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The optimal control theory — a scientific approach to fundamentally solving control problems

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The maximum principle has bridged mathematical optimization to optimal control, ushering in significant developments and refinements in optimal control theory, notably during the 1960s with the advent of linear quadratic (LQ) control and linear quadratic estimation (LQE). This progression propelled optimal control theory into further advancements, encompassing stochastic control, robust/H-infinity control, model predictive control (MPC), networked control, and reinforcement learning control. Optimal control, established upon a rigorous mathematical foundation, extends static optimization theory to dynamic systems, exhibiting scientific essence, unity, and perfection. Consequently, since its inception, optimal control theory has served as an indispensable core role across all control-related domains, including communication-constrained control in networked systems, consensus control, cooperative control, and reinforcement learning control.

The essence of optimal control lies in its essence as an analysis and synthesis of dynamic systems. Indeed, whether addressing theoretical aspects such as system feedback stabilization, system stability, or networked systems' consensus control, or practical applications like trajectory planning and tracking, industrial process optimization control, or economic investment portfolio optimization, the key and essence of these problems are inherently optimal control issues. Moreover, optimal control provides theoretically optimal solutions to these problems.

Optimal control exhibits unity, with nearly all optimal control problems solvable under a unified framework, involving the establishment of the maximum principle and forward/backward equation solving (decoupling). Specifically, the LQ control problems can be unified as linear quadratic regulation (LQR) problems, with controller design unified as Riccati equation solutions. Indeed, LQR problems were satisfactorily resolved in the 1960s, subsequently leading to investigations into linear quadratic Gaussian (LQG) problems, revealing the unification of LQG problems with LQR, as both designs employ the solution of the same standard Riccati equation to design controllers. Moreover, Linear optimal estimation and LQR are also unified, as the estimated gain matrix for the former precisely matches the control gain matrix for the dual inverse system of the LQR problem. Furthermore, after decades of research, the results of stochastic LQ control with multiplicative noise have been unified with LQR, with the Riccati equation transitioning into a generalized Riccati equation. In the 1980s, the H-infinite control/robust control gained widespread attention, with nearly two decades of research culminating in the unification of H-infinity control with LQR, with the Riccati equation transforming into an indefinite Riccati equation. Additionally, MPC represents a segmented optimal control problem, while the current hot research topic of consensus control can also be unified into decentralized LQ control.

Optimal control exhibits perfection in scientific terms, comprising solvable sufficient and necessary conditions, unique solutions under certain conditions, clear physical significance, and inclusiveness with other theories. Firstly, under the basic assumptions, the solvable conditions and analytical solutions of LQ control are characterized by a simple form of the Riccati equation. LQ control is dual to linear optimal estimation. Secondly, the Riccati equation possesses clear physical significance, with its solutions representing the magnitude of the weighted matrix of the optimal control performance or the optimal estimation error variance. Thirdly, the optimal performance is the best Lyapunov function, leading to necessary and sufficient conditions for system stability and stabilization. The Riccati equation can degenerate/transform into the Lyapunov equation.

The practical value of optimal control lies in its potential to be the only method for delivering the most effective and precise algorithms for practical applications. In fact, generalized predictive control, robust control, and adaptive control theories, which have been developed based on optimal control theory, provide important and effective precise control algorithms for practical applications.

Over the past two decades, our research results of non-standard LQ control have also precisely demonstrated the essence, unity, and perfection of LQ control. The following

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provides detailed explanations from three aspects.

(1) Stochastic control problems with time delays. The stochastic LQ control problem with multiplicative noise was effectively addressed in the 1970s. However, unlike deterministic systems or stochastic systems with additive noise, the results of stochastic LQ control with multiplicative noise have long been unable to be extended to situations involving time delays. Similarly, classical results in time-delayed system control cannot be generalized to stochastic systems with multiplicative noise. This limitation arises due to the inability to design controllers based on existing tools such as Riccati equations. Moreover, the deeper reason lies in the difficulty of decoupling and solving the forward-backward stochastic differential/difference equations (FBDEs) corresponding to this problem. To address this, a general method for decoupling and solving FBDEs was proposed. As a result, the stochastic LQ control problems with time delays were comprehensively resolved. Additionally, classical Smith predictor control theory and Reduction methods were extended to multiplicative noise systems.

As revealed in [1], the design of stochastic LQ controllers with time delays can be unified and accomplished using the Riccati-ZXL equation, or similar equations proposed by us:

$$\begin{aligned}&-\dot{Z}(t)=A'Z(t)+Z(t)A+\overline{A}'X(t)\overline{A}-L(t)+Q,\\&X(t)=Z(t)+\textstyle\int_t^{t+h}e^{A'(s-t)}L(s)e^{A(s-t)}\mathrm{d}s,\end{aligned}\tag{1}$$

where

$$\begin{aligned}&K(t)=-\Omega^{-1}(t)[B'Z(t)+\overline{B}'X(t)\overline{A}],\\&\Omega(t)=R+\overline{B}'X(t)\overline{B},\\&L(t)=K'(t)\Omega(t)K(t).\end{aligned}\tag{2}$$

The optimal controller is given by

$$u(t-h)=K(t)\hat{x}(t|t-h),\tag{3}$$

where

$$\hat{x}(t|t-h)=e^{Ah}x(t-h)+\int_{t-h}^{t}e^{A(s-t)}Bu(\theta-h)\mathrm{d}\theta.\tag{4}$$

Unity. The above results recover the existing results of LQ control. In fact,

(i) when $h=0$, the Riccati-ZXL equations (1)–(2) become:

$$-\dot{Z}(t)=A'Z(t)+Z(t)A+\overline{A}'X(t)\overline{A}-L(t)+Q,\tag{5}$$

which is in the classical generalized Riccati equation;

(ii) when $\overline{A}=0,\overline{B}=0$, Eqs. (1)–(2) are reduced into

$$-\dot{Z}(t)=A'Z(t)+Z(t)A-L(t)+Q.\tag{6}$$

It is the standard Riccati equation and the optimal controller in (3) becomes the well-known Smith prediction controller.

Perfection. Parallel to the classical results of LQR control, the necessary and sufficient conditions for the unique solution of stochastic LQ control with time delays have been obtained. Simultaneously, under the basic assumptions, the necessary and sufficient conditions for system stabilization have been derived.

Essence. The obtained results in [1] possess the essence of stochastic control with delay. Actually, with the proposed approach method in [1], the general cases of multiple input delays and state delay have been well solved, and more importantly, a long-standing challenging control problem in areas of NCSs of simultaneously packet losses and the delay was also well solved in a latter study [2].

(2) Irregular LQR problems. Irregular LQR problems, also known as singular control, are the only fundamental issues that classical LQ control theory has not yet fully resolved. This problem has garnered widespread attention since the 1970s. Significant progress has been made in addressing irregular control under special initial conditions, but substantial advancements regarding non-regular LQR problems with arbitrary initial conditions are rare to find [3]. Many issues, including solvable conditions, controller forms, and distinctions from standard LQR, remain to be elucidated. Over the past decade, we have explored the essential differences between irregular LQR and standard LQR, starting from foundational problems such as the maximum principle and forward-backward differential equations. Our research has revealed that the key to solving non-regular LQR lies in constructing analytical solutions for non-regular FBDEs. Thus, we obtained the necessary and sufficient conditions for solvability and analytical solutions for controllers by decoupling and solving irregular FBDEs. This research demonstrates the essence, unity, and perfection of LQ control.

It is revealed in [4] that the irregular LQR problem is solvable if and only if there exists a matrix $P_1(T)$ satisfying $B_2'(T)[P_1(T)+P(T)]=0$ such that

$$J_0(t_0,x_0,T)=J(t_0,x_0,T)+x'(T)P_1(T)x(T)\tag{7}$$

is regular. Moreover, the optimal controller is given as

$$u(t)=u^o(t)+[I-R^{\dagger}(t)R(t)]z(t),\tag{8}$$

where $u^o(t)=R^{\dagger}(t)B'(t)[P(t)+P_1(t)]x(t)$. When $z(t)$ results in $P_1(T)x(T)=0$, J_0 is minimized.

Unity. The above results unify the existing results of LQR. In fact, the regular condition is naturally satisfied for $R>0$ and thus $P_1(T)=0$. In this case, the optimal controller (8) becomes

$$u(t)=-R^{-1}(t)B'(t)P(t)x(t).\tag{9}$$

Perfection. Parallel to the classical results of LQR control, the necessary and sufficient conditions for the existence of a solution to irregular LQR control have been obtained. Simultaneously, under the basic assumptions, the necessary and sufficient conditions for system stabilization have been derived.

Essence. The presented results in [4] provide an essential method for general irregular LQ control problems, including stochastic control, H-infinity control, robust control, and so on.

(3) Optimization methods based on optimal control. As mentioned earlier, the maximum principle has bridged the gap between optimization and optimal control, leading to the resolution and development of optimal control. However, despite centuries of research, optimization problems themselves have yet to find completely satisfactory solutions.

It is well-known that commonly used optimization algorithms include gradient descent and Newton's iteration. However, both of these algorithms have recognized limitations and shortcomings. Gradient descent, for instance, is advantageous due to its computational simplicity but suffers from slow convergence. Although improved accelerated gradient descent partially alleviates this issue, it also increases the complexity of parameter selection. On the other hand, while Newton's iteration method converges quickly, it suffers from instability and susceptibility to divergence. While improved quasi-Newton methods enhance stability and complexity, they also slow down the convergence rate. Regularized Newton methods ensure the invertibility of the Hessian matrix but result in linear convergence.

It is evident that the improved optimization algorithms proposed in the past have not transcended the framework of gradient descent and Newton's iteration, thus still exhibiting shortcomings. In this context, in order to obtain better (faster, more stable) algorithms, we have conceived a new idea: solving optimization problems based on the principles of optimal control. Specifically, we treat the update terms of iterative algorithms as controllers and seek to minimize the sum of the optimization function values and the control energies. This innovative approach theoretically ensures that the algorithm converges to the extremum point in the fastest and most stable manner. The new optimization algorithm we have developed also demonstrates unity and perfection.

Consider the following optimization problem

$$\min_x f(x), \tag{10}$$

where $f(x) : R^n \mapsto R^1$ is twice continuously differentiable. Convert the problem (10) into an optimal control problem, we first get the implicit optimization algorithm

$$x_{k+1} = x_k - R^{-1} \sum_{i=k+1}^{N+1} f'(x_i). \tag{11}$$

Further, using Taylor expansion linearization and simplification, the iterative algorithm is obtained as

$$\begin{aligned} &x_{k+1} = x_k - g_k(x_k), \\ &g_l(x_k) = (R + f''(x_k))^{-1}(f'(x_k) + Rg_{l-1}(x_k)), l = 1, \ldots, k, \\ &g_0(x_k) = (R + f''(x_k))^{-1} f'(x_k). \end{aligned} \tag{12}$$

More details of our proposed optimization algorithm can be found in [5].

The iterative algorithm (12) is superlinearly convergent [5]. This algorithm unifies classical Newton's iteration algorithm and its improved variants. Setting $R = 0$, Eq. (12) reduces to Newton's iteration. If we set $g_{k-1}(x_k) = x_k - x_{k-1}$, Eq. (12) becomes the accelerated gradient descent algorithm:

$$x_{k+1} = x_k - \alpha_k f'(x_k) - \beta_k(x_k - x_{k-1}), \tag{13}$$

where $\alpha_k = (R + f''(x_k))^{-1}$, $\beta_k = (R + f''(x_k))^{-1} R$.

Algorithm (11) originates from the principles of optimal control, possessing global optimality, and therefore exhibits the fastest and most stable algorithmic structure. By setting $N = 0$ and using the first-order Taylor expansion of $f'(x_{N+1})$ in (11), we obtain the regularized Newton's iteration. If we then set $R = 0$, the iteration simplifies to Newton's method. Additionally, if $N = 0$ and we assume $f'(x_{N+1}) = f'(x_N)$, Eq. (11) reduces to gradient descent.

Thus, it can be observed that classical Newton's iteration is a locally optimal algorithm, while gradient descent is a locally suboptimal algorithm.

The algorithm (12) is essential to optimization because it is derived based on optimal control theory and it provides the theoretical basis for gradient descent, Newton's iteration, and their improved algorithms.

Conclusion. In summary, just like the results obtained by the authors in recent years, optimal control possesses fundamental scientific attributes: essence, perfection, and unity. Much like mathematics in science, optimal control maintains an irreplaceable core foundation in automation and information fields, serving as the fundamental approach to problem-solving. Similar to the widely used MPC algorithm, optimal control represents the best theoretical method for providing precise control algorithms for practical applications.

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