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• **Supplementary File** •

Exit options sustain altruistic punishment and decrease the second-order free-riders, but it is not a panacea

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Appendix A Stability analysis of the equilibria in infinite and well-mixed population

Solving Eq.7 in the main text, we obtain 12 equilibrium points: $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,0,1)$, $(0,0,1,0)$, $(\epsilon,0,0,1-\epsilon)$, $(0,\epsilon,0,1-\epsilon)$, $(x,1-x,0,0)$, $(x,\epsilon-x,0,1-\epsilon)$, $(\frac{-1+b}{\beta},\frac{1-b+\beta}{\beta},0,0)$, $(\frac{\epsilon}{b-\beta},0,\frac{\epsilon-\beta-\epsilon\beta}{(\beta\epsilon)\gamma},1-\frac{\epsilon(\gamma+1+\beta-b)}{(b-\beta)\gamma})$, $(\frac{(-1+b)\epsilon}{\beta},\frac{\epsilon-\beta+\beta\epsilon}{\beta},0,1-\epsilon)$, $(\frac{\gamma}{1-b+\beta+\gamma},0,\frac{-1+b-\beta}{-1+b-\beta+\gamma},0)$. To examine the stability of these equilibria, we calculate the eigenvalues of Jacobian matrix:

$$J=\begin{bmatrix}\frac{\partial f(x,y,z)}{\partial x} & \frac{\partial f(x,y,z)}{\partial y} & \frac{\partial f(x,y,z)}{\partial z}\\ \frac{\partial g(x,y,z)}{\partial x} & \frac{\partial g(x,y,z)}{\partial y} & \frac{\partial g(x,y,z)}{\partial z}\\ \frac{\partial h(x,y,z)}{\partial x} & \frac{\partial h(x,y,z)}{\partial y} & \frac{\partial h(x,y,z)}{\partial z}\end{bmatrix}. \tag{A1}$$

If the eigenvalues have negative real parts, Eq.7 will approach zero regardless of the initial states. Thus, when all eigenvalues have negative real parts, the corresponding equilibrium is stable. When some eigenvalues have positive real parts, the corresponding equilibrium is unstable. If some eigenvalues have negative real parts and the rest eigenvalues have zero real parts, the stability of equilibrium needs to be determined by the center manifold theorem [1–3]. The stability can be determined by analyzing a lower-order system whose order equals the number of eigenvalues with zero real parts.

Then we have the following conclusion.

Theorem 1. When $b<1+\beta$, and $\epsilon<0$, the equilibrium points $(x^*,1-x^*,0,0)$ and $(0,0,1,0)$ are stable, while the rest of others are unstable; When $b<1+\beta$, and $\epsilon>0$, the equilibrium points $(x^*,1-x^*,0,0)$ and $(0,0,0,1)$ are stable, and the others are unstable; When $b\geqslant 1+\beta$, only the equilibrium point $(0,0,1,0)$ is stable, and the rest of others are unstable. When $\epsilon>0$, only the equilibrium point $(0,0,0,1)$ is stable, and the rest of others are unstable.

Proof. (1). For K_1: $(x,y,z,w)=(1,0,0,0)$, the Jacobian matrix J_1 is

$$J_1=\begin{bmatrix}-1+\epsilon & -1+\epsilon & -b+\beta+\epsilon\\ 0 & 0 & 0\\ 0 & 0 & -1+b-\beta\end{bmatrix} \tag{A2}$$

and its corresponding eigenvalues are

$$\{\lambda_1,\lambda_2,\lambda_3\}=\{0,-1+b-\beta,-1+\epsilon\}. \tag{A3}$$

When $b>\beta+1$, K_1 is unstable because $-1+b-\beta$ is a positive eigenvalue. Otherwise, there is at least one zero eigenvalue. Thus, we use the center manifold theorem to analyze the stability of K_1. Using $b<\beta+1$ as an example. First, there is an invertible matrix whose column elements are the eigenvectors of J_1

$$P=\begin{bmatrix}-1 & -1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \tag{A4}$$

and J_1 can be diagonalized as

$$P^{-1}J_1P=\begin{bmatrix}0 & 0 & 0\\ 0 & -1+b-\beta & 0\\ 0 & 0 & -1+\epsilon\end{bmatrix}. \tag{A5}$$

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Then change of variable:

$$\begin{bmatrix} x_1' \\ y_1 \\ z_1 \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x+y+z \end{bmatrix} \tag{A6}$$

and the system becomes Eq.A7.

$$\begin{aligned}
\dot{x}_1' =& g(z_1 - x_1' - y_1, x_1', y_1) \\
=& x_1'((1-x_1')(-\epsilon - y_1 + z_1) - x_1'(-\epsilon + bx_1' + (b-\beta)(-x_1' - y_1 + z_1)) - y_1(-\epsilon + bx_1' + (b-\beta)(-x_1' - y_1 + z_1))) \\
\dot{y}_1 =& h(z_1 - x_1' - y_1, x_1', y_1) \\
=& y_1(-x_1'(-\epsilon - y_1 + z_1) - y_1(-\epsilon - y_1 - x_1'y_1 + z_1) + (1-y_1)(-\epsilon + bx_1' + (b-\beta)(-x_1' - y_1 + z_1))) \\
\dot{z}_1 =& f(z_1 - x_1' - y_1, x_1', y_1) + g(z_1 - x_1' - y_1, x_1', y_1) + h(z_1 - x_1' - y_1, x_1', y_1) \\
=& x_1'(\epsilon(-1 + 2x_1' + y_1) + (-1 + x_1' + bx_1' + by_1)(y_1 - z_1) - \beta(x_1' + y_1)(x_1' + y_1 - z_1)) + \\
& y_1((-1+y_1)(\epsilon + b(y_1 - z_1) - \beta(x_1' + y_1 - z_1)) + x_1'(\epsilon + y_1 - z_1) + y_1(\epsilon + y_1 + x_1'y_1 - z_1)) + \\
& (x_1' + y_1 - z_1)(\epsilon + (1 - b + \beta + x_1')y_1^2 - \epsilon z_1 + (-1+z_1)z_1 + y_1(1 + x_1'^2 + x_1'(1 + \beta - z_1) + (-2 + b - \beta)z_1)).
\end{aligned} \tag{A7}$$

Let $x_1' = x_1 + 1$ and the system becomes Eq.A8.

$$\begin{aligned}
\dot{x}_1 =& g(z_1 - x_1 - 1 - y_1, x_1 + 1, y_1) \\
=& (x_1+1)(-x_1(-\epsilon - y_1 + z_1) - (x_1+1)(-\epsilon + b(x_1+1) + (b-\beta)(-x_1 - 1 - y_1 + z_1)) - \\
& y_1(-\epsilon + b(x_1+1) + (b-\beta)(-x_1 - 1 - y_1 + z_1))) \\
\dot{y}_1 =& h(z_1 - x_1 - 1 - y_1, x_1 + 1, y_1) \\
=& y_1(-(x_1+1)(-\epsilon - y_1 + z_1) - y_1(-\epsilon - y_1 - (x_1+1)y_1 + z_1) + (1-y_1)(-\epsilon + b(x_1+1) + \\
& (b-\beta)(-x_1 - 1 - y_1 + z_1))) \\
\dot{z}_1 =& f(z_1 - x_1 - 1 - y_1, x_1 + 1, y_1) + g(z_1 - x_1 - 1 - y_1, x_1 + 1, y_1) + h(z_1 - x_1 - 1 - y_1, x_1 + 1, y_1) \\
=& (x_1+1)(\epsilon(-1 + 2(x_1+1) + y_1) + (x_1 + b(x_1+1) + by_1)(y_1 - z_1) - \beta(x_1 + 1 + y_1)(x_1 + 1 + y_1 - z_1)) + \\
& y_1((-1+y_1)(\epsilon + b(y_1 - z_1) - \beta(x_1 + 1 + y_1 - z_1)) + (x_1+1)(\epsilon + y_1 - z_1) + y_1(\epsilon + y_1 + (x_1+1)y_1 - z_1)) + \\
& (x_1 + 1 + y_1 - z_1)(\epsilon + (2 - b + \beta + x_1)y_1^2 - \epsilon z_1 + (-1+z_1)z_1 + y_1(1 + (x_1+1)^2 + (x_1+1)(1 + \beta - z_1) + \\
& (-2 + b - \beta)z_1)).
\end{aligned} \tag{A8}$$

Put the system into the form

$$\begin{aligned} \dot{\boldsymbol{X}} &= A\boldsymbol{X} + \mathbf{F}(\boldsymbol{X}, \boldsymbol{Y}) \\ \dot{\boldsymbol{Y}} &= B\boldsymbol{Y} + \boldsymbol{G}(\boldsymbol{X}, \boldsymbol{Y}) \end{aligned}, \tag{A9}$$

where $\boldsymbol{X} = [x_1]$, $\boldsymbol{Y} = \begin{bmatrix} y_1 \\ z_1 \end{bmatrix}$, and $A = [0]$, $B = \begin{bmatrix} -1 + b - \beta & 0 \\ 0 & -1 + \epsilon \end{bmatrix}$, whose eigenvalues have zero and negative real parts, respectively. $\boldsymbol{F}$ and $\boldsymbol{G}$ are the functions of $\boldsymbol{X}$ and $\boldsymbol{Y}$. They satisfy the condition $\boldsymbol{F(0,0) = 0, F'(0,0) = O}$. According to the existence theorem of the center manifold, the system has the center manifold $S = \{(\boldsymbol{X}, \boldsymbol{H(X)}) | \boldsymbol{H} : \mathbb{R}^1 \to \mathbb{R}^2\}$. We define a mapping

$$\begin{aligned} (M\varphi)(\boldsymbol{X}) =& \varphi'(\boldsymbol{X})(A\boldsymbol{X} + \boldsymbol{F}(\boldsymbol{X}, \varphi(\boldsymbol{X})) \\ & - B\varphi(\boldsymbol{X}) - \boldsymbol{G}(\boldsymbol{X}, \boldsymbol{\varphi}(\boldsymbol{X})) \end{aligned} \tag{A10}$$

Set $\varphi(\boldsymbol{Y}) = O(\boldsymbol{X}^2)$, we obtain

$$\begin{aligned} \dot{x}_1 =& (x_1+1)(-\epsilon x_1 - (x_1+1)(-\epsilon + b(x_1+1) \\ & - (x_1+1)(b-\beta))) + O(x_1^4) \end{aligned} \tag{A11}$$

Then we define $m(x_1) = (x_1+1)(-\epsilon x_1 - (x_1+1)(-\epsilon + b(x_1+1) - (x_1+1)(b-\beta)))$, and $m(x_1)' = \epsilon - b(x_1+1)^2 - (x_1+1)(b-\beta)) + (x_1+1)(-2bx_1 - b - \beta)$. Since $m(0) = \epsilon - 3b < 0$, then $x_1 = 0$ is asymptotically stable. Accordingly, we can conclude the point K_1 is stable when $b < \beta + 1$. When $b = \beta + 1$, K_1 is unstable in accordance with the center manifold theorem whose derivation process is similar to the above analysis.

(2). For K_2: $(x, y, z, w) = (0, 1, 0, 0)$, the corresponding eigenvalues of J are

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{0, -1 + b, -1 + \epsilon\}. \tag{A12}$$

K_2 is unstable since $-1 + b > 0$.

(3). For K_3: $(x, y, z, w) = (0, 0, 1, 0)$. Its corresponding eigenvalues of J are

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{0, \epsilon, -\gamma\}. \tag{A13}$$

When $\epsilon < 0$, K_3 has an eigenvalue with zero real part and other eigenvalues with negative real part. According to the center manifold theorem, K_3 is stable. When $\epsilon > 0$, K_3 is unstable because the eigenvalue ϵ has a positive real part.

(4). For $K_4 : (x, y, z, w) = (0, 0, 0, 1)$. Its corresponding eigenvalues of J are

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{-\epsilon, -\epsilon, -\epsilon\}. \tag{A14}$$

K_4 is stable when $\epsilon > 0$ because all eigenvalues have negative real parts. K_4 is unstable when $\epsilon < 0$ because all eigenvalues have positive real parts.

(5). For $K_5 : (x, y, z, w) = (\epsilon, 0, 0, 1 - \epsilon)$. Its corresponding eigenvalues of J are

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{0, \epsilon(-1 + b - \beta), \epsilon(1 - \epsilon)\}. \tag{A15}$$

When $0 < \epsilon < 1$ or $\epsilon < 0$ and $b < 1 + \beta$, K_5 is unstable because one of its eigenvalues has a positive real part. When $\epsilon < 0$ and $b \geqslant 1 + \beta$, K_5 has at least one eigenvalue with a zero real part and the others have negative real parts. According to the center manifold theorem, K_5 is unstable.

(6). For $K_6 : (x, y, z, w) = (0, \epsilon, 0, 1 - \epsilon)$. Its corresponding eigenvalues of J are

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{0, \epsilon(-1 + b), \epsilon(1 - \epsilon)\}. \tag{A16}$$

When $\epsilon > 0$, K_6 is unstable because eigenvalue $\epsilon(-1 + b) >$. When $\epsilon < 0$, there is one eigenvalue with a zero real part and two eigenvalues with negative real parts. According to the center manifold theorem, K_6 is unstable.

(7). For $K_7 : (x, y, z, w) = (x^*, 1 - x^*, 0, 0)$. Its corresponding eigenvalues of J are

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{0, -1 + \epsilon, -1 + b - \beta x^*\}. \tag{A17}$$

When $x^* > \frac{b-1}{\beta}$, namely $b < 1 + \beta$, there is one eigenvalue with a zero real part and others with negative real parts. According to the center manifold theorem, K_7 is stable. When $x^* < \frac{b-1}{\beta}$, K_7 is unstable because one of its eigenvalues has a positive real part.

(8). For $K_8 : (x, y, z, w) = (x^*, \epsilon - x^*, 0, 1 - \epsilon + x^*)$. Its corresponding eigenvalues of J are

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{0, \epsilon - \epsilon^2, -\epsilon + \beta - \beta x^*\}. \tag{A18}$$

When $\epsilon > 0$, K_8 is unstable because $\epsilon - \epsilon^2 > 0$. When $\epsilon < 0$, K_8 is unstable because $-\epsilon + \beta - \beta x^* > 0$.

(9). For $K_9 : (x, y, z, w) = (\frac{-1+b}{\beta}, \frac{1-b+\beta}{\beta}, 0, 0)$. Its corresponding eigenvalues of J are

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{0, 0, -1 + \epsilon\}. \tag{A19}$$

K_9 exists only when $b < 1 + \beta$. When K_9 exists, there is one eigenvalue with a negative real part and two eigenvalues with zero real parts. According to the center manifold theorem, k_9 is unstable.

(10). For $K_{10} : (x, y, z, w) = (\frac{\epsilon}{b-\beta}, 0, \frac{\epsilon-\beta-\epsilon\beta}{(b-\beta)\gamma}, 1 - \frac{\epsilon(\gamma+1+\beta-b)}{(b-\beta)\gamma})$. Its corresponding eigenvalues of J are

$$\begin{aligned}&\{\lambda_1, \lambda_2, \lambda_3\} = \\ &\{-\frac{\epsilon(-1 + b - \beta)}{b - \beta}, -\frac{\epsilon(-1 + b - \beta)}{b - \beta}, \epsilon + \frac{\epsilon^2(-1 + b - \beta + \gamma)}{(b - \beta)\gamma}\}.\end{aligned} \tag{A20}$$

K_{10} exists when $1 - \frac{\epsilon(\gamma+1+\beta-b)}{(b-\beta)\gamma}) < 1$, namely $b > \beta + \epsilon\frac{1-\gamma}{\gamma+\epsilon}$. Then its eigenvalue $\epsilon + \frac{\epsilon^2(-1+b-\beta+\gamma)}{(b-\beta)\gamma} > 0$. Thus, K_{10} is unstable.

(11). For $K_{11} : (x, y, z, w) = (\frac{(-1+b)\epsilon}{\beta}, \frac{\epsilon-\beta+\beta\epsilon}{\beta}, 0, 1 - \epsilon)$. Its corresponding eigenvalues of J are

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{0, 0, \epsilon(1 - \epsilon)\}. \tag{A21}$$

K_{11} exists when $\epsilon > 0$, then eigenvalue $\epsilon(1 - \epsilon) > 0$. Thus, K_{11} is unstable.

(12). For $K_{12} : (x, y, z, w) = (\frac{\gamma}{1-b+\beta+\gamma}, 0, \frac{1-b+\beta}{1-b+\beta+\gamma}, 0)$. Its corresponding eigenvalues of J are

$$\begin{aligned}&\{\lambda_1, \lambda_2, \lambda_3\} = \\ &\{\frac{(1 - b + \beta)\gamma}{1 - b + \beta + \gamma}, \frac{(1 - b + \beta)\gamma}{1 - b + \beta + \gamma}, \epsilon + \frac{(-b + \beta)\gamma}{1 - b + \beta + \gamma}\}.\end{aligned} \tag{A22}$$

K_{12} exists when $b < 1 + \beta$, then eigenvalue $\frac{(1-b+\beta)\gamma}{1-b+\beta+\gamma} > 0$. Thus K_{12} is unstable.

References

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