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Stabilities of delay stochastic McKean-Vlasov equations in the G-framework

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Abstract This article focuses on a type of stochastic McKean-Vlasov equation in the G-framework (G-SMVE) and addresses the stability issue for delay stochastic McKean-Vlasov equations in the G-framework (G-SMVDEs). Distribution dependence and uncertainty prevent us from directly applying the stochastic analysis method for stochastic McKean-Vlasov equations (SMVEs) to G-SMVEs directly. To overcome this difficulty, we introduce definitions, including the derivative of a function with a law under the G-expectation and the Lions derivatives. Then we construct a new G-Itô formula for G-SMVEs according to the G-Itô formula for stochastic differential equations in the G-framework (G-SDEs) and the Itô formula for SMVEs. Using the new G-Itô formula and the Lyapunov functional method, we investigate the moment exponential stability and almost sure asymptotic stability of G-SMVDEs. To overcome the difficulty in obtaining the distribution dependence of the exact solution to the G-SMVDE, we introduce the empirical measure and the corresponding interacting particle system, and then prove the stability equivalence between the underlying G-SMVDE and the corresponding interacting particle system. Two examples are used to confirm our theoretical results.

Keywords delay stochastic McKean-Vlasov equations, Wasserstein distance, G-Brownian motion, Lions derivative, stability

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1 Introduction

Stochastic differential equations driven by classical Brownian motion (b-SDEs) are often used to describe uncertain phenomena and play important roles in many science and industry fields. The most vital assumption condition imposed on b-SDEs may be the distribution certainty in the sense that each relevant random variable has a certain probability distribution. However, in the real world, it is difficult to find an ideal model that the probability can be exactly determined, and hence, the probability uncertainty becomes a hard and interesting topic. In the economy and finance field, to model Knightian uncertainty, which describes the probability uncertainty in events, Peng [1] introduced the theory of sublinear expectation and developed the theory of G-expectation, and then he proved that it can well characterize the Knightian uncertainty. Under the G-expectation framework, a new type of Brownian motion, the so-called G-Brownian motion, has been introduced, and the stochastic calculus with respect to the G-Brownian motion has been developed. More recently, stochastic differential equations in the G-framework (G-SDEs) and their stabilities become the research highlight. See Peng [1] for the existence and uniqueness of the solutions to G-SDEs, see Zhang et al. [2], Ren et al. [3], and Hu et al. [4] for the stability analysis for G-SDEs, and see Ren et al. [3], Shen et al. [5], and Liu et al. [6] for the stabilization of unstable G-SDEs with continuous and discrete-time feedback controls.

Stochastic McKean-Vlasov equations (SMVEs, also known as mean-field SDEs) play important roles in various fields, such as game theory, mathematical finance, oil resources management, and complex networked systems. They were first introduced by Kac [7, 8] to study the Boltzmann equation for the particle density in diluted monatomic gases and the stochastic toy model for the Vlasov kinetic equation of plasma. Then, they were applied in Lasry et al. [9], Huang et al. [10], and Bensoussan et al. [11] to

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solve the existence of an approximate Nash equilibrium for differential games that had many players. Recently, SMVE theory has renewed interest, and the following SMVEs:

$$\begin{cases} \mathrm{d}X(t) = f(X(t), \mu_t)\mathrm{d}t + g(X(t), \mu_t)\mathrm{d}B(t), \\ \mu_t = \mathcal{L}_{X(t)} = \text{the probability law of } X(t), \end{cases} \tag{1}$$

have been studied by increasingly more researchers. See [12] for more background on SMVEs, see Sznitman [13], Carmona et al. [14], Bahlali et al. [15], Wang [16, 17], and Reis et al. [18, 19] for existence and uniqueness of the strong solutions to SMVEs, and see Bahlali et al. [15], Govindan et al. [20], and Ding et al. [21] for various stability properties of SMVEs.

More recently, SMVEs have found applications in systemic risk modeling with Knightian uncertainty (see [22,23]), and SMVEs with coefficients involving distribution uncertainty have evolved as a prospective research highlight. However, to our knowledge, there is little study on the stability of this type of SMVE, particularly the SMVEs with coefficients dependent on not only the law of the current state but also the law of the past state (SMVDEs). Consequently, in this paper, we study the SMVDEs in the frame of G-expectation (G-SMVDEs) with the following form:

$$\begin{aligned} \mathrm{d}X(t) =& f(X(t), X(t-\tau), \mu_t, \mu_{t-\tau})\mathrm{d}t + g(X(t), X(t-\tau), \mu_t, \mu_{t-\tau})\mathrm{d}B(t) \\ &+ h(X(t), X(t-\tau), \mu_t, \mu_{t-\tau})\mathrm{d}\langle B\rangle(t),\ t \geqslant 0, \end{aligned} \tag{2}$$

where $\mu_t = \mathcal{L}_{X_t}$ represents the law of $X(t)$. $B(t)$ is one-dimensional G-Brownian motion, and $\langle B\rangle(t)$ is its quadratic variation process. $f : \boldsymbol{R}^d \times \boldsymbol{R}^d \times \mathcal{P} \times \mathcal{P} \rightarrow \boldsymbol{R}^d$, $g : \boldsymbol{R}^d \times \boldsymbol{R}^d \times \mathcal{P} \times \mathcal{P} \rightarrow \boldsymbol{R}^d$, and $h : \boldsymbol{R}^d \times \boldsymbol{R}^d \times \mathcal{P} \times \mathcal{P} \rightarrow \boldsymbol{R}^d$ are Borel measurable functions. $\mathcal{P}$ denotes a weakly compact set of probability measures, which will be defined in Section 2.

The main contributions of this paper are as follows:

• We first establish the G-Itô formula for G-SMVEs by generalizing the G-Itô formula for G-SDEs and the Itô formula for SMVEs established in [21].

• The Lyapunov functionals used in this article contain not only the current and past state of variables but also the laws of these variables, while the previous Lyapunov functions used for G-SDEs contain only the current state variable. This feature is essential.

• We study three types of stability, the mean square exponential stability, the quasi sure exponential stability, and the almost sure asymptotic stability of the G-SMVDE, by defining a new lower expectation and a new Itô's operator.

• Because the distribution of the exact solution to the G-SMVDE is difficult to obtain, while the empirical distribution can be observed more easily, we study the corresponding particle system. Moreover, we show that the exponential stabilities of the G-SMVDE and the corresponding particle system are equivalent.

The remaining article is arranged as follows. In Section 2, we introduce necessary notations, assumptions and lemmas, and then establish the G-Itô formula for G-SMVEs. In Section 3, we study two types of exponential stability, the mean square exponential stability and the quasi sure exponential stability of G-SMVDE (2). In Section 4, we study the almost sure asymptotic stability of G-SMVDE (2). In Section 5, we introduce the interacting particle systems and show the stability equivalence between the G-SMVDE and the corresponding interacting particle system. Finally, two examples are used to illustrate our theoretical results.

Notations. Throughout this paper, we use the following notations. Let $\boldsymbol{R}^{d\times n}$ denote the space of $d\times n$ real matrices and $\boldsymbol{R}^d$ denote the space of d-dimensional real column vectors. Let $\mathbb{S}(d)$ denote the collection of all $d\times d$ symmetric matrices. Let $|\cdot|$ denote the Euclidean norm in $\boldsymbol{R}^d$ or trace norm in $\boldsymbol{R}^{d\times d}$. Let $\Omega = C\big([0,\infty);\boldsymbol{R}^d\big)$ be the space of all $\boldsymbol{R}^d$-valued continuous paths $(\omega_t)_{t\geqslant 0}$, endowed with the distance $\rho_d(\omega^1,\omega^2) = \sum_{i=1}^{\infty}\frac{1}{2^i}(|\omega^1-\omega^2|_{C([0,i];\boldsymbol{R}^d)}\wedge 1)$, where $|\omega^1-\omega^2|_{C([0,T];\boldsymbol{R}^d)} = \max_{t\in[0,T]}|\omega_t^1-\omega_t^2|$, for $T>0$. Given any $T>0$, we also define $\Omega_T = \{(\omega_{t\wedge T})_{t\geqslant 0} : \omega\in\Omega\}$. $\mathcal{B}(\Omega)$ denotes the Borel σ-algebra over (Ω,ρ_d) and $\mathcal{H}$ denotes a linear space of real-valued functions on Ω. E denotes the linear expectation. $\hat{\boldsymbol{E}}$ denotes the G-expectation and $\boldsymbol{G}(A) = \frac{1}{2}\hat{\boldsymbol{E}}(A)$ for $A\in\mathbb{S}(d)$. $B(t)$ denotes the G-Brownian motion. $\overline{\sigma}^2 = \hat{\boldsymbol{E}}[B^2(t_0)]$ and $\underline{\sigma}^2 = -\hat{\boldsymbol{E}}[-B^2(t_0)]$. Let $\mathcal{F}_t$ be the formal filtration. Assume that $\{\mathcal{F}_t : t\geqslant 0\}$ assures the typical properties. Let $C_{b,L_{\rm ip}}(\boldsymbol{R}^{d\times n})$ denote the space of bounded and Lipschitz functions on $\boldsymbol{R}^{d\times n}$. Let $L_G^p(\Omega)$ denote the completion of $L_{\rm ip}(\Omega)$ under the natural norm $|\cdot|_p = (\hat{\boldsymbol{E}}|\cdot|^p)^{\frac{1}{p}}$, where $L_{\rm ip}(\Omega) = \bigcup_{t_j=0}^{\infty} L_{\rm ip}(\Omega_{t_j})$ and $L_{\rm ip}(\Omega_{t_j}) = \{\phi(B(t_1),\ldots,B(t_j)) : 0\leqslant t_1\leqslant\cdots\leqslant t_j, \phi\in C_{b,L_{\rm ip}}(\boldsymbol{R}^{d\times j})\}$.

$M_G^p([0,T])$ denotes the completion of $M_G^{p,0}([0,T])$ with the norm $|\eta|_{M_G^{p,0}([0,T])} = (\frac{1}{T}\int_0^T \hat{\boldsymbol{E}}|\eta_t|^p \mathrm{d}t)^{\frac{1}{p}}$, where $M_G^{p,0}([0,T]) = \{\eta_t = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t),\ \xi_j \in L_G^p(\Omega_{t_j})\}$. Let $\mathcal{P}(\boldsymbol{R}^d)$ denote the space of all probability measures over $(\boldsymbol{R}^d, \mathcal{B}(\boldsymbol{R}^d))$ and $\mathcal{P}(\boldsymbol{R}^d \times [0,T])$ denote the space of all probability measures over $(\boldsymbol{R}^d \times [0,T], \mathcal{B}(\boldsymbol{R}^d \times [0,T]))$.

2 Preliminaries

For $p \geqslant 1$, let $\mathcal{P}_p(\boldsymbol{R}^d)$ denote the subspace of $\mathcal{P}(\boldsymbol{R}^d)$ as follows:

$$\mathcal{P}_p(\boldsymbol{R}^d) = \left\{\mu \in \mathcal{P}(\boldsymbol{R}^d) : \int_{\boldsymbol{R}^d} |x|^p \mu(\mathrm{d}x) < \infty\right\}.$$

For $\mu, v \in \mathcal{P}_p(\boldsymbol{R}^d)$ with $\mu = \mathcal{L}_X$, $v = \mathcal{L}_Y$, we define the 2-Wasserstein distance $W_2(\mu, v)$ by

$$W_2(\mu, v) = \inf_{\pi \in \mathscr{L}(\mu,v)} \left\{\left(\int_{\boldsymbol{R}^d \times \boldsymbol{R}^d} |x-y|^2 \pi(\mathrm{d}x, \mathrm{d}y)\right)^{\frac{1}{2}}\right\},$$

where $\mathscr{L}(\mu, v)$ denotes all probability measure couplings between μ and v on $\boldsymbol{R}^d \times \boldsymbol{R}^d$, i.e., $\pi \in \mathscr{L}(\mu, v)$ if and only if $\pi(\cdot \times \boldsymbol{R}^d) = \mu$ and $\pi(\boldsymbol{R}^d \times \cdot) = v$.

Proposition 1. It can be easily deduced that

$$\begin{aligned} W_2(\mu, v)^2 &= \inf_{\pi \in \mathscr{L}(\mu,v)} \left[\int_{\boldsymbol{R}^d \times \boldsymbol{R}^d} |x-y|^2 \pi(\mathrm{d}x, \mathrm{d}y)\right] \\ &\leqslant \int_{\boldsymbol{R}^d \times \boldsymbol{R}^d} |x-y|^2 \mathrm{d}(P_{(X,Y)}(x,y)) \\ &\leqslant \hat{\boldsymbol{E}}[|X-Y|^2]. \end{aligned} \tag{3}$$

2.1 Upper expectation and upper probability

We introduce the following results in [1, 24] to define the derivative of a function with a law under the G-expectation.

Theorem 1 ([1, 24])**.** Let

$$\mathcal{P} = \{P \text{ probability on } (\Omega, \mathcal{B}(\Omega)) : E_P[X] \leqslant \hat{\boldsymbol{E}}[X], \text{for all } X \in L_G^1(\Omega)\}.$$

Then, $\mathcal{P} \neq \varnothing$ is a convex, weakly compact subset of the space $\mathcal{P}(\boldsymbol{R}^d \times [0,T])$ endowed with the topology of weak convergence, and

$$\hat{\boldsymbol{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi], \text{ for all } \xi \in L_G^1(\Omega).$$

The set $\mathcal{P}$ is said to represent $\hat{\boldsymbol{E}}$.

Proposition 2 ([1, 24])**.** The supremum in Theorem 1 is in fact a maximum: For all $\xi \in L_G^1(\Omega)$, there exists a probability measure $P \in \mathcal{P}$ such that $\hat{\boldsymbol{E}}[\xi] = E_P[\xi]$. Consequently, the set

$$\mathcal{P}_{\{\xi\}} = \{P \in \mathcal{P} : \hat{\boldsymbol{E}}[\xi] = E_P[\xi]\}$$

is nonempty.

Let $\mathcal{P}^2_{\{\xi\}}$ denote the subset of $\mathcal{P}_{\{\xi\}}$ with the probability measures having a finite second order moment, that is, for any $\mu \in \mathcal{P}^2_{\{\xi\}}$, $\int_{\boldsymbol{R}^d} |x|^2 \mu(\mathrm{d}x) < \infty$. To avoid symbol confusion, we henceforth use $\mathcal{P}_{\{\xi\}}$ to denote $\mathcal{P}^2_{\{\xi\}}$.

The following lemma is introduced to ensure the weak compactness of $\mathcal{P}_{\{\xi\}}$.

Theorem 2 ([24])**.** Let $\xi, \eta \in L_G^1(\Omega)$ and $0 < \epsilon_l \downarrow 0$ $(l \to \infty)$, and let $P_l \in \mathcal{P}_{\{\xi+\epsilon_l\eta\}}$, $l \geqslant 1$, then, we have the following results:

(1) There exist a subsequence of (P_l), denoted by (P_{l_k}), and a probability measure $P \in \mathcal{P}$, such that $P_{l_k} \to P$, as $l_k \to \infty$ (weak convergence of probability measures);

(2) If $P_{l_k} \to P$, as $\epsilon_l \downarrow 0$ $(l \to \infty)$, for some $P \in \mathcal{P}$, then $P \in \mathcal{P}_{\{\xi\}}$.

Corollary 1. $\mathcal{P}_{\{\xi\}}$ is in fact a weakly compact subset.

Remark 1. For any $\xi \in L_G^1(\Omega)$, we can find a weakly compact set of probability measures in which the nonlinear expectation of ξ is equal to its linear expectation.

Lemma 1 ([24])**.** Let $\xi, \eta \in L_G^1(\Omega)$ and define $F(\lambda) = \hat{\boldsymbol{E}}[\xi + \lambda\eta]$, $\lambda \in \boldsymbol{R}$. Thus, F is convex and

$$F'_+(0) = \lim_{\lambda \downarrow 0} \frac{\hat{\boldsymbol{E}}[\xi + \lambda\eta] - \hat{\boldsymbol{E}}[\xi]}{\lambda} = \max_{p \in \mathcal{P}_{\{\xi\}}} E_P[\xi] = \hat{\boldsymbol{E}}_{\{\xi\}}[\eta],$$

$$F'_-(0) = \lim_{0<\lambda \downarrow 0} \frac{\hat{\boldsymbol{E}}[\xi - \lambda\eta] - \hat{\boldsymbol{E}}[\xi]}{-\lambda} = \max_{p \in \mathcal{P}_{\{\xi\}}} E_P[\xi] = -\hat{\boldsymbol{E}}_{\{\xi\}}[-\eta],$$

where $\hat{\boldsymbol{E}}_{\{\xi\}}[\eta] = \sup_{P \in \mathcal{P}_{\{\xi\}}} E_P[\eta]$ is a new sublinear expectation called upper expectation. $\hat{\boldsymbol{E}}_{\{\xi\}}$ is dominated by $\hat{\boldsymbol{E}}$, i.e., $\hat{\boldsymbol{E}}_{\{\xi\}}[\cdot] \leqslant \hat{\boldsymbol{E}}[\cdot]$. $F(\lambda)$ is said to be differentiable at $\lambda = 0$ if and only if $\hat{\boldsymbol{E}}_{\{\xi\}}[\eta] = -\hat{\boldsymbol{E}}_{\{\xi\}}[-\eta]$.

Now, we define the upper probability, which is important in the stochastic stability analysis.

Definition 1. We define

$$\hat{\mathbb{C}}_{\{\xi\}}(B) = \sup_{P \in \mathcal{P}_{\{\xi\}}} P(B) \text{ for } B \in \mathcal{B}(\Omega).$$

Then, the set function $\hat{\mathbb{C}}_{\{\xi\}}$ is called an upper probability (or a capacity) associated with $\mathcal{P}_{\{\xi\}}$. For symbol consistency, we henceforth use $\hat{\mathbb{C}}$ to represent $\hat{\mathbb{C}}_{\{\xi\}}$.

If $\hat{\mathbb{C}}(B) = 0$, B is called a polar set, and if a characteristic exists outside a polar set, it is called quasi surely (q.s.).

Corollary 2. It follows from the weak compactness of $\mathcal{P}_{\{\xi\}}$ and Theorem 2 that there exists a probability $\widetilde{P} \in \mathcal{P}_{\{\xi\}}$ such that $\hat{\mathbb{C}}(B) = \widetilde{P}(B)$ for $B \in \mathcal{B}(\Omega)$.

2.2 Lions derivatives

Now, we proceed to define the Lions derivatives.

Definition 2. We say that $h : \mathcal{P} \mapsto \boldsymbol{R}$ is differentiable at $\mu \in \mathcal{P}$, if there exist some $\zeta \in L^2(\Omega; \boldsymbol{R}^d)$ such that $\mathcal{L}_\zeta = \mu$ and the lifted function $\widetilde{h}$ which has the form $\widetilde{h}(\zeta) = h(\mathcal{L}_\zeta)$, is Fréchet differentiable at ζ.

Recall that $\widetilde{h}$ is Fréchet differentiable at ζ, which means that there exists a linear continuous mapping $D\widetilde{h}(\zeta) : L^2(\Omega; \boldsymbol{R}^d) \to \boldsymbol{R}$ such that for any $\eta \in L^2(\Omega; \boldsymbol{R}^d)$,

$$\widetilde{h}(\zeta + \eta) - \widetilde{h}(\zeta) = D\widetilde{h}(\zeta)(\eta) + o(|\eta|_{L^2}), \ |\eta|_{L^2} \to 0.$$

As for $D\widetilde{h}(\zeta) \in L^1(L^2(\Omega; \boldsymbol{R}^d)\boldsymbol{R})$, by the Riesz representation theorem and the representation of G-expectation, there exist a weakly compact set $\mathcal{P}_{\{\xi\}}$ and a variable $\varepsilon \in L^2(\Omega; \boldsymbol{R}^d)$ such that for all $\eta \in L^2(\Omega; \boldsymbol{R}^d)$,

$$D\widetilde{h}(\zeta)(\eta) = (\varepsilon, \eta)_{L^2} = E[\varepsilon \cdot \eta].$$

Cardaliaguet [25] proved that there exists a Borel measurable function $h_d : \boldsymbol{R}^d \to \boldsymbol{R}^d$ that depends on the law $\mathcal{L}_\zeta$ rather than on ζ itself, such that $\varepsilon = h_d(\zeta)$. Therefore, for $\eta \in L^2(\Omega; \boldsymbol{R}^d)$,

$$h(\mathcal{L}_\eta) - h(\mathcal{L}_\zeta) = E[h_d(\zeta)(\eta - \zeta)] + o(|\eta - \zeta|_{L^2}).$$

We call $h_d(y) = \partial_\mu h(\mathcal{L}_\zeta)(y)$ the derivative of $h : \mathcal{P} \to \boldsymbol{R}$ at $\mathcal{L}_\zeta, \zeta \in L^2(\Omega; \boldsymbol{R}^d)$.

The following spaces will be used later.

Let $C_b^{1,1}(\mathcal{P})$ denote the space of differentiable functions $h : \mathcal{P} \to \boldsymbol{R}$ with $\partial_\mu h(\cdot)(\cdot)$ bounded and Lipschitz continuous, i.e., there exists a positive constant $C > 0$ such that

(1) $|\partial_\mu h(\mu)(y)| \leqslant C, \mu \in \mathcal{P}, y \in \boldsymbol{R}^d$;

(2) $|\partial_\mu h(\mu)(y) - \partial_\mu h(\mu')(y')| \leqslant C(W_2(\mu, \mu') + |y - y'|), \mu, \mu' \in \mathcal{P}, y, y' \in \boldsymbol{R}^d$.

Let $C_b^{2,1}(\mathcal{P})$ denote the space of all functions $h \in C_b^{1,1}(\mathcal{P})$ such that

(1) $(\partial_\mu h)(\cdot, y) \in C_b^{1,1}(\mathcal{P})$ for all $y \in \boldsymbol{R}^d$, and $\partial_\mu^2 h : \mathcal{P} \times \boldsymbol{R}^d \times \boldsymbol{R}^d \to \boldsymbol{R}^d \times \boldsymbol{R}^d$ is bounded and Lipschitz continuous;

(2) $\partial_\mu h(\mu,\cdot):\boldsymbol{R}^d\to\boldsymbol{R}^d$ is differentiable for every $\mu\in\mathcal{P}$, and its derivative $\partial_y\partial_\mu h(\mathcal{P}\times\boldsymbol{R}^d)\to\boldsymbol{R}^d\times\boldsymbol{R}^d$ is bounded and Lipschitz continuous.

Let $C_b^{2,2;1}(\mathcal{P}\times\boldsymbol{R}^d)$ denote the space of all functions $\Xi\in C_b^{2,1}(\mathcal{P})$ such that

(1) Ξ is bicontinuous in (μ,y);

(2) $\Xi(\cdot,y)\in C_b^{2,1}(\mathcal{P})$ for any $y\in\boldsymbol{R}^d$, and $\Xi(\mu,\cdot)\in C^2(\boldsymbol{R}^d)$ for any $\mu\in\mathcal{P}$;

(3) $\partial_y\Xi(\mu,\cdot)$ is bounded for any $\mu\in\mathcal{P}$.

2.3 Itô formula

Define the segment X_t by $X_t(\theta)=\{X(t+\theta):-\tau\leqslant\theta\leqslant 0\}$ and denote $X_t=X_t(0)=X(t)$, and then define a projection operator

$$\Psi_\theta(\phi)=\phi(\theta),$$

for $\theta\in[-\tau,0]$ and $\phi\in C([-\tau,0];\boldsymbol{R}^d)$. Define

$$\Psi(\phi)=(\Psi_0(\phi),\Psi_{-\tau}(\phi))\text{ and }\mathcal{L}_{\Psi(\phi)}=(\mathcal{L}_{\Psi_0(\phi)},\mathcal{L}_{\Psi_{-\tau}(\phi)}),$$

for $\phi\in C([-\tau,0];\boldsymbol{R}^d)$. Then G-SMVDE (2) can be rewritten as

$$\mathrm{d}X_t=f(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})\mathrm{d}t+g(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})\mathrm{d}B(t)+h(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})\mathrm{d}\langle B\rangle(t),\ t\geqslant 0,\tag{4}$$

with the initial data $\{X_0(\theta):\theta\in[-\tau,0]\}=\xi\in C([-\tau,0];\boldsymbol{R}^d)$ and the uniform norm $|\xi|=\sup_{-\tau\leqslant\theta\leqslant 0}|\xi(\theta)|$.

Let $y^{(2)}=(y_1,y_2)$, $z^{(2)}=(z_1,z_2)$ for $y_i,z_i\in\boldsymbol{R}^d$ with $i=1,2$. Let $\mu^{(2)}=(\mu_1,\mu_2)$, $v^{(2)}=(v_1,v_2)$ for $\mu_j,v_j\in\mathcal{P}$ with $j=1,2$.

We impose the following assumptions.

Assumption 1. There exists a constant K_1 such that

$$|\zeta(y^{(2)},\mu^{(2)})-\zeta(z^{(2)},v^{(2)})|^2\leqslant K_1\sum_{i=1}^2\left(|y_i-z_i|^2+W_2^2(\mu_i,v_i)\right),\ \zeta=f,g,h,$$

for all $y_i,z_i\in\boldsymbol{R}^d$, $\mu_i,v_i\in\mathcal{P}$ with $i=1,2$.

Assumption 2. There exists a constant K_2 such that

$$|\zeta(y^{(2)},\mu^{(2)})|^2\leqslant K_2\sum_{i=1}^2\left(|y_i|^2+W_2^2(\mu_i,\delta_0)\right)+k,\ \zeta=f,g,h,\tag{5}$$

for any $y_i\in\boldsymbol{R}^d$ and $\mu_i\in\mathcal{P}$ with $i=1,2$, where

$$k=2K_2\left(\left|f(0^{(2)},\delta_0^{(2)})\right|^2\vee\left|g(0^{(2)},\delta_0^{(2)})\right|^2\vee\left|h(0^{(2)},\delta_0^{(2)})\right|^2\right),$$

and

$$0^{(2)}=(0,0),\ \delta_0^{(2)}=(\delta_0,\delta_0),$$

where δ_0 denotes the Dirac measure at 0.

Assumption 3. There exists some $p\geqslant 2$ such that f,g, and h are in $M_G^p([-\tau,T];\boldsymbol{R}^d)$.

Remark 2. Under Assumptions 1–3, it follows from Theorem 2.1 in [16] and Theorem 2.2 in [1] that G-SMVDE (2) with the initial data ξ admits a unique solution $X(t;\xi)$ ($X(t)$ for short), which belongs to $M_G^2([-\tau,T];\boldsymbol{R}^d)$.

The following lemmas are crucial in stochastic calculus.

Lemma 2 ([26]). For any $0\leqslant r\leqslant T<\infty$, $p\geqslant 1$,

(1) $\hat{\boldsymbol{E}}\big[\int_0^T\eta_r\mathrm{d}B(r)\big]=0$; $\hat{\boldsymbol{E}}\big[\int_0^T\eta_r\mathrm{d}\langle B\rangle(r)\big]\leqslant\overline{\sigma}^2\hat{\boldsymbol{E}}[\int_0^T|\eta_r|\mathrm{d}r]$, $\forall\eta_r\in M_G^1([0,T])$,

(2) $\hat{\boldsymbol{E}}[\big(\int_0^T\eta_r\mathrm{d}B(r)\big)^2]=\hat{\boldsymbol{E}}[\int_0^T\eta_r^2\mathrm{d}\langle B\rangle(r)],\forall\eta_r\in M_G^2([0,T])$,

(3) $\hat{\boldsymbol{E}}[\int_0^T|\eta_r|^p\mathrm{d}r]\leqslant\int_0^T\hat{\boldsymbol{E}}[|\eta_r|^p]\mathrm{d}r,\forall\eta_r\in M_G^p([0,T])$.

Lemma 3 ([27]). For any $p\geqslant 1$ and $\eta\in M_G^p([0,T])$,

$$\hat{\boldsymbol{E}}\left[\sup_{0\leqslant u\leqslant T}\left|\int_0^u\eta(r)\mathrm{d}\langle B\rangle(r)\right|^p\right]\leqslant\overline{\sigma}^{2p}T^{p-1}\int_0^T\hat{\boldsymbol{E}}[|\eta(r)|^p]\mathrm{d}r.$$

Lemma 4 ([27]). For any $p \geqslant 2$ and $\eta \in M_G^p([0,T])$,

$$\hat{\boldsymbol{E}}\left(\sup_{0\leqslant u\leqslant T}\left|\int_0^u \eta(r)\mathrm{d}B(r)\right|^p\right) \leqslant \overline{\sigma}^p T^{\frac{p}{2}-1}\int_0^T \hat{\boldsymbol{E}}[|\eta(r)|^p]\mathrm{d}r.$$

We introduce the following notations and lemmas to establish the G-Itô formula for G-SMVEs.

Let $x_t = X(t)$, $f_t = f(\Psi(X_t), \mathcal{L}_{\Psi(X_t)})$, $g_t = g(\Psi(X_t), \mathcal{L}_{\Psi(X_t)})$, $h_t = h(\Psi(X_t), \mathcal{L}_{\Psi(X_t)})$ and $B_t = B(t)$. Then, system (4) can be rewritten as

$$\mathrm{d}x_t = f_t\mathrm{d}t + g_t\mathrm{d}B_t + h_t\mathrm{d}\langle B\rangle_t,\ x_0 \in L_G^2(\mathcal{F}_0;\boldsymbol{R}^d), \tag{6}$$

which satisfies $x_t \in \boldsymbol{R}^d$ for all t a.s.

Lemma 5. Under Assumption 2, it holds that $\hat{\boldsymbol{E}}\int_0^T(|f_t|^2+|g_t|^4+|h_t|^4)\mathrm{d}t < \infty$ for any $T>0$.

Proof. Let x_t be the solution of (6). It follows from the Hölder inequality, Lemmas 3 and 4 and Assumption 2 that for any $0 \leqslant t \leqslant T$,

$$\begin{aligned}
\hat{\boldsymbol{E}}\sup_{0\leqslant s\leqslant t}|x_s|^2 \leqslant & 4\hat{\boldsymbol{E}}|x_0|^2 + 4\hat{\boldsymbol{E}}\sup_{0\leqslant s\leqslant t}\left|\int_0^s f_r\mathrm{d}r\right|^2 + 4\hat{\boldsymbol{E}}\sup_{0\leqslant s\leqslant t}\left|\int_0^s g_r\mathrm{d}F_r\right|^2 + 4\hat{\boldsymbol{E}}\sup_{0\leqslant s\leqslant t}\left|\int_0^s h_r\mathrm{d}\langle B\rangle_r\right|^2 \\
\leqslant & 4\hat{\boldsymbol{E}}|x_0|^2 + 4t\hat{\boldsymbol{E}}\int_0^t|f_r|^2\mathrm{d}r + 4\overline{\sigma}^2\hat{\boldsymbol{E}}\int_0^t|g_r|^2\mathrm{d}r + 4\overline{\sigma}^4 t\hat{\boldsymbol{E}}\int_0^t|h_r|^2\mathrm{d}r \\
\leqslant & (4+4K_2t^2+4K_2\overline{\sigma}^2t+4K_2\overline{\sigma}^4t^2)\hat{\boldsymbol{E}}|x_0|^2 + 4k(t+\overline{\sigma}^2+\overline{\sigma}^4t)t \\
& + 8K_2(t+\overline{\sigma}^2+\overline{\sigma}^4t)\hat{\boldsymbol{E}}\int_0^t\sup_{0\leqslant s\leqslant r}|x_s|^2\mathrm{d}r \\
& + 4K_2(t+\overline{\sigma}^2+\overline{\sigma}^4t)\hat{\boldsymbol{E}}\int_0^t\sup_{0\leqslant s\leqslant r}[W_2^2(\mu_s,\delta_0)+W_2^2(\mu_{s-\tau},\delta_0)]\mathrm{d}r.
\end{aligned} \tag{7}$$

Eq. (7) together with (3) yields

$$\begin{aligned}
\hat{\boldsymbol{E}}\sup_{0\leqslant s\leqslant t}|x_s|^2 \leqslant & (4+8K_2t^2+8K_2\overline{\sigma}^2t+8K_2\overline{\sigma}^4t^2)\hat{\boldsymbol{E}}|x_0|^2 + 4k(t+\overline{\sigma}^2+\overline{\sigma}^4t)t \\
& + 16K_2(t+\overline{\sigma}^2+\overline{\sigma}^4t)\hat{\boldsymbol{E}}\int_0^t\sup_{0\leqslant s\leqslant r}|x_s|^2\mathrm{d}r.
\end{aligned} \tag{8}$$

The Gronwall formula gives

$$\begin{aligned}
\hat{\boldsymbol{E}}\sup_{0\leqslant t\leqslant T}|x_t|^2 \leqslant & \big[(4+8K_2T^2+8K_2\overline{\sigma}^2T+8K_2\overline{\sigma}^4T^2)\hat{\boldsymbol{E}}|x_0|^2 \\
& + 4k(T+\overline{\sigma}^2+\overline{\sigma}^4T)T\big]\cdot \mathrm{e}^{16K_2(T+\overline{\sigma}^2+\overline{\sigma}^4T)T}.
\end{aligned} \tag{9}$$

Similarly, we can derive

$$\hat{\boldsymbol{E}}\sup_{0\leqslant t\leqslant T}|x_t|^4 \leqslant C, \tag{10}$$

where C is a constant dependent on K_2, $\overline{\sigma}$, and T. Thus, it follows from Assumption 2 and (9) that

$$\begin{aligned}
\hat{\boldsymbol{E}}\int_0^T(|f_t|^2+|g_t|^4+|h_t|^4)\mathrm{d}t \leqslant & K_2\hat{\boldsymbol{E}}\int_0^T\big[|x_t|^2+|x_{t-\tau}|^2+W_2^2(\mu_t,\delta_0)+W_2^2(\mu_{t-\tau},\delta_0)+k\big]\mathrm{d}t \\
& + 10K_2^2\hat{\boldsymbol{E}}\int_0^T\big[|x_t|^4+|x_{t-\tau}|^4+W_2^4(\mu_t,\delta_0)+W_2^4(\mu_{t-\tau},\delta_0)+k^2\big]\mathrm{d}t \\
< & \infty.
\end{aligned}$$

Lemma 6. Let $\hat{\boldsymbol{E}}\int_0^T(|f_t|^2+|g_t|^4+|h_t|^4)\mathrm{d}t < \infty$ hold. Let $v \in C^{1,1}(\mathcal{P})$ such that for any compact subset $\kappa \subseteq \mathcal{P}$,

$$\sup_{\mu\in\kappa}\int_{\boldsymbol{R}^d}\big[|\partial_\mu v(\mu)(y)|^2+|\partial_y\partial_\mu v(\mu)(y)|^2\big]\mu(\mathrm{d}y) < \infty.$$

Then, for $\mu_t = \mathcal{L}_{x_t}$, it holds that

$$\begin{aligned} v(\mu_t) =& v(\mu_0) + \int_0^t \int_{\boldsymbol{R}^d} f_r \partial_\mu v(\mu_r)(y)\mu(\mathrm{d}y)\mathrm{d}r + \int_0^t \int_{\boldsymbol{R}^d} h_r \partial_\mu v(\mu_r)(y)\mu(\mathrm{d}y)\mathrm{d}\langle B\rangle_r \\ &+ \frac{1}{2}\int_0^t \int_{\boldsymbol{R}^d} g_r g_r^{\mathrm{T}} \partial_y(\partial_\mu v)(\mu_r)(y)\mu(\mathrm{d}y)\mathrm{d}\langle B\rangle_r. \end{aligned} \tag{11}$$

Proof. Let $\big((x_t^l)_{t\geqslant 0}\big)_{l\geqslant 1}$ be a sequence of independent and identically distributed (i.i.d.) copies of $(x_t)_{t\geqslant 0}$. That is, for any $l \geqslant 1$,

$$\mathrm{d}x_t^l = f_t^l \mathrm{d}t + g_t^l \mathrm{d}B_t^l + h_t^l \mathrm{d}\langle B_t^l\rangle,\ t \geqslant 0,$$

where $\big((f_t^l, g_t^l, h_t^l, B_t^l)_{t\geqslant 0}, x_0^l\big)_{l\geqslant 0}$ are i.i.d. copies of $\big((f_t, g_t, h_t, B_t)_{t\geqslant 0}, x_0\big)$. Recall the definition of the flow of marginal empirical measures:

$$\bar{\mu}_t^N = \frac{1}{N}\sum_{l=1}^N \delta_{x_t^l}.$$

Denote $u^N(x_t^1, x_t^2, \ldots, x_t^N) = v(\bar{\mu}_t^N) = v(\frac{1}{N}\sum_{l=1}^N \delta_{x_t^l})$. The G-Itô formula together with Proposition 3.1 in [28] yields

$$\begin{aligned} u^N(x_t^1, x_t^2, \ldots, x_t^N) =& u^N(x_0^1, x_0^2, \ldots, x_0^N) + \frac{1}{N}\sum_{l=1}^N \int_0^t f_r^l \partial_\mu v(\bar{\mu}_r^N)(x_r^l)\mathrm{d}r \\ &+ \frac{1}{N}\sum_{l=1}^N \int_0^t g_r^l \partial_\mu v(\bar{\mu}_r^N)(x_r^l)\mathrm{d}B_r^l + \frac{1}{N}\sum_{l=1}^N \int_0^t h_r^l \partial_\mu v(\bar{\mu}_r^N)(x_r^l)\mathrm{d}\langle B_r^l\rangle \\ &+ \frac{1}{2N}\sum_{l=1}^N \int_0^t g_r^l (g_r^l)^{\mathrm{T}} \partial_y(\partial_\mu v)(\bar{\mu}_r^N)(x_r^l)\mathrm{d}\langle B_r^l\rangle \\ &+ \frac{1}{2N^2}\sum_{l=1}^N \int_0^t g_r^l (g_r^l)^{\mathrm{T}} (\partial_\mu^2 v)(\bar{\mu}_r^N)(x_r^l)(x_r^l)\mathrm{d}\langle B_r^l\rangle. \end{aligned} \tag{12}$$

Taking the G-expectation on both sides of (12), we have

$$\begin{aligned} \hat{\boldsymbol{E}}[v(\bar{\mu}_t^N)] \leqslant& \hat{\boldsymbol{E}}[v(\bar{\mu}_0^N)] + \frac{1}{N}\sum_{l=1}^N \hat{\boldsymbol{E}}\left[\int_0^t f_r^l \partial_\mu v(\bar{\mu}_r^N)(x_r^l)\mathrm{d}r\right] \\ &+ \frac{1}{N}\sum_{l=1}^N \hat{\boldsymbol{E}}\left[\int_0^t h_r^l \partial_\mu v(\bar{\mu}_r^N)(x_r^l)\mathrm{d}\langle B_r^l\rangle\right] \\ &+ \frac{1}{2N}\sum_{l=1}^N \hat{\boldsymbol{E}}\left[\int_0^t g_r^l (g_r^l)^{\mathrm{T}} \partial_y(\partial_\mu v)(\bar{\mu}_r^N)(x_r^l)\mathrm{d}\langle B_r^l\rangle\right] \\ &+ \frac{1}{2N^2}\sum_{l=1}^N \hat{\boldsymbol{E}}\left[\int_0^t g_r^l (g_r^l)^{\mathrm{T}} (\partial_\mu^2 v)(\bar{\mu}_r^N)(x_r^l, x_r^l)\mathrm{d}\langle B_r^l\rangle\right]. \end{aligned}$$

Noting that the processes $\big((f_r^l, g_r^l, h_r^l, x_r^l)_{r\in[0,t]}\big)_{l\in\{1,2,\ldots,N\}}$ are i.i.d., we deduce that

$$\hat{\boldsymbol{E}}[v(\bar{\mu}_t^N)] \leqslant \hat{\boldsymbol{E}}[v(\bar{\mu}_0^N)] + \hat{\boldsymbol{E}}\left[\int_0^t f_r^1 \partial_\mu v(\bar{\mu}_r^N)(x_r^1)\mathrm{d}r\right] \tag{13}$$

$$+ \hat{\boldsymbol{E}}\left[\int_0^t h_r^1 \partial_\mu v(\bar{\mu}_r^N)(x_r^1)\mathrm{d}\langle B_r^1\rangle\right] \tag{14}$$

$$+ \frac{1}{2}\hat{\boldsymbol{E}}\left[\int_0^t g_r^1 (g_r^1)^{\mathrm{T}} \partial_y(\partial_\mu v)(\bar{\mu}_r^N)(x_r^1)\mathrm{d}\langle B_r^1\rangle\right] \tag{15}$$

$$+\frac{1}{2N}\hat{\boldsymbol{E}}\left[\int_0^t g_r^1(g_r^1)^{\mathrm{T}}(\partial_\mu^2 v)(\bar{\mu}_r^N)(x_r^1,x_r^1)\mathrm{d}\langle B_r^1\rangle\right]. \tag{16}$$

We see that (16) converges to 0 as $N\to\infty$. Recalling Theorem 10.2.7 in [29], we have

$$\lim_{N\to\infty}\hat{\boldsymbol{E}}\left[\sup_{0\leqslant r\leqslant t}W_2^2(\bar{\mu}_r^N,\mu_r)\right]=0. \tag{17}$$

It follows from the uniform continuity of v with respect to the Wasserstein distance that $\hat{\boldsymbol{E}}[v(\bar{\mu}_t^N)]$ converges to $\hat{\boldsymbol{E}}[v(\mu_t)]$. Combining the uniform continuity of $\partial_\mu v$ on $\mathcal{P}$ with (17), the term on the right-hand side of (13) and the terms (14) and (15) converge. The proof is completed.

Lemma 6 can be used to derive the G-Itô formula for a function that depends on (t,x,μ). We introduce another probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{P})$, $(\widetilde{\mathcal{F}}_t)_{t\geqslant 0}$ is a filtration, $\widetilde{B}$, $\widetilde{b}$, $\widetilde{h}$, and $\widetilde{g}$ are defined on this probability space, and they have the same laws as B, b, h, and g. Assume that $\widetilde{B}$ is G-Brownian motion. Then,

$$\mathrm{d}\widetilde{x}_t=\widetilde{b}_t\mathrm{d}t+\widetilde{g}_t\mathrm{d}\widetilde{B}_t+\widetilde{h}_t\mathrm{d}\langle\widetilde{B}\rangle_t,\ \widetilde{x}_0\in L_G^2(\widetilde{\mathcal{F}}_0,\boldsymbol{R}^d)$$

is another G-Itô process that satisfies $\widetilde{x}_t\in\boldsymbol{R}^d$ for all t a.s. Moreover, if we consider the probability space $(\Omega\otimes\widetilde{\Omega},F\otimes\widetilde{\mathcal{F}},P\otimes\widetilde{P})$, we see that the processes x_t and $\widetilde{x}_t$ are independent in this new space. Thus, we have the following G-Itô formula.

Theorem 3. Assume that $\hat{\boldsymbol{E}}\int_0^T(|f_t|^2+|g_t|^4+|h_t|^4)\mathrm{d}t<\infty$ holds. Let $v\in C_b^{2,2;1}([0,T]\times\boldsymbol{R}^d\times\mathcal{P})$ such that for any compact subset $\kappa\subseteq\mathcal{P}$,

$$\sup_{t,x,\mu\in\kappa}\int_{\boldsymbol{R}^d}\left(|\partial_\mu v(t,x,\mu)(y)|^2+|\partial_y\partial_\mu v(t,x,\mu)(y)|^2\right)\mu(\mathrm{d}y)<\infty.$$

Then, it holds that for $\mu_t=\mathcal{L}_{\widetilde{x}_t}$,

$$\begin{aligned}v(t,x_t,\mu_t)=&v(0,x_0,\mu_0)+\int_0^t\partial_r v(r,x_r,\mu_r)\mathrm{d}r+\int_0^t f_r\partial_x v(r,x_r,\mu_r)\mathrm{d}r\\&+\int_0^t\left(h_r\partial_x v(r,x_r,\mu_r)+\frac{1}{2}g_rg_r^{\mathrm{T}}\partial_x^2 v(r,x_r,\mu_r)\right)\mathrm{d}\langle B\rangle_r\\&+\int_0^t g_r\partial_x v(r,x_r,\mu_r)\mathrm{d}B_r+\int_0^t\int_{\boldsymbol{R}^d}f_r\partial_\mu v(r,x_r,\mu_r)(y)\mu(\mathrm{d}y)\mathrm{d}r\\&+\frac{1}{2}\int_0^t\int_{\boldsymbol{R}^d}g_rg_r^{\mathrm{T}}\partial_x(\partial_\mu v)(r,x_r,\mu_r)(y)\mu(\mathrm{d}y)\mathrm{d}\langle B\rangle_r\\&+\int_0^t\int_{\boldsymbol{R}^d}h_r\partial_\mu v(r,x_r,\mu_r)(y)\mu(\mathrm{d}y)\mathrm{d}\langle B\rangle_r.\end{aligned} \tag{18}$$

Proof. Fix $(\bar{t},\bar{x})$ and apply Lemma 6 to the function $v(\bar{t},\bar{x},\mu)$ with the law $\mu_t=\mathcal{L}_{\widetilde{x}_t}$. Then

$$\begin{aligned}v(\bar{t},\bar{x},\mu_t)=&v(\bar{t},\bar{x},\mu_0)+\int_0^t\int_{\boldsymbol{R}^d}\widetilde{b}_r\partial_\mu v(\bar{t},\bar{x},\mu_r)(\widetilde{x}_r)\mu(\mathrm{d}\widetilde{x}_r)\mathrm{d}r\\&+\int_0^t\int_{\boldsymbol{R}^d}\widetilde{h}_r\partial_\mu v(\bar{t},\bar{x},\mu_r)(\widetilde{x}_r)\mu(\mathrm{d}\widetilde{x}_r)\mathrm{d}\langle\widetilde{B}\rangle_r\\&+\frac{1}{2}\int_0^t\int_{\boldsymbol{R}^d}\widetilde{g}_r\widetilde{g}_r^{\mathrm{T}}\partial_y(\partial_\mu v)(\bar{t},\bar{x},\mu_r)(\widetilde{x}_r)\mu(\mathrm{d}\widetilde{x}_r)\mathrm{d}\langle\widetilde{B}\rangle_r.\end{aligned}$$

We thus see that the map $t\to v(\bar{t},\bar{x},\mu_t)$ is absolutely continuous for all $(\bar{t},\bar{x})$ and so for almost all t we have

$$\begin{aligned}\partial_t v(\bar{t},\bar{x},\mu_t)=&\mathrm{d}\left(\int_0^t\int_{\boldsymbol{R}^d}\widetilde{b}_r\partial_\mu v(r,x_r,\mu_r)(\widetilde{x}_r)\mu(\mathrm{d}\widetilde{x}_r)\mathrm{d}r\right.\\&+\frac{1}{2}\int_0^t\int_{\boldsymbol{R}^d}\widetilde{g}_r\widetilde{g}_r^{\mathrm{T}}\partial_x(\partial_\mu v)(r,x_r,\mu_r)(\widetilde{x}_r)\mu(\mathrm{d}\widetilde{x}_r)\mathrm{d}\langle\widetilde{B}\rangle_r\end{aligned}$$

$$+\int_0^t \int_{\boldsymbol{R}^d} \widetilde{h}_r \partial_\mu v(r, x_r, \mu_r)(\widetilde{x}_r)\mu(\mathrm{d}\widetilde{x}_r)\mathrm{d}\langle \widetilde{B}\rangle_r \Bigg) \Big/ \mathrm{d}t.$$

Let $\bar{v}(t,x) = v(t,x,\mu_t)$, then

$$\begin{aligned}\partial_t \bar{v}(t,x) =& (\partial_t v)(t,x,\mu_t) + \mathrm{d}\left(\int_0^t \int_{\boldsymbol{R}^d} \widetilde{b}_r \partial_\mu v(r, x_r, \mu_r)(\widetilde{x}_r)\mu(\mathrm{d}\widetilde{x}_r)\mathrm{d}r\right.\\ &+ \frac{1}{2}\int_0^t \int_{\boldsymbol{R}^d} \widetilde{g}_r \widetilde{g}_r^{\mathrm{T}} \partial_x(\partial_\mu v)(r, x_r, \mu_r)(\widetilde{x}_r)\mu(\mathrm{d}\widetilde{x}_r)\mathrm{d}\langle \widetilde{B}\rangle_r\\ &\left.+\int_0^t \int_{\boldsymbol{R}^d} \widetilde{h}_r \partial_\mu v(r, x_r, \mu_r)(\widetilde{x}_r)\mu(\mathrm{d}\widetilde{x}_r)\mathrm{d}\langle \widetilde{B}\rangle_r\right) \Big/ \mathrm{d}t.\end{aligned}$$

The completeness of $C^{2,2;1}$ functions and the limiting argument together with the classical Itô formula yield

$$\begin{aligned}\bar{v}(t,x) - \bar{v}(0,x_0) =& \int_0^t \partial_r v(r,x_r,\mu_r)\mathrm{d}r + \int_0^t f_r \partial_x v(r,x_r,\mu_r)\mathrm{d}r\\ &+ \int_0^t \left(h_r \partial_x v(r,x_r,\mu_r) + \frac{1}{2} g_r g_r^{\mathrm{T}} \partial_x^2 v(r,x_r,\mu_r)\right)\mathrm{d}\langle B\rangle_r\\ &+ \int_0^t g_r \partial_x v(r,x_r,\mu_r)\mathrm{d}B(r) + \int_0^t \int_{\boldsymbol{R}^d} f_r \partial_\mu v(r,x_r,\mu_r)(y)\mu(\mathrm{d}y)\mathrm{d}r\\ &+ \frac{1}{2}\int_0^t \int_{\boldsymbol{R}^d} g_r g_r^{\mathrm{T}} \partial_x(\partial_\mu v)(r,x_r,\mu_r)(y)\mu(\mathrm{d}y)\mathrm{d}\langle B\rangle_r\\ &+ \int_0^t \int_{\boldsymbol{R}^d} h_r \partial_\mu v(r,x_r,\mu_r)(y)\mu(\mathrm{d}y)\mathrm{d}\langle B\rangle_r. \end{aligned} \tag{19}$$

The proof is completed.

Remark 3. The G-Itô formula (18) for G-SMVEs involves the derivatives with respect to the law, which makes it novel and different from the Itô formulas for G-SDEs and SMVEs. We establish it by generalizing the Itô formula for SMVEs and the G-Itô formula for G-SDEs, both of which can be considered special cases of the G-Itô formula (18). We can also establish it using the Taylor expansion technique, which is identical to the approach to obtaining the G-Itô formula for G-SDEs, and one can see Peng [1] for the details.

We define the following operator for future use.

Definition 3. For $\mathscr{V} \in C_b^{2,2;1}(\boldsymbol{R}^d \times \boldsymbol{R}^d \times \mathcal{P} \times \mathcal{P}; \boldsymbol{R}^+)$, the operator $L^\mu \mathscr{V} : \boldsymbol{R}^d \times \boldsymbol{R}^d \times \mathcal{P} \times \mathcal{P} \to \boldsymbol{R}$ for G-SMVDE (4) is defined as

$$\begin{aligned}L^\mu \mathscr{V}(x,\mu) =& \partial_x \mathscr{V}(x,\mu) f(\Psi(X_t), \mathcal{L}_{\Psi(X_t)}) + \int_{\boldsymbol{R}^d} f(y, \mathcal{L}_{\Psi(X_t)}) \partial_\mu \mathscr{V}(x,\mu)(y)\mu(\mathrm{d}y)\\ &+ \boldsymbol{G}\left(\partial_x^2 \mathscr{V}(x,\mu)(gg^{\mathrm{T}})(\Psi(X_t), \mathcal{L}_{\Psi(X_t)}) + \int_{\boldsymbol{R}^d} (gg^{\mathrm{T}})(y, \mathcal{L}_{\Psi(X_t)}) \partial_x(\partial_\mu \mathscr{V})(x,\mu)(y)\mu(\mathrm{d}y)\right.\\ &\left.+2\int_{\boldsymbol{R}^d} h(y, \mathcal{L}_{\Psi(X_t)}) \partial_\mu \mathscr{V}(x,\mu)(y)\mu(\mathrm{d}y) + 2\partial_x \mathscr{V}(x,\mu) h(\Psi(X_t), \mathcal{L}_{\Psi(X_t)})\right). \end{aligned} \tag{20}$$

3 Exponential stabilities

It is well-known that the stability properties of stochastic dynamical systems are crucial for studying such systems. In this section, we will study two types of exponential stability for G-SMVDE (4). We first introduce the corresponding definitions.

Definition 4. G-SMVDE (4) is said to be mean square exponentially stable if there exists a pair of positive constants γ and C such that, for any ξ in $M_G^2([-\tau,0]; \boldsymbol{R}^d)$,

$$\hat{\boldsymbol{E}}|X_t|^2 \leqslant C\hat{\boldsymbol{E}}|\xi|^2 \mathrm{e}^{-\gamma t}.$$

Definition 5. G-SMVDE (4) is said to be quasi sure exponentially stable if there exists a constant $\lambda > 0$ such that, for any ξ in $M_G^2([-\tau, 0]; \boldsymbol{R}^d)$,

$$\lim_{t\to\infty} \sup \frac{1}{t} \ln |X_t| \leqslant -\lambda, \text{ q.s.} \tag{21}$$

Theorem 4. Let Assumption 1 hold. Assume that there exists a function $\mathscr{V}(x, \mu) \in C_b^{2,2;1}(\boldsymbol{R}^d \times \boldsymbol{R}^d \times \mathcal{P} \times \mathcal{P}; \boldsymbol{R}^+)$ and three positive constants γ, c_1, c_2 such that

$$c_1 \int_{\boldsymbol{R}^d} |x|^2 \mu(\mathrm{d}x) \leqslant \int_{\boldsymbol{R}^d} \mathscr{V}(x, \mu)\mu(\mathrm{d}x) \leqslant c_2 \int_{\boldsymbol{R}^d} |x|^2 \mu(\mathrm{d}x), \tag{22}$$

and

$$\int_{\boldsymbol{R}^d} (L^\mu \mathscr{V}(x, \mu) + \gamma \mathscr{V}(x, \mu))\, \mu(\mathrm{d}x) \leqslant 0 \tag{23}$$

hold for any $(x, \mu) \in \boldsymbol{R}^d \times \boldsymbol{R}^d \times \mathcal{P} \times \mathcal{P}$. Then the solution to G-SMVDE (4) satisfies

$$\hat{\boldsymbol{E}}|X_t|^2 \leqslant \frac{c_2}{c_1} \mathrm{e}^{-\gamma t} \hat{\boldsymbol{E}}|\xi|^2, \ t \geqslant 0. \tag{24}$$

Furthermore, let $\lambda = \frac{\gamma}{2}$. Then, for any $t \geqslant 0$,

$$\lim_{t\to\infty} \sup \frac{1}{t} \ln |X_t| \leqslant -\lambda, \text{ q.s.} \tag{25}$$

Proof. Using G-Itô's formula (18), we have

$$\begin{aligned}
&\mathrm{e}^{\gamma t}\mathscr{V}(\Psi(X_t), \mathcal{L}_{\Psi(X_t)}) - \mathscr{V}(\Psi(X_0), \mathcal{L}_{\Psi(X_0)}) \\
&= \int_0^t \gamma \mathrm{e}^{\gamma r}\mathscr{V}(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\mathrm{d}r + \int_0^t \mathrm{e}^{\gamma r} f(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\partial_x \mathscr{V}(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\mathrm{d}r \\
&\quad + \int_0^t \mathrm{e}^{\gamma r} g(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\partial_x \mathscr{V}(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\mathrm{d}B_r \\
&\quad + \int_0^t \mathrm{e}^{\gamma r} h(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\partial_x \mathscr{V}(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\mathrm{d}\langle B\rangle_r \\
&\quad + \int_0^t \int_{\boldsymbol{R}^d} \mathrm{e}^{\gamma r} f(y, \mathcal{L}_{\Psi(X_r)})\partial_\mu \mathscr{V}(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})(y)\mu(\mathrm{d}y)\mathrm{d}r \\
&\quad + \frac{1}{2}\int_0^t \mathrm{e}^{\gamma r} (gg^{\mathrm{T}})(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\partial_x^2 \mathscr{V}(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\mathrm{d}\langle B\rangle_r \\
&\quad + \frac{1}{2}\int_0^t \int_{\boldsymbol{R}^d} \mathrm{e}^{\gamma r} (gg^{\mathrm{T}})(y, \mathcal{L}_{\Psi(X_r)})(\partial_x \partial_\mu \mathscr{V})(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})(y)\mu(\mathrm{d}y)\mathrm{d}\langle B\rangle_r \\
&\quad + \int_0^t \int_{\boldsymbol{R}^d} \mathrm{e}^{\gamma r} h(y, \mathcal{L}_{\Psi(X_r)})\partial_\mu \mathscr{V}(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})(y)\mu(\mathrm{d}y)\mathrm{d}\langle B\rangle_r.
\end{aligned} \tag{26}$$

Thus, we have

$$\begin{aligned}
\mathrm{e}^{\gamma t}\mathscr{V}(\Psi(X_t), \mathcal{L}_{\Psi(X_t)}) =& \mathscr{V}(\Psi(X_0), \mathcal{L}_{\Psi(X_0)}) + \int_0^t \mathrm{e}^{\gamma r}\big(\gamma \mathscr{V}(\Psi(X_r), \mathcal{L}_{\Psi(X_r)}) + L^\mu \mathscr{V}(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\big)\mathrm{d}r \\
&+ \int_0^t \mathrm{e}^{\gamma r} g(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\partial_x \mathscr{V}(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\mathrm{d}B_r + M_0^t,
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
M_0^t = \int_0^t \mathrm{e}^{\gamma r} \Bigg(& h(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\partial_x \mathscr{V}(\Psi(X_r), \mathcal{L}_{\Psi(X_r)}) + \frac{1}{2}(gg^{\mathrm{T}})(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})\partial_x^2 \mathscr{V}(\Psi(X_r), \mathcal{L}_{\Psi(X_r)}) \\
&+ \frac{1}{2}\int_{\boldsymbol{R}^d} (gg^{\mathrm{T}})(y, \mathcal{L}_{\Psi(X_r)})(\partial_x \partial_\mu \mathscr{V})(\Psi(X_r), \mathcal{L}_{\Psi(X_r)})(y)\mu(\mathrm{d}y)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\boldsymbol{R}^d} h(y,\mathcal{L}_{\Psi(X_r)})\partial_\mu\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})(y)\mu(\mathrm{d}y)\Big)\,\mathrm{d}\langle B\rangle_r\\
&-\int_0^t \mathrm{e}^{\gamma r}\boldsymbol{G}\Big(\partial_x^2\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})(gg^{\mathrm{T}})(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\\
&+r\int_{\boldsymbol{R}^d}(gg^{\mathrm{T}})(y,\mathcal{L}_{\Psi(X_r)})\partial_x(\partial_\mu\mathscr{V})(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})(y)\mu(\mathrm{d}y)\\
&+2\int_{\boldsymbol{R}^d}\partial_\mu\mathscr{V}(x,\mu)(y)h(y,\mathcal{L}_{\Psi(X_r)})\mu(\mathrm{d}y)+2\partial_x\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})h(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\Big)\,\mathrm{d}r.
\end{aligned}\tag{28}
$$

It follows from Theorem 13 in [1] that M_0^t is a G-martingale. Taking the G-expectation on both sides of (27) gives

$$
\begin{aligned}
\hat{\boldsymbol{E}}\big[\mathrm{e}^{\gamma t}\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})\big]\leqslant&\hat{\boldsymbol{E}}\big[\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})\big]+\hat{\boldsymbol{E}}\bigg[\int_0^t\mathrm{e}^{\gamma r}\big(\gamma\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\\
&+L^\mu\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\big)\,\mathrm{d}r\bigg].
\end{aligned}
$$

It follows from the Jensen inequality and (23) that

$$
\hat{\boldsymbol{E}}\big[\mathrm{e}^{\gamma t}\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})\big]\leqslant\hat{\boldsymbol{E}}[\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})],
$$

which yields

$$
\hat{\boldsymbol{E}}[\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})]\leqslant\mathrm{e}^{-\gamma t}\hat{\boldsymbol{E}}[\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})].
$$

The definition of $\Psi(X_t)$ and inequality (22) yield

$$
\begin{aligned}
c_1\hat{\boldsymbol{E}}|X_t|^2\leqslant&c_1\hat{\boldsymbol{E}}|\Psi(X_t)|^2\leqslant\hat{\boldsymbol{E}}[\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})]\leqslant\mathrm{e}^{-\gamma t}\hat{\boldsymbol{E}}[\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})]\\
\leqslant&c_2\mathrm{e}^{-\gamma t}\hat{\boldsymbol{E}}|\Psi(X_0)|^2\leqslant 2c_2\mathrm{e}^{-\gamma t}\hat{\boldsymbol{E}}|\xi|^2.
\end{aligned}
$$

Thus, we have

$$
\hat{\boldsymbol{E}}|X_t|^2\leqslant\frac{2c_2}{c_1}\mathrm{e}^{-\gamma t}\hat{\boldsymbol{E}}|\xi|^2.\tag{29}
$$

For any $t\geqslant 0$, we find a positive integer n such that $n-1\leqslant t\leqslant n$. Thus, it follows from (29) that

$$
\hat{\boldsymbol{E}}\left[\sup_{n-1\leqslant t\leqslant n}|X_t|^2\right]\leqslant C\mathrm{e}^{-\gamma n},\tag{30}
$$

where $C=\frac{2c_2}{c_1}\mathrm{e}^{\gamma}\hat{\boldsymbol{E}}|\xi|^2$. Hence, for any $\epsilon>0$,

$$
\hat{\mathbb{C}}\left\{\omega:\sup_{n-1\leqslant t\leqslant n}|X_t|^2>\mathrm{e}^{(-\gamma+\epsilon)n}\right\}\leqslant C\mathrm{e}^{-\epsilon n}.
$$

Given the well-known Borel-Cantelli lemma, one sees that for almost all $\omega\in\Omega$, there is a random integer $n_0=n_0(\omega)$ such that

$$
\sup_{n-1\leqslant t\leqslant n}|X_t|^2\leqslant\mathrm{e}^{(-\gamma+\epsilon)n},\ \text{as } n\geqslant n_0.
$$

Thus, for almost all $\omega\in\Omega$, if $n-1\leqslant t\leqslant n$ and $n\geqslant n_0$,

$$
\frac{1}{t}\log|X_t|\leqslant\frac{(-\gamma+\epsilon)n}{2(n-1)}.
$$

Then

$$
\lim_{t\to\infty}\sup\frac{1}{t}\log|X_t|\leqslant\frac{-\gamma+\epsilon}{2}=-\frac{\gamma}{2}+\frac{\epsilon}{2}.
$$

The assertion (21) holds by letting $\epsilon\to 0$.

Remark 4. As we all know, the Lyapunov function and the Rszumikhin technique play important roles in the stochastic stability analysis (e.g., [30, 31]). We construct a Lyapunov functional that contains not only the current and past state variables but also the law of the variables for G-SMVDE (4), whereas the previous Lyapunov functionals used in the stochastic stability analysis only contain the current and past state variables. This feature is essential for our functional.

4 Almost sure asymptotic stability

We know that the p-moment stability and the trajectory stability are two types of classical stability in the stochastic stability analysis. Consequently, we consider the almost sure asymptotic stability of G-SMVDE (4) in this section. Let us first introduce the corresponding definition.

Definition 6. G-SMVDE (4) is said to be almost surely asymptotically stable if, for any initial data $X_0 \in \boldsymbol{R}^d$, it holds that

$$\hat{\mathbb{C}}\left\{\lim_{t\to\infty}|X_t|=0\right\}=1. \tag{31}$$

Proposition 3 ([1])**.** Denote a lower expectation $\hat{\hat{\boldsymbol{E}}}[X] = -\hat{\boldsymbol{E}}[-X]$ for each $X \in \mathcal{H}$. Then, $\hat{\hat{\boldsymbol{E}}}[\cdot]$ is a nonlinear expectation.

Now, we strengthen (22) and (23) to prove the almost sure asymptotic stability of G-SMVDE (4). We introduce a function class. Let κ denote the family of functions $\gamma : \boldsymbol{R}^+ \to \boldsymbol{R}^+$, which are continuous, strictly increasing, and $\gamma(0) = 0$. Let κ_∞ denote the family of functions $\gamma \in \kappa$ with $\gamma(x) \to \infty$ as $x \to \infty$.

Theorem 5. Suppose that Assumption 1 holds. If there exist a function $\mathscr{V}(x,\mu) \in C_b^{2,2;1}(\boldsymbol{R}^d \times \boldsymbol{R}^d \times \mathcal{P} \times \mathcal{P}; \boldsymbol{R}^+)$, a nonnegative constant β, and two class functions $\gamma_1,\ \gamma_2 \in \kappa_\infty$ such that

$$L^\mu \mathscr{V}(x,\mu) + \beta\mathscr{V}(x,\mu) \leqslant 0, \tag{32}$$

and

$$\gamma_1(|x|) \leqslant \mathscr{V}(x,\mu) \leqslant \gamma_2(|x|) \tag{33}$$

hold, then, G-SMVDE (4) is almost surely asymptotically stable.

Proof. By considering (33) and the norm of $\Psi(X_t)$, we only need to prove $\mathscr{V}(\Psi(X_t), \mathcal{L}_{\Psi(X_t)}) = 0$. The G-Itô formula (18) gives

$$\begin{aligned}
&\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)}) - \mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})\\
&= \int_0^t f(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\partial_x\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\mathrm{d}r + \int_0^t g(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\partial_x\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\mathrm{d}B_r\\
&\quad + \int_0^t h(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\partial_x\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\mathrm{d}\langle B\rangle_r\\
&\quad + \frac{1}{2}\int_0^t (gg^{\mathrm{T}})(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\partial_x^2\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\mathrm{d}\langle B\rangle_r\\
&\quad + \int_0^t\int_{\boldsymbol{R}^d} f(y,\mathcal{L}_{\Psi(X_r)})\partial_\mu\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})(y)\mu(\mathrm{d}y)\mathrm{d}r\\
&\quad + \int_0^t\int_{\boldsymbol{R}^d} h(y,\mathcal{L}_{\Psi(X_r)})\partial_\mu\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})(y)\mu(\mathrm{d}y)\mathrm{d}\langle B\rangle_r\\
&\quad + \frac{1}{2}\int_0^t\int_{\boldsymbol{R}^d} (gg^{\mathrm{T}})(y,\mathcal{L}_{\Psi(X_r)})(\partial_x\partial_\mu\mathscr{V})(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})(y)\mu(\mathrm{d}y)\mathrm{d}\langle B\rangle_r.
\end{aligned} \tag{34}$$

Thus, it yields

$$\begin{aligned}
\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)}) =& \mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)}) + \int_0^t L^\mu\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\mathrm{d}r\\
&+ \int_0^t g(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\partial_x\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\mathrm{d}B_r + M_0^t\\
\leqslant& \mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)}) - \int_0^t \beta\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\mathrm{d}r\\
&+ \int_0^t g(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\partial_x\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\mathrm{d}B_r + M_0^t\\
\leqslant& \mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)}) + M_0^t
\end{aligned}$$

$$+\int_0^t g(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\partial_x\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\mathrm{d}B_r, \tag{35}$$

where M_0^t is defined in (28). Considering that X_t is the solution to G-SMVDE (4), we see that $\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})$ is adapted to $\{\mathcal{F}_t\}_{t\geqslant 0}$ according to the Doob-Dynkin lemma. Taking the conditional G-expectation on both sides of (35), we have

$$\begin{aligned}\hat{\boldsymbol{E}}[\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})|\mathcal{F}_0] \leqslant&\hat{\boldsymbol{E}}[\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})|\mathcal{F}_0]+\hat{\boldsymbol{E}}[M_0^t|\mathcal{F}_0]\\&+\hat{\boldsymbol{E}}\left[\int_0^t g(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\partial_x\mathscr{V}(\Psi(X_r),\mathcal{L}_{\Psi(X_r)})\mathrm{d}B_r|\mathcal{F}_0\right]\\ \leqslant&\hat{\boldsymbol{E}}[\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})],\end{aligned} \tag{36}$$

which implies that $\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})$ is a supermartingale with respect to $\{\mathcal{F}_t\}_{t\geqslant 0}$.

When $X_0=0$, (33) together with (36) yields $\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})=0$ a.s. for $t\geqslant 0$. Thus, $\hat{\mathbb{C}}\{\lim_{t\to\infty}|X_t|=0\}=1$ holds.

When $X_0\neq 0$, we decompose the sample space into the following three mutually incompatible sets:

(1) $E_1=\{\omega:\lim_{t\to\infty}\sup\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})=0\}$;

(2) $E_2=\{\omega:\lim_{t\to\infty}\inf\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})>0\}$;

(3) $E_3=\{\omega:\lim_{t\to\infty}\inf\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})=0 \text{ and} \lim_{t\to\infty}\sup\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})>0\}$.

Thus, to prove the asymptotical stability, we turn to prove that $\hat{\mathbb{C}}\{E_2\}=\hat{\mathbb{C}}\{E_3\}=0$ and $\hat{\mathbb{C}}\{E_1\}=1$.

Let $\tau_r=\inf\{t\geqslant 0:|X_t|\geqslant r\}$. Using G-Itô's formula (18), we have

$$\begin{aligned}\hat{\boldsymbol{E}}[\mathscr{V}(\Psi(X_{t\wedge\tau_r}),\mathcal{L}_{\Psi(X_{t\wedge\tau_r})})]=&\hat{\boldsymbol{E}}\big[\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})\big]+\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}L^\mu\mathscr{V}(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})\mathrm{d}s\right]\\ \leqslant&\hat{\boldsymbol{E}}\big[\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})\big]+\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}-\beta\mathscr{V}(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})\mathrm{d}s\right]\\ =&\hat{\boldsymbol{E}}\big[\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})\big]-\widehat{\hat{\boldsymbol{E}}}^2_{\{\xi\}}\left[\int_0^{t\wedge\tau_r}\beta\mathscr{V}(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})\mathrm{d}s\right].\end{aligned} \tag{37}$$

Thus,

$$\widehat{\hat{\boldsymbol{E}}}\left[\int_0^{t\wedge\tau_r}\beta\mathscr{V}(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})\mathrm{d}s\right]\leqslant\hat{\boldsymbol{E}}[\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})]. \tag{38}$$

Letting $t\to\infty$ and $r\to\infty$, we have

$$\widehat{\hat{\boldsymbol{E}}}\left[\int_0^{\infty}\mathscr{V}(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})\mathrm{d}s\right]\leqslant\frac{\hat{\boldsymbol{E}}[\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})]}{\beta}. \tag{39}$$

Eq. (39) together with the Fatou lemma yields

$$\lim_{t\to\infty}\inf\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})=0 \text{ a.s.}$$

Thus, $\hat{\mathbb{C}}\{E_2\}=0$ holds.

The proof by contradiction will be used to prove $\hat{\mathbb{C}}\{E_3\}=0$. Suppose that $\hat{\mathbb{C}}\{E_3\}>0$, and there exists a pair of constants $\varepsilon_1>0$ and $\epsilon_0>0$ such that

$$\hat{\mathbb{C}}\big\{\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)}) \text{ crosses from below } \varepsilon_1 \text{ to above } 2\varepsilon_1 \text{ and back infinitely many times}\big\}\geqslant\epsilon_0. \tag{40}$$

For any $t\in[0,T]$, using G-Itô's formula (18), it follows from the elementary inequality, Lemma 2, and Assumption 1 that

$$\begin{aligned}&\hat{\boldsymbol{E}}\left[\sup_{0\leqslant s\leqslant t}|\Psi(X_{s\wedge\tau_r})-\Psi(X_0)|^2\right]\\ &\leqslant 2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}f(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})|\Psi(X_s)-\Psi(X_0)|\mathrm{d}s\right]+2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}h(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})|\Psi(X_s)-\Psi(X_0)|\mathrm{d}\langle B\rangle_s\right]\end{aligned}$$

$$
\begin{aligned}
&+\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}(gg^{\rm T})(\Psi(X_s),\mathcal{L}_{\Psi(X_s)}){\rm d}\langle B\rangle_s\right]+2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}g(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})|\Psi(X_s)-\Psi(X_0)|{\rm d}B_s\right]\\
&\leqslant 2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|f(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})||\Psi(X_s)-\Psi(X_0)|{\rm d}s\right]+2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|h(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})||\Psi(X_s)-\Psi(X_0)|{\rm d}\langle B\rangle_s\right]\\
&+\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|g(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})|^2{\rm d}\langle B\rangle_s\right]+2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|g(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})||\Psi(X_s)-\Psi(X_0)|{\rm d}B_s\right]\\
&\leqslant\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|f(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})|^2{\rm d}s\right]+\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|\Psi(X_s)-\Psi(X_0)|^2{\rm d}s\right]+\overline{\sigma}^2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|h(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})|^2{\rm d}s\right]\\
&+\overline{\sigma}^2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|\Psi(X_s)-\Psi(X_0)|^2{\rm d}s\right]+\overline{\sigma}^2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|g(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})|^2{\rm d}s\right]\\
&+\frac{1}{2}\hat{\boldsymbol{E}}\left[\sup_{0\leqslant s\leqslant t}|\Psi(X_{s\wedge\tau_r})-\Psi(X_0)|^2\right]+8\overline{\sigma}^2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|g(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})|^2{\rm d}s\right]\\
&\leqslant\big(1+10\overline{\sigma}^2\big)K_2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}\big(W_2(\mu_i,\delta_0)^2+W_2(\mu_{i-\tau},\delta_0)^2+k\big){\rm d}s\right]\\
&+(1+\overline{\sigma}^2)\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|\Psi(X_s)-\Psi(X_0)|^2{\rm d}s\right]+\frac{1}{2}\hat{\boldsymbol{E}}\left[\sup_{0\leqslant s\leqslant t}|\Psi(X_{s\wedge\tau_r})-\Psi(X_0)|^2\right]\\
&\leqslant\big(1+10\overline{\sigma}^2\big)K_2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}\big(|X_s|^2+|X_{s-\tau}|^2+k\big){\rm d}s\right]\\
&+(1+\overline{\sigma}^2)\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|\Psi(X_s)-\Psi(X_0)|^2{\rm d}s\right]+\frac{1}{2}\hat{\boldsymbol{E}}\left[\sup_{0\leqslant s\leqslant t}|\Psi(X_{s\wedge\tau_r})-\Psi(X_0)|^2\right],
\end{aligned}
$$

which yields

$$
\begin{aligned}
\hat{\boldsymbol{E}}\left[\sup_{0\leqslant s\leqslant t}|\Psi(X_{s\wedge\tau_r})-\Psi(X_0)|^2\right]\leqslant&\,2\big(1+10\overline{\sigma}^2\big)K_2\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}(|X_s|^2+|X_{s-\tau}|^2+k){\rm d}s\right]\\
&+2(1+\overline{\sigma}^2)\hat{\boldsymbol{E}}\left[\int_0^{t\wedge\tau_r}|\Psi(X_s)-\Psi(X_0)|^2{\rm d}s\right].
\end{aligned}
$$

It follows from (29) that

$$
\hat{\boldsymbol{E}}\left[\sup_{0\leqslant s\leqslant t}|\Psi(X_{s\wedge\tau_r})-\Psi(X_0)|^2\right]\leqslant C\hat{\boldsymbol{E}}|X_t|^2(t\wedge\tau_r)\leqslant Ct,
$$

where $C>0$ depends on $\overline{\sigma}$, c_1, c_2, K_2 and $\hat{\boldsymbol{E}}|\xi|^2$.

For any $\eta>0$, apply the Doob martingale inequality to obtain

$$
\hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t}|\Psi(X_{s\wedge\tau_r})-\Psi(X_0)|>\eta\right\}\leqslant\frac{\hat{\boldsymbol{E}}\big[|\Psi(X_{t\wedge\tau_r})-\Psi(X_0)|^2\big]}{\eta^2}\leqslant\frac{Ct}{\eta^2}.
$$

For any given $\epsilon_1>0$, we find a function $\phi(\cdot)\in\kappa_\infty$ such that $\frac{\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})}{\beta\phi(\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)}))}\leqslant\epsilon_1$. Denote that $\rho(\cdot)=\gamma_1^{-1}\circ\phi\circ\gamma_2(\cdot)$, where γ_1,γ_2 are defined in (33). Noting that $\sup_{0\leqslant s\leqslant t}|\Psi(X_s)|\geqslant\rho(|\Psi(X_0)|)$ implies $\sup_{0\leqslant s\leqslant t}\mathscr{V}(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})\geqslant\phi(\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)}))$, it follows from (39) that

$$
\begin{aligned}
&\hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t}|\Psi(X_s)|\geqslant\rho(|\Psi(X_0)|)\right\}\\
&\leqslant\hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t}\mathscr{V}(\Psi(X_s),\mathcal{L}_{\Psi(X_s)})\geqslant\phi(\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)}))\right\}\\
&\leqslant\frac{\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)})}{\beta\phi(\mathscr{V}(\Psi(X_0),\mathcal{L}_{\Psi(X_0)}))}\leqslant\epsilon_1,\ \ \forall t\geqslant 0.
\end{aligned}\tag{41}
$$

By the uniform continuity of $\mathscr{V}(\Psi(X_t),\mathcal{L}_{\Psi(X_t)})$ on $B=\{(\Psi(\cdot),\mathcal{L}_{\Psi(\cdot)})\in\boldsymbol{R}^d\times\boldsymbol{R}^d\times\mathcal{P}\times\mathcal{P}:|\Psi(X_t)|<\rho(|\Psi(X_0)|),\ |\mu|_{W_2}<\sqrt{2+2\rho^2(|\Psi(X_0)|)}\}$, for $\forall\varepsilon_2>0$, we find a function $\widetilde{\phi}\in\kappa_\infty$ such that for any

$(\Psi(X_1), \mathcal{L}_{\Psi(X_1)}), (\Psi(X_2), \mathcal{L}_{\Psi(X_2)}) \in B$ with $|\Psi(X_1) - \Psi(X_2)| < \widetilde{\phi}(\varepsilon_2)$ and $W_2(\mathcal{L}_{\Psi(X_1)}, \mathcal{L}_{\Psi(X_2)}) < \widetilde{\phi}(\varepsilon_2)$,

$$\left|\mathscr{V}(\Psi(X_1), \mathcal{L}_{\Psi(X_1)}) - \mathscr{V}(\Psi(X_2), \mathcal{L}_{\Psi(X_2)})\right| \leqslant \varepsilon_2. \tag{42}$$

Thus,

$$\begin{aligned}
&\hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t} |\mathscr{V}(\Psi(X_s), \mathcal{L}_{\Psi(X_s)}) - \mathscr{V}(\Psi(X_0), \mathcal{L}_{\Psi(X_0)})| > \varepsilon_2\right\} \\
&\leqslant \hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t} |\mathscr{V}(\Psi(X_s), \mathcal{L}_{\Psi(X_s)}) - \mathscr{V}(\Psi(X_0), \mathcal{L}_{\Psi(X_0)})| > \varepsilon_2 \text{ and } \sup_{0\leqslant s\leqslant t} |\Psi(X_s)| < \rho(|\Psi(X_0)|)\right\} \\
&\quad + \hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t} |\mathscr{V}(\Psi(X_s), \mathcal{L}_{\Psi(X_s)}) - \mathscr{V}(\Psi(X_0), \mathcal{L}_{\Psi(X_0)})| > \varepsilon_2 \text{ and } \sup_{0\leqslant s\leqslant t} |\Psi(X_s)| \geqslant \rho(|\Psi(X_0)|)\right\} \\
&\leqslant \hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t} |\mathscr{V}(\Psi(X_s), \mathcal{L}_{\Psi(X_s)}) - \mathscr{V}(\Psi(X_0), \mathcal{L}_{\Psi(X_0)})| > \varepsilon_2 \text{ and } \sup_{0\leqslant s\leqslant t} |\Psi(X_s)| < \rho(|\Psi(X_0)|)\right\} \\
&\quad + \hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t} |\Psi(X_s)| \geqslant \rho(|\Psi(X_0)|)\right\} \\
&\leqslant \hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t} |\Psi(X_s) - \Psi(X_0)| > \widetilde{\phi}(\varepsilon_2)\right\} + \hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t} |\Psi(X_s)| \geqslant \rho(|\Psi(X_0)|)\right\} \leqslant \frac{Ct}{\widetilde{\phi}^2(\varepsilon_2)} + \epsilon_1.
\end{aligned} \tag{43}$$

Set $\epsilon_1 = \frac{1}{2}$ and choose a t^* such that $\frac{Ct^*}{\widetilde{\phi}^2(\varepsilon_2)} \leqslant \frac{1}{4}$. Then, we obtain

$$\hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t^*} |\mathscr{V}(\Psi(X_s), \mathcal{L}_{\Psi(X_s)}) - \mathscr{V}(\Psi(X_0), \mathcal{L}_{\Psi(X_0)})| > \varepsilon_2\right\} \leqslant \frac{3}{4},$$

which implies

$$\hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t^*} |\mathscr{V}(\Psi(X_s), \mathcal{L}_{\Psi(X_s)}) - \mathscr{V}(\Psi(X_0), \mathcal{L}_{\Psi(X_0)})| \leqslant \varepsilon_2\right\} \geqslant \frac{1}{4}.$$

Let $T^1_{\varepsilon_1} = \inf\left\{t \geqslant 0 : \mathscr{V}(\Psi(X_t), \mathcal{L}_{\Psi(X_t)}) \leqslant \varepsilon_1\right\}$, $T^1_{2\varepsilon_1} = \inf\left\{t \geqslant T^1_{\varepsilon_1} : \mathscr{V}(\Psi(X_t), \mathcal{L}_{\Psi(X_t)}) > 2\varepsilon_1\right\}$, and, similarly, $T^i_{\varepsilon_1} = \inf\left\{t \geqslant T^{i-1}_{2\varepsilon_1} : \mathscr{V}(\Psi(X_t), \mathcal{L}_{\Psi(X_t)}) \leqslant \varepsilon_1\right\}$, $T^i_{2\varepsilon_1} = \inf\left\{t \geqslant T^i_{\varepsilon_1} : \mathscr{V}(\Psi(X_t), \mathcal{L}_{\Psi(X_t)}) > 2\varepsilon_1\right\}$, for all $i \geqslant 2$. Because of the continuity of $\mathscr{V}(\Psi(\cdot), \mathcal{L}_{\Psi(\cdot)})$, we have $T^i_{\varepsilon_1}, T^i_{2\varepsilon_1} \to \infty$ a.s. $i \to \infty$. Thus, by (38), we obtain

$$\begin{aligned}
\mathscr{V}\big(\Psi(X_0), \mathcal{L}_{\Psi(X_0)}\big) &\geqslant \beta \hat{\hat{\boldsymbol{E}}}\left\{\int_0^\infty \mathscr{V}(\Psi(X_s), \mathcal{L}_{\Psi(X_s)}) \mathrm{d}s\right\} \\
&\geqslant \beta \sum_{i=1}^{\infty} \hat{\hat{\boldsymbol{E}}}\left\{I_{\{T^i_{2\varepsilon_1} < \tau_r\}} \int_{T^i_{2\varepsilon_1}}^{T^{i+1}_{2\varepsilon_1}} \mathscr{V}(\Psi(X_s), \mathcal{L}_{\Psi(X_s)}) \mathrm{d}s\right\} \\
&\geqslant \beta \sum_{i=1}^{\infty} \hat{\hat{\boldsymbol{E}}}\left\{I_{\{T^i_{2\varepsilon_1} < \tau_r\}} \varepsilon_1 (T^{i+1}_{2\varepsilon_1} - T^i_{2\varepsilon_1})\right\} \\
&= \beta \varepsilon_1 \sum_{i=1}^{\infty} \hat{\hat{\boldsymbol{E}}}\left\{I_{\{T^i_{2\varepsilon_1} < \tau_r\}} \hat{\hat{\boldsymbol{E}}}\big(T^{i+1}_{2\varepsilon_1} - T^i_{2\varepsilon_1} | \mathcal{F}_{T^i_{2\varepsilon_1}}\big)\right\}.
\end{aligned} \tag{44}$$

Setting $\widetilde{X}_s = X_{s+T^i_{2\varepsilon_1}}$ and $\varepsilon_2 = \frac{\beta\varepsilon_1}{2}$, by the strong Markov property of the solution to G-SMVDE (4) (see Lemma 5.3 in [21] and Theorem 4.2 in [32]), similar to Deng [33], we obtain the following estimation of $\{T^i_{2\varepsilon_1} < \tau_r\}$:

$$\begin{aligned}
\hat{\hat{\boldsymbol{E}}}\Big[T^{i+1}_{2\varepsilon_1} - T^i_{2\varepsilon_1} | \mathcal{F}_{T^i_{2\varepsilon_1}}\Big] &\geqslant \hat{\hat{\boldsymbol{E}}}\left\{(T^{i+1}_{2\varepsilon_1} - T^i_{2\varepsilon_1}) \times I_{\sup_{0\leqslant s\leqslant t^*} |\mathscr{V}(\Psi(\widetilde{X}_s), \mathcal{L}_{\Psi(\widetilde{X}_s)}) - \mathscr{V}(\Psi(\widetilde{X}_0), \mathcal{L}_{\Psi(\widetilde{X}_0)})| \leqslant \frac{\beta\varepsilon_1}{2}} \Big| \mathcal{F}_{T^i_{2\varepsilon_1}}\right\} \\
&\geqslant t^* \hat{\mathbb{C}}\left\{\sup_{0\leqslant s\leqslant t^*} |\mathscr{V}(\Psi(\widetilde{X}_s), \mathcal{L}_{\Psi(\widetilde{X}_s)}) - \mathscr{V}(\Psi(\widetilde{X}_0), \mathcal{L}_{\Psi(\widetilde{X}_0)})| \leqslant \frac{\beta\varepsilon_1}{2} \Big| \mathcal{F}_{T^i_{2\varepsilon_1}}\right\}
\end{aligned}$$

$$= t^* \hat{\mathbb{C}} \left\{ \sup_{0 \leqslant s \leqslant t^*} |\mathscr{V}(\Psi(X_s), \mathcal{L}_{\Psi(X_s)}) - \mathscr{V}(\Psi(X_0), \mathcal{L}_{\Psi(X_0)})| \leqslant \varepsilon_2 \right\} \geqslant \frac{t^*}{4}. \tag{45}$$

Eq. (44) together with (45) yields

$$\frac{t^*}{4} \beta \varepsilon_1 \sum_{i=1}^{\infty} \hat{\hat{\boldsymbol{E}}} \big\{ \hat{\mathbb{C}} \{ T^i_{2\varepsilon_1} < \tau_r \} \big\} < \infty. \tag{46}$$

The Borel-Cantelli lemma implies that

$$\hat{\mathbb{C}} \big\{ T^i_{2\varepsilon_1} < \tau_r \text{ for infinitely many } i \big\} = 0.$$

Thus, we have

$$\hat{\mathbb{C}} \big\{ T^i_{2\varepsilon_1} < \infty \text{ for infinitely many } i \text{ and } \tau_r = \infty \big\} = 0,$$

and

$$\hat{\mathbb{C}} \big\{ T^i_{2\varepsilon_1} < \infty \text{ for infinitely many } i \text{ and } \tau_r < \infty \big\} = 0.$$

We know that τ_r increases with r. If $\hat{\mathbb{C}}\{\tau_r = \infty\} \to 1$ as $r \to \infty$, then it holds that

$$\hat{\mathbb{C}} \big\{ T^i_{2\varepsilon_1} < \infty \text{ for infinitely many } i \big\} = 0, \tag{47}$$

which contradicts (40). Consequently, we only need to prove $\hat{\mathbb{C}}\{\tau_r = \infty\} \to 1$ as $r \to \infty$.

By Theorem 3.15 in [34], we obtain the convergence of $\mathscr{V}(\Psi(\cdot), \mathcal{L}_{\Psi(\cdot)})$. Thus,

$$\mathscr{V}(\Psi(X_\infty), \mathcal{L}_{\Psi(X_\infty)}) = \lim_{t \to \infty} \mathscr{V}(\Psi(X_t), \mathcal{L}_{\Psi(X_t)}) = C,$$

and

$$\hat{\hat{\boldsymbol{E}}} \mathscr{V}(\Psi(X_\infty), \mathcal{L}_{\Psi(X_\infty)}) < \infty$$

hold. By the supermartingale inequality in [35], we obtain

$$\hat{\mathbb{C}} \left\{ \sup_{t \geqslant 0} \mathscr{V}(\Psi(X_t), \mathcal{L}_{\Psi(X_t)}) \geqslant \gamma_1(r) \right\} \leqslant \frac{\mathscr{V}(\Psi(X_0), \mathcal{L}_{\Psi(X_0)})}{\beta \gamma_1(r)}, \ \forall r > 0. \tag{48}$$

Hence,

$$\begin{aligned} \hat{\mathbb{C}}\{\tau_r = \infty\} = \hat{\mathbb{C}} \left\{ \sup_{t \geqslant 0} |X_t| < r \right\} &\geqslant \hat{\mathbb{C}} \left\{ \sup_{t \geqslant 0} \mathscr{V}(\Psi(X_t), \mathcal{L}_{\Psi(X_t)}) < \gamma_1(r) \right\} \\ &\geqslant 1 - \frac{\mathscr{V}(\Psi(X_0), \mathcal{L}_{\Psi(X_0)})}{\beta \gamma_1(r)}, \end{aligned} \tag{49}$$

which implies that $\hat{\mathbb{C}}\{\tau_r = \infty\} \to 1$ as $r \to \infty$. This result completes the proof.

5 Interacting particle systems

The drift and diffusion coefficients of G-SMVDE (4) depend on not only the current and past state variables but also their distributions. Because the distribution of the exact solution is difficult to observe, while the empirical distribution can be observed more easily, we use the stochastic particle method to analyze the stability of the G-SMVDE.

Assume that $(\xi^1, B^1(t)), (\xi^2, B^2(t)), \ldots$ are i.i.d. of $(\xi, B(t))$. Now, we focus on the following noninteracting particle system as a bridge:

$$\begin{aligned} \mathrm{d}X^i(t) =& f(\Psi(X^i_t), \mathcal{L}_{\Psi(X^i_t)})\mathrm{d}t + g(\Psi(X^i_t), \mathcal{L}_{\Psi(X^i_t)})\mathrm{d}B(t) \\ &+ h(\Psi(X^i_t), \mathcal{L}_{\Psi(X^i_t)})\mathrm{d}\langle B \rangle(t), \ t \geqslant 0, \end{aligned} \tag{50}$$

with the initial value ξ^i $(i=1,2,\ldots,N)$, where $\mathcal{L}_{\Psi(X_t^i)}$ is the law of $\Psi(X_t^i)$ and $\Psi(X_t^i)=(X^i(t),X^i(t-\tau))$, $\mathcal{L}_{\Psi(X_t^i)}=(\mathcal{L}_{X^i(t)},\mathcal{L}_{X^i(t-\tau)})$. By i.i.d., $\hat{\boldsymbol{E}}X^i(t)=\hat{\boldsymbol{E}}X_t=\hat{\boldsymbol{E}}X(t)$.

Consider the corresponding interacting particle system:

$$\begin{aligned}\mathrm{d}X^{i,N}(t)=&f(\Psi(X_t^{i,N}),\mathcal{L}_{\Psi(X_t^N)})\mathrm{d}t+g(\Psi(X_t^{i,N}),\mathcal{L}_{\Psi(X_t^N)})\mathrm{d}B(t)\\&+h(\Psi(X_t^{i,N}),\mathcal{L}_{\Psi(X_t^N)})\mathrm{d}\langle B\rangle(t),\ t\geqslant 0,\end{aligned}\tag{51}$$

with the initial value ξ^i $(i=1,2,\ldots,N)$, where

$$\Psi(X_t^{i,N})=(X^{i,N}(t),X^{i,N}(t-\tau)),\ \mathcal{L}_{\Psi(X_t^N)}=(\mathcal{L}_{X^N(t)},\mathcal{L}_{X^N(t-\tau)}),$$

and

$$\mathcal{L}_{X^N(t)}=\frac{1}{N}\sum_{j=1}^N\delta_{X^{j,N}(t)},\ \mathcal{L}_{X^N(t-\tau)}=\frac{1}{N}\sum_{j=1}^N\delta_{X^{j,N}(t-\tau)}.$$

We know that when N becomes large, the solution of (51) approximates that of G-SMVDE (4) in an appropriate sense, which is called the propagation of chaos. Consequently, in the case that the distribution of the exact solution is difficult to obtain while that of the corresponding interacting particle system can be obtained, we need to explore the relationship between the stability of G-SMVDE (4) and that of the corresponding interacting particle system (51).

The following lemmas are needed.

Lemma 7. Let Assumption 1 hold. Then, for any $T>0$, (51) admits a unique strong solution $X^{i,N}(t)$ that satisfies

$$\sup_{0\leqslant t\leqslant T}\hat{\boldsymbol{E}}|X^{i,N}(t)|^2\leqslant C_{\xi,\overline{\sigma},T},i=1,2,\ldots,N,$$

where $C_{\xi,\overline{\sigma},T}$ is a constant dependent on ξ, $\overline{\sigma}$, and T.

Proof. The proof can be obtained by combining the proofs of Theorem 2.1 in [16] and Theorem 3.3 in [36] as well as Theorem 5.1.3 in [1], and so we omit it here.

Lemma 8 ([37])**.** Assume that $\{X^n\}_{n\geqslant 1}$ is a sequence of i.i.d. random variables in $\boldsymbol{R}^d$ with common distribution $\mu\in\mathcal{P}$. For any $N\in\boldsymbol{N}$, we recall the empirical measure $\mu^N=\frac{1}{N}\sum_{i=j}^N\delta_{X^j}$. Let $\mathcal{P}_q$ denote the subset of $\mathcal{P}$ with the probability measures having a finite q-th order moment. If $\mu\in\mathcal{P}_q$ with $q>4$, then, there exists a constant $C=C(d,q,W_q(\mu))$ such that for any $N\geqslant 2$,

$$\hat{\boldsymbol{E}}[W_2^2(\mu^N,\mu)]\leqslant C(N)=C\begin{cases}N^{-\frac{1}{2}}, & 1\leqslant d<4,\\ N^{-\frac{1}{2}}\ln(N), & d=4,\\ N^{-\frac{2}{d}}, & 4<d.\end{cases}$$

Now, we present the following theorem to show the propagation of chaos.

Theorem 6. Let Assumption 1 hold. For any $T>0$, we have

$$\max_{1\leqslant j\leqslant N}\sup_{0\leqslant t\leqslant T}\hat{\boldsymbol{E}}[|X^j(t)-X^{j,N}(t)|^2]\leqslant C_{T,\overline{\sigma},K_1}N^{-\frac{1}{4}},$$

where $C_{T,\overline{\sigma},K_1}$ is a positive constant dependent on T, $\overline{\sigma}$ and K_1.

Proof. Using the G-Itô formula, we have

$$\begin{aligned}\hat{\boldsymbol{E}}|X^i(t)-X^{i,N}(t)|^2\leqslant&\hat{\boldsymbol{E}}\bigg[\int_0^t 2|X^i(s)-X^{i,N}(s)|\big[f(\Psi(X_s^i),\mathcal{L}_{\Psi(X_s^i)})-f(\Psi(X_s^{i,N}),\mathcal{L}_{\Psi(X_s^N)})\big]\mathrm{d}s\\&+\int_0^t 2|X^i(s)-X^{i,N}(s)|\big[g(\Psi(X_s^i),\mathcal{L}_{\Psi(X_s^i)})-g(\Psi(X_s^{i,N}),\mathcal{L}_{\Psi(X_s^N)})\big]\mathrm{d}B_s\\&+\int_0^t\Big(2|X^i(s)-X^{i,N}(s)|\big[h(\Psi(X_s^i),\mathcal{L}_{\Psi(X_s^i)})-h(\Psi(X_s^{i,N}),\mathcal{L}_{\Psi(X_s^N)})\big]\\&+\big[g(\Psi(X_s^i),\mathcal{L}_{\Psi(X_s^i)})-g(\Psi(X_s^{i,N}),\mathcal{L}_{\Psi(X_s^N)})\big]^2\Big)\,\mathrm{d}\langle B\rangle_s\bigg].\end{aligned}\tag{52}$$

It follows from the elementary inequality, Assumption 1 and Lemma 2 that

$$\hat{\boldsymbol{E}}|X^i(t)-X^{i,N}(t)|^2 \leqslant \hat{\boldsymbol{E}}\bigg[\int_0^t \big((1+K_1+\overline{\sigma}^2+2\overline{\sigma}^2K_1)|X^i(s)-X^{i,N}(s)|^2 \\ +(1+2\overline{\sigma}^2)K_1\big[|X^i(s-\tau)-X^{i,N}(s-\tau)|^2+W_2^2(\mathcal{L}_{\Psi(X_s^i)},\mathcal{L}_{\Psi(X_s^N)})\big]\big)\,\mathrm{d}s\bigg]. \quad (53)$$

We turn to define another empirical measure to handle the Wasserstein distance. Define

$$\mathcal{L}_{X_t^{*N}}=\frac{1}{N}\sum_{j=1}^N\delta_{X^j(t)},$$

and

$$\mathcal{L}_{\Psi(X_t^{*N})}=\frac{1}{N}\sum_{j=1}^N\delta_{\Psi(X_t^j)}=\left(\frac{1}{N}\sum_{j=1}^N\delta_{X^j(t)},\frac{1}{N}\sum_{j=1}^N\delta_{X^j(t-\tau)}\right).$$

Thus, we have

$$W_2^2(\mathcal{L}_{\Psi(X_t^i)},\mathcal{L}_{\Psi(X_t^N)}) \leqslant 2W_2^2(\mathcal{L}_{\Psi(X_t^i)},\mathcal{L}_{\Psi(X_t^{*N})})+2W_2^2(\mathcal{L}_{\Psi(X_t^{*N})},\mathcal{L}_{\Psi(X_t^N)}) \\ \leqslant 2W_2^2(\mathcal{L}_{\Psi(X_t^i)},\mathcal{L}_{\Psi(X_t^{*N})})+\frac{2}{N}\sum_{j=1}^N\big[\hat{\boldsymbol{E}}|X^j(t)-X^{j,N}(t)|^2 \\ +\hat{\boldsymbol{E}}|X^j(t-\tau)-X^{j,N}(t-\tau)|^2\big],$$

and

$$\frac{1}{N}\sum_{j=1}^N\hat{\boldsymbol{E}}|X^j(t)-X^{j,N}(t)|^2\leqslant\hat{\boldsymbol{E}}|X^i(t)-X^{i,N}(t)|^2.$$

For any $t>0$, these facts together with Lemma 8 yield

$$\sup_{0\leqslant s\leqslant t}\hat{\boldsymbol{E}}|X^i(s)-X^{i,N}(s)|^2 \leqslant \int_0^t\bigg((1+2K_1+\overline{\sigma}^2+4\overline{\sigma}^2K_1)\sup_{0\leqslant r\leqslant s}\hat{\boldsymbol{E}}|X^i(r)-X^{i,N}(r)|^2 \\ +(1+2\overline{\sigma}^2)K_1\sup_{0\leqslant r\leqslant s}\hat{\boldsymbol{E}}W_2^2(\mathcal{L}_{\Psi(X_r^i)},\mathcal{L}_{\Psi(X_r^N)})\bigg)\,\mathrm{d}s \\ \leqslant \int_0^t\bigg((1+7K_1+\overline{\sigma}^2+14\overline{\sigma}^2K_1)\sup_{0\leqslant r\leqslant s}\hat{\boldsymbol{E}}|X^i(r)-X^{i,N}(r)|^2 \\ +(2+4\overline{\sigma}^2)K_1\sup_{0\leqslant r\leqslant s}\hat{\boldsymbol{E}}W_2^2(\mathcal{L}_{\Psi(X_r^i)},\mathcal{L}_{\Psi(X_r^{*N})})\bigg)\,\mathrm{d}s. \quad (54)$$

Thanks to the Gronwall inequality and Lemma 8, we have

$$\sup_{0\leqslant t\leqslant T}\hat{\boldsymbol{E}}|X^i(t)-X^{i,N}(t)|^2\leqslant C_{T,\overline{\sigma},K_1}N^{-\frac{1}{4}}.$$

Now, we present the main result that illustrates the relationship between the stability of the G-SMVDE and that of the corresponding interacting particle system.

Theorem 7. The exact solution to the G-SMVDE (4) is mean square exponentially stable if and only if the solution to (51) is mean square exponentially stable.

Proof. The proof can be completed using the following two inequalities in the case of $N\to\infty$:

$$\hat{\boldsymbol{E}}|X^{i,N}(t)|^2 \leqslant 3\hat{\boldsymbol{E}}|X^{i,N}(t)-X^i(t)|^2+3\hat{\boldsymbol{E}}|X^i(t)-X(t)|^2+3\hat{\boldsymbol{E}}|X(t)|^2 \\ \leqslant 3C_{T,\overline{\sigma},K_1}N^{-\frac{1}{4}}+3\hat{\boldsymbol{E}}|X(t)|^2, \\ \hat{\boldsymbol{E}}|X(t)|^2 \leqslant 3\hat{\boldsymbol{E}}|X^i(t)-X^{i,N}(t)|^2+3\hat{\boldsymbol{E}}|X(t)-X^i(t)|^2+3\hat{\boldsymbol{E}}|X^{i,N}(t)|^2 \\ \leqslant 3C_{T,\overline{\sigma},K_1}N^{-\frac{1}{4}}+3\hat{\boldsymbol{E}}|X^{i,N}(t)|^2.$$

Remark 5. System (51) is sometimes called an interacting particle system. Recently, Eq. (51) has been referred as a system with mean-field terms or mean-field interactions by interests in control systems theory. Essentially, as N becomes large, the solution to (51) appropriately approximates the solution to G-SMVDE (4), which is called the propagation of chaos. Some researchers use (51) as a reference system to build the numerical schemes. See [36, 38] for the numerical schemes for SMVEs.

6 Examples

In this section, we use two examples to illustrate the results.

Example 1. Let $B(t)$ be a 1-dimensional G-Brownian motion with $B(t) \sim N\big(0; [\underline{\sigma}^2, \overline{\sigma}^2]t\big)$. Consider the following G-SMVDE:

$$\begin{cases} \mathrm{d}X(t) = \left(-6X(t) - 6X(t-1) - \displaystyle\int_{\boldsymbol{R}} 2z\mu_t(\mathrm{d}z)\right)\mathrm{d}t + X(t)\mathrm{d}B(t) + X(t)\mathrm{d}\langle B\rangle(t),\ t \geqslant 0, \\ X(t) = x_0,\ -1 \leqslant t \leqslant 0, \end{cases} \tag{55}$$

where x_0 is a positive constant.

Letting $\mathscr{V}(x,\mu) = |x|^2 + \int_{\boldsymbol{R}} |z|^2\mu(\mathrm{d}z)$ and considering that $\partial_\mu\left(\int_{\boldsymbol{R}} |z|^2\mu(\mathrm{d}z)\right)(y) = 2y$, we have

$$\begin{aligned} L^\mu\mathscr{V}(X(t),\mu) = &\left(-6X(t) - 6X(t-1) - \int_{\boldsymbol{R}} 2z\mu(\mathrm{d}z)\right)2|X(t)| \\ &+ \int_{\boldsymbol{R}}\left(-6y(t) - 6y(t-1) - \int_{\boldsymbol{R}} 2z\mu(\mathrm{d}z)\right)2y\mu(\mathrm{d}y) \\ &+ 3|X(t)|^2 + \int_{\boldsymbol{R}} 2y^2\mu(\mathrm{d}y) + \int_{\boldsymbol{R}} y^2\mu(\mathrm{d}y) \\ \leqslant &-19|X(t)|^2 - 21\int_{\boldsymbol{R}}|z|^2\mu(\mathrm{d}z) - 2\left(\int_{\boldsymbol{R}} z\mu(\mathrm{d}z)\right)^2, \end{aligned}$$

and

$$\int_{\boldsymbol{R}} |x|^2\mu(\mathrm{d}x) \leqslant \int_{\boldsymbol{R}} \mathscr{V}(x,\mu)\mu(\mathrm{d}x) \leqslant 2\int_{\boldsymbol{R}} |x|^2\mu(\mathrm{d}x).$$

Obviously, Eqs. (22) and (23) hold. Moreover, $\frac{1}{2}|x|^2 \leqslant \mathscr{V}(x,\mu) \leqslant 2|x|^2$. These facts mean that Theorems 4 and 5 hold. Therefore, we conclude that G-SMVDE (55) is mean square exponentially stable and almost surely asymptotically stable. Thus, the corresponding interacting particle system is mean square exponentially stable.

Now, we perform a numerical experiment to confirm our theoretical results. Because the exact solution of the G-SMVDE is difficult to obtain, we use the numerical solution of the EM method under $M = 10^3$ as the "exact solution". To simulate the G-Brownian motion $(B(t), t \in [0,T])$, we take a sequence of random variables $\zeta_M^k = B(t_M^k) - B(t_M^{k-1}) \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2]\Delta_t)$, $k = 1, \ldots, M$, and equal-step points $\sigma_s (s = 1, \ldots, I)$ such that $\underline{\sigma} = \sigma_1 < \sigma_s < \cdots < \sigma_I = \overline{\sigma}$. Thus, in the sth-round random sampling $(s = 1, \ldots, I)$, $\zeta_k^{sj} (k = 1, 2, \ldots, M; j = 1, 2, \ldots, J)$ obeys the classical normal distribution $N(0, \sigma_s^2\Delta_t)$. By applying the EM method to (55), we have

$$\begin{cases} X^{i,N,M}(t_M^{k+1}) = \left(-6X^{i,N,M}(t_M^k) - 6X^{i,N,M}(t_M^{k-100}) - \dfrac{1}{N}\displaystyle\sum_{r=1}^{N} 2X^{r,N,M}(t_M^k)\right)\Delta_t \\ \qquad\qquad + X^{i,N,M}(t_M^k)\zeta_k^{sj} + X^{i,N,M}(t_M^k)\sigma_s^2\Delta_t, \quad k \geqslant 0, \\ X^{i,N,M}(t_M^0) = 1, \end{cases}$$

for $1 \leqslant s \leqslant I$, $1 \leqslant j \leqslant J$, where $T = 10$ and $\mathcal{L}_{X^{i,N,M}(t_M^k)} \approx \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N,M}(t_M^k)}$. Taking $J = N = 100$, $I = 10$, and $[\underline{\sigma}^2, \overline{\sigma}^2] = [0.5, 1]$, we plot the experimental results in Figure 1.

In Figure 1, we plot the sample paths of $X(t)$ for G-SMVDE (55) with different variance intervals. We take $[\underline{\sigma}^2, \overline{\sigma}^2] = [0.5, 1]$, $[\underline{\sigma}^2, \overline{\sigma}^2] = [2, 2.3]$, and $[\underline{\sigma}^2, \overline{\sigma}^2] = [2.3, 2.6]$, with the same initial data $x_0 = \xi = 1$

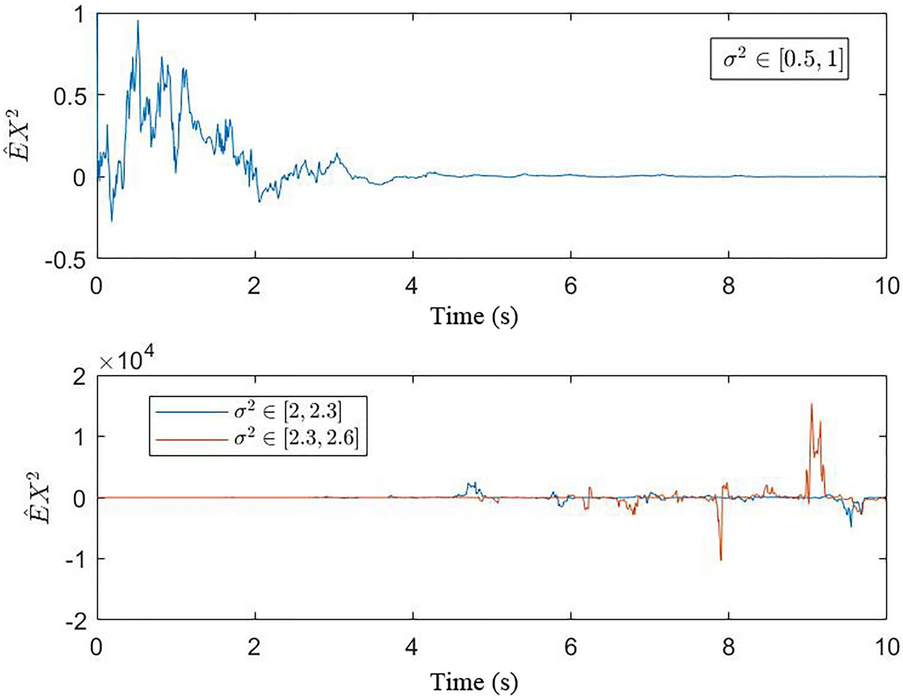

Figure 1 (Color online) States of the solution to G-SMVDE (55) with different variance intervals.

and step-size $\Delta_t = 0.001$. We observe that the "exact solution" performs well when $[\underline{\sigma}^2, \overline{\sigma}^2]$ falls within $[0, 1]$, and the smaller the variance interval the better the stability. The "exact solution" oscillates severely when $[\underline{\sigma}^2, \overline{\sigma}^2]$ falls outside the interval $[0, 1]$.

Example 2. Consider the following scalar G-SMVDE:

$$\begin{aligned} \mathrm{d}X(t) = &\left(-3X(t) - 3X(t-1) - \int_{\boldsymbol{R}} 2z\mu_t(\mathrm{d}z) + \int_{\boldsymbol{R}} z\mu_{t-1}(\mathrm{d}z)\right)\mathrm{d}t \\ &+ \big(X(t) + X(t-1)\big)\mathrm{d}B(t) + \big(X(t) + X(t-1)\big)\mathrm{d}\langle B\rangle(t) \end{aligned} \tag{56}$$

with the initial data $X(t) = \xi(t)$, $-1 \leqslant t \leqslant 0$.

Denote that $\Psi(X_t) = (X(t), X(t-1))$ and $\mathcal{L}_{\Psi(X_t)} = (\mathcal{L}_{X(t)}, \mathcal{L}_{X(t-1)})$. Letting $\mathscr{V}(\Psi(X_t), \mu) = |\Psi(X_t)|^2 + \int_{\boldsymbol{R}} |z|^2\mu(\mathrm{d}z)$, we have

$$\begin{aligned} L^{\mu}\mathscr{V}(\Psi(X_t), \mu) = &\left(-3X(t) - 3X(t-1) - \int_{\boldsymbol{R}} 2z\mu(\mathrm{d}z)\right) 2|\Psi(X_t)| \\ &+ \int_{\boldsymbol{R}} \left(-3y(t) - 3y(t-1) - \int_{\boldsymbol{R}} 2z\mu(\mathrm{d}z)\right) 2y\mu(\mathrm{d}y) \\ &+ 3|X(t) + X(t-1)|^2 + \int_{\boldsymbol{R}} \big(y(t) + y(t-1)\big)^2\mu(\mathrm{d}y) \\ &+ \int_{\boldsymbol{R}} 2y(t)\big(y(t) + y(t-1)\big)\mu(\mathrm{d}y) \\ \leqslant &- |\Psi(X_t)|^2 - 4\int_{\boldsymbol{R}} |z|^2\mu(\mathrm{d}z) - 2\left(\int_{\boldsymbol{R}} z\mu(\mathrm{d}z)\right)^2 \\ \leqslant &- \left(|\Psi(X_t)|^2 + \int_{\boldsymbol{R}} |z|^2\mu(\mathrm{d}z)\right). \end{aligned}$$

Moreover, we have

$$\int_{\boldsymbol{R}} |X(t)|^2\mu(\mathrm{d}z) \leqslant \int_{\boldsymbol{R}} \mathscr{V}(\Psi(X_t), \mu)\mu(\mathrm{d}z) \leqslant 5\int_{\boldsymbol{R}} |X(t)|^2\mu(\mathrm{d}z),$$

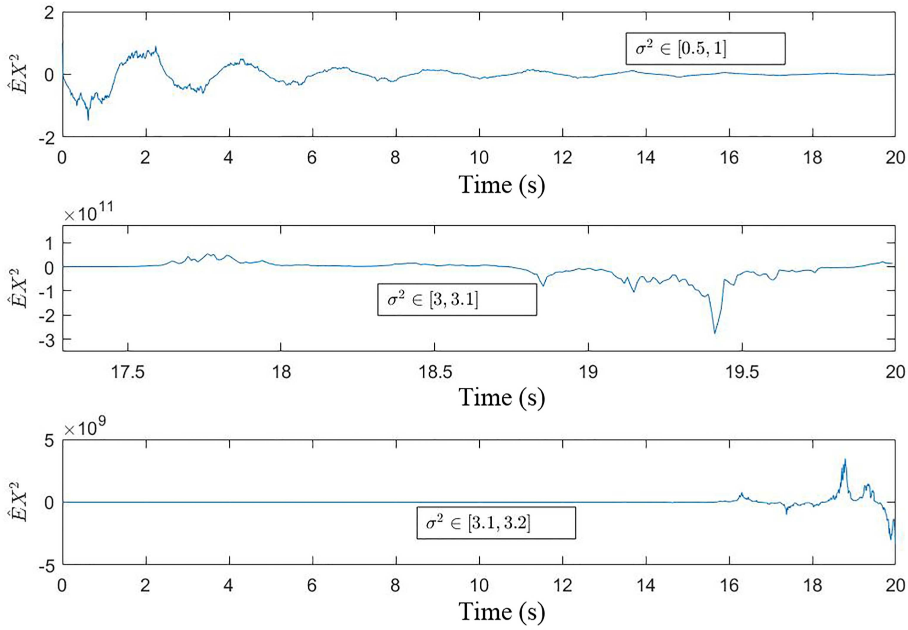

Figure 2 (Color online) States of the solution to G-SMVDE (56) with different variance intervals.

and

$$|X(t)|^2 \leqslant \mathscr{V}(\Psi(X_t), \mu) \leqslant 5|X(t)|^2.$$

Obviously, the conditions of Theorems 4 and 5 hold. Therefore, G-SMVDE (56) is mean square exponentially stable and almost surely asymptotically stable, and correspondingly, the interacting particle system is mean square exponentially stable.

We use the techniques in Example 1 to confirm our theoretical results. We use the numerical solution of the EM method under $M = 10^3$ as the "exact solution". To simulate the G-Brownian motion $(B(t), t \in [0, T])$, we take a sequence of random variables $\zeta_M^k = B(t_M^k) - B(t_M^{k-1}) \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2]\Delta_t)$, $k = 1, \ldots, M$, and equal-step points $\sigma_s (s = 1, \ldots, I)$ such that $\underline{\sigma} = \sigma_1 < \sigma_s < \cdots < \sigma_I = \overline{\sigma}$. Consequently, in the sth-round random sampling $(s = 1, \ldots, I)$, $\zeta_k^{sj} (k = 1, 2, \ldots, M; j = 1, 2, \ldots, J)$ obeys the classical normal distribution $N(0, \sigma_s^2 \Delta_t)$. By applying the EM method to (56), we have

$$\begin{cases} X^{i,N,M}(t_M^{k+1}) = \left(-3X^{i,N,M}(t_M^k) - 3X^{i,N,M}(t_M^{k-100}) - \dfrac{1}{N}\displaystyle\sum_{r=1}^{N} 2X^{r,N,M}(t_M^k) + \dfrac{1}{N}\displaystyle\sum_{r=1}^{N} X^{r,N,M}(t_M^{k-100}) \right) \\ \qquad \times \Delta_t + \left(X^{i,N,M}(t_M^k) + X^{i,N,M}(t_M^{k-100}) \right) \zeta_k^{sj} + \left(X^{i,N,M}(t_M^k) + X^{i,N,M}(t_M^{k-100}) \right) \sigma_s^2 \Delta_t, \\ \qquad k \geqslant 0, \\ X^{i,N,M}(t_M^k) = 1, k < 0, \end{cases}$$

for $1 \leqslant s \leqslant I$, $1 \leqslant j \leqslant J$, where $T = 20$ and $\mathcal{L}_{X^{i,N,M}(t_M^k)} \approx \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N,M}(t_M^k)}$. Taking $J = N = 100$, $I = 10$, and $[\underline{\sigma}^2, \overline{\sigma}^2] = [0.5, 1]$, we plot the experimental results in Figure 2.

In Figure 2, we plot the sample paths of $X(t)$ for G-SMVDE (56) with different variance intervals. We take $[\underline{\sigma}^2, \overline{\sigma}^2] = [0.5, 1]$, $[\underline{\sigma}^2, \overline{\sigma}^2] = [3, 3.1]$, and $[\underline{\sigma}^2, \overline{\sigma}^2] = [3.1, 3.2]$, with the same initial data $X(t) = \xi(t) = 1$, $-1 \leqslant t \leqslant 0$, and step-size $\Delta_t = 0.001$. We observe that the "exact solution" performs well when $[\underline{\sigma}^2, \overline{\sigma}^2]$ falls within $[0, 1]$, and the smaller the variance interval the better the stability. The "exact solution" oscillates severely when $[\underline{\sigma}^2, \overline{\sigma}^2]$ falls outside the interval $[0, 1]$. This finding is consistent with our theoretical results.

7 Concluding remark

In this paper, we studied three types of stability of the SMVDE driven by G-Brownian motion. The distribution dependence and probability uncertainty prevent us from directly using the stochastic analysis methods for G-SDEs and SMVEs directly. To overcome these difficulties, we defined the derivative of a function with a law under the G-expectation, then introduced the upper probability and the Lions derivatives. Next, we established the G-Itô formula for G-SMVEs by expanding the G-Itô formula for G-SDEs and the Itô formula for SMVEs. Thanks to the new G-Itô formula, we obtained the mean square exponential stability and the almost sure asymptotic stability of the G-SMVDEs by using the Lyapunov functional method and defining a new lower expectation. Notably, G-SMVDEs have been widely used to describe the finance and systemic risk model with Knightian uncertainty. However, the stability of G-SMVDEs is unreported. Therefore, our work is novel and important.

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