SCIENCE CHINA
Information Sciences

• LETTER •

December 2024, Vol. 67, Iss. 12, 229202:1–229202:2
https://doi.org/10.1007/s11432-023-4179-0

Controllability of neighborhood Corona product networks

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Received 30 November 2023/Revised 22 June 2024/Accepted 6 September 2024/Published online 19 November 2024

Controlling networks aims to study the models, structures, and related dynamics of complex networks. The primary problem of controlling networks is to determine whether they are controllable. Nowadays, controllability has been widely studied and applied to system engineering and control theory, power systems, aerospace, and quantum systems. Various classical criteria include the Gram matrix criterion, Kalman rank criterion, and PBH test. Directly applying these criteria to larger-scale networks will cause high complexity due to intricate topology structures and large amounts of calculation.

Various graph products [1] have been widely used to model graphs that have similar properties to real-world network systems. These products help estimate the properties of generated graphs and describe specific composite network topologies through using a "small" network to build a "large" network. Composite networks, consisting of smaller-scale factor networks, can be analyzed and verified for characteristics like stability, consensus, and controllability by examining the properties of the factors. Chapman et al. [2] investigated the controllability of composite networks via Cartesian products and established a sufficient and necessary controllable condition. Corona products, including Corona operation, neighborhood Corona operation, and hybrid operation, can usually produce more complex composite graphs with the advantage of tractable analysis and rigorous inference, and the process of Corona graph generation also has a good natural interpretation and argument. In [3], some descriptive indicators of the Corona network topologies are obtained, including the spectrum of Corona graphs such as generalized neighbors, connection spectrum, and Laplacian spectrum. More recently, the controllability of complex networks via Corona product was studied in [4, 5]. In particular, the controllability of Corona product networks with N-duplication and Laplacian dynamics was investigated, and the controllability conditions were established. However, to the best of our knowledge, the controllability of graph-product networks is still in its early stages. There are questions about how to properly model large-scale composite networks via graph products, how to select analytical methods to characterize the controllability of graph-product networks, and how to reveal the controllability relationship between the whole network and its factors. In the context of neighborhood Corona product networks (NCPNs), the controllability problem bears new difficulties and challenges.

This study investigates the controllability problem of NCPNs. We formulate the neighborhood Corona product network model and establish controllable criteria. Compared to relevant studies (e.g., [4, 5]), the contributions of this work are threefold. (i) Different from the Corona product network model [4, 5] with Laplacian matrix consisting of its factors' Laplacian matrices, the Laplacian matrix of such model involves not only the factors' Laplacian matrices but also adjacency matrices, which makes it more difficult to formulate and characterize the features of the NCPN by its factors and find the appropriately analytical techniques. (ii) This study considers the connectivity of factors and discusses its influence on the NCPN controllability, including connected and disconnected factor graphs, which differs from the case of only considering the connected factor graphs in [4, 5]. (iii) Taking advantage of the good features of regular graphs, the controllability criteria of NCPNs can be derived in terms of the determinants, eigenvalues, and eigenvectors of the low-dimensional factors, rather than just using the classical controllability criteria such as the PBH test. Here, the new criteria established significantly reduce computational complexity and improve the efficiency in applications compared with directly utilizing the PBH test. This work will help us recognize and understand the synergistic mechanism of networks.

The dynamics of two multiagent networks $\mathcal{G}_1$ and $\mathcal{G}_2$ under a leader-follower framework are described as follows:

$$\dot{x}_{\mathcal{G}_1} = -\mathcal{L}_1 x_{\mathcal{G}_1} + \mathcal{B}_1 u_1, \tag{1a}$$
$$\dot{x}_{\mathcal{G}_2} = -\mathcal{L}_2 x_{\mathcal{G}_2} + \mathcal{B}_2 u_2, \tag{1b}$$

where $x_{\mathcal{G}_\alpha} \in \mathcal{R}^{n_\alpha}$, $\mathcal{L}_\alpha \in \mathcal{R}^{n_\alpha \times n_\alpha}$, $u_\alpha \in \mathcal{R}^{n_\alpha}$, $\mathcal{B}_\alpha = [b_{\alpha\beta}] \in \mathcal{R}^{n_\alpha \times n_\alpha}$ for $\alpha = 1, 2$ are the states, Laplacian matrices, control inputs, control matrices of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. $b_{\alpha\beta} = e_{\alpha\beta}$ if agent β is a leader, otherwise $b_{\alpha\beta} = \mathbf{0}_{n_\alpha}$, where $e_{\alpha\beta}$ denotes the column vector with all zero entries except for $[e_{\alpha\beta}]_\beta = 1$, $\beta = 1, 2, \cdots, n_\alpha$. Then

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we have a compact form of the NCPN:

$$\Sigma:\ \dot{x} = -\mathcal{L}x + \mathcal{B}u \tag{2}$$

generated by the r-regular undirected and unweighted $\mathcal{G}_1$ and arbitrary undirected and unweighted $\mathcal{G}_2$ with

$$\mathcal{L}=\begin{bmatrix}\mathcal{L}_1+n_2rI_{n_1} & -\mathcal{A}_1\otimes\mathbf{1}_{n_2}^T\\ -\mathcal{A}_1\otimes\mathbf{1}_{n_2} & I_{n_1}\otimes(\mathcal{L}_2+rI_{n_2})\end{bmatrix},\ \mathcal{B}=\begin{bmatrix}\mathcal{B}_1 & 0\\ 0 & I_{n_1}\otimes\mathcal{B}_2\end{bmatrix}, \tag{3}$$

where $\mathcal{L},\mathcal{B}\in\mathcal{R}^{n_1(1+n_2)\times n_1(1+n_2)}$, and $\mathbf{1}_{n_2}$ is the column vector with all 1.

Lemma 1 ([3]). Let $\mathcal{G}=\mathcal{G}_1\bigstar\mathcal{G}_2$, where $\mathcal{G}_1$ is a connected r-regular graph on $n_1\geqslant 1$ vertices and $\mathcal{G}_2$ is arbitrary graph on $n_2\geqslant 2$ vertices. Let $\sigma(\mathcal{A}_1)=\{\ \mu_1=\mu_{max}=r,\mu_2,\cdots,\mu_{n_1}\ \}$, $\sigma(\mathcal{L}_1)=\{\ \theta_1=r-\mu_1=0,\theta_2,\cdots,\theta_{n_1}\}$ and $\sigma(\mathcal{L}_2)=\{\eta_1=0,\eta_2,\cdots,\eta_{n_2}\ \}$. Then the eigenvalues of the NCPN can be expressed as $\sigma(\mathcal{L})=V_1\cup V_2\cup V_3$, where $V_1=\{\ \lambda_i|\lambda_i=\frac{(n_2+1)r+\theta_i+\sqrt{\Delta_i}}{2}, i=1,2,\cdots,n_1\ \}=\{\ \lambda_i|\lambda_i=\frac{(n_2+1)r+\theta_i+\sqrt{\Delta_i}}{2},\ \theta_i\neq r\ \}\cup\{\lambda_i|\lambda_i=\frac{(n_2+1)r+\theta_i+\sqrt{\Delta_i}}{2}=(n_2+1)r,\theta_i=r\}=V_{1_1}\cup V_{1_2}$, $V_2=\{\ \hat{\lambda}_i|\hat{\lambda}_i=\frac{(n_2+1)r+\theta_i-\sqrt{\Delta_i}}{2}, i=1,2,\cdots,n_1\ \}=\{\ \hat{\lambda}_i|\hat{\lambda}_i=\frac{(n_2+1)r+\theta_i-\sqrt{\Delta_i}}{2},\theta_i\neq r\ \}\cup\{\ \hat{\lambda}_i|\hat{\lambda}_i=\frac{(n_2+1)r+\theta_i-\sqrt{\Delta_i}}{2}=r,\theta_i=r\ \}=V_{2_1}\cup V_{2_2}$, $V_3=\{\ r+\eta_j,\ j=2,\cdots,n_2\ \}$, and $\Delta_i\triangleq((n_2+1)r+\theta_i)^2-4\theta_i((2n_2+1)r-n_2\theta_i)$.

Proposition 1. The eigenvalues of $\mathcal{L}$ have the following properties: (i) $V_1\cap V_2=\emptyset$; $V_{1_2}\cap V_3=\emptyset$; $V_{1_2}=\emptyset\Leftrightarrow\theta_i\neq r$ $(i=1,2,\cdots,n_1)$; $V_{1_1}\cap V_{1_2}\neq\emptyset\Leftrightarrow\theta_i=r$; $V_{1_1}\cap V_3\neq\emptyset\Leftrightarrow r=1$, $\eta_j=n_2$, $\theta_i=0$ with multiplicity-1; $V_{1_1}\cap V_3=\emptyset$ if $\mathcal{G}_2$ is disconnected; (ii) $V_{2_1}\cap V_{2_2}=\emptyset$; $V_{2_1}\cap V_3=\emptyset$; $V_{2_2}=\emptyset\Leftrightarrow\theta_i\neq r$ $(i=1,2,\cdots,n_1)$; $V_{2_2}\cap V_3=\emptyset\Leftrightarrow\mathcal{G}_2$ is connected.

Let $X_1=\mathbf{1}_{n_1},X_2,\cdots,X_{n_1}$ be the orthogonal eigenvectors corresponding to eigenvalues $\mu_1,\mu_2,\cdots,\mu_{n_1}$ of $\mathcal{A}_1$; $Z_1=\mathbf{1}_{n_2},Z_2,\cdots,Z_{n_2}$ be the linearly independent eigenvectors corresponding to eigenvalues $\eta_1,\eta_2,\cdots,\eta_{n_2}$ of $\mathcal{L}_2$, where Z_1 is orthogonal to $Z_2,\cdots,Z_{n_2}$. Then

$$\begin{bmatrix}\frac{\lambda_i-r}{\theta_i-r}X_i\\ X_i\otimes\mathbf{1}_{n_2}\end{bmatrix},\begin{bmatrix}X_i\\ \mathbf{0}_{n_1n_2}\end{bmatrix},\begin{bmatrix}\frac{\hat{\lambda}_i-r}{\theta_i-r}X_i\\ X_i\otimes\mathbf{1}_{n_2}\end{bmatrix},\begin{bmatrix}\mathbf{0}_{n_1}\\ X_i\otimes\mathbf{1}_{n_2}\end{bmatrix},\begin{bmatrix}\mathbf{0}_{n_1}\\ e_q\otimes Z_j\end{bmatrix}$$

$(i\in\underline{n_1}\triangleq\{1,2,\cdots,n_1\}, j=2,\cdots,n_2,\ q=1,2,\cdots,n_1)$ are the eigenvectors corresponding to $\lambda_i\in V_{1_1}$, $(n_2+1)r\in V_{1_2}$, $\hat{\lambda}_i\in V_{2_1}$, $r\in V_{2_2}$ and $r+\eta_j\in V_3$, respectively.

Theorem 1. Let $\mathcal{G}_2$ be connected. Then NCPN (2) is controllable if and only if the following conditions are satisfied. (i) $(\mathcal{L}_2,\mathcal{B}_2)$ is controllable; (ii) If $V_{1_2}\neq\emptyset$, then $\sum_{p=1}^{m_1}a_{i_p}X_{i_p}^T\mathcal{B}_1\neq 0$ $(i\in\underline{n_1})$ holds, where m_1 is the multiplicity of $(n_2+1)r$ in V_{1_2} and arbitrary constants $a_{i_1},a_{i_2},\cdots,a_{i_{m_1}}$ are not all zero; (iii) If $V_{1_1}\cap V_3\neq\emptyset$, then $X_1^T\mathcal{B}_1\neq 0$ or $a_1X_1^T\otimes(\mathbf{1}_{n_2}^T\mathcal{B}_2)+\sum_{p=1}^{m_2}\sum_{q=1}^{n_1}a_{qj_p}e_q^T\otimes(Z_{j_p}^T\mathcal{B}_2)\neq 0$ $(j\in\underline{n_2}\triangleq\{2,\cdots,n_2\})$ holds, where m_2 is the multiplicity of η_j and arbitrary constants $a_1,a_{1j_1},\cdots,a_{1j_{m_2}},\cdots,a_{n_1j_{m_2}}$ are not all zero.

Theorem 2. Let $\mathcal{G}_2$ be disconnected. Then NCPN (2) is controllable if and only if the following conditions are satisfied. (i) If $r+\eta_{j_1}=r+\eta_{j_2}=\cdots=r+\eta_{j_l}\in V_3$, where l is the multiplicity of η_j, then $\sum_{q=1}^{n_1}\sum_{p=1}^{l}a_{qj_p}e_q^T\otimes(Z_{j_p}^T\mathcal{B}_2)\neq 0$ $(j=2,\cdots,n_2)$ holds, where arbitrary constants $a_{1j_1},\cdots,a_{n_1j_1},\cdots,a_{n_1j_l}$ are not all zero; (ii) If $V_{1_2}\neq\emptyset$, then $\sum_{p=1}^{m_1}a_{i_p}X_{i_p}^T\mathcal{B}_1\neq 0$ $(i\in\underline{n_1})$ holds, where m_1 is the multiplicity of $(n_2+1)r$ in V_{1_2} and arbitrary constants $a_{i_1},a_{i_2},\cdots,a_{i_{m_1}}$ are not all zero; (iii) If $V_{2_2}\neq\emptyset$, then $\sum_{s=1}^{m_1}a_{i_s}X_{i_s}^T\otimes(\mathbf{1}_{n_2}^T\mathcal{B}_2)+\sum_{q=1}^{n_1}\sum_{p=1}^{l'}a_{qj_p}e_q^T\otimes(Z_{j_p}^T\mathcal{B}_2)\neq 0$ $(i\in\underline{n_1},\ j\in\underline{n_2})$ holds, where m_1 and l' are the multiplicity of r in V_{2_2} and in V_3 (as $\eta_j=0$), respectively, and arbitrary constants $a_{i_1},\cdots,a_{i_{m_1}},a_{1j_1},\cdots,a_{n_1j_1},\cdots,a_{n_1j_{l'}}$ are not all zero.

Lemma 2. $(\mathcal{A}_1,\mathcal{B}_1)$ is controllable if and only if $(\mathcal{L}_1,\mathcal{B}_1)$ is controllable.

Corollary 1. If $\mathcal{G}_2$ is connected and NCPN (2) is controllable, then $(\mathcal{L}_2,\mathcal{B}_2)$ is controllable.

Corollary 2. If $\mathcal{G}_2$ is connected, and both $(\mathcal{L}_1,\mathcal{B}_1)$ and $(\mathcal{L}_2,\mathcal{B}_2)$ are controllable, then NCPN (2) is controllable.

Corollary 3. If $\mathcal{G}_2$ is connected and $\det(\mathcal{A}_1)\neq 0$ & $r\neq 1$, then NCPN (2) is controllable if and only if $(\mathcal{L}_2,\mathcal{B}_2)$ is controllable.

Corollary 4. If $\mathcal{G}_2$ is disconnected and $\det(\mathcal{A}_1)\neq 0$ & $r+\eta_{j_1}=r+\eta_{j_2}=\cdots=r+\eta_{j_l}\in V_3$, where l is the multiplicity of η_j, then NCPN (2) is controllable if and only if $\sum_{q=1}^{n_1}\sum_{p=1}^{l}a_{qj_p}e_q^T\otimes(Z_{j_p}^T\mathcal{B}_2)\neq 0$ $(j=2,\cdots,n_2)$ holds, where arbitrary constants $a_{1j_1},\cdots,a_{n_1j_1},\cdots,a_{n_1j_l}$ are not all zero.

Theorems 1 and 2 and Corollaries 1–4 revealed the controllability relationship between the NCPN and its factors, gave effective criteria and easier approaches to determine and check the controllability, and further provided insights to model a composite network. Moreover, the computational complexity of selecting the new criteria obtained is approximately $O(n_1^4)+O(n_2^4)$, while directly using the PBH test it is approximately $O(n_1^4n_2^4)$ operations. Obviously, the new criteria greatly reduce the computational cost.

The preliminaries are included in Appendix A. The proofs of Theorems 1 and 2, Proposition 1, Lemma 2 and some remarks are included in Appendixes B–E, respectively. Illustrating examples and simulations are presented in Appendix F.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grant Nos. 62425602, 62173355, 12461027) and the Program for Innovative Research Team in Universities of Inner Mongolia (Grant No. NMGIRT2317).

Supporting information Appendixes A–F. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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