bmatrix} = \begin{bmatrix} AX_1 + BU - \lambda_i X_1 \\ CX_1 + DU \end{bmatrix} = \begin{bmatrix} A - \lambda_i I & B \\ C & D \end{bmatrix} \begin{bmatrix} X_1 \\ U \end{bmatrix}. \tag{17}$$

Under Assumption 2, the pair (A, B) is controllable. Hence, the pair $(A, [B, E])$ is controllable. Note that system (1a) can be put into the following form:

$$x(k+1) = Ax(k) + Bu(k) + Ev(k) = Ax(k) + \begin{bmatrix} B & E \end{bmatrix} \begin{bmatrix} u(k) \\ v(k) \end{bmatrix}.$$

Under Assumption 5, the input sequence $\tilde{u}(0), \ \ldots \ , \tilde{u}(T-1)$ is persistently exciting of order $n+1$. Then, according to Corollary 2 in [30], matrix $[\, X_1^{\mathrm{T}} \ U^{\mathrm{T}} \ V^{\mathrm{T}} \,]^{\mathrm{T}}$ has a full row rank. Thus,

$$\operatorname{rank} \begin{bmatrix} X_1 \\ U \end{bmatrix} = n + m. \tag{18}$$

Under Assumption 3, $\left[\begin{smallmatrix} A - \lambda_i I & B \\ C & D \end{smallmatrix}\right]$ has a full row rank. Together with (18), we obtain that $\lambda_i \mathcal{L} - \bar{\mathcal{A}}$ has a full row rank. Therefore, Eq. (13) admits a solution.

Second, from (11), we derive

$$\begin{aligned} X_2 - EV &= AX_1 + BU, \\ \mathcal{E} - FV &= CX_1 + DU. \end{aligned} \tag{19}$$

Substituting (19) into (13) yields

$$\begin{aligned} X_1 Y S &= AX_1 Y + BUY + E, \\ \mathbf{0} &= CX_1 Y + DUY + F. \end{aligned} \tag{20}$$

Let $M = X_1 Y$ and $N = UY$. Then from (20), the pair $(X_1 Y, UY)$ is a solution to the regulator equation (4).

Remark 5. Lemma 1 introduces a data-driven method for solving the regulator equation (4), featuring two distinctive features. First, the proof within Lemma 1 demonstrates the existence of a solution to the data-based regulator equation (13) under Assumptions 3 and 5, rather than assuming its existence as in [15]. Second, the method in Lemma 1 is designed to handle scenarios where all the system matrices (A, B, C, D) are unknown, whereas existing methods, like those in [14], typically manage cases where only (A, B) are unknown.

Remark 6. Our method can be extended to deal with the unknown matrix S, with the strategy being to infer S from the collected data. In practice, matrix S can be deduced from

$$\operatorname{vec}(S^{\mathrm{T}}) = (I \otimes VV^{\mathrm{T}})^{-1} \sum_{i=0}^{T-1} \left[(I \otimes v(i)) v(i+1) \right], \tag{21}$$

where $v(i)$ is the state of exosystem (2) and V is defined in (9).

3.3 Design of control parameters K and L

Using the collected data from (10) and the solution pair (M, N) from (12), we design the control gains K and L as outlined in (5). For this purpose, we first process the data in (10) as follows:

$$\bar{U}_1 = \left[u(0) - Nv(0) \ \cdots \ u(T-1) - Nv(T-1) \right], \tag{22a}$$

$$\bar{X}_1 = \left[x(0) - Mv(0) \ \cdots \ x(T-1) - Mv(T-1) \right], \tag{22b}$$

Algorithm 1 Data-driven state feedback control (5).

Input: Data X, V and U in (9).

Output: The state feedback control (5).

1: Process data X according to (10b) and (10c) to derive data X_1 and X_2;
2: Determine Y from (13), and calculate matrices M, N from (12);
3: Process data X, V and U according to (22a)–(22c) to obtain data $\bar{U}_1$, $\bar{X}_1$ and $\bar{X}_2$;
4: Select a constant $\lambda \in [0, 1)$, obtain $\bar{x}_i$ according to (27), select matrices H, G and vectors Θ_1, Θ_2 from (8), determine $\bar{X}_1^\dagger$ and matrix $P \geqslant \mathbf{0}$ from

$$PH = H\bar{X}_2\bar{X}_1^\dagger, \tag{28a}$$

$$P\Theta_1 \leqslant \lambda\Theta_1, \tag{28b}$$

$$G\bar{U}_1\bar{X}_1^\dagger\bar{x}_i \leqslant \Theta_2, \quad i = 1, \ldots, \alpha; \tag{28c}$$

5: Select $K = \bar{U}_1\bar{X}_1^\dagger$, $L = N - KM$, and determine the state feedback control (5).

$$\bar{X}_2 = \left[x(1) - Mv(1) \ \cdots \ x(T) - Mv(T)\right]. \tag{22c}$$

Then, K and L in (5) can be designed as

$$K = \bar{U}_1\bar{X}_1^\dagger, \tag{23a}$$

$$L = N - KM, \tag{23b}$$

where $\bar{X}_1^\dagger$ is a right inverse of matrix $\bar{X}_1$.

Then, the closed-loop system is composed of (1), (2), and (5) with K and L provided by (23). Performing the following state and input transformation:

$$\bar{x}(k) = x(k) - Mv(k), \quad \bar{u}(k) = u(k) - Nv(k), \tag{24}$$

yields the following error dynamics:

$$\begin{aligned} \bar{x}(k+1) &= A\bar{x}(k) + B\bar{u}(k), \\ e(k) &= C\bar{x}(k) + D\bar{u}(k). \end{aligned} \tag{25}$$

Based on (25), we define a polytope set

$$\Omega_3 = \{\bar{x} \in \mathbb{R}^n : H\bar{x} \leqslant \Theta_1\}. \tag{26}$$

Since $\Theta_1 \geqslant 0$, $H\mathbf{0} \leqslant \Theta_1$, implying the origin is an interior point of set Ω_3. Note that Ω_3 is a polytope set, and thus, it is convex and compact. By Definition A1, Ω_3 is C-set. Therefore, Ω_3 is inherently bounded, and according to [25, pp. 107–108], the convex polytope set Ω_3 admits a vertex representation of the following form:

$$\Omega_3 = \left\{\bar{x} = \sum_{i=1}^{\alpha}\beta_i\bar{x}_i : \mathbf{1}^{\mathrm{T}}\beta = 1, \ \beta \geqslant 0\right\}, \tag{27}$$

where $\bar{x}_1, \cdots, \bar{x}_\alpha$ are the vertices of set Ω_3.

Based on the collected data X, V and U in (9), we provide an algorithm for the design of control parameters K and L in the state feedback control (5).

Then, we demonstrate that the state feedback control (5) with parameters K, L provided by (23) can solve Problem 1.

Theorem 1. If Assumptions 1–4 and 5 hold, the linear constrained output regulation problem of the discrete-time systems (1) and (2) is solvable by the data-driven state feedback control law (5) where parameters K, L are designed by following Algorithm 1.

Proof: First, prove that the right inverse of matrix $\bar{X}_1$ exists. From (10b) and (22b),

$$\bar{X}_1 = X_1 - MV = \left[I \ M\right]\begin{bmatrix} X_1 \\ V \end{bmatrix}. \tag{29}$$

Under Assumption 5, $[\, X_1^{\mathrm{T}} \; U^{\mathrm{T}} \; V^{\mathrm{T}} \,]^{\mathrm{T}}$ has full row rank, so does matrix $[\, X_1^{\mathrm{T}} \; V^{\mathrm{T}} \,]^{\mathrm{T}}$. Therefore, matrix $\bar{X}_1$ has full row rank, and thus, the right inverse of $\bar{X}_1$ exists.

The second step involves demonstrating that Ω_3 in (26) is a λ-contractive set of the error dynamics (25). According to Definition A2, we need to prove that for each $\bar{x}(k) \in \Omega_3$, $k \in \mathbb{N}$, the following condition holds:

$$\inf \left\{ \bar{\lambda} \geqslant 0 : A\bar{x}(k) + B\bar{u}(k) \in \bar{\lambda}\Omega_3 \right\} \leqslant \lambda. \tag{30}$$

For $\bar{x}(0)$, under Assumption 4,

$$H\bar{x}(0) = Hx(0) - HMv(0) \leqslant |H(x(0) - Mv(0))| \leqslant \Theta_1. \tag{31}$$

From (22) and (25), we have

$$\bar{X}_2 = A\bar{X}_1 + B\bar{U}_1. \tag{32}$$

From (23a), (32) and $I_n = \bar{X}_1\bar{X}_1^{\dagger}$, we obtain

$$A + BK = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} I_n \\ K \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{U}_1 \end{bmatrix} \bar{X}_1^{\dagger} = \bar{X}_2\bar{X}_1^{\dagger}. \tag{33}$$

Combining (5), (23b) with (24), $\bar{u}(k) = K\bar{x}(k)$.

According to (28a), (28b), (31) and (33), we obtain

$$H[A\bar{x}(0) + B\bar{u}(0)] = H[(A + BK)\bar{x}(0)] = H\bar{X}_2\bar{X}_1^{\dagger}\bar{x}(0) = PH\bar{x}(0) \leqslant P\Theta_1 \leqslant \lambda\Theta_1, \tag{34}$$

implying $A\bar{x}(0) + B\bar{u}(0) \in \lambda\Omega_3$. For $\lambda \in [0, 1)$, $\lambda \in \left\{\bar{\lambda} \geqslant 0 : A\bar{x}(0) + B\bar{u}(0) \in \bar{\lambda}\Omega_3\right\}$, and then $\lambda \geqslant \inf\left\{\bar{\lambda} \geqslant 0 : A\bar{x}(0) + B\bar{u}(0) \in \bar{\lambda}\Omega_3\right\}$. Thus, Eq. (30) holds for $k = 0$.

Next, we prove (30) using mathematical induction. Assume that Eq. (30) holds for some $k \in \mathbb{N}$, i.e.,

$$A\bar{x}(k) + B\bar{u}(k) = \bar{x}(k + 1) \in \lambda\Omega_3. \tag{35}$$

Then, we prove (30) holds for $k + 1$. According to (35) and

$$\lambda\Omega_3 = \{\lambda\bar{x} \in \mathbb{R}^n : H\bar{x} \leqslant \Theta_1\} = \{\lambda\bar{x} \in \mathbb{R}^n : H\lambda\bar{x} \leqslant \lambda\Theta_1\},$$

we obtain

$$H\bar{x}(k + 1) \leqslant \lambda\Theta_1. \tag{36}$$

From (28), (33), and (36),

$$H[A\bar{x}(k+1)+B\bar{u}(k+1)] = H[(A+BK)\bar{x}(k+1)] = H\bar{X}_2\bar{X}_1^{\dagger}\bar{x}(k+1) = PH\bar{x}(k+1) \leqslant P\Theta_1 \leqslant \lambda\Theta_1, \tag{37}$$

implying $A\bar{x}(k+1)+B\bar{u}(K+1) \in \lambda\Omega_3$. Then Eq. (30) holds for $k+1$. Therefore, set Ω_3 is a λ-contractive set for the system (25), which implies $H\bar{x}(k) \leqslant \Theta_1$ for all $k \in \mathbb{N}$. According to (24) and Assumption 4,

$$Hx(k) = H\bar{x}(k) + HMv(k) \leqslant \Theta_1 + \zeta_1 - \Theta_1 = \zeta_1.$$

Thus,

$$x(k) \in \Omega_1, \; \forall k \in \mathbb{N}. \tag{38}$$

The third step involves proving $u(k) \in \Omega_2$ for all $k \in \mathbb{N}$. From (27), any point $\bar{x}$ in set Ω_3 can be expressed as a convex combination of the vertices of set Ω_3 with coefficients β_i. Combing (23a), (28c) and $\bar{u}(k) = K\bar{x}(k)$, we obtain, for $\forall k \in \mathbb{N}$,

$$G\bar{u}(k) = GK\bar{x}(k) = GK\sum_{i=1}^{\alpha}\beta_i\bar{x}_i(k) = \sum_{i=1}^{\alpha}\beta_i G\bar{U}_1\bar{X}_1^{\dagger}\bar{x}_i(k) \leqslant \sum_{i=1}^{\alpha}\beta_i\Theta_2 = \Theta_2. \tag{39}$$

From (8b) and (24), we obtain

$$Gu(k) = G\bar{u}(k) + GNv(k) \leqslant \Theta_2 + \zeta_2 - \Theta_2 = \zeta_2, \tag{40}$$

implying

$$u(k) \in \Omega_2,\ \forall k \in \mathbb{N}. \tag{41}$$

Since $\bar{u}(k) = K\bar{x}(k)$, we derive $\bar{x}(k+1) = (A + BK)\bar{x}(k)$. Since Ω_3 is a λ-contractive set for the system (25), by Lemma A1, all the eigenvalues of $A + BK$ have modulus less than or equal to λ, which implies $\lim_{k\to\infty} \bar{x}(k) = 0$. Thus,

$$\lim_{k\to\infty} e(k) = C\bar{x}(k) + D\bar{u}(k) = (C + DK)\bar{x}(k) = 0. \tag{42}$$

From (38), (41) and (42), Problem 1 is solved. □

Remark 7. According to (22) and (23), the design of the state feedback controller (5) is based on system data X, V, and U in (9) since control parameters are determined by (23). State feedback control (5) offers two advantages. First, unlike traditional output regulation problems in [1,20,21], our design can handle cases where system dynamics are unknown and subject to input and state constraints (3). Second, unlike reinforcement learning-based methods in [9,12], our design is easy to implement and computationally efficient since it leverages finite input-state data directly without iterative procedures. This approach, while not optimal, provides a feasible solution to the constrained output regulation problem.

4 Output feedback control design

In this section, we provide the specific form of the output feedback control (6) by constructing a refined Luenberger observer.

4.1 Refined Luenberger observer design

To satisfy the constraints (3), the traditional Luenberger observer (6b) is no longer suitable. We have developed a refined one that ensures that the observational error consistently stays within a predefined set. Based on this refinement, the output feedback control is designed as follows:

$$u(k) = K_1\xi(k) + K_2v(k), \tag{43a}$$

$$\xi(k+1) = A_\xi\xi(k) + B_\xi u(k) + Ev(k) + L_\xi[C_\xi\xi(k) + D_\xi u(k) + Fv(k) - e(k)], \tag{43b}$$

where $\xi(k) \in \mathbb{R}^n$ is the estimate of the state $x(k)$ satisfying $\xi(0) = Mv(0)$, and

$$\begin{aligned} A_\xi &= (X_2 - E\mathcal{V})V_x,\ B_\xi = (X_2 - E\mathcal{V})V_y, \\ C_\xi &= (\mathcal{E} - F\mathcal{V})V_x,\ D_\xi = (\mathcal{E} - F\mathcal{V})V_y, \end{aligned} \tag{44}$$

with V_x and V_y satisfying

$$\begin{bmatrix} X_1 \\ U \end{bmatrix} \begin{bmatrix} V_x & V_y \end{bmatrix} = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}. \tag{45}$$

It should be noted that from (18), the matrix $[V_x\ \ V_y]$ always exists, and from (19) and (45),

$$\begin{bmatrix} X_2 - E\mathcal{V} \\ \mathcal{E} - F\mathcal{V} \end{bmatrix} \begin{bmatrix} V_x & V_y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{46}$$

Let $\theta(k) = x(k) - \xi(k)$. Then, the error dynamics of the observer (43b) is denoted by

$$\theta(k+1) = (A_\xi + L_\xi C_\xi)\theta(k). \tag{47}$$

We also define a polyhedral C-set Ω_4 for (47) as follows:

$$\Omega_4 = \{\theta \in \mathbb{R}^n : H\theta \leqslant \zeta_5\}, \tag{48}$$

where $\zeta_5 = |H(x(0) - \xi(0))| \geqslant 0$. It should be noted that Ω_4 possesses two properties. First, it is convex and compact. Second, the origin is an interior point of this set. Demonstrating that Ω_4 serves as a λ-contractive set for the estimation error $\theta(k)$ allows us to prove that $\theta(k)$ remains bounded and asymptotically converges to zero.

The next step involves establishing a lemma for designing L_ξ using linear programming and demonstrating the properties of (43b) by applying the principle of λ-contractivity.

Lemma 2. Under Assumptions 2 and 5, if there exists L_ξ satisfying

$$Q\zeta_5 \leqslant \lambda\zeta_5, \tag{49a}$$
$$QH = H(A_\xi + L_\xi C_\xi), \tag{49b}$$

for matrix $Q \geqslant \mathbf{0}$, then

$$\theta(k) \in \Omega_4,\ \forall k \in \mathbb{N}, \tag{50a}$$
$$\lim_{k\to+\infty} \theta(k) = 0. \tag{50b}$$

Proof. To demonstrate (50), we need to prove that Ω_4 is a λ-contractive set of the system (47). According to Definition A2, we need to prove that for each $\theta(k) \in \Omega_4$, the following condition holds:

$$\inf\left\{\bar{\lambda} \geqslant 0 : (A_\xi + L_\xi C_\xi)\theta(k) \in \bar{\lambda}\Omega_4\right\} \leqslant \lambda. \tag{51}$$

For $\theta(0)$, we obtain $H\theta(0) = Hx(0) - H\xi(0) \leqslant \zeta_5$. From (49b), we obtain

$$H[(A_\xi + L_\xi C_\xi)\theta(0)] = QH\theta(0) \leqslant Q\zeta_5 \leqslant \lambda\zeta_5,$$

implying $(A_\xi + L_\xi C_\xi)\theta(0) \in \lambda\Omega_4$. For $\lambda \in [0,1)$, $\lambda \in \left\{\bar{\lambda} \geqslant 0 : (A_\xi + L_\xi C_\xi)\theta(0) \in \bar{\lambda}\Omega_4\right\}$, and then $\lambda \geqslant \inf\left\{\bar{\lambda} \geqslant 0 : (A_\xi + L_\xi C_\xi)\theta(0) \in \bar{\lambda}\Omega_4\right\}$. Thus, (51) holds for $k = 0$.

Next, we prove (51) by mathematical induction. Assume that Eq. (51) holds for some $k \in \mathbb{N}$, i.e.,

$$(A_\xi + L_\xi C_\xi)\theta(k) = \theta(k+1) \in \lambda\Omega_4. \tag{52}$$

According to (52) and

$$\lambda\Omega_4 = \{\lambda\theta \in \mathbb{R}^n : H\theta \leqslant \zeta_5\} = \{\lambda\theta \in \mathbb{R}^n : H\lambda\theta \leqslant \lambda\zeta_5\},$$

we obtain

$$H\theta(k+1) \leqslant \lambda\zeta_5. \tag{53}$$

Then, we prove (51) holds for $k+1$. From (49) and (53), we obtain

$$H[(A_\xi + L_\xi C_\xi)\theta(k+1)] = QH\theta(k+1) \leqslant Q\zeta_5 \leqslant \lambda\zeta_5, \tag{54}$$

implying $(A_\xi + L_\xi C_\xi)\theta(k+1) \in \lambda\Omega_4$, and then condition (51) holds for $k+1$. Therefore, the set Ω_4 is λ-contractive set for observer error $\theta(k)$, and

$$\theta(k) \in \Omega_4,\ \forall k \in \mathbb{N}.$$

According to Lemma A1, all eigenvalues of $A_\xi + L_\xi C_\xi$ have a modulus less than or equal to λ, which implies that the observer error $\theta(k)$ asymptotically approaches zero.

Remark 8. The observer (43b) is designed by combining linear programming, data-driven control techniques, and the principle of λ-contractivity, setting it apart from the traditional Luenberger observer in two aspects. First, Eq. (43b) is designed to ensure that the estimation error $\theta(k)$ remains within a predefined set. Consequently, the control parameter L_ξ (43b) needs to satisfy (49), differing from the Luenberger observer's requirement to make the matrix $A_\xi + L_\xi C_\xi$ is Schur. Second, the design of (43b) depends on the collected system data. Specifically, since system matrices A, B, C, and D are unknown, parameters A_ξ, B_ξ, C_ξ, and D_ξ in (43b) are determined by (44), which utilizes the data X, V, U, $\mathcal{E}$ collected in (9).

4.2 Design of control parameters K_1 and K_2

Based on the pair (M, N) from Lemma 1, we design the control parameters K_1 and K_2 in (43) using linear programming, i.e., K_1 and K_2 are chosen such that

$$\begin{bmatrix} H(A_\xi + B_\xi K_1) & -HL_\xi C_\xi \end{bmatrix} = W_\xi \begin{bmatrix} H & \mathbf{0} \\ \mathbf{0} & H \end{bmatrix}, \tag{55a}$$

Algorithm 2 Data-driven output feedback control (43).

Input: Data $\mathcal{E}$, X, V and U in (9).
Output: The output feedback control (43).
1: Process data X according to (10b) and (10c) to obtain data X_1 and X_2;
2: Determine V_x and V_y from (45), and then calculate A_ξ, B_ξ, C_ξ and D_ξ from (44);
3: Determine L_ξ, K_1 from (49) and (55), respectively;
4: Determine Y from (13), calculate matrices M, N from (12) and obtain $K_2 = N - K_1 M$;
5: Select the initial state $\xi(0) = Mv(0)$ for the observer (43b), and determine the output feedback control (43).

$$W_\xi \begin{bmatrix} \eta \\ \zeta_5 \end{bmatrix} \leqslant \lambda\eta, \tag{55b}$$

$$GK_1 = W_u H,\ W_u \eta \leqslant \Theta_2, \tag{55c}$$

$$K_2 = N - K_1 M, \tag{55d}$$

where $\eta = \Theta_1 - \zeta_5 = \Theta_1 - |H(x(0) - Mv(0))| \geqslant 0$, and each entry of $W_\xi \in \mathbb{R}^{n_x \times 2n_x}$ and $W_u \in \mathbb{R}^{n_u \times n_x}$ is nonnegative.

The closed-loop system is composed of (1), (2) and (43) with K_1 and K_2 given by (55) and L_ξ given by (49). Performing the following state transformation:

$$\bar{\xi}(k) = \xi(k) - Mv(k), \tag{56}$$

yields

$$\bar{\xi}(k+1) = (A_\xi + B_\xi K_1)\bar{\xi}(k) - L_\xi C_\xi \theta(k), \tag{57a}$$

$$e(k) = (C_\xi + D_\xi K_1)\bar{\xi}(k) + C_\xi \theta(k). \tag{57b}$$

We define a polyhedral C-set for $\bar{\xi}$ as follows:

$$\Omega_5 = \left\{ \bar{\xi} \in \mathbb{R}^n : H\bar{\xi} \leqslant \eta \right\}. \tag{58}$$

Using the system data $\mathcal{E}$, X, V, and U, we provide an algorithm to design the control parameters L_ξ, A_ξ, B_ξ, C_ξ and D_ξ in the output feedback control (43).

Next, we can present the main result for the output feedback control (43).

Theorem 2. If Assumptions 1–4 and 5 hold, for all $x(0)$ and $\xi(0) = Mv(0)$, the linear constrained output regulation problem of systems (1) and (2) can be solved by the data-driven output feedback control (43) with parameters given by Algorithm 2.

Proof. First, we prove that Ω_5 is a λ-contractive set of the system (57a). This is equivalent to showing that

$$\bar{\xi}(k+1) \in \lambda\Omega_5,\ \forall \bar{\xi}(k) \in \Omega_5 \text{ and } \theta \in \Omega_4. \tag{59}$$

For $\bar{\xi}(0)$ and $\theta(0)$, we obtain

$$H\bar{\xi}(0) = H\xi(0) - HMv(0) = 0 \leqslant \eta,\ H\theta(0) \leqslant \zeta_5. \tag{60}$$

Then,

$$\begin{bmatrix} H & \mathbf{0} \\ \mathbf{0} & H \end{bmatrix} \begin{bmatrix} \bar{\xi}(0) \\ \theta(0) \end{bmatrix} \leqslant \begin{bmatrix} \eta \\ \zeta_5 \end{bmatrix}. \tag{61}$$

Using Lemma A2, Eqs. (55a) and (55b) are equivalent to

$$\begin{gathered} \begin{bmatrix} H(A_\xi + B_\xi K_1) & -HL_\xi C_\xi \end{bmatrix} \begin{bmatrix} \bar{\xi} \\ \theta \end{bmatrix} \leqslant \lambda\eta, \\ \forall \begin{bmatrix} H & \mathbf{0} \\ \mathbf{0} & H \end{bmatrix} \begin{bmatrix} \bar{\xi} \\ \theta \end{bmatrix} \leqslant \begin{bmatrix} \eta \\ \zeta_5 \end{bmatrix}. \end{gathered} \tag{62}$$

Combining (48), (57a), (58) with (62), we can conclude that (59) holds. Consequently, the set Ω_5 is a λ-contractive set of the system (57a), and $\bar{\xi}(k) \in \Omega_5$, $\forall k \in \mathbb{N}$. Since $\theta(k) \in \Omega_4$, $\forall k \in \mathbb{N}$, under Assumption 4, we obtain

$$Hx(k) = H\theta(k) + H\xi(k) = H\theta(k) + H\bar{\xi}(k) + HMv(k) \leqslant \zeta_5 + \eta + \zeta_1 - \Theta_1 = \zeta_1,\ \forall k \in \mathbb{N}. \tag{63}$$

Second, from (60), $\bar{\xi}(0)$ satisfies $H\bar{\xi}(0) \leqslant \eta$, and based on Lemma A2, Eq. (55c) is equivalent to

$$GK_1\bar{\xi}(k) \leqslant \Theta_2, \quad \text{for all} \quad H\bar{\xi}(k) \leqslant \eta. \tag{64}$$

Since the set Ω_5 is a λ-contractive set of the system (57), $H\bar{\xi}(k) \leqslant \eta$, $\forall k \in \mathbb{N}$. Thus,

$$GK_1\bar{\xi}(k) \leqslant \Theta_2, \ \forall k \in \mathbb{N}. \tag{65}$$

Substituting (55d) and (56) into (43), $u(k) = K_1\bar{\xi}(k) + Nv(k)$. Under Assumption 4 and from (65), for all $k \in \mathbb{N}$,

$$Gu(k) = GK_1\bar{\xi}(k) + GNv(k) \leqslant \Theta_2 + \zeta_2 - \Theta_2 \leqslant \zeta_2. \tag{66}$$

Finally, since Ω_4 and Ω_5 are the λ-contractive sets for the systems (47) and (57), respectively, Lemma A1 ensures that all the eigenvalues of $A_\xi + B_\xi K_1$ and $A_\xi + L_\xi C_\xi$ have a modulus less than or equal to λ. Thus,

$$\lim_{k\to\infty} \theta(k) = 0, \ \lim_{k\to\infty} \bar{\xi}(k) = 0. \tag{67}$$

Then from (57b), we derive

$$\lim_{k\to\infty} e(k) = (C_\xi + D_\xi K_1)\bar{\xi}(k) + C_\xi\theta(k) = 0. \tag{68}$$

From (63), (66), and (68), Problem 1 is solved.

Remark 9. The technical difficulties of designing (43) mainly lie in three aspects. First, the observer (43b) is improved based on the traditional Luenberger observer. This involves determining the condition (49) for designing the parameter L_ξ to simultaneously ensure the invariance of set Ω_4 and the convergence of observer error $\theta(k)$. Second, designing the parameter K_1 requires solving a linear programming problem (55). Meanwhile, parameter K_2 is derived using a data-driven approach to obtain the solution pair (M, N) as presented in Lemma 1. This approach is more complicated compared to those reported in [20, 21]. Third, the data-driven approach alters the stabilization procedure, making traditional steady-state behavior analysis for closed-loop systems characterized by (M, N) [31, 32] no longer applicable. Instead, we define Ω_5 for closed-loop systems (57a) and rigorously demonstrate that Ω_5 is the λ-contractive set for (57a) based on Lemma A2, which also fundamentally differs from the Lyapunov-based stabilization procedure reported in [33].

5 Example

We consider a mobile robot in the form of (1) with

$$\begin{aligned} A &= \begin{bmatrix} 0.8 & 0.5 \\ -0.4 & 1.2 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 2 \end{bmatrix}, \ D = 0, \ F = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}. \end{aligned} \tag{69}$$

The exogenous signal $v(k)$ is generated by (2) with

$$S = \begin{bmatrix} \cos(0.1) & \sin(0.1) & 0 \\ -\sin(0.1) & \cos(0.1) & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It can be verified that Assumptions 1–3 are satisfied. Sets Ω_1 and Ω_2 in (3) are determined by

$$H = \begin{bmatrix} 0.2 & 0.4 \\ -0.2 & -0.4 \\ -0.15 & 0.2 \\ 0.15 & -0.2 \end{bmatrix}, \ \zeta_1 = \begin{bmatrix} 1.5 \\ 1.5 \\ 1.5 \\ 1.5 \end{bmatrix}, \ G = \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}, \ \zeta_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{70}$$

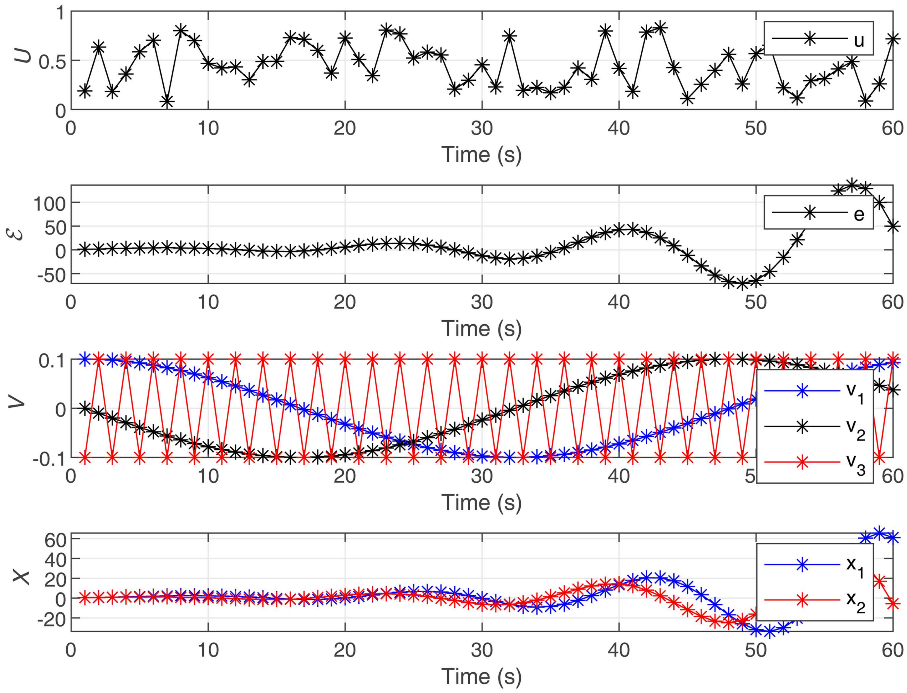

Figure 1 (Color online) Collected data U, $\mathcal{E}$, V and X in (9) from the open-loop experiment.

5.1 State feedback control design

We first design the state feedback control (5). We conduct an open-loop experiment with $u(k)$ being a random variable uniformly distributed on $[0,1]$. We set

$$T=60,\ x(0)=\begin{bmatrix}0.4\ 0.4\end{bmatrix}^{\mathrm{T}},\ v(0)=\begin{bmatrix}0.1\ 0\ -0.1\end{bmatrix}^{\mathrm{T}}.$$

Then, we collect the data U, $\mathcal{E}$, V and X in (9) generated from the experiment, as illustrated in Figure 1.

Based on the collected data, we solve the regulation equation (4) using Lemma 1, and obtain

$$M=\begin{bmatrix}2.1394 & -0.4800 & -0.1611\\ -1.0698 & 0.2400 & 0.5806\end{bmatrix},\ N=\begin{bmatrix}1.0509 & -1.3489 & -1.3421\end{bmatrix}.$$

To demonstrate the performance of the state feedback control (5), we provide the initial conditions as follows:

$$x(0)=\begin{bmatrix}-2\ 3.5\end{bmatrix}^{\mathrm{T}},\ v(0)=\begin{bmatrix}-0.4\ 0\ 1.5\end{bmatrix}^{\mathrm{T}}.$$

Let $\Theta_1=[1\ 1\ 1\ 1]^{\mathrm{T}}$ and $\Theta_2=[0.5\ 0.5]^{\mathrm{T}}$. It can be verified that Assumption 4 is also satisfied.

We utilize the YALMIP tool in MATLAB to solve the linear programming (28) with $\lambda=0.8$. The parameters in (23) are determined by $K=[0.2920\ -0.5510]$, $L=[-0.1633\ -1.0765\ -0.9751]$. Figure 2 presents the trajectory of the closed-loop system (1), (2), and (5). It is evident that the system's state remains in the set Ω_1. Figure 3 depicts the control input $u(k)$ and the tracking error $e(k)$, highlighting that $u(k)$ satisfies the constraint $-5\leqslant u(k)\leqslant 5$ and $e(k)$ asymptotically converges to zero.

This example cannot be solved by using existing data-driven state feedback control schemes for the unconstrained linear output regulation problems. For example, if utilizing the method in [14], the feedback and feedforward control gains are computed as $K=[0.3420\ -0.2010]$ and $L=[0.1044\ -1.1356\ -1.1700]$, but the control (5) fails to satisfy the constraints (3), as shown in Figure 2.

5.2 Output feedback control design

To design the output feedback control (43), we employ YALMIP to solve the linear programming (49) with $\zeta_5=[0.7\ 0.7\ 0.57\ 0.57]^{\mathrm{T}}$, and the parameters L_ξ can be determined as $L_\xi=[0.41\ -1.09]^{\mathrm{T}}$. The

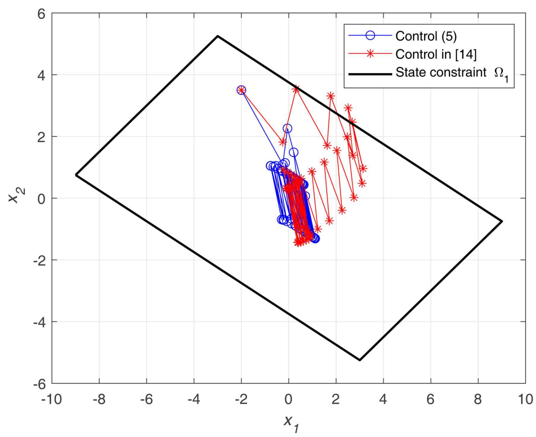

Figure 2 (Color online) Trajectory $x(k)$ of the closed-loop system (1), (2), and (5).

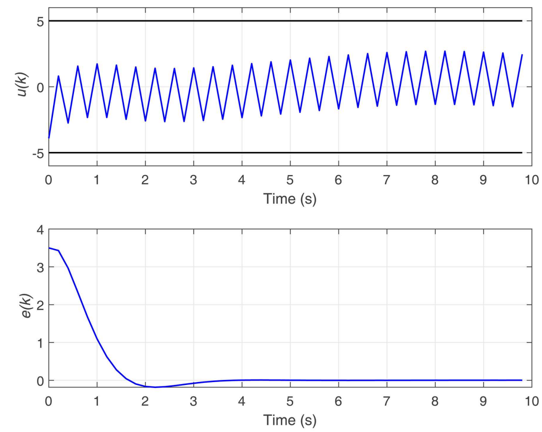

Figure 3 (Color online) Control input $u(k)$ and the tracking error $e(k)$ under (5).

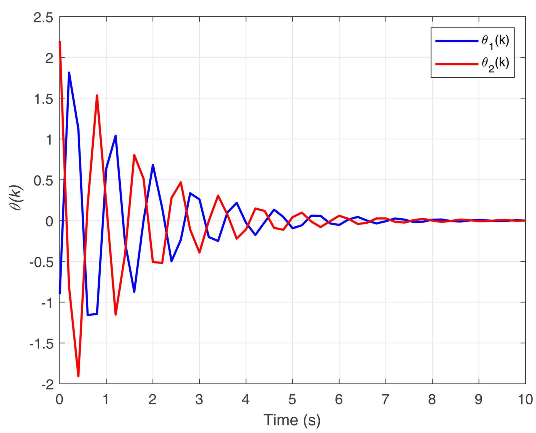

Figure 4 (Color online) Estimation error $\theta(k)$ of the observer (43b).

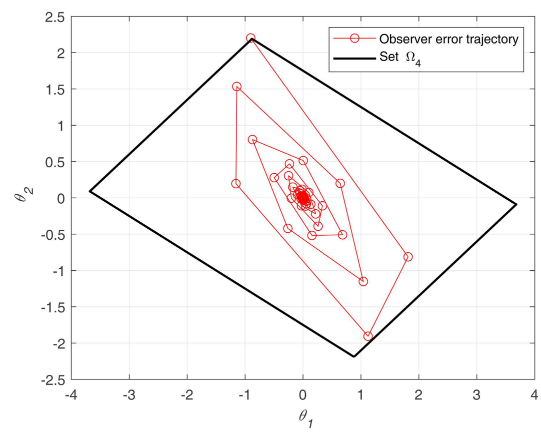

Figure 5 (Color online) Trajectory of the observer (43b).

estimation error $\theta(k)$ of the observer (43b) is shown in Figures 4 and 5. Figure 4 demonstrates that $\theta(k)$ asymptotically converges to zero, while Figure 5 shows that $\theta(k)$ is confined within the set Ω_4.

According to Theorem 2, the gain matrices of the output feedback control (43) can be obtained by solving the linear programming (55) with $\eta = [0.3\ 0.3\ 0.43\ 0.43]^{\mathrm{T}}$. K_1 and K_2 are determined as

$$K_1 = \begin{bmatrix} 0.3935 & -0.9538 \end{bmatrix},\ K_2 = \begin{bmatrix} -0.8113 & -0.9411 & -0.7249 \end{bmatrix}.$$

Figures 6 and 7 illustrate the performance of the output feedback control (43). Figure 6 displays that the system's state remains in the set Ω_1, while Figure 7 shows that the control input $u(k)$ satisfies $-5 \leqslant u(k) \leqslant 5$ and the tracking error $e(k)$ asymptotically converges to zero.

6 Conclusion

This study addresses the linear constrained output regulation problem for an unknown discrete-time system using data-driven state and output feedback control laws. These ensure that tracking errors converge to zero asymptotically while satisfying the state and input constraints. By integrating the output regulation theory with data-driven techniques, our design is easy to implement and computationally efficient, as it can be implemented directly from system data by solving feasible linear programming

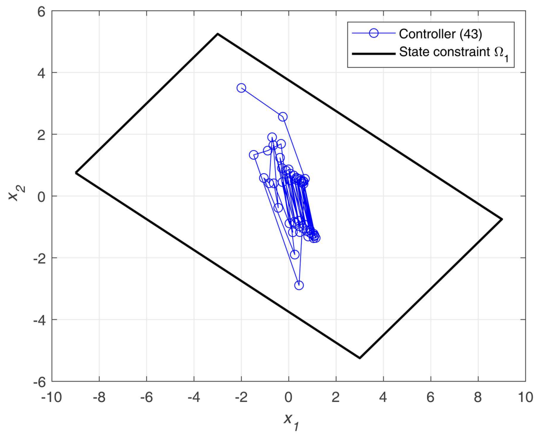

Figure 6 (Color online) Trajectory $x(k)$ of the closed-loop system (1), (2) and (43).

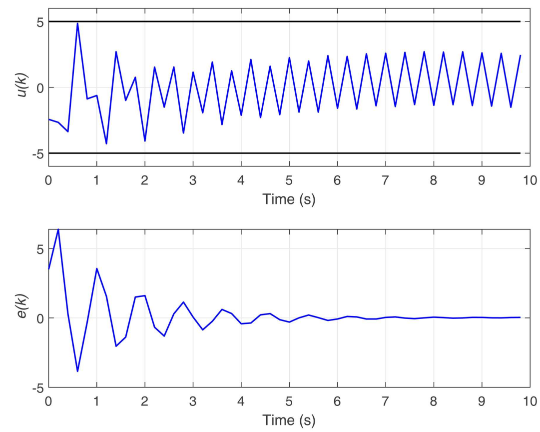

Figure 7 (Color online) Control input $u(k)$ and the tracking error $e(k)$ under (43).

under a verifiable rank condition. In future work, we aim to explore the linear constrained output regulation problem for multi-agent systems facing external disturbances.

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Appendix A

We provide key definitions and lemmas for contractive sets.

Definition A1 ([25], Def 3.10)**.** A C-set is a convex and compact subset of $\mathbb{R}^n$ including the origin as an interior point.

Definition A2. The C-set is λ-contractive set of system $z(k+1) = A_z z(k) + B_z u(k)$, if and only if there exists a controller $u : \Omega \to \mathbb{R}^m$ and a constant $\lambda \in [0, 1)$ such that for each $z(k) \in \Omega$, the following condition holds:

$$\inf \left\{ \bar{\lambda} \geqslant 0 : A_z z(k) + B_z u(k) \in \bar{\lambda}\Omega \right\} \leqslant \lambda \ .$$

Lemma A1 ([25], Cor 4.52)**.** For a discrete-time linear time-invariant system $z(k+1) = A_z z(k)$, there exists a polyhedral C-set which is λ-contractive if and only if all the eigenvalues of A_z have modulus less or equal to λ and all the eigenvalues for which the equality holds have phases that are rational multiple of π.

Lemma A2 ([16], Fact 2)**.** Let $V \in \mathbb{R}^{p\times\ell}, W \in \mathbb{R}^{q\times\ell}, r \in \mathbb{R}^p$, and $s \in \mathbb{R}^q$, and assume that there exists $z \in \mathbb{R}^\ell$ satisfying $Vz \leqslant r$.

$$Wx \leqslant s \text{ for all } x \in \mathbb{R}^\ell \text{ such that } Vx \leqslant r$$

if and only if there exists $E \in \mathbb{R}^{q\times p}$ such that $E \geqslant 0, W = EV, Er \leqslant s$.