SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

February 2025, Vol. 68, Iss. 2, 122203:1–122203:14
https://doi.org/10.1007/s11432-024-4118-x

New results on finite-time stability and instability theorems for stochastic nonlinear time-varying systems

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Received 7 January 2024/Revised 19 March 2024/Accepted 6 August 2024/Published online 17 January 2025

Abstract This paper studies finite-time stability and instability theorems in the probability sense for stochastic nonlinear time-varying systems. Firstly, a new sufficient condition is proposed to guarantee that the considered system has a global solution. Secondly, we propose new finite-time stability and instability criteria that relax the constraints on $\mathcal{L}V$ (the infinitesimal operator of Lyapunov function V) by the uniformly asymptotically stable function. On the one hand, these obtained results make up for the shortcomings of the existing results. On the other hand, the new finite-time stability theorems can be viewed as natural extensions of the existing results and also allow $\mathcal{L}V$ to be indefinite (negative or positive) rather than just only allow $\mathcal{L}V < 0$. Finally, some simulation examples verify the validity of the theoretical results.

Keywords stochastic nonlinear systems, finite-time stability, finite-time instability, uniformly asymptotically stable function

Citation Zhang W H, Yao L Q. New results on finite-time stability and instability theorems for stochastic nonlinear time-varying systems. Sci China Inf Sci, 2025, 68(2): 122203, https://doi.org/10.1007/s11432-024-4118-x

1 Introduction

Stability plays a central role in systems theory and engineering applications, and is always the most fundamental consideration in system analysis and synthesis. The two most commonly used concepts in stability theory are asymptotic stability [1–3] and finite-time stability [4–9]. The asymptotic stability describes the asymptotic behavior of the system state when time tends to infinity, however, the finite-time stability illustrates the transient performance of the system state trajectory within a finite time interval. For many engineering problems, one often pays more attention to finite-time stability instead of asymptotic stability, e.g., the tracking control of robotic manipulators [10], the position and orientation control of underwater vehicles [11]. Specially, the finite-time stable system usually shows better robustness and faster convergence rate. So finite-time stability and its related control problems have evoked great interest during the past two decades, see [12–18].

Many practical systems are subject to stochastic perturbations such as environmental noise. These systems in stochastic environments are often modeled as stochastic systems, which can be studied by the stochastic analysis method. Many stochastic asymptotic stabilization problems of stochastic nonlinear systems are considered, such as [19,20]. Meanwhile, the finite-time stability and stabilization of stochastic systems also have attracted a great deal of attention. For example, Ref. [21] presented the definition of finite-time stability and established finite-time stability and instability criteria in probability sense under the hypothesis that the considered system has a unique strong solution. Generally speaking, it is difficult to ensure that there exists a unique strong solution to stochastic nonlinear systems without Lipschitz condition. So researchers have to study finite-time stability and stabilization problems for stochastic nonlinear systems under the framework of strong solution or weak solution. Ref. [22] generalized the finite-time stability criteria of [21] in strong solution or weak solution sense and allowed $\mathcal{L}V$ (the infinitesimal generator of Lyapunov function) to be negative semi-definitive. Recently, Ref. [23] has presented new

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finite-time stability results by multiple Lyapunov functions for stochastic nonlinear time-varying systems. A fast finite-time stability criterion was given in [24] and a fast finite-time stabilization problem was also considered. In addition, Ref. [25] studied the finite-time feedback stabilization of weak solutions for non-local Lipschitzian stochastic nonlinear systems. Refs. [26, 27] discussed the finite-time stabilization for two classes of stochastic nonlinear high-order systems with switching, respectively. Ref. [28] investigated finite-time stabilization for a class of stochastic nonlinear high-order systems with output constraints. Ref. [29] considered in depth finite-time stabilization for stochastic nonlinear systems with stochastic inverse dynamics and discussed the relationship among three types of stochastic stability (i.e., finite-time stochastic integral input-to-state stability, finite-time stochastic input-to-state stability and stochastic input-to-state stability). We can also find that a stronger definition than finite-time stabilization called predefined-time stabilization has been introduced in [4, 5].

However, these existing finite-time stability results leave much space for improvement. For example, we consider the following stochastic nonlinear time-varying system:

$$\mathrm{d}x(t) = \frac{1}{2}\mu(t)x^{\frac{1}{3}}(t)\mathrm{d}t - \frac{1}{2}x(t)\mathrm{d}t + x(t)\cos(x(t))\mathrm{d}W(t), \tag{1}$$

where $\mu(t)$ is a piecewise continuous function and can take positive/negative value. If we take Lyapunov function $V(x) = x^2$, then by a simple computation, we can get that

$$\mathcal{L}V(x(t)) = \mu(t)V^{\frac{2}{3}}(x(t)). \tag{2}$$

Note that $\mu(t)$ can take positive value or negative value. Hence, the existing finite-time stability criteria cannot be used to analyze the finite-time stability of the system (1). In fact, system (1) may be finite-time stability in probability by the proposed stability results in this paper if $\mu(t)$ satisfies some conditions, and the detailed analysis can be found in Example 1 in Section 3. This is one motivation for us to present new finite-time stability results. On the other hand, the existing finite-time stability theorem about stochastic nonlinear time-varying systems given in [23] cannot degenerate to the corresponding stability result of stochastic nonlinear systems in [21]. This is another motivation for us to improve the existing finite-time stability results.

In this paper, we focus on the existence of the system solution and its finite-time stability and instability for stochastic time-varying systems. A new sufficient condition is given to guarantee that the considered system has a global solution, which is weaker than the sufficient condition appeared in Lemma 1 of [23]. It can be found that, in existing literature, most finite-time stability theorems are based on $\mathcal{L}V$ to be negative (e.g., [21,23–25]) or non-positive (e.g., [22]), where V is a Lyapunov function. Moreover, Ref. [21] gave a finite-time instability theorem of stochastic nonlinear systems which requires $\mathcal{L}V$ to be negative. In order to relax the constraints on the Lyapunov function, this paper proposes new finite-time stability and instability theorems. By the uniformly asymptotically stable function (UASF) proposed in [30, 31], we establish some new finite-time stability/instability theorems for stochastic nonlinear time-varying systems in which $\mathcal{L}V$ can be negative definite or positive definite rather than just only negative definite. On the other hand, the obtained results in this paper contain or cover many existing sufficient conditions about finite-time stability and instability criteria as special cases. In other words, these obtained results are natural extensions of the existing theorems in time-varying cases.

The rest of this paper is organized as follows. Mathematical preliminaries are given in Section 2. Sections 3 and 4 discuss finite-time stability and instability of stochastic nonlinear systems, respectively. In Section 5, an example is given to show the application of our presented finite-time stability theorem. Section 6 concludes this paper.

Notations. The set of all natural numbers is denoted by $\mathcal{N}$. $\mathcal{R}_+$ denotes the family of all nonnegative real numbers and $\mathcal{R}^r$ denotes the real r-dimensional space. $|x|$ is the Euclidean norm of a vector x. The Frobenius norm of a real matrix X is $||X|| = [\mathrm{Tr}(X^{\mathrm{T}}X)]^{1/2}$. $C^{1,2}([t_0,\infty)\times\mathcal{R}^r;\mathcal{R}_+)$ stands for the set of all nonnegative functions $w(t,x)$ that are C^1 in t and C^2 in x. $C_0^{1,2}([t_0,\infty)\times\mathcal{R}^r;\mathcal{R}_+)$ denotes the set of all nonnegative functions $w(t,x)\in C^{1,2}([t_0,\infty)\times\mathcal{R}^r;\mathcal{R}_+)$ except possibly at the point $x=0$. $\mathcal{K}$ represents the set of all strictly increasing, continuous functions $\gamma(t)$ with $\gamma(0)=0$. $\mathcal{K}_\infty$ denotes the set of all unbounded functions $\gamma(t)$ with $\gamma(t)\in\mathcal{K}$. $\mathcal{KL}$ denotes the set of all functions $\beta(s,t)\in\mathcal{K}$ with t being fixed and $\lim_{t\to\infty}\beta(s,t)=0$ with s being fixed.

2 Preliminaries

Consider the stochastic nonlinear Itô system

$$\mathrm{d}x(t) = f(t, x(t))\mathrm{d}t + g(t, x(t))\mathrm{d}W(t), \tag{3}$$

in which system state $x(t) \in \mathcal{R}^r$ and $x(t_0) = x_0$, $W(t) \in \mathcal{R}^d$, defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant t_0}, P)$, is a standard Wiener process. The continuous functions $f : [t_0, \infty) \times \mathcal{R}^r \to \mathcal{R}^r$ and $g : [t_0, \infty) \times \mathcal{R}^r \to \mathcal{R}^{r\times d}$ satisfy $f(t, 0) = 0$ and $g(t, 0) = 0$ with $t \in [t_0, \infty)$.

According to Lemma 3.2 of Chapter 4 in [3], almost all the sample paths of any solution of system (3) starting from a non-zero state will never reach the origin in a finite time, if $f(t, x)$ and $g(t, x)$ satisfy local Lipschitz condition in x. This implies that $f(t, x)$ or $g(t, x)$ does not satisfy local Lipschitz condition if the solution of (3) can arrive at the origin in a finite time. In this case, the uniqueness of the solution for system (3) cannot be ensured. As mentioned in Remark 2.1 of [13], it suffices for the stochastic nonlinear system to guarantee the existence of a solution when studying finite-time stability. Hence, most existing finite-time stability results are obtained based on the fact that system (3) has a continuous solution; see [22–25]. In this paper, we study the finite-time stability under the precondition that system (3) has a continuous solution.

The following Lemmas 1 and 2 give some sufficient conditions, which ensure that system (3) has a continuous strong solution. Lemma 1 comes from Theorem 5.2 in [32] and the proof of Lemma 2 can be found in Appendix A.

Lemma 1. For any $T < \infty$, if system (3) satisfies **H0**: $|f(t, x)|^2 + ||g(t, x)||^2 \leqslant H(1 + |x|^2)$, where $H > 0$ is a constant and $t \in [t_0, T]$, then system (3) has a continuous solution with probability 1.

Lemma 2. For system (3), assume that $f(t, x)$ and $g(t, x)$ are locally bounded in x and uniformly bounded in t. If there is a function $U(t, x) \in C^{1,2}([t_0, \infty) \times \mathcal{R}^r; \mathcal{R}_+)$, a $\mathcal{K}_\infty$ function $\gamma(\cdot)$, a continuous function $l(t)$ and a constant $d_U \geqslant 0$ such that $\gamma(|x|) \leqslant U(t, x)$ and $\mathcal{L}U(t, x) \leqslant l(t)U(t, x) + d_U$ hold for any $x \in \mathcal{R}^r$, then system (3) has a continuous solution with probability 1.

Remark 1. Lemma 2 is an improved version of Lemma 1 in [23]. In Lemma 1 of [23], $l(t)$ is a nonnegative function with $\int_{t_0}^{\infty} |l(t)|\mathrm{d}t < \infty$ and $d_U = 0$. However, Lemma 2 removes these strict constraints and allows $l(t)$ to be a continuous function and $d_U \geqslant 0$.

Definition 1. If system (3) has a solution $x(t)$ and satisfies that

(i) For any $x_0 \in \mathcal{R}^r\backslash\{0\}$, the stochastic settling time $\rho_{x_0} = \inf\{t \geqslant t_0 : x(t) = 0\}$ is a finite time a.s., i.e., $P\{\rho_{x_0} < \infty\} = 1$.

(ii) There is a positive constant $\delta(\varepsilon, R)$ such that $P\{|x(t)| < R, \quad t \geqslant t_0\} \geqslant 1 - \varepsilon$ with $|x_0| < \delta(\varepsilon, R)$ for any $0 < \varepsilon < 1$ and $R > 0$.

Then, the solution of system (3) is called finite-time stable in probability.

Remark 2. From [13], if the solution of system (3) is finite-time stable in probability and ρ_{x_0} is the stochastic settling time, then $x(\rho_{x_0} + t) = 0$ a.s. for any $t \geqslant 0$. This implies that the solution of system (3) will stay at the equilibrium point for ever after the stochastic settling time almost surely as soon as it arrives at the equilibrium point.

Definition 2. In Definition 1, if (i) or (ii) cannot be satisfied, then the solution of system (3) is finite-time instable in probability.

Definition 3. For system $\dot{z}(t) = \mu(t)z(t)$ with $t \in [t_0, \infty)$, if there exists a function $\beta(\cdot, \cdot) \in \mathcal{KL}$ such that $|z(t)| \leqslant \beta(|z(t_0)|, t - t_0)$, then this system is globally uniformly asymptotically stable and the piecewise continuous function $\mu(t)$ is called a UASF.

Remark 3. (i) Definitions 1–3 come from [21,23,30], respectively. (ii) According to [31], $\mu(t)$ is a UASF if and only if there exist constants $c_\mu > 0$ and $d_\mu \geqslant 0$ such that $\int_{t_0}^{t} \mu(s)\mathrm{d}s \leqslant d_\mu - c_\mu(t - t_0)$.

For any given $V(t, x) \in C^{1,2}([t_0, \infty) \times \mathcal{R}^r; \mathcal{R}_+)$ associated with system (3), the infinitesimal operator $\mathcal{L}$ is defined as $\mathcal{L}V(t, x) = V_t(t, x) + V_x(t, x)f(t, x) + \frac{1}{2}\mathrm{Tr}\{g^{\mathrm{T}}(t, x)V_{xx}(t, x)g(t, x)\}$.

The main goal of this paper is to present new finite-time stability and instability theorems under the condition that system (3) has a continuous solution. By stochastic analysis technique and UASF, the established finite-time stability and instability theorems can break through some strict constraints imposed on the existing finite-time stability and instability results.

3 Finite-time stability theorems

In this section, our purpose is to build some new finite-time stability theorems of stochastic nonlinear time-varying systems. Compared with some existing related results, the new proposed finite-time stability theorems weaken the condition on $\mathcal{L}V$ and permit $\mathcal{L}V$ to be indefinite. For this end, we need the following Lemma 3, which plays an important role in the proof of Theorem 1. Appendix B shows the detailed proof of Lemma 3.

Lemma 3. For system (3), if there is a function $V^*(t,x) \in C_0^{1,2}([t_0,\infty)\times U_k;\mathcal{R}_+)$ and a UASF $\mu^*(t)$ such that for any $\varepsilon \in (0,k)$,

$$\mathcal{L}V^*(t,x) \leqslant \mu^*(t),\ \forall x \in U_{k,\varepsilon}, \tag{4}$$

where $U_k = \{x \in \mathcal{R}^r : |x| < k\}$ and $U_{k,\varepsilon} = \{x \in \mathcal{R}^r : \varepsilon < |x| < k\}$. Then, the solution of system (3) with $x_0 \in U_{k,\varepsilon}$ first reaches the boundary of $U_{k,\varepsilon}$ in finite time a.s.

Theorem 1. Suppose that there exists a global solution to system (3). If there are functions $\underline{\gamma}(\cdot) \in \mathcal{K}_\infty$, $\bar{\gamma}(\cdot) \in \mathcal{K}_\infty$, $V(t,x) \in C^{1,2}([t_0,\infty)\times\mathcal{R}^r;\mathcal{R}_+)$ and a UASF $\mu(t)$, and a constant $\kappa \in (0,1)$ such that

$$\underline{\gamma}(|x|) \leqslant V(t,x) \leqslant \bar{\gamma}(|x|), \tag{5}$$

$$\mathcal{L}V(t,x) \leqslant \mu(t)[V(t,x)]^\kappa, \tag{6}$$

then the solution of system (3) is finite-time stable in probability.

Proof. If system initial value $x_0 = 0$, then $x(t) \equiv 0$ is the solution of system (3), where $f(0,t) \equiv 0$ and $g(0,t) \equiv 0$ are considered. In the following, we only discuss the case that $x(t_0) = x_0 \in \mathcal{R}^r \backslash \{0\}$.

Defining $W(V) = \int_0^V s^{-\kappa}\mathrm{d}s$ with $V \in (0,+\infty)$ along with system (3) and using Lemma 3.1 in [33], then

$$\mathcal{L}W(V(t,x)) = \frac{\mathcal{L}V}{V^\kappa} - \frac{\kappa}{2}\frac{\mathrm{Tr}\{(V_x g)^{\mathrm{T}} V_x g\}}{V^{\kappa+1}} \leqslant V^{-\kappa}\mathcal{L}V \leqslant \mu(t),\quad x \in \mathcal{R}^r\backslash\{0\}. \tag{7}$$

For any given $x(t_0) = x_0 \in \mathcal{R}^r\backslash\{0\}$, there must exists a k ($k \in \{2,3,\ldots\}$) such that $x(t_0) \in U_{k,\frac{1}{k}}$, where $U_{k,\frac{1}{k}} = \{x \in \mathcal{R}^r : \frac{1}{k} < |x| < k\}$. Define $\rho_k = \varrho_{k,\frac{1}{k}} = \inf\{t \geqslant t_0 : |x(t)| \notin U_{k,\frac{1}{k}}\}$, and then by (7) and Dynkin's formula, we have

$$-W(V(t_0,x_0)) \leqslant EW(V(t\wedge\rho_k, x(t\wedge\rho_k))) - W(V(t_0,x_0)) \leqslant E\int_{t_0}^{t\wedge\rho_k}\mu(s)\mathrm{d}s. \tag{8}$$

Because $\mu(t)$ is a UASF, there exist constants $c_\mu > 0$ and $d_\mu \geqslant 0$ such that $\int_{t_0}^t \mu(s)\mathrm{d}s \leqslant d_\mu - c_\mu(t-t_0)$. Hence, Eq. (8) can be turned into

$$-\frac{1}{1-\kappa}V^{1-\kappa}(t_0,x_0) = -\int_0^{V(t_0,x_0)} s^{-\kappa}\mathrm{d}s \leqslant -c_\mu E(t\wedge\rho_k) + c_\mu t_0 + d_\mu,$$

that is to say,

$$E(t\wedge\rho_k) \leqslant t_0 + \frac{d_\mu}{c_\mu} + \frac{\bar{\gamma}^{1-\kappa}(|x_0|)}{c_\mu(1-\kappa)}, \tag{9}$$

where Eq. (5) is considered.

Let $W = V^*$ and $\mu(t) = \mu^*(t)$ in (7), and then we have $\rho_k \to \rho_\infty = \rho_{x_0}$ a.s. as $k \to \infty$ and $P\{\rho_{x_0} < \infty\} = 1$ by Lemma 3 and Definition 1. In fact, ρ_k denotes the moment when the system state first reaches the boundary of $U_{k,\frac{1}{k}}$ and $\rho_k < \infty$ a.s. according to Lemma 3. Together with $\rho_k \to \rho_\infty = \rho_{x_0} = \inf\{t \geqslant t_0 : x(t) = 0\} < \infty$ a.s. or $\rho_k \to \rho_\infty = \inf\{t \geqslant t_0 : x(t) = \infty\} < \infty$ a.s. as $k \to \infty$, if $\rho_\infty = \inf\{t \geqslant t_0 : x(t) = \infty\} < \infty$ a.s. as $k \to \infty$, which leads to a contradiction with the fact that system (3) has a global solution. Hence $\rho_\infty = \rho_{x_0}$ a.s. as $k \to \infty$ and $P\{\rho_{x_0} < \infty\} = 1$.

Further, let $t = k$ and $k \to \infty$, and then $k \wedge \rho_k \to \rho_{x_0}$ a.s. Accordingly, Eq. (9) can be changed into

$$E(\rho_{x_0}) \leqslant t_0 + \frac{d_\mu}{c_\mu} + \frac{\bar{\gamma}^{1-\kappa}(|x_0|)}{c_\mu(1-\kappa)}. \tag{10}$$

This implies that the solution of system (3) is finite-time attractive in probability.

Applying Dynkin's formula for the function $W(V)=\int_0^V s^{-\kappa}\mathrm{d}s$, together with (5) and (7), leads to

$$E\left[\underline{\gamma}^{1-\kappa}(|x(t)|)I_{[t_0,\rho_{x_0})}(t)\right]\leqslant E\left[V^{1-\kappa}(t,x(t))I_{[t_0,\rho_{x_0})}(t)\right]\leqslant\bar{\gamma}^{1-\kappa}(|x_0|)+d_\mu(1-\kappa). \tag{11}$$

Meanwhile,

$$E\left[\underline{\gamma}^{1-\kappa}(|x(t)|)I_{[\rho_{x_0},\infty)}(t)\right]\equiv 0. \tag{12}$$

Hence, we have

$$\begin{aligned}E\left[\underline{\gamma}^{1-\kappa}(|x(t)|)\right]=&E\left[\underline{\gamma}^{1-\kappa}(|x(t)|)I_{[t_0,\rho_{x_0})}(t)+\underline{\gamma}^{1-\kappa}(|x(t)|)I_{[\rho_{x_0},\infty)}(t)\right]\\=&E\left[\underline{\gamma}^{1-\kappa}(|x(t)|)I_{[t_0,\rho_{x_0})}(t)\right]+E\left[\underline{\gamma}^{1-\kappa}(|x(t)|)I_{[\rho_{x_0},\infty)}(t)\right]\\\leqslant&\bar{\gamma}^{1-\kappa}(|x_0|)+d_\mu(1-\kappa),\quad t\geqslant t_0\end{aligned} \tag{13}$$

from (11) and (12).

For any $\varepsilon\in(0,1)$, let $\bar{R}>\varepsilon^{-1}\bar{\gamma}^{1-\kappa}(|x_0|)+\varepsilon^{-1}d_\mu(1-\kappa)$, and then from (13) and Chebyshev's inequality, we can get that

$$P\{\underline{\gamma}^{1-\kappa}(|x(t)|)\geqslant\bar{R},t\geqslant t_0\}\leqslant E\left[\underline{\gamma}^{1-\kappa}(|x(t)|)\right]\bar{R}^{-1}<\varepsilon,$$

which means that

$$P\{|x(t)|<R,\quad t\geqslant t_0\}\geqslant 1-\varepsilon,$$

where $R=\underline{\gamma}^{-1}(\bar{R}^{\frac{1}{1-\kappa}})$, $|x_0|<\delta(\varepsilon,R)$ and $\delta(\varepsilon,R)=\bar{\gamma}^{-1}([\varepsilon\underline{\gamma}^{1-\kappa}(R)-d_\mu(1-\kappa)]^{\frac{1}{1-\kappa}})$. This signifies that the solution of system (3) is stable in probability.

Theorem 2. Suppose that there is a solution to system (3). If there are functions $\underline{\gamma}\in\mathcal{K}_\infty$, $\bar{\gamma}\in\mathcal{K}_\infty$, a UASF $\mu(t)$ and $V(t,x)\in C^{1,2}([t_0,\infty)\times\mathcal{R}^r;\mathcal{R}_+)$ such that

$$\underline{\gamma}(|x|)\leqslant V(t,x)\leqslant\bar{\gamma}(|x|), \tag{14}$$

$$\mathcal{L}V(t,x)\leqslant\mu(t), \tag{15}$$

then the solution of system (3) is finite-time stable in probability.

Proof. Similar to Theorem 1, we need to only consider the case that $x(t_0)=x_0\in\mathcal{R}^r\backslash\{0\}$ in the following.

From Dynkin's formula, (15) and the property of the UASF $\mu(t)$, we have

$$EV(t\wedge\sigma_n\wedge\rho_{x_0},x(t\wedge\sigma_n\wedge\rho_{x_0}))\leqslant V(t_0,x_0)-c_\mu E(t\wedge\sigma_n\wedge\rho_{x_0}-t_0)+d_\mu, \tag{16}$$

where $\sigma_n=\inf\{t\geqslant t_0:|x(t)|>n\}$ and ρ_{x_0} represents the stochastic settling time. Note that $\sigma_n\to\infty$ as $n\to\infty$. So by Fatou's lemma, (14) and (16), we have

$$c_\mu E(t\wedge\rho_{x_0})\leqslant EV(t\wedge\rho_{x_0},x(t\wedge\rho_{x_0}))+c_\mu E(t\wedge\rho_{x_0})\leqslant\bar{\gamma}(|x_0|)+c_\mu t_0+d_\mu,$$

which indicates that

$$E(t\wedge\rho_{x_0})\leqslant\frac{\bar{\gamma}(|x_0|)}{c_\mu}+\frac{d_\mu}{c_\mu}+t_0. \tag{17}$$

Applying Fatou's lemma for (17), we have

$$E(\rho_{x_0})\leqslant\frac{\bar{\gamma}(|x_0|)}{c_\mu}+\frac{d_\mu}{c_\mu}+t_0, \tag{18}$$

which signifies that $P\{\rho_{x_0}<\infty\}=1$.

The proof of stability in probability is similar to that of Theorem 1 and is thus omitted here.

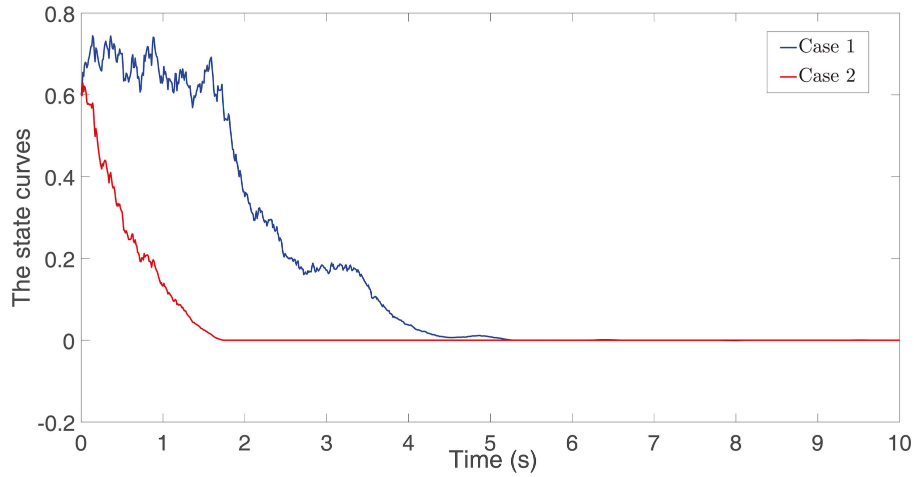

Figure 1 (Color online) State curves of system (19) with different $\mu(t)$.

Remark 4. (i) Theorem 2 signifies that Theorem 1 also holds for $\kappa = 0$, that is to say, Theorem 1 holds for any $\kappa \in [0, 1)$. In the proof of Theorem 1, Eq. (10) also holds for $\kappa \in [0, 1)$ since Eq. (18) is consistent with (10) with $\kappa = 0$. (ii) If $\mu(t) = -c(c > 0)$ that is a UASF, then Theorems 1 and 2 come down to Theorems 3.1 and 3.2 in [21], respectively. Furthermore, if $\mu(t) = -c(c > 0)$ and $g(t, x) = 0$, then Theorem 1 reduces to Theorem 4.2 in [7]. In other words, Theorem 1/Theorem 2 can be viewed as a further generalized version of Theorem 3.1/Theorem 3.2 in [21]. This extension is very necessary. On the one hand, Theorems 1 and 2 are necessary supplements for the existing finite-time stability results. For example, we can analyze the finite-time stability of system (1) by Theorem 1, which can be found in Case 1 of Example 1. On the other hand, the existing finite-time stability theorems (such as Theorems 3.1 and 3.2 in [21]) are special cases of our obtained stability theorems. This generalization is valuable because the structural characteristics of existing finite-time stability criteria are not destroyed. (iii) Note that Theorem 3.1 in [21] does not hold for $\kappa = 1$ according to Example 3.3 in [21]. Therefore, Theorem 1 also does not hold for $\kappa = 1$. (iv) From Remark 3, $\int_{t_0}^{t}(-\mu(s))\mathrm{d}s \to +\infty$ as $t \to +\infty$. The role of $\mu(t)$ is similar to that of $-c(t)$ in [23], but $\mu(t)$ is indefinite and $-c(t) \leqslant 0$.

Remark 5. In some existing results, $\mathcal{L}V$ must be negative definite such as [21, 23, 25] or negative semi-definite such as [22]. Theorems 1 and 2 break through these constraints about $\mathcal{L}V$ and permit $\mathcal{L}V$ to take negative value or positive value. This indicates that the given finite-time stability conditions in this paper are weaker than the existing sufficient conditions.

Example 1. For stochastic nonlinear time-varying system

$$\mathrm{d}x(t) = \frac{1}{2}\mu(t)x^{\frac{1}{3}}(t)\mathrm{d}t - \frac{1}{2}x(t)\mathrm{d}t + x(t)\cos(x(t))\mathrm{d}W(t), \tag{19}$$

where $\mu(t)$ is a piecewise continuous function, $x(t) \in \mathcal{R}$ and $W(t) \in \mathcal{R}$ are system state and standard Wiener process, respectively.

Case 1. If $\mu(t) = \mu_{11}(t) = 2/(1 + t) - |\sin 2t|$, then it is a UASF by [34] and system (19) is a time-varying system. Note that $|f(t, x)|^2 + |g(t, x)|^2 \leqslant x^{2/3}(t) + \frac{5}{4}x^2(t) + x^{4/3}(t) \leqslant \frac{1}{3}|x(t)|^2 + \frac{5}{4}|x(t)|^2 + \frac{2}{3}|x(t)|^2 + 1 \leqslant H_1(|x(t)|^2 + 1)$, where $H_1 = 9/4$, Lemma 3 in [26] and $\mu_{11}(t) = 2/(1 + t) - |\sin 2t| \leqslant 2$ are used. Hence, system (19) has a continuous solution by Lemma 1. Let $V_{11}(x) = x^2$, and then we have $\mathcal{L}V_{11}(x(t)) = 2x(t)(0.5\mu_{11}(t)x^{1/3}(t) - 0.5x(t)) + x^2(t)\cos^2(x(t)) \leqslant \mu_{11}(t)V_{11}^{2/3}(x(t))$. This implies that the solution of system (19) is finite-time stable in probability by Theorem 1.

Case 2. If $\mu(t) = \mu_{12}(t) = -1$, then system (19) is not a time-varying system. From Lemma 1, we know that $|f(t, x)|^2 + |g(t, x)|^2 \leqslant H_2(|x(t)|^2 + 1)$, where $H_2 = 5/3$. This indicates that system (19) has a continuous solution. Let $V_{12}(x) = x^2$, and then $\mathcal{L}V_{12}(x(t)) \leqslant -2V_{12}^{2/3}(x(t))$, which also means that the solution of system (19) is finite-time stable in probability by Theorem 1.

In the simulation, let $x_0 = 0.6$, and then the state curves of system (19) with different $\mu(t)$ are shown in Figure 1, which show that system state trajectories beginning from non-zero initial value converge to the origin in finite time.

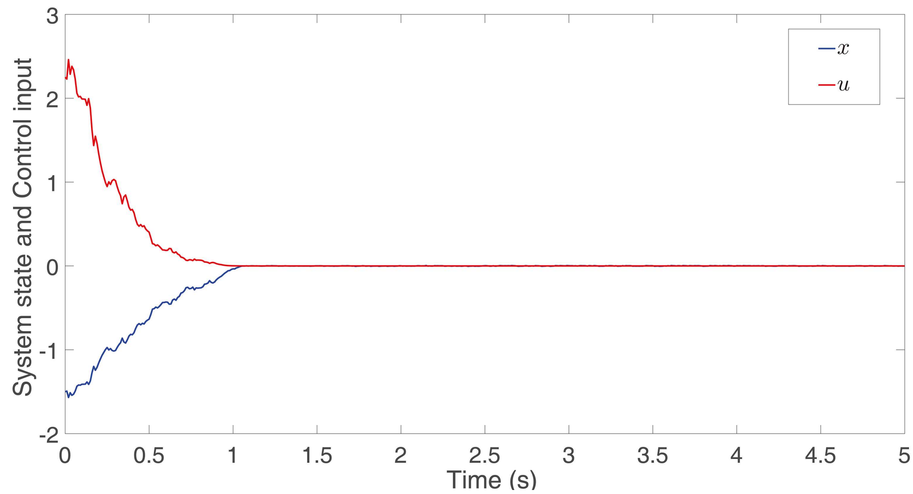

Figure 2 (Color online) Input signal and system state in system (20).

From this example, $\mathcal{L}V_{12}(x) \leqslant -2V_{12}^{2/3}(x)$ in **Case 2** implies that Lyapunov function $V_{12}(x)$ is a decreasing function in **Case 2**, but Lyapunov function $V_{11}(x)$, which satisfies $\mathcal{L}V_{11}(x) \leqslant \mu_{11}(t)V_{11}^{2/3}(x(t))$ with $\mu_{11}(t)$ being a UASF in **Case 1**, is not a decreasing function in **Case 1**. This is also illustrated in Figure 1. However, the condition that $\mathcal{L}V \leqslant -cV^{\kappa} < 0$ with $0 \leqslant k < 1$ and $c > 0$ sometimes cannot be easily satisfied for some stochastic nonlinear time-varying systems, see the following example.

Example 2. For stochastic time-varying system

$$\mathrm{d}x(t) = \frac{t\cos t}{1+t}x^{\frac{1}{5}}(t)\mathrm{d}t - x^3(t)\mathrm{d}t + x(t)u(t)\mathrm{d}t - 2x^{\frac{1}{5}}(t)\mathrm{d}t + x^{\frac{3}{5}}(t)\mathrm{d}W(t), \tag{20}$$

where $W(t) \in \mathcal{R}$ is a standard Wiener process, $x(t) \in \mathcal{R}$ is system state and $u(t) \in \mathcal{R}$ is system input.

For system (20), let $u(t) = x^2(t)$, and then the closed-loop system can be stated as

$$\mathrm{d}x(t) = \mu_2(t)x^{\frac{1}{5}}(t)\mathrm{d}t + x^{\frac{3}{5}}(t)\mathrm{d}W(t), \tag{21}$$

where $\mu_2(t) = t\cos t/(1+t) - 2$. For the closed-loop system (21), we have $|\mu_2(t)x^{1/5}(t)|^2 + |x^{3/5}(t)|^2 \leqslant 9x^{2/5}(t) + x^{6/5}(t) \leqslant \frac{38}{5}(|x(t)|^2 + 1)$, which means that system (21) has a continuous solution by Lemma 1. Let $V_2(x) = x^2$, and then we have $\mathcal{L}V_2(x(t)) \leqslant \tilde{\mu}_2(t)V_2^{3/5}$, where $\tilde{\mu}_2(t) = 2\mu_2(t) + 1$ is a UASF because $\int_{t_0}^{t} \tilde{\mu}_2(s)\mathrm{d}s \leqslant -3(t - t_0) + 10$. This indicates that the solution of the closed-loop system (21) is finite-time stable in probability by Theorem 1. Figure 2 illustrates this fact.

Example 3. For stochastic nonlinear time-varying system

$$\begin{cases} \mathrm{d}x_1(t) = f_1(t,x)\mathrm{d}t + g_1(t,x)\mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) = f_2(t,x)\mathrm{d}t + g_2(t,x)\mathrm{d}W_2(t), \end{cases} \tag{22}$$

where $f_1(t,x) = -x_1(t) + (\psi(t) - 0.5)x_1^{4/5}(t)$, $g_1(t,x) = \sqrt{2}x_2(t)\cos(x_1(t))$, $f_2(t,x) = -x_2(t) + (\psi(t) - 0.5)x_2^{4/5}(t)$, $g_2(t,x) = \sqrt{2}x_1(t)\sin(x_2(t))$, $\psi(t) = t\sin t/(1+t)$, $W_i(t) \in \mathcal{R}(i = 1,2)$ are mutually independent standard Wiener processes. Let $U_3(x) = x^{\mathrm{T}}x = |x|^2$, and then $\mathcal{L}U_3(x) \leqslant 2^{1/10}3U_3^{9/10}(x) \leqslant 2^{1/10}(2.7U_3(x)+0.3)$, where Lemma 4 in [26] is considered. This means that system (22) has a continuous solution by Lemma 2.

Let $V_3(x) = x_1^2 + x_2^2$, and then $\mathcal{L}V_3(x) \leqslant 2(\psi(t) - 0.5)(x_1^{9/5} + x_2^{9/5}) = 2\bar{\mu}_3(t)[(x_1^2)^{9/10} + (x_2^2)^{9/10}] \leqslant \mu_3(t)V_3^{9/10}(x)$, where Lemma 4 in [26] is used, $\bar{\mu}_3(t) = \psi(t) - 0.5$ and

$$\mu_3(t) = \begin{cases} 2^{\frac{11}{10}}\bar{\mu}_2(t), & \text{if } \psi(t) \geqslant 0.5, \\ 2\bar{\mu}_3(t), & \text{if } \psi(t) < 0.5. \end{cases}$$

Since $\bar{\mu}_3(t) = \psi(t) - 0.5$ is a UASF and $\int_{t_0}^{t} \bar{\mu}_3(s)\mathrm{d}s \leqslant 5 - 0.5(t - t_0)$, it is easy to verify that $\mu_3(t)$ is also a UASF. Therefore, the solution of system (20) with $x_0 \in \mathcal{R}^2\backslash\{0\}$ is finite-time stable in probability by

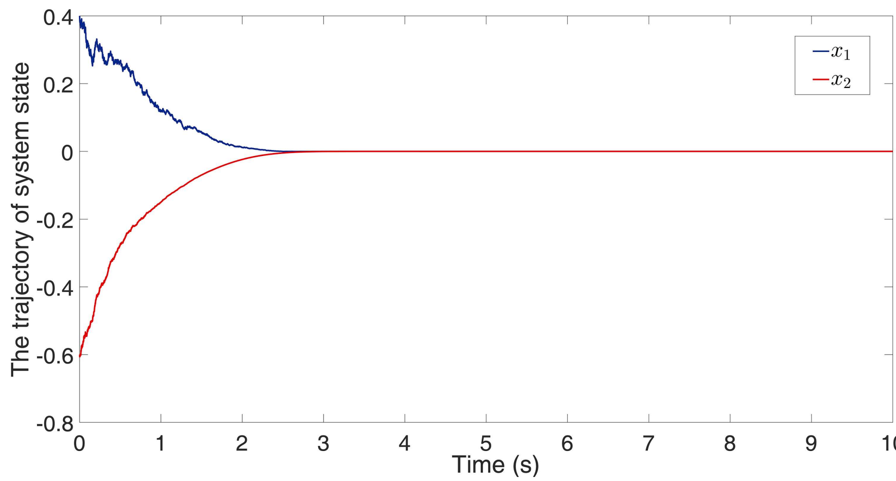

Figure 3 (Color online) Trajectory of state $x(t)$ in (22).

Theorem 1 with $\kappa = 9/10$. For simulation, let $x_0 = (0.4, -0.6)^{\mathrm{T}}$, and then the trajectories of states $x_1(t)$ and $x_2(t)$ are described in Figure 3. This signifies that the solution of system (22) is finite-time stable in probability.

Remark 6. (i) Since system (19) and (21) satisfy the condition of Lemma 1, system (19) and (21) have a continuous solution, respectively. We can also prove that Examples 1 and 2 (with $u = x^2$) satisfy Lemma 2. For example, we know that $f(x,t)$ and $g(x,t)$ in Example 1 are locally bounded in x and are uniformly bounded in t. Furthermore, let $U_1(x) = x^2$ in Example 1, and then $\mathcal{L}U_1(x(t)) = \mu_1(t)U_1^{2/3}(x(t)) \leqslant 2U_1^{2/3}(x(t)) \leqslant 4/3U_1 + 2/3 \triangleq l(t)U_1 + 2/3$.

Remark 7. Examples 1 and 3 show that the finite-time stability of stochastic nonlinear time-varying systems can be analyzed by Theorem 1. Example 2 illustrates finite-time stabilization control for a stochastic nonlinear time-varying system. Although most examples given in this paper are numerical examples, this does not mean that the application of our obtained finite-time stability theorems is narrow or that these finite-time stability results cannot be applied in engineering. In fact, we can design a terminal sliding mode controller such that the system state arrives at a sliding mode surface in finite time (i.e., arrival stage) and the system state slides to the system equilibrium point along the sliding mode surface in finite time (i.e., sliding stage). Since the sliding mode function is artificially designed according to the control objective, the corresponding closed-loop system in the arrival stage may appear some characteristics of these numerical examples. The sliding mode control strategy is a kind of variable structure control methods. Hence, these numerical examples are only to show the feasibility and effectiveness of our obtained finite-time stability criteria. How we stabilize the actual complex physical system in finite time by sliding mode control strategy and the obtained finite-time stability criteria is our future work. In order to further show the validity of Theorem 1, we study the finite-time stabilization problem for a class of stochastic nonlinear time-varying systems in Section 5.

4 Finite-time instability theorem

In this section, we focus on some sufficient conditions which ensure that a stochastic nonlinear time-varying system (3) is finite-time instable in probability. From item (iii) of Remark 4, we know that Theorem 1 does not hold for $\kappa = 1$. This implies a potential direction to study finite-time instability of stochastic nonlinear time-varying system (3).

Theorem 3. Suppose there exists a solution to system (3) with any given non-zero initial value. If there exist a function $V(t,x) \in C^{1,2}([t_0-\tau,\infty)\times\mathcal{R}^r;\mathcal{R}_+)$, $\mathcal{K}_\infty$ functions $\underline{\gamma}$ and $\bar{\gamma}$, and a UASF $\mu(t)$ such that

$$\underline{\gamma}(|x|) \leqslant V(t,x) \leqslant \bar{\gamma}(|x|), \tag{23}$$

$$\mathcal{L}V(t,x) = \mu(t)V(t,x), \tag{24}$$

$$|V_x(t,x)g(t,x)|^2 \leqslant a(t)V^2(t,x), \tag{25}$$

where $a(t) \geqslant 0$ and $\int_{t_0}^{\infty} a(t)\mathrm{d}t < \infty$, then the solution of system (3) is finite-time instable in probability.

Proof. Firstly, let $\tilde{V}(t,x) = \exp\{-\int_{t_0}^{t} \mu(s)\mathrm{d}s\}V(t,x)$ along system (3) and apply Itô's formula for $\tilde{V}$, and then we have

$$\mathrm{d}\tilde{V} = \mathcal{L}\tilde{V}\mathrm{d}t + \tilde{V}_x^{\mathrm{T}} g(t,x(t))\mathrm{d}W(t), \tag{26}$$

where $\mathcal{L}\tilde{V} = \exp\{-\int_{t_0}^{t} \mu(s)\mathrm{d}s\}(\mathcal{L}V(t,x) - \mu(t)V(t,x)) = 0$ and $\tilde{V}_x = \exp\{-\int_{t_0}^{t} \mu(s)\mathrm{d}s\}V_x(t,x)$. It follows from (26) that

$$\mathrm{e}^{-\int_{t_0}^{t} \mu(s)\mathrm{d}s} V(t,x(t)) = \int_{t_0}^{t} \mathrm{e}^{-\int_{t_0}^{s} \mu(v)\mathrm{d}v} V_x^{\mathrm{T}} g \mathrm{d}W(s) + V(t_0,x_0), \tag{27}$$

which, together with (25) and Burkholder-Davis-Gundy inequality (Theorem 7.3 of Chapter 1 in [3]), leads to

$$\begin{aligned} E\left[\sup_{t_0 \leqslant t \leqslant T} \left(\mathrm{e}^{-\int_{t_0}^{t} \mu(s)\mathrm{d}s} V(t,x(t))\right)^2\right] \leqslant & 8E\int_{t_0}^{T} a(t)\left(\mathrm{e}^{-\int_{t_0}^{t} \mu(v)\mathrm{d}v} V(t,x(t))\right)^2 \mathrm{d}t + 2V^2(t_0,x_0) \\ \leqslant & 8\int_{t_0}^{T} a(t) E\left[\sup_{t_0 \leqslant s \leqslant T} \left(\mathrm{e}^{-\int_{t_0}^{s} \mu(v)\mathrm{d}v} V(s,x(s))\right)^2\right] \mathrm{d}t \\ & + 2V^2(t_0,x_0). \end{aligned} \tag{28}$$

Further, by Gronwall's inequality (Theorem 8.1 of Chapter 1 in [3]), it can be derived from (28) that

$$E\left[\sup_{t_0 \leqslant t \leqslant T} \left(\mathrm{e}^{-\int_{t_0}^{t} \mu(s)\mathrm{d}s} V(t,x(t))\right)^2\right] \leqslant H_0, \tag{29}$$

where $H_0 = 2\mathrm{e}^{\int_{t_0}^{\infty} 8a(t)\mathrm{d}t} \bar{\gamma}^2(|x_0|)$. Let $T \to \infty$, and then by Fatou's lemma, we have

$$E\left[\sup_{t \in [t_0,\infty)} \left(\mathrm{e}^{-\int_{t_0}^{t} \mu(s)\mathrm{d}s} V(t,x(t))\right)^2\right] \leqslant H_0 < \infty,$$

which means that

$$E\left[\sup_{t \in [t_0,\infty)} \left(\mathrm{e}^{-\int_{t_0}^{t} \mu(s)\mathrm{d}s} V(t,x(t))\right)\right] < \infty. \tag{30}$$

For any given $x_0 \in \mathcal{R}^r \backslash \{0\}$, let $\sigma_n = \inf\{t \geqslant t_0 : |x(t)| > n\}$. It follows from (27) that

$$E\left\{\left[\mathrm{e}^{-\int_{t_0}^{t} \mu(s)\mathrm{d}s} V(t,x(t))\right]\Big|_{t=\sigma_n \wedge \sigma_b}\right\} = V(t_0,x(t_0)), \tag{31}$$

where σ_b represents any bounded stopping time. Moreover,

$$0 \leqslant \mathrm{e}^{-\int_{t_0}^{\sigma_n \wedge \sigma_b} \mu(s)\mathrm{d}s} V(\sigma_n \wedge \sigma_b, x(\sigma_n \wedge \sigma_b)) \leqslant \sup_{t \in [t_0,\infty)} \left(\mathrm{e}^{-\int_{t_0}^{t} \mu(s)\mathrm{d}s} V(t,x(t))\right). \tag{32}$$

From (30) and (32), apply Lebesgue's dominated convergence theorem for (31), and then we have

$$E\left[\mathrm{e}^{-\int_{t_0}^{\sigma_b} \mu(s)\mathrm{d}s} V(\sigma_b, x(\sigma_b))\right] = V(t_0,x(t_0)),$$

where $\sigma_n \to \infty$ as $n \to \infty$ is used. This signifies that $\mathrm{e}^{-\int_{t_0}^{t} \mu(s)\mathrm{d}s} V(t,x(t))$ is a uniformly integrable martingale. Hence, by martingale convergence theorem (Theorem 7.11 in [35]), we have

$$0 \leqslant \lim_{t \to \infty} \mathrm{e}^{-\int_{t_0}^{t} \mu(s)\mathrm{d}s} V(t,x(t)) = \vartheta(\omega) < \infty, \quad \text{a.s.}, \tag{33}$$

where $\vartheta(\omega)$ is a random variable.

Secondly, the conditions (23) and (24) ensure that there is a stable solution to system (3) in probability. In fact, for every given $x_0 \in \mathcal{R}^r\backslash\{0\}$, let $T^* = \max\{T \geqslant t_0 : \int_{t_0}^{T} \mu(t)\mathrm{d}t = 0\}$. If $t \in [t_0, T^*]$, from (23), (27) and Remark 3, we obtain that

$$E\left[\underline{\gamma}(|x(t)|)\right] \leqslant EV(t, x(t)) \leqslant \bar{\gamma}(|x_0|)\mathrm{e}^{d_\mu}. \tag{34}$$

If $t \in (T^*, \infty)$, by (27), we have

$$E\left[\underline{\gamma}(|x(t)|)\right] \leqslant EV(t, x(t)) = V(t_0, x_0)\mathrm{e}^{\int_{t_0}^{t}\mu(s)\mathrm{d}s} \leqslant \bar{\gamma}(|x_0|), \tag{35}$$

where $\int_{t_0}^{t}\mu(s)\mathrm{d}s = \int_{t_0}^{T^*}\mu(s)\mathrm{d}s + \int_{T^*}^{t}\mu(s)\mathrm{d}s = \int_{T^*}^{t}\mu(s)\mathrm{d}s < 0$ is used.

Hence, it follows from (34) and (35) that for any $t \geqslant t_0$, $E[\underline{\gamma}(|x(t)|)] \leqslant \bar{\gamma}(|x_0|)\mathrm{e}^{d_\mu}$. From Chebyshev's inequality, for any $\varepsilon \in (0, 1)$ and $R > 0$, we have

$$P\{|x(t)| \geqslant R, t \geqslant t_0\} = P\{\underline{\gamma}(|x(t)|) \geqslant \underline{\gamma}(R), t \geqslant t_0\} \leqslant \underline{\gamma}^{-1}(R)E[\underline{\gamma}(|x(t)|)] \leqslant \underline{\gamma}^{-1}(R)\bar{\gamma}(|x_0|)\mathrm{e}^{d_\mu} \leqslant \varepsilon \tag{36}$$

as $|x_0| < \tilde{\delta}(\varepsilon, R) = \bar{\gamma}^{-1}(\varepsilon\underline{\gamma}(R)\mathrm{e}^{-d_\mu})$. This implies that $P\{|x(t)| < R, t \geqslant t_0\} \geqslant 1 - \varepsilon$ as $|x_0| < \tilde{\delta}(\varepsilon, R)$.

Finally, considering (33), we can prove that it is not finite-time attractive in probability for the solution of system (3) by the same method as Theorem 4.1 in [21].

Remark 8. In Theorem 4.1 of [21], $\mathcal{L}V(t, x) = -c_3V(t, x)$ with $c_3 > 0$ is used. It is clear that $\mu(t) = -c_3(c_3 > 0)$ is a UASF. In Theorem 3, the constraint condition of $\mathcal{L}V(t, x) = -c_3V(t, x)$ is replaced by $\mathcal{L}V(t, x) = \mu(t)V(t, x)$. Theorem 3 gives a looser constraint for $\mathcal{L}V(t, x)$ compared with Theorem 4.1 in [21]. Similarly, Theorem 3 is also an improvement of Theorem 3 with $\alpha = 1$ in [23]. Hence, Theorem 3 can be viewed as a natural extension of some instability criteria for stochastic nonlinear time-varying systems in the existing literature.

Remark 9. Theorem 3 gives sufficient conditions of finite-time instability for stochastic nonlinear time-varying systems. Note that finite-time stability in probability indicates globally asymptotical stability in probability. However, Theorem 3 shows that globally asymptotical stability in probability is not equivalent to finite-time stability for some stochastic systems.

5 A simulation example

Example 4. For stochastic nonlinear time-varying system

$$\sum\begin{cases}\mathrm{d}\chi(t) = \varphi(t)\chi^{\beta_1}(t)\mathrm{d}t + \cos(x_1(t))\chi^{\beta_2}(t)\mathrm{d}B_0(t),\\ \mathrm{d}x_1(t) = x_2(t)\mathrm{d}t,\\ \mathrm{d}x_2(t) = u(t)\mathrm{d}t + x_2^{\beta_3}(t)\sin(\chi(t))\mathrm{d}B(t),\end{cases} \tag{37}$$

where $\beta_1 = (2l-1)/(2l+1) \in (0.5, 1)$, $\beta_2 = 2l/(2l+1) \in (0.5, 1)$ and $\beta_3 = (2l-2)/(2l-1) \in (0.5, 1)$ with $l \in \mathcal{N}$ and $l \geqslant 2$. $B_0(t) \in \mathcal{R}$ and $B(t) \in \mathcal{R}$ are mutually independent standard Brownian motions (Wiener processes). $\varphi(t) = 0.5(t\cos t/(1+t) - 1.5)$. For χ-subsystem, we select $V_0 = \chi^2$, and then

$$\mathcal{L}V_0 = 2\varphi(t)\chi^{\beta_1+1} + \chi^{2\beta_2}\cos^2(x_1) \leqslant (2\varphi(t) + 1)\chi^{2\beta_2} = \mu_0(t)V_0^{\beta_2},$$

where $\beta_1 + 1 = 2\beta_2$ and $\mu_0(t) = 2\varphi(t) + 1$ is a UASF since $\int_{t_0}^{t}\mu_0(s)\mathrm{d}s \leqslant -0.5(t - t_0) + 5$. For system Σ, we introduce $V_1 = V_0 + W_0$ with $W_0 = 0.5z_1^2$ and $z_1 = x_1$, and then

$$\mathcal{L}V_1 = \mathcal{L}V_0 + z_1x_2 \leqslant \mu_0(t)V_0^{\beta_2} + z_1(x_2 - \alpha) + z_1\alpha. \tag{38}$$

Let the stabilizing function $\alpha = -c_1z_1^{\lambda}$ with $\lambda = \beta_1$ and $c_1 > 0$ being a constant to be designed, and then Eq. (38) can be changed into

$$\mathcal{L}V_1 \leqslant \mu_0(t)V_0^{\beta_2} + z_1(x_2 - \alpha) - c_1z_1^{1+\lambda}. \tag{39}$$

Further, we introduce $V_2 = V_1 + W_1, W_1 = \int_{\alpha}^{x_2}(v^{\frac{1}{\lambda}} - \alpha^{\frac{1}{\lambda}})^{2-\lambda}\mathrm{d}v$ and $z_2 = x_2^{\frac{1}{\lambda}} - \alpha^{\frac{1}{\lambda}}$, and then it follows from Propositions B.1 and B.2 in [36] that V_2 is a positive definite function and $W_0 + W_1 \leqslant 2(z_1^2 + z_2^2)$. Meanwhile, we can deduce that

$$\mathcal{L}V_2 = \mathcal{L}V_1 + \frac{\partial W_1}{\partial x_1}x_2 + \frac{\partial W_1}{\partial x_2}u + \frac{1}{2}\frac{\partial^2 W_1}{\partial x_2^2}x_2^{2\beta_3}\sin^2\chi. \tag{40}$$

Note that

$$\begin{aligned}
\frac{\partial W_1}{\partial x_1}x_2 =& -(2-\lambda)\frac{\partial \alpha^{\frac{1}{\lambda}}}{\partial x_1}x_2\int_{\alpha}^{x_2}(v^{\frac{1}{\lambda}} - \alpha^{\frac{1}{\lambda}})^{1-\lambda}\mathrm{d}v\\
=& c_1^{\frac{1}{\lambda}}(2-\lambda)x_2\int_{\alpha}^{x_2}(v^{\frac{1}{\lambda}} - \alpha^{\frac{1}{\lambda}})^{1-\lambda}\mathrm{d}v\\
\leqslant& c_1^{\frac{1}{\lambda}}(2-\lambda)|z_2|^{1-\lambda}|x_2-\alpha||x_2|\\
\leqslant& c_1^{\frac{1}{\lambda}}(2-\lambda)2^{1-\lambda}|z_2|(|x_2-\alpha|+|\alpha|)\\
\leqslant& c_1^{\frac{1}{\lambda}}(2-\lambda)2^{1-\lambda}(2^{1-\lambda}z_2^{1+\lambda} + c_1|z_2||z_1|^{\lambda})\\
\leqslant& c_1^{\frac{1}{\lambda}}(2-\lambda)2^{1-\lambda}\left(2^{1-\lambda}z_2^{1+\lambda} + \frac{c_1h_1^{-\lambda}}{1+\lambda}z_2^{1+\lambda} + \frac{c_1h_1\lambda}{1+\lambda}z_1^{1+\lambda}\right)\\
=& c_1^{\frac{1}{\lambda}}(2-\lambda)\left[2^{2(1-\lambda)}z_2^{1+\lambda} + \frac{2^{1-\lambda}c_1h_1\lambda}{1+\lambda}z_1^{1+\lambda}\right] + \frac{2^{1-\lambda}c_1^{1+\frac{1}{\lambda}}h_1^{-\lambda}}{1+\lambda}(2-\lambda)z_2^{1+\lambda},
\end{aligned} \tag{41}$$

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 W_1}{\partial x_2^2}x_2^{2\beta_3}\sin^2(\chi) =& \frac{2-\lambda}{2}z_2^{1-\lambda}\frac{1}{\lambda}x_2^{\frac{1}{\lambda}-1}x_2^{2\beta_3}\sin^2(\chi)\\
\leqslant& \frac{2-\lambda}{2\lambda}z_2^{1-\lambda}(x_2^{\frac{1}{\lambda}})^{2\lambda}\\
\leqslant& \frac{2-\lambda}{2\lambda}z_2^{1-\lambda}2^{2\lambda-1}(z_2^{2\lambda} + \alpha^2)\\
\leqslant& \frac{2-\lambda}{\lambda}z_2^{1-\lambda}(z_2^{2\lambda} + \alpha^2)\\
\leqslant& \frac{2-\lambda}{\lambda}z_2^{1+\lambda} + \frac{2-\lambda}{\lambda}c_1^2\left(\frac{2\lambda h_2}{1+\lambda}z_1^{1+\lambda} + \frac{1-\lambda}{1+\lambda}h_2^{-\frac{2\lambda}{1-\lambda}}z_2^{1+\lambda}\right)\\
=& \frac{2-\lambda}{\lambda}z_2^{1+\lambda} + \frac{2(2-\lambda)}{1+\lambda}c_1^2h_2z_1^{1+\lambda} + \frac{2-\lambda}{\lambda}\frac{1-\lambda}{1+\lambda}c_1^2h_2^{-\frac{2\lambda}{1-\lambda}}z_2^{1+\lambda},
\end{aligned} \tag{42}$$

$$\frac{\partial W_1}{\partial x_2}u = z_2^{2-\lambda}u, \tag{43}$$

where $1-\lambda = 2/(2l+1)$ and $1+\lambda = 4l/(2l+1)$, $h_1 > 0$ and $h_2 > 0$ are to be designed constants and Lemma 4 in [26] is used. Substituting (39), (41)–(43) into (40), we can obtain that

$$\mathcal{L}V_2 \leqslant \mu_0(t)V_0^{\beta_2} + z_1(x_2-\alpha) - c_1z_1^{1+\lambda} + \tilde{d}_2z_2^{1+\lambda} + \tilde{d}_1z_1^{1+\lambda} + z_2^{2-\lambda}u, \tag{44}$$

where $\tilde{d}_1 = (2-\lambda)[2c_1^2h_2/(1+\lambda) + 2^{1-\lambda}c_1^{1+1/\lambda}h_1\lambda/(1+\lambda)]$, $\tilde{d}_2 = (2-\lambda)[2^{1-\lambda}c_1^{1+1/\lambda}h_1^{-\lambda}/(1+\lambda) + c_1^{1/\lambda}2^{2(1-\lambda)} + 1/\lambda + (1-\lambda)c_1^2h_2^{-2\lambda/(1-\lambda)}/\lambda(1+\lambda)]$. In addition,

$$z_1(x_2-\alpha) \leqslant |z_1|\left|\left(x_2^{\frac{1}{\lambda}}\right)^{\lambda} - \left(\alpha^{\frac{1}{\lambda}}\right)^{\lambda}\right| \leqslant \frac{2^{1-\lambda}}{1+\lambda}\left[h_3z_1^{1+\lambda} + \lambda h_3^{-\frac{1}{\lambda}}z_2^{1+\lambda}\right], \tag{45}$$

where $h_3 > 0$ is a constant that will be designed, and Lemma 2.3 in [37] is applied. Hence, Eqs. (44) and (45) mean that

$$\mathcal{L}V_2 \leqslant \mu_0(t)V_0^{\beta_2} - c_1z_1^{1+\lambda} + d_2z_2^{1+\lambda} + d_1z_1^{1+\lambda} + z_2^{2-\lambda}u, \tag{46}$$

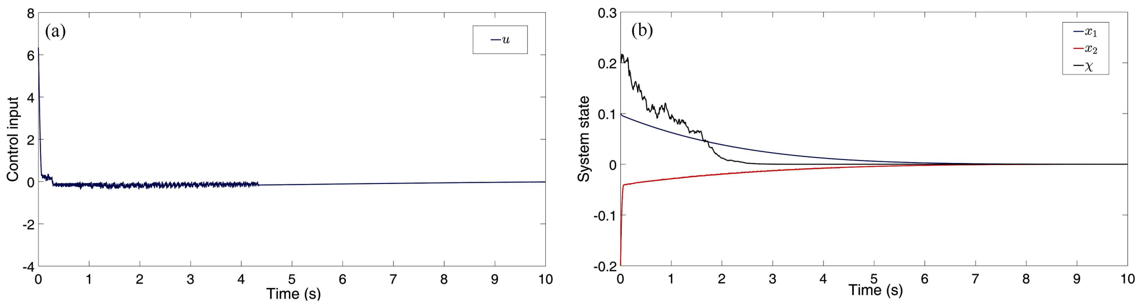

Figure 4 (Color online) (a) Control input trajectory and (b) system state curves of (37).

where $d_1 = \tilde{d}_1 + 2^{1-\lambda} h_3/(1+\lambda)$ and $d_2 = 2^{1-\lambda}\lambda h_3^{-1/\lambda}/(1+\lambda) + \tilde{d}_2$. Let $h_1 = (1+\lambda)2^{\lambda-1}c_1^{-1/\lambda}/6\lambda(2-\lambda)$, $h_2 = (1+\lambda)c_1^{-1}/12(2-\lambda)$ and $h_3 = 2^{\lambda}c_1(1+\lambda)/12$, and then $d_1 = c_1/2$ and Eq. (46) can be rewritten as

$$\mathcal{L}V_2 \leqslant \mu_0(t)V_0^{\beta_2} - 0.5c_1 z_1^{1+\lambda} + d_2 z_2^{1+\lambda} + z_2^{2-\lambda}u. \tag{47}$$

Lastly, we choose the controller

$$u = -d_2 z_2^{2\lambda-1} - 0.5c_2 z_2^{2\lambda-1}, \tag{48}$$

where c_2 is a positive constant to be designed. Then, together with $1+\lambda = 1+\beta_1 = 2\beta_2$, Eqs. (47) and (48) lead to

$$\begin{aligned}\mathcal{L}V_2 \leqslant & \mu_0(t)V_0^{\beta_2} - 0.5c_0((z_1^2)^{\beta_2} + (z_2^2)^{\beta_2}) \\ \leqslant & \mu_0(t)V_0^{\beta_2} - 0.5c_0(z_1^2 + z_2^2)^{\beta_2} \\ \leqslant & \mu_0(t)V_0^{\beta_2} - 0.25c_0(W_0 + W_1)^{\beta_2} \\ \leqslant & \tilde{\mu}(t)(V_0^{\beta_2} + (W_0 + W_1)^{\beta_2}) \\ \leqslant & \mu(t)V_2^{\beta_2},\end{aligned} \tag{49}$$

where

$$\mu(t) = \begin{cases} 2^{1-\beta_2}\tilde{\mu}(t), & \text{if } \tilde{\mu}(t) \geqslant 0, \\ \tilde{\mu}(t), & \text{if } \tilde{\mu}(t) < 0, \end{cases} \quad \tilde{\mu}(t) = \begin{cases} \mu_0(t), & \text{if } \mu_0(t) \geqslant -0.25c_0, \\ -0.25c_0, & \text{if } \mu_0(t) < -0.25c_0, \end{cases} \quad c_0 = \min\{c_1, c_2\}.$$

Moreover, Lemma 4 in [26], $W_0 + W_1 \leqslant 2(z_1^2 + z_2^2)$ and $0.5^{\beta_2} > 0.5$ are used in (49). Note that $\mu_0(t)$ is a UASF, so $\tilde{\mu}(t)$ and $\mu(t)$ are UASFs by Remark 3. Therefore, let $\xi = (\chi, z_1, z_2)^{\mathrm{T}}$ and Lyapunov function $V = V_0 + W_0 + W_1$ for system Σ, and then the positive function $V = V_0 + W_0 + W_1 \leqslant \chi^2 + 2(z_1^2 + z_2^2) \leqslant 2|\xi|^2$ and $\mathcal{L}V \leqslant \mu(t)V^{\beta_2}$. By Theorem 1, stochastic nonlinear system (37) with the controller (48) is finite-time stable in probability. In simulation, we choose the design parameter $l = 4$, $c_1 = 0.3$ and $c_2 = 0.3$, system initial value $\chi(0) = 0.2$, $x_1(0) = 0.1$ and $x_2(0) = -0.2$, and then system control input trajectory and system state curves are given in Figure 4. This implies that system (37) can be finite-time stabilized in probability by introducing controller (48).

6 Conclusion

This paper has further studied the finite-time stability and instability in probability for stochastic nonlinear time-varying systems. A weaker sufficient condition, which ensures that the considered system has a global solution, has been presented. Some new finite-time stability and instability theorems have been given by the UASF. These obtained results are natural extensions of the existing results in time-varying case and also relax the strict constraint conditions on $\mathcal{L}V$ existing in previous references. Some examples are given to illustrate that the obtained stability theorems can be used to analyze and synthesize stochastic nonlinear time-varying systems including autonomous systems and forced systems.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 62373229), Shandong Provincial Natural Science Foundation of China (Grant No. ZR2020MA033), and Research Fund for the Taishan Scholar Project of Shandong Province of China.

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Appendix A Proof of Lemma 2

Proof. For each integer $k \geqslant 1$, let

$$f_k(t,x) = \begin{cases} f\left(t, \dfrac{kx}{|x|}\right), & \text{if } |x| \geqslant k, \\ f(t,x), & \text{if } |x| < k, \end{cases} \qquad g_k(t,x) = \begin{cases} g\left(t, \dfrac{kx}{|x|}\right), & \text{if } |x| \geqslant k, \\ g(t,x), & \text{if } |x| < k, \end{cases}$$

together with $f(t,x)$ and $g(t,x)$ being locally bounded in x and uniformly bounded in t. Then $|f_k(t,x)| \leqslant \sup_{|y|\leqslant k, t\geqslant t_0} |f(t,y)| \triangleq c_{1k} < \infty$, $|g_k(t,x)| \leqslant \sup_{|y|\leqslant k, t\geqslant t_0} |g(t,y)| \triangleq c_{2k} < \infty$. According to Theorem 5.2 in [32],

$$\mathrm{d}x_k(t) = f_k(t, x_k(t))\mathrm{d}t + g_k(t, x_k(t))\mathrm{d}W(t) \tag{A1}$$

has a continuous solution $x_k(t)$ on $[t_0,\infty)$. Further, define $\tau_k=\inf\{t\geqslant t_0:|x_k(t)|\geqslant k\}$, and then Eq. (A1) can be rewritten as

$$\mathrm{d}x_k(t)=f(t,x_k(t))\mathrm{d}t+g(t,x_k(t))\mathrm{d}W(t),\quad t\in[t_0,\tau_k). \tag{A2}$$

Applying Dynkin formula for $U(t,x)\exp\{-\int_{t_0}^t l(s)\mathrm{d}s\}$ along system (A2), for any $T\in[t_0,\infty)$, we have

$$\begin{aligned}E[U(\tau_k\wedge T,x(\tau_k\wedge T))]\leqslant&E\left[U(t_0,x_0)\mathrm{e}^{\int_{t_0}^{\tau_k\wedge T}l(v)\mathrm{d}v}\right]+d_UE\left[\int_{t_0}^{\tau_k\wedge T}\mathrm{e}^{\int_{t_0}^{\tau_k\wedge T}l(v)\mathrm{d}v-\int_{t_0}^{s}l(v)\mathrm{d}v}\mathrm{d}s\right]\\\leqslant&U(t_0,x_0)\mathrm{e}^{\int_{t_0}^{T}|l(v)|\mathrm{d}v}+d_U\int_{t_0}^{T}\mathrm{e}^{\int_{s}^{T}|l(v)|\mathrm{d}v}\mathrm{d}s\\\leqslant&U(t_0,x_0)\mathrm{e}^{M_T(T-t_0)}+d_UT\mathrm{e}^{M_T(T-t_0)},\end{aligned} \tag{A3}$$

where $M_T=\max_{t_0\leqslant t\leqslant T}\{|l(t)|\}$. Note that $E[U(\tau_k\wedge T,x(\tau_k\wedge T))]\geqslant\inf_{|x|=k,t\geqslant t_0}U(t,x)P\{\tau_k\leqslant T\}$, and then Eq. (A3) can be turned into

$$P\{\tau_k\leqslant T\}\leqslant\frac{U(t_0,x_0)+d_UT}{\inf_{|x|=k,t\geqslant t_0}U(t,x)}\mathrm{e}^{M_T(T-t_0)}. \tag{A4}$$

Let $k\to\infty$ in (A4), together with $\lim_{k\to\infty}\inf_{|x|=k,t\geqslant t_0}U(t,x)=\infty$, and then we have $\lim_{k\to\infty}P\{\tau_k\leqslant T\}=0$ for any $T\in[t_0,\infty)$. This means that $\lim_{k\to\infty}\tau_k=\infty$ a.s. Hence, for any $t\in[t_0,\tau_k)$, if we define $x(t)=x_k(t)$ in (A2), then $x(t)$ is the global solution of system (3).

Appendix B Proof of Lemma 3

Proof. Let $\varpi(t)=-\int_{t_0}^t\mu^*(s)\mathrm{d}s$, and then $\varpi(t)\geqslant c_{\mu^*}(t-t_0)-d_{\mu^*}$ and $\lim_{t\to\infty}\varpi(t)=+\infty$ by Remark 3. Moreover, note that the function $\mu^*(t)$ may have the oscillation property. Let $T_0=\max\{T\geqslant t_0:\int_{t_0}^T\mu^*(t)\mathrm{d}t=0\}$, and then $\varpi(T_0)=0$ with $t_0\leqslant T_0<\infty$ and $\varpi(t)>0$ as $t>T_0$. From the definition of $\varpi(t)$, we know that $\int_{t_0}^t\mu^*(s)\mathrm{d}s\leqslant-c_{\mu^*}(t-t_0)+d_{\mu^*}$. If $d_{\mu^*}>0$, then $T_0>t_0$ by the continuity of $-\varpi(t)$. Otherwise, $T_0=t_0$. The definition of T_0, the continuity of $\varpi(t)$ and $\lim_{t\to\infty}\varpi(t)=+\infty$ ensure that T_0 exists and is unique.

Let $\varrho^*_{k,\varepsilon}=\inf\{t\geqslant t_0:|x(t)|\notin U_{k,\varepsilon}\}$, and then we need to prove that $P\{\varrho^*_{k,\varepsilon}<\infty\}=1$, i.e.,

$$P\left\{\varrho^*_{k,\varepsilon}=\bar{\varrho}_{k,\varepsilon}I_{\{\varrho^*_{k,\varepsilon}\leqslant T_0\}}+\varrho_{k,\varepsilon}I_{\{\varrho^*_{k,\varepsilon}>T_0\}}<\infty\right\}=1,$$

where $\bar{\varrho}_{k,\varepsilon}=\inf\{t_0\leqslant t\leqslant T_0:|x(t)|\notin U_{k,\varepsilon}\}$ and $\varrho_{k,\varepsilon}=\inf\{t>T_0:|x(t)|\notin U_{k,\varepsilon}\}$.

First, we prove the case that $P\{\varrho^*_{k,\varepsilon}=\varrho_{k,\varepsilon}<\infty\}=1$. From Dynkin's formula and (4), we can get that for any $x\in U_{k,\varepsilon}$,

$$\begin{aligned}EV^*(\varrho_{k,\varepsilon}\wedge t,x(\varrho_{k,\varepsilon}\wedge t))=&EV^*(T_0,x(T_0))+E\int_{T_0}^{\varrho_{k,\varepsilon}\wedge t}\mathcal{L}V^*(s,x(s))\mathrm{d}s\\\leqslant&EV^*(T_0,x(T_0))+E\int_{T_0}^{\varrho_{k,\varepsilon}\wedge t}\mu^*(s)\mathrm{d}s,\end{aligned} \tag{B1}$$

where $t>T_0$. Define $\varpi_1(t)=-\int_{T_0}^t\mu^*(s)\mathrm{d}s$, and then $\varpi_1(t)=\varpi(t)>0$ as $t>T_0$ and $\lim_{t\to\infty}\varpi_1(t)=\lim_{t\to\infty}\varpi(t)=+\infty$. Further, Eq. (B1) can be turned into $EV^*(\varrho_{k,\varepsilon}\wedge t,x(\varrho_{k,\varepsilon}\wedge t))\leqslant EV^*(T_0,x(T_0))-E\varpi_1(\varrho_{k,\varepsilon}\wedge t)$, which means that

$$E\varpi_1(\varrho_{k,\varepsilon}\wedge t)\leqslant EV^*(T_0,x(T_0)),\quad t>T_0. \tag{B2}$$

Note that for any $t>T_0$,

$$\varpi_1(t)P\{\varrho_{k,\varepsilon}\geqslant t\}=\int_{\{\varrho_{k,\varepsilon}\geqslant t\}}\varpi_1(t)\mathrm{d}P\leqslant E\varpi_1(\varrho_{k,\varepsilon}\wedge t). \tag{B3}$$

Substituting (B3) into (B2) arrives at

$$P\{\varrho_{k,\varepsilon}\geqslant t\}\leqslant\frac{1}{\varpi_1(t)}EV^*(T_0,x(T_0)),\quad t>T_0. \tag{B4}$$

In (B4), let $t\to\infty$, and then $P\{\varrho_{k,\varepsilon}<\infty\}=1$, that is to say, $P\{\varrho^*_{k,\varepsilon}=\varrho_{k,\varepsilon}<\infty\}=1$ can be obtained, where $EV^*(T_0,x(T_0))<\infty$ is considered. In fact, from (4) and Dynkin's formula, together with the definition of T_0, we also have $EV^*(T_0,x(T_0))=V^*(t_0,x(t_0))+E\int_{t_0}^{T_0}\mathcal{L}V^*(s,x(s))\mathrm{d}s\leqslant V^*(t_0,x(t_0))+E\int_{t_0}^{T_0}\mu^*(s)\mathrm{d}s=V^*(t_0,x(t_0))<\infty$.

Second, we prove another case that $P\{\varrho^*_{k,\varepsilon}=\bar{\varrho}_{k,\varepsilon}<\infty\}=1$. In fact, it is natural because $P\{\varrho^*_{k,\varepsilon}=\bar{\varrho}_{k,\varepsilon}\leqslant T_0\}=1$. Hence, we have $P\{\varrho^*_{k,\varepsilon}=\bar{\varrho}_{k,\varepsilon}I_{\{\varrho^*_{k,\varepsilon}\leqslant T_0\}}+\varrho_{k,\varepsilon}I_{\{\varrho^*_{k,\varepsilon}>T_0\}}<\infty\}=1$, i.e., $P\{\varrho^*_{k,\varepsilon}<\infty\}=1$. This means that the solution of system (3) with $x_0\in U_{k,\varepsilon}$ firstly arrives at the boundary of $U_{k,\varepsilon}$ in finite time almost surely.