SCIENCE CHINA
Information Sciences

• **Supplementary File** •

Controllability of neighborhood corona product networks

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Appendix A Mathematical preliminaries

A triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ consists of a vertex set $\mathcal{V}$, an edge set $\mathcal{E}$ and the adjacency matrix $\mathcal{A} = [a_{ij}] \in R^{n\times n}$, where $\mathcal{D}$ and $\mathcal{L} \triangleq \mathcal{D} - \mathcal{A}$ are the degree matrix and the Laplacian matrix of $\mathcal{G}$, respectively. Throughout this work, we consider undirected and unweighted graphs, where $a_{ij} = a_{ji} = 1$ if $(i, j) \in \mathcal{E}$, otherwise 0.

The neighborhood corona product graph $\mathcal{G} \triangleq \mathcal{G}_1 \bigstar \mathcal{G}_2$ is a class of composite graphs generated by two smaller factor subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$, all vertex-disjoint, with n_1 and n_2 vertices, n_1' and n_2' edges, respectively, which can be obtained by taking one copy of $\mathcal{G}_1$ and n_1 copies of $\mathcal{G}_2$, and for each i $(i = 1, 2, \ldots, n_1)$, connecting each neighbourhood of the i-th vertex of $\mathcal{G}_1$ to each vertex in the i-th copy of $\mathcal{G}_2$ by a new edge. It is easy to see that the graph $\mathcal{G}_1 \bigstar \mathcal{G}_2$ has $n_1(1 + n_2)$ vertices and $n_1'(1 + 2n_2) + n_1 n_2'$ edges. Generally speaking, operation $\bigstar$ is not commutative, that is, $\mathcal{G}_1 \bigstar \mathcal{G}_2 \neq \mathcal{G}_2 \bigstar \mathcal{G}_1$. And the connectivity of $\mathcal{G}_1 \bigstar \mathcal{G}_2$ is only determined by that of $\mathcal{G}_1$. A visual example of the neighborhood corona product graph is illustrated as Fig. A1.

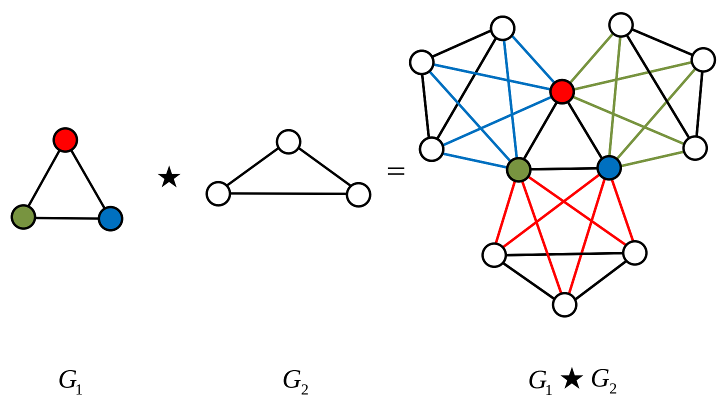

Figure A1 Neighborhood corona product graph of $\mathcal{G}_1$ and $\mathcal{G}_2$.

Lemma A1. (PBH Test) [4] System $(\mathcal{G}, \Sigma)$ is uncontrollable if and only if there exists a left eigenvector ξ corresponding to eigenvalue λ of $\mathcal{L}$ such that $\xi^T \mathcal{B} = 0$.

Appendix B Proof of Theorem 1

Sufficiency. From Lemma A1, to prove the controllability of the neighborhood corona product network (NCPN) (2)-(3), we need to prove that $\xi^T \mathcal{B} \neq 0$ for all the left eigenvectors of $\mathcal{L}$. Three cases will be discussed here.

Case (1). Consider the eigenvalues in V_1. Firstly, if each λ_i in V_{1_1} is single, its corresponding eigenvector is $\xi_i = \begin{bmatrix} \frac{\lambda_i - r}{\theta_i - r} X_i \\ X_i \otimes \mathbf{1}_{n_2} \end{bmatrix}$ for $i \in \underline{n_1}$. Then, from Lemma A1, we can have

$$\xi_i^T \mathcal{B} = \begin{bmatrix} \frac{\lambda_i - r}{\theta_i - r} X_i \\ X_i \otimes \mathbf{1}_{n_2} \end{bmatrix}^T \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_{n_1} \otimes \mathcal{B}_2 \end{bmatrix} = \begin{bmatrix} \frac{\lambda_i - r}{\theta_i - r} X_i^T \mathcal{B}_1 & X_i^T \otimes (\mathbf{1}_{n_2}^T \mathcal{B}_2) \end{bmatrix} \neq 0, \tag{B1}$$

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due to $X_i^T \neq 0$ and $\mathbf{1}_{n_2}^T \mathcal{B}_2 \neq 0$ (since $(\mathcal{L}_2, \mathcal{B}_2)$ is controllable and $\mathbf{1}_{n_2}^T (\neq 0)$ is the left eigenvector corresponding to zero eigenvalue of $\mathcal{L}_2$).

Secondly, if λ_i in V_{1_1} is k-repeated, that is, $\lambda_{i_1} = \lambda_{i_2} = \cdots = \lambda_{i_k}$ in V_{1_1}, $\xi_i^k = \sum_{p=1}^k a_{ip} \xi_{ip}$ is the eigenvector corresponding to $\lambda_{i_1} = \lambda_{i_2} = \cdots = \lambda_{i_k}$ $(i \in \underline{n_1})$, where arbitrary constants $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$ are not all zero. So

$$\begin{aligned}
{\xi_i^k}^T \mathcal{B} &= \sum_{p=1}^k a_{ip} \begin{bmatrix} \frac{\lambda_{ip}-r}{\theta_{ip}-r} X_{ip} \\ X_{ip} \otimes \mathbf{1}_{n_2} \end{bmatrix}^T \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_{n_1} \otimes \mathcal{B}_2 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{p=1}^k a_{ip} \frac{\lambda_{ip}-r}{\theta_{ip}-r} X_{ip} \\ \sum_{p=1}^k a_{ip} X_{ip} \otimes \mathbf{1}_{n_2} \end{bmatrix}^T \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_{n_1} \otimes \mathcal{B}_2 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{p=1}^k a_{ip} \frac{\lambda_{ip}-r}{\theta_{ip}-r} X_{ip}^T \mathcal{B}_1 & \sum_{p=1}^k a_{ip} X_{ip}^T \otimes (\mathbf{1}_{n_2}^T \mathcal{B}_2) \end{bmatrix} \\
&\neq 0,
\end{aligned} \tag{B2}$$

due to $\sum_{p=1}^k a_{ip} X_{ip}^T \neq 0$ (since $X_{i_1}^T, \cdots, X_{i_k}^T$ are the orthogonal left eigenvectors) and $\mathcal{B}_2 \neq 0$ (since $(\mathcal{L}_2, \mathcal{B}_2)$ is controllable).

Thirdly, if $V_{1_2} \neq \emptyset$, eigenvalue $(n_2+1)r$ in V_{1_1} (as $\theta_1 = 0$) is single and in V_{1_2} is m_1-repeated. Thus $\xi_i^{(m_1+1)} = a_1 \begin{bmatrix} -n_2 X_1 \\ X_1 \otimes \mathbf{1}_{n_2} \end{bmatrix} + \sum_{p=1}^{m_1} a_{ip} \begin{bmatrix} X_{ip} \\ \mathbf{0}_{n_1 n_2} \end{bmatrix}$ $(i \in \underline{n_1})$ is the eigenvector corresponding to (m_1+1)-repeated eigenvalue $(n_2+1)r$ in V_1, where arbitrary constants $a_1, a_{i_1}, a_{i_2}, \cdots, a_{i_{m_1}}$ are not all zero. So

$$\begin{aligned}
{\xi_i^{(m_1+1)}}^T \mathcal{B} &= \begin{bmatrix} -a_1 n_2 X_1 + \sum_{p=1}^{m_1} a_{ip} X_{ip} \\ a_1 X_1 \otimes \mathbf{1}_{n_2} \end{bmatrix}^T \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_{n_1} \otimes \mathcal{B}_2 \end{bmatrix} \\
&= \begin{bmatrix} -a_1 n_2 X_1^T \mathcal{B}_1 + \sum_{p=1}^{m_1} a_{ip} X_{ip}^T \mathcal{B}_1 & a_1 X_1^T \otimes (\mathbf{1}_{n_2}^T \mathcal{B}_2) \end{bmatrix} \\
&\neq 0,
\end{aligned} \tag{B3}$$

due to $\sum_{p=1}^{m_1} a_{ip} X_{ip}^T \mathcal{B}_1 \neq 0$, $X_i^T \neq 0$ and $\mathcal{B}_2 \neq 0$ (since $(\mathcal{L}_2, \mathcal{B}_2)$ is controllable).

Fourthly, if $V_{1_1} \cap V_3 \neq \emptyset$, from proposition 1, there must exist a common eigenvalue $\tau = \eta_{j_1} + r = \eta_{j_2} + r = \cdots = \eta_{j_{m_2}} + r = n_2 + 1 \in V_{1_1} \cap V_3$. Then $\xi_j^{(m_2+1)} = a_1 \begin{bmatrix} \frac{\lambda_1 - r}{\theta_1 - r} X_1 \\ X_1 \otimes \mathbf{1}_{n_2} \end{bmatrix} + \sum_{p=1}^{m_2} \sum_{q=1}^{n_1} a_{qjp} \begin{bmatrix} \mathbf{0}_{n_1} \\ e_q \otimes Z_{jp} \end{bmatrix} = \begin{bmatrix} -a_1 n_2 X_1 \\ a_1 X_1 \otimes \mathbf{1}_{n_2} + \sum_{p=1}^{m_2} \sum_{q=1}^{n_1} a_{qjp} e_q \otimes Z_{jp} \end{bmatrix}$ $(j \in \underline{n_2})$ is the eigenvectors corresponding to eigenvalue $n_2 + 1$, where arbitrary constants $a_1, a_{1j_1}, \cdots, a_{1j_{m_2}}, \cdots, a_{n_1 j_{m_2}}$ are not all zero. So

$$\begin{aligned}
{\xi_j^{(m_2+1)}}^T \mathcal{B} &= \begin{bmatrix} -a_1 n_2 X_1 \\ a_1 X_1 \otimes \mathbf{1}_{n_2} + \sum_{p=1}^{m_2} \sum_{q=1}^{n_1} a_{qjp} e_q \otimes Z_{jp} \end{bmatrix}^T \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_{n_1} \otimes \mathcal{B}_2 \end{bmatrix} \\
&= \begin{bmatrix} -a_1 n_2 X_1^T \mathcal{B}_1 & a_1 X_1^T \otimes (\mathbf{1}_{n_2}^T \mathcal{B}_2) + \sum_{p=1}^{m_2} \sum_{q=1}^{n_1} a_{qjp} e_q^T \otimes (Z_{jp}^T \mathcal{B}_2) \end{bmatrix} \\
&\neq 0,
\end{aligned} \tag{B4}$$

since $X_1^T \mathcal{B}_1 \neq 0$ or $a_1 X_1^T \otimes (\mathbf{1}_{n_2}^T \mathcal{B}_2) + \sum_{p=1}^{m_2} \sum_{q=1}^{n_1} a_{qjp} e_q^T \otimes (Z_{jp}^T \mathcal{B}_2) \neq 0$.

Case (2). Consider the eigenvalues in V_2. Firstly, if each $\hat{\lambda}_i$ in V_{2_1} is single, its corresponding eigenvector is $\hat{\xi}_i = \begin{bmatrix} \frac{\hat{\lambda}_i - r}{\theta_i - r} X_i \\ X_i \otimes \mathbf{1}_{n_2} \end{bmatrix}$ for $i \in \underline{n_1}$. Then

$$\hat{\xi}_i^T \mathcal{B} = \begin{bmatrix} \frac{\hat{\lambda}_i - r}{\theta_i - r} X_i \\ X_i \otimes \mathbf{1}_{n_2} \end{bmatrix}^T \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_{n_1} \otimes \mathcal{B}_2 \end{bmatrix} = \begin{bmatrix} \frac{\hat{\lambda}_i - r}{\theta_i - r} X_i^T \mathcal{B}_1 & X_i^T \otimes (\mathbf{1}_{n_2}^T \mathcal{B}_2) \end{bmatrix} \neq 0, \tag{B5}$$

due to $X_i^T \neq 0$ and $\mathcal{B}_2 \neq 0$ (since $(\mathcal{L}_2, \mathcal{B}_2)$ is controllable).

Secondly, if $\hat{\lambda}_i$ in V_{2_1} is k'-repeated, that is, $\lambda_{i_1} = \lambda_{i_2} = \cdots = \lambda_{i_{k'}}$ in V_{2_1}, its corresponding eigenvector is $\hat{\xi}_i^{k'} = \sum_{p=1}^{k'} a_{ip} \hat{\xi}_{ip}$ $(i \in \underline{n_1})$, where arbitrary constants $a_{i_1}, a_{i_2}, \cdots, a_{i_{k'}}$ are not all zero. So

$$\begin{aligned}
{\hat{\xi}_i^{k'}}{}^T \mathcal{B} &= \sum_{p=1}^{k'} a_{ip} \begin{bmatrix} \frac{\hat{\lambda}_{ip}-r}{\theta_{ip}-r} X_{ip} \\ X_{ip} \otimes \mathbf{1}_{n_2} \end{bmatrix}^T \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_{n_1} \otimes \mathcal{B}_2 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{p=1}^{k'} a_{ip} \frac{\hat{\lambda}_{ip}-r}{\theta_{ip}-r} X_{ip}^T \mathcal{B}_1 & \sum_{p=1}^{k'} a_{ip} X_{ip}^T \otimes (\mathbf{1}_{n_2}^T \mathcal{B}_2) \end{bmatrix} \\
&\neq 0,
\end{aligned} \tag{B6}$$

due to $\sum_{p=1}^{k'} a_{ip} X_{ip}^T \neq 0$ (since $X_{i_1}^T, \cdots, X_{i_{k'}}^T$ are the orthogonal left eigenvectors) and $\mathcal{B}_2 \neq 0$ (since $(\mathcal{L}_2, \mathcal{B}_2)$ is controllable).

Thirdly, if $V_{2_2} \neq \emptyset$, eigenvalue r in V_{2_2} is m_1-repeated. Thus $\hat{\xi}_i^{m_1} = \sum_{p=1}^{m_1} a_{i_p} \begin{bmatrix} \mathbf{0}_{n_1} \\ X_{i_p} \otimes \mathbf{1}_{n_2} \end{bmatrix}$ $(i \in \underline{n_1})$ is the eigenvector corresponding to m_1-repeated eigenvalue r in V_{2_2}, where arbitrary constants $a_1, a_{i_1}, a_{i_2}, \cdots, a_{i_{m_1}}$ are not all zero. So

$$\hat{\xi}_i^{m_1\,T} \mathcal{B} = \sum_{p=1}^{m_1} a_{i_p} \begin{bmatrix} \mathbf{0}_{n_1} \\ X_{i_p} \otimes \mathbf{1}_{n_2} \end{bmatrix}^T \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_{n_1} \otimes \mathcal{B}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n_1}^T & \sum_{p=1}^{m_1} a_{i_p} X_{i_p}^T \otimes (\mathbf{1}_{n_2}^T \mathcal{B}_2) \end{bmatrix} \neq 0, \tag{B7}$$

due to $\sum_{p=1}^{m_1} a_{i_p} X_{i_p}^T \neq 0$ (since $X_{i_1}^T, \cdots, X_{i_{m_1}}^T$ are the orthogonal left eigenvectors) and $\mathcal{B}_2 \neq 0$ (since $(\mathcal{L}_2, \mathcal{B}_2)$ is controllable).

Case (3). Consider the eigenvalues in V_3. Firstly, if each $r + \eta_j$ in V_3 is single, its corresponding eigenvector is $\xi_j = \begin{bmatrix} \mathbf{0}_{n_1} \\ e_q \otimes Z_j \end{bmatrix}$ $(j = 2, 3, \ldots, n_2)$. Then

$$\xi_j^T \mathcal{B} = \begin{bmatrix} \mathbf{0}_{n_1} \\ e_q \otimes Z_j \end{bmatrix}^T \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_{n_1} \otimes \mathcal{B}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n_1}^T & e_q^T \otimes (Z_j^T \mathcal{B}_2) \end{bmatrix} \neq 0, \tag{B8}$$

since $(\mathcal{L}_2, \mathcal{B}_2)$ is controllable.

Secondly, if $r + \eta_j$ in V_3 is k''-repeated, that is, $r + \eta_{j_1} = r + \eta_{j_2} = \cdots = r + \eta_{j_{k''}} \in V_3$, its corresponding eigenvector is $\xi_j^{k''} = \sum_{q=1}^{n_1} \sum_{p=1}^{k''} a_{qj_p} \begin{bmatrix} \mathbf{0}_{n_1} \\ e_q \otimes Z_{j_p} \end{bmatrix}$ $(j = 2, \cdots, n_2)$, where arbitrary constants $a_{1j_1}, \cdots, a_{1j_{k''}}, \cdots, a_{n_1 j_{k''}}$ are not all zero. So

$$\begin{aligned} \xi_j^{k''\,T} \mathcal{B} &= \sum_{q=1}^{n_1} \sum_{p=1}^{k''} a_{qj_p} \begin{bmatrix} \mathbf{0}_{n_1} \\ e_q \otimes Z_{j_p} \end{bmatrix}^T \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_{n_1} \otimes \mathcal{B}_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0}_{n_1}^T & \sum_{q=1}^{n_1} \sum_{p=1}^{k''} a_{qj_p} e_q^T \otimes (Z_{j_p}^T \mathcal{B}_2) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0}_{n_1}^T & \sum_{p=1}^{k''} a_{1i_p} Z_{j_p}^T \mathcal{B}_2 & \sum_{p=1}^{k''} a_{2i_p} Z_{j_p}^T \mathcal{B}_2 & \cdots & \sum_{p=1}^{k''} a_{n_1 i_p} Z_{j_p}^T \mathcal{B}_2 \end{bmatrix} \\ &\neq 0, \end{aligned} \tag{B9}$$

since $(\mathcal{L}_2, \mathcal{B}_2)$ is controllable.

In summary, the NCPN (2)-(3) is controllable combining with equations (B1)-(B9).

Necessity. If the NCPN (2)-(3) is controllable, then $\xi^T \mathcal{B} \neq 0$ for all the left eigenvectors of $\mathcal{L}$ and $\mathcal{B} \neq 0$. From equation (B8) and equation (B9), if $\xi_j^T \mathcal{B} \neq 0$ and $\xi_j^{k''\,T} \mathcal{B} \neq 0$ for $j = 2, \cdots, n_2$, obviously, $Z_j^T \mathcal{B}_2 \neq 0$ and $Z_{j_p}^T \mathcal{B}_2 \neq 0$. As $j = 1$, we can have $Z_1^T \mathcal{B}_2 = 1_{n_2}^T \mathcal{B}_2 \neq 0$. Therefore, $Z_j^T \mathcal{B}_2 \neq 0$ and $Z_{j_p}^T \mathcal{B}_2 \neq 0$ for $j = 1, 2, \cdots, n_2$, Thus, $(\mathcal{L}_2, \mathcal{B}_2)$ is controllable. From equation (B3), if $V_{1_2} \neq \emptyset$ and $\xi_i^{(m_1+1)\,T} \mathcal{B} \neq 0$, then $-a_1 n_2 X_1^T \mathcal{B}_1 + \sum_{p=1}^{m_1} a_{i_p} X_{i_p}^T \mathcal{B}_1 \neq 0$ or $a_1 X_1^T \otimes (\mathbf{1}_{n_2}^T \mathcal{B}_2) \neq 0$, for arbitrary constants a_1 and $a_{i_1}, \cdots, a_{i_{m_1}}$ (not all zero), which implies that $\sum_{p=1}^{m_1} a_{i_p} X_{i_p}^T \mathcal{B}_1 \neq 0$ as $a_1 = 0$. Finally, from equation (B4), if $V_{1_1} \cap V_3 \neq \emptyset$, and $\xi_j^{(m_2+1)\,T} \mathcal{B} \neq 0$, then $X_1^T \mathcal{B}_1 \neq 0$ or $a_1 X_1^T \otimes (\mathbf{1}_{n_2}^T \mathcal{B}_2) + \sum_{p=1}^{m_2} \sum_{q=1}^{n_1} a_{qj_p} e_q^T \otimes (Z_{j_p}^T \mathcal{B}_2) \neq 0$ $(j \in \underline{n_2})$.

Appendix C Proof of Theorem 2

Sufficiency. Similar to the proof of Theorem 1, three cases will also be discussed in the following.

Case (1). Consider the eigenvalues in V_3. If $\varsigma = r + \eta_{j_1} = r + \eta_{j_2} = \cdots = r + \eta_{j_l} \in V_3$, its corresponding eigenvector is $\xi_j^l = \sum_{q=1}^{n_1} \sum_{p=1}^{l} a_{qj_p} \begin{bmatrix} \mathbf{0}_{n_1} \\ e_q \otimes Z_{j_p} \end{bmatrix}$ $(j = 2, \cdots, n_2)$, where arbitrary constants $a_{1j_1}, \cdots, a_{1j_l}, \cdots, a_{n_1 j_l}$ are not all zero. So

$$\begin{aligned} \xi_j^{l\,T} \mathcal{B} &= \sum_{q=1}^{n_1} \sum_{p=1}^{l} a_{qj_p} \begin{bmatrix} \mathbf{0}_{n_1} \\ e_q \otimes Z_{j_p} \end{bmatrix}^T \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_{n_1} \otimes \mathcal{B}_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0}_{n_1}^T & \sum_{q=1}^{n_1} \sum_{p=1}^{l} a_{qj_p} e_q^T \otimes (Z_{j_p}^T \mathcal{B}_2) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0}_{n_1}^T & \sum_{p=1}^{l} a_{1i_p} Z_{j_p}^T \mathcal{B}_2 & \sum_{p=1}^{l} a_{2i_p} Z_{j_p}^T \mathcal{B}_2 & \cdots & \sum_{p=1}^{l} a_{n_1 i_p} Z_{j_p}^T \mathcal{B}_2 \end{bmatrix} \\ &\neq 0, \end{aligned} \tag{C1}$$

and $\mathcal{B}_2 \neq 0$, since $\sum_{q=1}^{n_1} \sum_{p=1}^{l} a_{qj_p} e_q^T \otimes (Z_{j_p}^T \mathcal{B}_2) \neq 0$.

Case (2). Consider the eigenvalues in V_1. If $V_{1_2} \neq \emptyset$, then $\sum_{p=1}^{m_1} a_{i_p} X_{i_p}^T \mathcal{B}_1 \neq 0$ $(i \in \underline{n_1})$ holds, which can be proved similar to that of Theorem 1, here omitted.

Case (3). Consider the eigenvalues in V_2. If $\hat{\lambda}_i$ is in V_{2_1}, the proof is similar to that of Theorem 1. If $V_{2_2} \neq \emptyset$, then eigenvalue r in V_{2_2} is m_1-repeated and in V_3 (as $\eta_j = 0$) is l'-repeated, and hence $\xi_{ij}^{m_1 l'} = \sum_{s=1}^{m_1} a_{i_s} \begin{bmatrix} \mathbf{0}_{n_1} \\ X_{i_s} \otimes \mathbf{1}_{n_2} \end{bmatrix} + \sum_{q=1}^{n_1} \sum_{p=1}^{l'} a_{qj_p} \begin{bmatrix} \mathbf{0}_{n_1} \\ e_q \otimes Z_{j_p} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n_1} \\ \sum_{s=1}^{m_1} a_{i_s} X_{i_s} \otimes \mathbf{1}_{n_2} + \sum_{q=1}^{n_1} \sum_{p=1}^{l'} a_{qj_p} e_q \otimes Z_{j_p} \end{bmatrix}$ $(i \in \underline{n_1},\ j \in \underline{n_2})$ is the eigenvector corresponding to $(m_1 + l')$-repeated eigenvalue r in $V_{2_2}\cap V_3$, where arbitrary constants $a_{i_1},\cdots,a_{i_{m_1}},a_{1j_1},\cdots,a_{n_1j_1},\cdots,a_{n_1j_{l'}}$ are not all zero. So

$$\begin{aligned}\xi_{ij}^{m_1l'T}\mathcal{B}&=\begin{bmatrix}\mathbf{0}_{n_1}\\ \sum_{s=1}^{m_1}a_{i_s}X_{i_s}\otimes\mathbf{1}_{n_2}+\sum_{q=1}^{n_1}\sum_{p=1}^{l'}a_{qj_p}e_q\otimes Z_{j_p}\end{bmatrix}^T\begin{bmatrix}\mathcal{B}_1 & 0\\ 0 & I_{n_1}\otimes\mathcal{B}_2\end{bmatrix}\\ &=\begin{bmatrix}\mathbf{0}_{n_1}^T & \sum_{s=1}^{m_1}a_{i_s}X_{i_s}^T\otimes(\mathbf{1}_{n_2}^T\mathcal{B}_2)+\sum_{q=1}^{n_1}\sum_{p=1}^{l'}a_{qj_p}e_q^T\otimes(Z_{j_p}^T\mathcal{B}_2)\end{bmatrix}\\ &\neq 0,\end{aligned}\tag{C2}$$

due to $\sum_{s=1}^{m_1}a_{i_s}X_{i_s}^T\otimes(\mathbf{1}_{n_2}^T\mathcal{B}_2)+\sum_{q=1}^{n_1}\sum_{p=1}^{l'}a_{qj_p}e_q^T\otimes(Z_{j_p}^T\mathcal{B}_2)\neq 0$.

In summary, the NCPN (2)-(3) is controllable combining with equations (C1)-(C2).

Necessity. If the NCPN (2)-(3) is controllable, then $\xi^T\mathcal{B}\neq 0$ for all the left eigenvectors of $\mathcal{L}$ and $\mathcal{B}\neq 0$. From equation (C1), obviously, $\xi_j^{l\,T}\mathcal{B}\neq 0$ $(j=2,\cdots,n_2)$, then $\sum_{q=1}^{n_1}\sum_{p=1}^{l}a_{qj_p}e_q^T\otimes(Z_{j_p}^T\mathcal{B}_2)\neq 0$ $(j=2,\cdots,n_2)$. From equation (B3), if $V_{1_2}\neq\emptyset$ and $\xi_i^{(m_1+1)T}\mathcal{B}\neq 0$, then $-a_1n_2X_1^T\mathcal{B}_1+\sum_{p=1}^{m_1}a_{i_p}X_{i_p}^T\mathcal{B}_1\neq 0$ or $a_1X_1^T\otimes(\mathbf{1}_{n_2}^T\mathcal{B}_2)\neq 0$, for arbitrary constants a_1 and $a_{i_1},\cdots,a_{i_{m_1}}$ (not all zero), which implies that $\sum_{p=1}^{m_1}a_{i_p}X_{i_p}^T\mathcal{B}_1\neq 0$ as $a_1=0$. Finally, from equation (C2), if $V_{2_2}\neq\emptyset$, and $\xi_{ij}^{m_1l'T}\mathcal{B}\neq 0$, then $\sum_{s=1}^{m_1}a_{i_s}X_{i_s}^T\otimes(\mathbf{1}_{n_2}^T\mathcal{B}_2)+\sum_{q=1}^{n_1}\sum_{p=1}^{l'}a_{qj_p}e_q^T\otimes(Z_{j_p}^T\mathcal{B}_2)\neq 0$ for arbitrary constants $a_{i_1},\cdots,a_{i_{m_1}},a_{1j_1},\cdots,a_{n_1j_1},\cdots,a_{n_1j_{l'}}$ (not all zero).

Remark 1. Comparing with the conditions of Theorems 1-2, we can find that when $\mathcal{G}_2$ is connected, if $(\mathcal{L}_2,\mathcal{B}_2)$ is controllable, the condition (i) of Theorem 2 holds, vice versa.

Remark 2. Theorems 1-2 give simple methods to analyze the controllability of larger-scale composite networks generated by lower-dimensional factor networks via the neighbourhood corona product, which can help us to understand the properties of NCPNs and be applied into the real scenarios.

Appendix D Proof of Proposition 1

(i) First of all, if $V_1\cap V_2\neq\emptyset$, there must be $\lambda_i=\hat{\lambda}_j\in V_1\cap V_2$, which implies that $\theta_i+\sqrt{\Delta_i}=\theta_j-\sqrt{\Delta_j}$. It is obvious to see $\sqrt{\Delta_i}+\theta_i\geqslant 0$. And $\theta_j-\sqrt{\Delta_j}<0$, since

$$\begin{aligned}\sqrt{\Delta_j}=&\sqrt{((n_2+1)r+\theta_j)^2-4\theta_j((2n_2+1)r-n_2\theta_j)}\\ =&\sqrt{(n_2+1)^2r^2-2\theta_jr(3n_2+1)+(4n_2+1)\theta_j^2}\\ =&\sqrt{\left((n_2+1)r-\frac{3n_2+1}{n_2+1}\theta_j\right)^2+\left(4n_2+1-\frac{(3n_2+1)^2}{(n_2+1)^2}\right)\theta_j^2}\\ >&\theta_j.\end{aligned}\tag{D1}$$

Therefore, $V_1\cap V_2=\emptyset$.

Secondly, if $V_{1_2}\cap V_3\neq\emptyset$, then $(n_2+1)r=\eta_j+r\in V_{1_2}\cap V_3$, and hence $n_2r=\eta_j\leqslant n_2\Rightarrow r\leqslant 1\Rightarrow\theta_i=r=1$, which contradicts with the fact $\theta_i=0$ or $\theta_i=2$, since the Laplacian matrix of $\mathcal{G}_1$ must be $\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ as $r=1$. Therefore, $V_{1_2}\cap V_3=\emptyset$.

Thirdly, it is obvious to know that $V_{1_2}=\emptyset\Leftrightarrow\theta_i\neq r$ $(i=1,2,\cdots,n_1)$.

Fourthly, if $V_{1_1}\cap V_{1_2}\neq\emptyset$, then $\theta_i=r$ from the definition of V_{1_2}. Conversely, if $\theta_i=r$, it is obvious to know $(n_2+1)r\in V_{1_2}$. Since $\theta_i=0\in\sigma(\mathcal{L}_1)$ for some i, $\lambda_i=\frac{(n_2+1)r+\theta_i+\sqrt{\Delta_i}}{2}=(n_2+1)r\in V_{1_1}$. Therefore, $V_{1_1}\cap V_{1_2}\neq\emptyset$.

Fifthly, if $V_{1_1}\cap V_3\neq\emptyset$, there exist $\lambda_i\in V_{1_1}$ and $\eta_j+r\in V_3$, such that $\lambda_i=\frac{(n_2+1)r+\theta_i+\sqrt{\Delta_i}}{2}=r+\eta_j$. And we can know

$$\begin{aligned}\sqrt{\Delta_i}=&\sqrt{((n_2+1)r+\theta_i)^2-4\theta_i((2n_2+1)r-n_2\theta_i)}\\ =&\sqrt{((n_2-1)r+\theta_i)^2+4n_2(r-\theta_i)^2}\\ \geqslant&(n_2-1)r+\theta_i>0.\end{aligned}\tag{D2}$$

Based on equation (D2), we can have $2\eta_j=(n_2-1)r+\theta_i+\sqrt{\Delta_i}>2(n_2-1)r+2\theta_i$, and $(n_2-1)r+\theta_i<\eta_j\leqslant n_2\Rightarrow(n_2-1)r+\theta_i<n_2\Rightarrow(n_2-1)r<n_2\Rightarrow r<\frac{n_2}{n_2-1}\Rightarrow r=1$, since $0\leqslant\eta_j\leqslant n_2$ and $0\leqslant\theta_i$. Because $\mathcal{G}_1$ is connected and $r=1$, $\sigma(\mathcal{L}(\mathcal{G}_1))=\{0,2\}$, but $\theta_i=2$ should be given up, which contradicts with the fact that $(n_2-1)r+2=(n_2-1)1+2<n_2$,therefore, $\theta_i=0$. Furthermore, put $r=1$ and $\theta_i=0$ into $\lambda_i=\frac{(n_2+1)r+\theta_i+\sqrt{\Delta_i}}{2}=r+\eta_j$, we can have $\lambda_i=n_2+1=1+\eta_j$, so $\eta_j=n_2$. Therefore, the result holds. Conversely, $V_{1_1}\cap V_3\neq\emptyset$ is clearly true if $r=1$, $\eta_j=n_2$ and $\theta_i=0$ with multiplicity-1. Specially, if $\mathcal{G}_2$ is a disconnected graph, it is easy to get $0\leqslant\eta_j<n_2$, which implies that $V_{1_1}\cap V_3=\emptyset$.

(ii) Firstly, If $V_{2_1}\cap V_{2_2}\neq\emptyset$, then $\exists\hat{\lambda}_i\in V_{2_1}$, such that $\hat{\lambda}_i=\frac{(n_2+1)r+\theta_i-\sqrt{\Delta_i}}{2}=r\in V_{2_1}\cap V_{2_2}$, and hence $(n_2-1)r+\theta_i-\sqrt{\Delta_i}=0$, which contradicts with equation (D2).

Secondly, if $V_{2_1}\cap V_3\neq\emptyset$, $\exists\hat{\lambda}_i\in V_{2_1}$ and $r+\eta_j\in V_3$, such that $\hat{\lambda}_i=\frac{(n_2+1)r+\theta_i-\sqrt{\Delta_i}}{2}=r+\eta_j\in V_{2_1}\cap V_3$, which implies that $2\eta_j=(n_2-1)r+\theta_i-\sqrt{\Delta_i}<0$ from equation (D2). It contradicts with the fact $\eta_j\geqslant 0$.

Thirdly, from the definition of V_{2_2}, it is easy to get $V_{2_2}=\emptyset\Leftrightarrow\theta_i\neq r$ $(i=1,2,\cdots,n_1)$.

Fourthly, if $V_{2_2}\cap V_3\neq\emptyset$, $\exists\hat{\lambda}_i\in V_{2_2}$ and $\eta_j+r\in V_3$, such that $\hat{\lambda}_i=\frac{(n_2+1)r+\theta_i-\sqrt{\Delta_i}}{2}=r=\eta_j+r\in V_{2_2}\cap V_3$, then $\eta_j=0$ for $j\in\underline{n_2}$, so $\mathcal{G}_2$ is disconnected, and we can get $V_{2_2}\cap V_3=\emptyset$ when $\mathcal{G}_2$ is connected. Conversely, if $\mathcal{G}_2$ is disconnected, then $\eta_j=0$ for $j\in\underline{n_2}$ and $r=\eta_j+r\in V_{2_2}\cap V_3\neq\emptyset$, therefore we can get $\mathcal{G}_2$ is connected when $V_{2_2}\cap V_3=\emptyset$. In summary, $V_{2_2}\cap V_3=\emptyset\Leftrightarrow\mathcal{G}_2$ is connected.

Appendix E Proof of Lemma 2

Since $X_1, X_2, \cdots, X_{n_1}$ are the orthogonal eigenvectors of $\mathcal{A}_1$ corresponding to eigenvalues $\mu_1, \mu_2, \cdots, \mu_{n_1}$, we can have $X_i^T \mathcal{A}_1 = \mu_i X_i^T$. Then

$$\begin{aligned} rX_i^T - X_i^T \mathcal{A}_1 &= X_i^T r I_{n_1} - \mu_i X_i^T \\ &= X_i^T (rI_{n_1} - \mathcal{A}_1) \\ &= (r - \mu_i) X_i^T, \end{aligned} \tag{E1}$$

and hence $X_i^T \mathcal{L}_1 = (r - \mu_i) X_i^T = \theta_i X_i^T$, which implies that $X_1, X_2, \cdots, X_{n_1}$ are also the eigenvectors of $\mathcal{L}_1$ corresponding to eigenvalues $\theta_1, \theta_2, \cdots, \theta_{n_1}$. Therefore, $\mathcal{A}_1$ and $\mathcal{L}_1$ have the same orthogonal eigenvectors, which means that the controllability of $(\mathcal{A}_1, \mathcal{B}_1)$ is equivalent to that of $(\mathcal{L}_1, \mathcal{B}_1)$.

Remark 3. Note that if $det(\mathcal{A}_1) \neq 0 \Leftrightarrow \mu_i \neq 0 \Leftrightarrow \theta_i \neq r$, then we can know that $V_{1_2} = \emptyset$, $V_{2_2} = \emptyset$ and $V_{1_1} \cap V_3 = \emptyset$ (as $r \neq 1$) according to proposition 1. Based on these, combining with Lemma 2, Theorems 1-2, some simpler controllable conditions of the NCPN (2)-(3) can be obtained, which are easier to check, compute and design the network structures.

Appendix F Examples and simulations

Example 1. A NCPN shown is shown in Fig. F1, where $\mathcal{G}_2$ is connected.

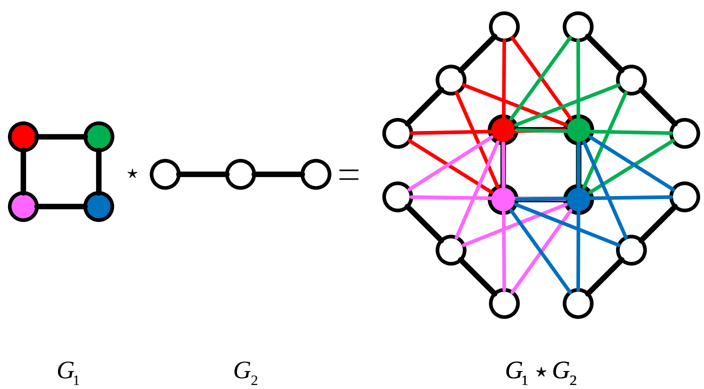

Figure F1 A NCPN of $\mathcal{G}_1$ and $\mathcal{G}_2$ for Example 1.

Let

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \mathcal{L}_1 = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}, \quad \mathcal{L}_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \mathcal{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 + 6I_4 & -\mathcal{A}_1 \otimes \mathbf{1}_3^T \\ -\mathcal{A}_1 \otimes \mathbf{1}_3 & I_4 \otimes (\mathcal{L}_2 + 2I_3) \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_4 \otimes \mathcal{B}_2 \end{bmatrix}.$$

By calculation, the eigenvalues and the corresponding eigenvectors of $\mathcal{L}_1$ and $\mathcal{L}_2$ are respectively

$$\begin{cases} \theta_1 = 0, \\ \theta_{2_1} = 2, \\ \theta_{2_2} = 2, \\ \theta_3 = 4, \end{cases} \qquad \begin{cases} X_1 = e_1 + e_2 + e_3 + e_4, \\ X_{2_1} = -e_1 + e_2 - e_3 + e_4, \\ X_{2_2} = e_1 + e_2 - e_3 - e_4, \\ X_3 = e_1 - e_2 - e_3 + e_4, \end{cases}$$

and

$$\begin{cases} \eta_1 = 0, \\ \eta_2 = 1, \\ \eta_3 = 3, \end{cases} \qquad \begin{cases} Z_1 = e_1 + e_2 + e_3, \\ Z_2 = e_1 - e_3, \\ Z_3 = e_1 - 2e_2 + e_3. \end{cases}$$

Furthermore, $V_{1_1} = \{8, 6 + 2\sqrt{7}\}$, $V_{1_2} = \{8, 8\}$, $V_{2_1} = \{0, 6 - 2\sqrt{7}\}$, $V_{2_2} = \{2, 2\}$, $V_3 = \{3, 3, 3, 3, 5, 5, 5, 5\}$ and rank$(\mathcal{L}_2, \mathcal{B}_2) = 3$, hence $V_{1_1} \cap V_{1_2} \neq \emptyset$, $V_{1_1} \cap V_3 = \emptyset$, $(\mathcal{L}_2, \mathcal{B}_2)$ is controllable, and $(a_{2_1} X_{2_1}^T + a_{2_2} X_{2_2}^T)\mathcal{B}_1 = \begin{bmatrix} -a_{2_1} + a_{2_2} & a_{2_1} + a_{2_2} & 0 & 0 \end{bmatrix} \neq 0$, which satisfy the conditions of Theorem 1, therefore, this NCPN is controllable.

Fig. F2 represents the all agents' movement trajectories from the arbitrary initial state to the desired state, where agents are denoted as ★ in $\mathcal{G}_1$ and ∘ in $\mathcal{G}_2$, respectively, and Letter 'N' shows the final configuration (the desired state, denoted as ▷).

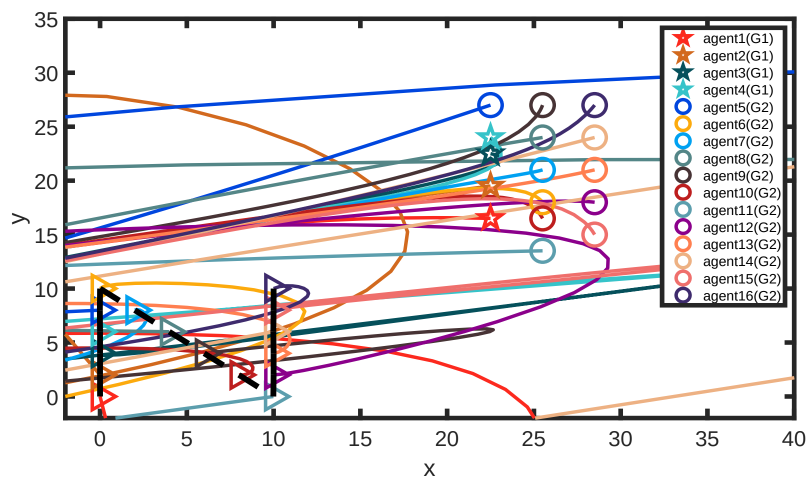

Figure F2 Letter 'N' configuration for $\mathcal{G}$ in Example 1.

Example 2. A NCPN shown is shown in Fig. F3 where $\mathcal{G}_2$ is connected. Let

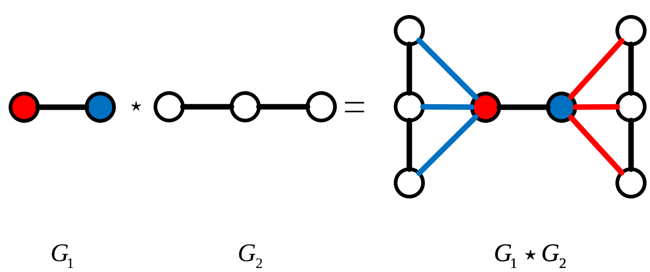

Figure F3 A NCPN of $\mathcal{G}_1$ and $\mathcal{G}_2$ for Example 2.

$$\mathcal{A}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{L}_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathcal{L}_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \mathcal{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 + 3I_2 & -\mathcal{A}_1 \otimes \mathbf{1}_3^T \\ -\mathcal{A}_1 \otimes \mathbf{1}_3 & I_2 \otimes (\mathcal{L}_2 + I_3) \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_2 \otimes \mathcal{B}_2 \end{bmatrix}.$$

By calculation, the eigenvalues and the corresponding eigenvectors of $\mathcal{L}_1$ and $\mathcal{L}_2$ are respectively

$$\begin{cases} \theta_1 = 0, \\ \theta_2 = 2, \end{cases} \qquad \begin{cases} X_1 = e_1 + e_2, \\ X_2 = -e_1 + e_2, \end{cases}$$

and

$$\begin{cases} \eta_1 = 0, \\ \eta_2 = 1, \\ \eta_3 = 3, \end{cases} \qquad \begin{cases} Z_1 = e_1 + e_2 + e_3, \\ Z_2 = e_1 - e_3, \\ Z_3 = e_1 - 2e_2 + e_3. \end{cases}$$

Furthermore, $V_{1_1}=\{4,3+\sqrt{7}\}$, $V_{2_1}=\{0,3-\sqrt{7}\}$, $V_3=\{2,2,4,4\}$ and rank$(\mathcal{L}_2,\mathcal{B}_2)=3$, hence $V_{1_2}=\emptyset, V_{2_2}=\emptyset$, $V_{1_1}\cap V_3\neq\emptyset$. So we can know that $(\mathcal{L}_2,\mathcal{B}_2)$ is controllable, $X_1^T\mathcal{B}_1=[1\ 2]\neq 0$ and

$$\begin{aligned}
&a_1X_1^T\otimes(\mathbf{1}_3^T\mathcal{B}_2)+\sum_{q=1}^{2}a_{q3}e_q^T\otimes(Z_3^T\mathcal{B}_2)\\
&=\begin{bmatrix}a_1 & a_1 & 0 & a_1 & a_1 & 0 & a_1 & a_1 & 0\end{bmatrix}+\begin{bmatrix}a_{13} & -2a_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0 & a_{23} & -2a_{23} & 0 & 0 & 0 & 0\end{bmatrix}\\
&=[a_1+a_{13}\ \ a_1-2a_{13}\ \ 0\ \ a_1+a_{23}\ \ a_1-2a_{23}\ \ 0\ \ a_1\ \ a_1\ \ 0]\\
&\neq 0,
\end{aligned}$$

which satisfy the conditions of Theorem 1, therefore, this NCPN is controllable.

Fig. F4 represents the all agents' movement trajectories from the arbitrary initial state to the desired state, where agents are denoted as ★ in $\mathcal{G}_1$ and ∘ in $\mathcal{G}_2$, respectively, and a 'rectangle' shows the final configuration (the desired state, denoted as ▷).

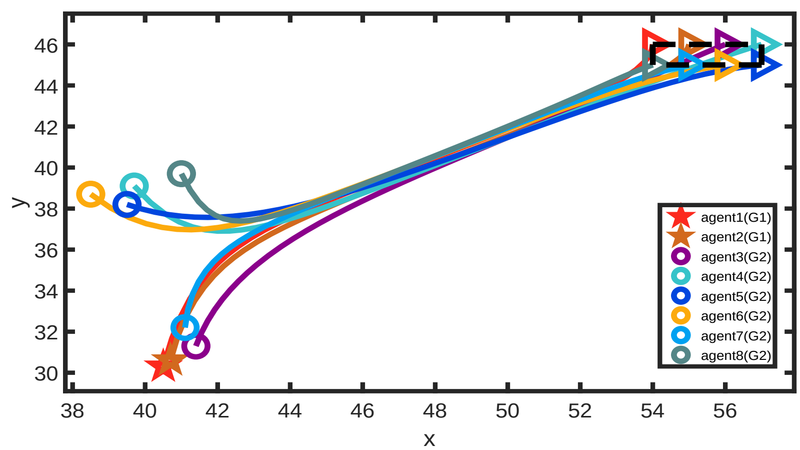

Figure F4 A rectangle configuration for $\mathcal{G}$ in Example 2.

Example 3. A NCPN is shown in Fig. F5, where $\mathcal{G}_2$ is disconnected.

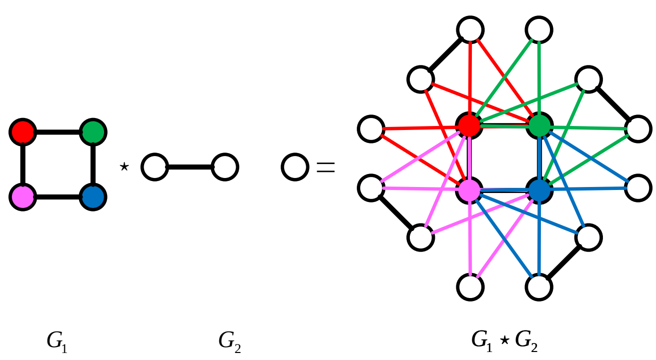

Figure F5 A NCPN of $\mathcal{G}_1$ and $\mathcal{G}_2$ for Example 3.

Let

$$\mathcal{A}_1=\begin{bmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{bmatrix},\quad \mathcal{L}_1=\begin{bmatrix}2&-1&-1&0\\-1&2&0&-1\\-1&0&2&-1\\0&-1&-1&2\end{bmatrix},\quad \mathcal{L}_2=\begin{bmatrix}1&-1&0\\-1&1&0\\0&0&0\end{bmatrix},\quad \mathcal{B}_1=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \mathcal{B}_2=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix},$$

then

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 + 6I_4 & -\mathcal{A}_1 \otimes \mathbf{1}_3^T \\ -\mathcal{A}_1 \otimes \mathbf{1}_3 & I_4 \otimes (\mathcal{L}_2 + 2I_3) \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & I_4 \otimes \mathcal{B}_2 \end{bmatrix}.$$

By calculation, the eigenvalues and the corresponding eigenvectors of $\mathcal{L}_1$ and $\mathcal{L}_2$ are respecively

$$\begin{cases} \theta_1 = 0, \\ \theta_{2_1} = 2, \\ \theta_{2_2} = 2, \\ \theta_3 = 4, \end{cases} \qquad \begin{cases} X_1 = e_1 + e_2 + e_3 + e_4, \\ X_{2_1} = -e_1 + e_2 - e_3 + e_4, \\ X_{2_2} = e_1 + e_2 - e_3 - e_4, \\ X_3 = e_1 - e_2 - e_3 + e_4, \end{cases}$$

and

$$\begin{cases} \eta_1 = 0, \\ \eta_2 = 0, \\ \eta_3 = 2, \end{cases} \qquad \begin{cases} Z_1 = e_1 + e_2 + e_3, \\ Z_2 = \frac{1}{2}e_1 + \frac{1}{2}e_2 - e_3, \\ Z_3 = -e_1 + e_2. \end{cases}$$

Furthermore, $V_{1_1} = \{8, 6+2\sqrt{7}\}$, $V_{1_2} = \{8, 8\}$, $V_{2_1} = \{0, 6-2\sqrt{7}\}$, $V_{2_2} = \{2, 2\}$, $V_3 = \{2,2,2,2,4,4,4,4\}$, so $V_{1_1} \cap V_{1_2} \neq \emptyset$ and $V_{2_2} \cap V_3 \neq \emptyset$. We can get $Z_2\mathcal{B}_2 = [\frac{1}{2}\ 0\ 0] \neq 0$, $Z_3\mathcal{B}_2 = [-1\ 0\ 0] \neq 0$, $(a_{2_1}X_{2_1}^T + a_{2_2}X_{2_2}^T)\mathcal{B}_1 = \begin{bmatrix} -a_{2_1} + a_{2_2} & a_{2_1} + a_{2_2} & 0 & 0 \end{bmatrix} \neq 0$, and

$$\begin{aligned}
&\sum_{s=1}^{2} a_{2_s} X_{2_s}^T \otimes (\mathbf{1}_{n_2}^T \mathcal{B}_2) + \sum_{q=1}^{4} a_{q2} e_q^T \otimes (Z_2^T \mathcal{B}_2) \\
&= \begin{bmatrix} -a_{2_1} & 0 & -a_{2_1} & a_{2_1} & 0 & a_{2_1} & -a_{2_1} & 0 & -a_{2_1} & a_{2_1} & 0 & a_{2_1} \end{bmatrix} + \begin{bmatrix} a_{2_2} & 0 & a_{2_2} & a_{2_2} & 0 & a_{2_2} & -a_{2_2} & 0 & -a_{2_2} & -a_{2_2} & 0 & -a_{2_2} \end{bmatrix} \\
&+ \begin{bmatrix} \frac{1}{2}a_{12} & 0 & -a_{12} & \frac{1}{2}a_{22} & 0 & -a_{22} & \frac{3}{2}a_{12} & 0 & -a_{32} & \frac{1}{2}a_{42} & 0 & -a_{42} \end{bmatrix} \\
&= \Big[-a_{2_1} + a_{2_2} + \frac{1}{2}a_{12} \quad 0 \quad -a_{2_1} + a_{2_2} - a_{12} \quad a_{2_1} + a_{2_2} + \frac{1}{2}a_{22} \quad 0 \quad a_{2_1} + a_{2_2} - a_{22} \quad -a_{2_1} - a_{2_2} \\
&\quad -\frac{1}{2}a_{12} \quad 0 \quad -a_{2_1} - a_{2_2} - a_{32} \quad a_{2_1} - a_{2_2} + \frac{1}{2}a_{42} \quad 0 \quad a_{2_1} - a_{2_2} - a_{42} \Big] \\
&\neq 0,
\end{aligned}$$

which satisfy the conditions of Theorem 2, therefore, this NCPN is controllable.

Fig. F6 represents the all agents' movement trajectories from the arbitrary initial state to the desired state, where agents are denoted as ★ in $\mathcal{G}_1$ and ∘ in $\mathcal{G}_2$, respectively, and a 'rectangle' shows the final configuration (the desired state, denoted as ▹).

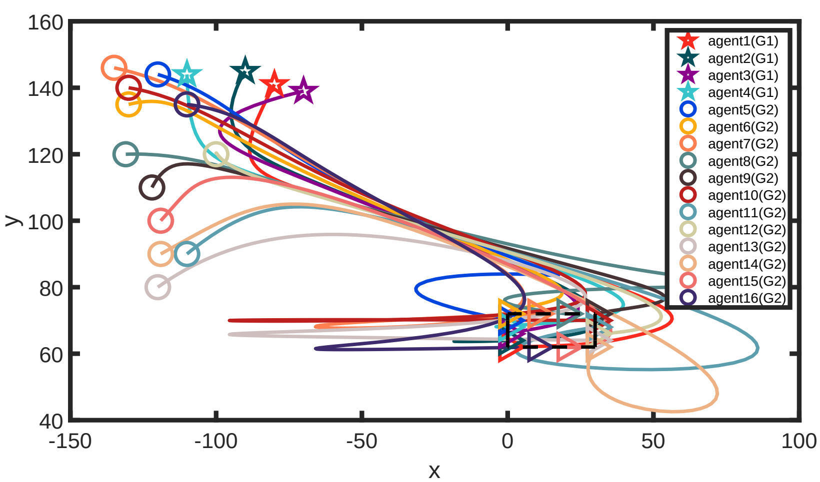

Figure F6 A rectangle configuration for $\mathcal{G}$ in Example 3.

Example 4. A NCPN is shown in Fig. F7.

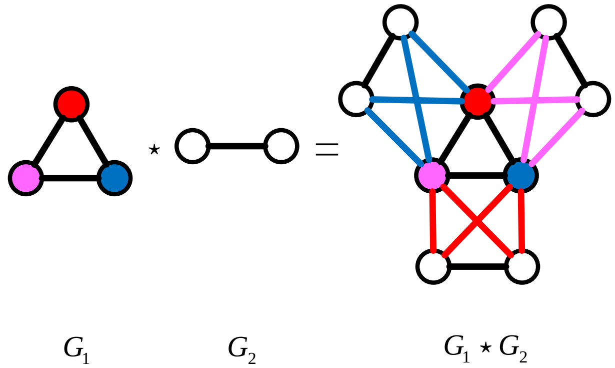

Figure F7 A NCPN of $\mathcal{G}_1$ and $\mathcal{G}_2$ for Example 4.

Let

$$\mathcal{A}_1=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix},\quad \mathcal{L}_1=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix},\quad \mathcal{L}_2=\begin{bmatrix}1&-1\\-1&1\end{bmatrix},\quad \mathcal{B}_1=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad \mathcal{B}_2=\begin{bmatrix}1&0\\0&1\end{bmatrix},$$

then

$$\mathcal{L}=\begin{bmatrix}\mathcal{L}_1+4I_3 & -\mathcal{A}_1\otimes\mathbf{1}_2^T\\ -\mathcal{A}_1\otimes\mathbf{1}_2 & I_3\otimes(\mathcal{L}_2+2I_2)\end{bmatrix},\quad \mathcal{B}=\begin{bmatrix}\mathcal{B}_1 & 0\\ 0 & I_3\otimes\mathcal{B}_2\end{bmatrix}.$$

By computing, $\det(\mathcal{A}_1)=2\neq 0 \Leftrightarrow \theta_i\neq r$, $r=2\neq 1$, so $V_{1_2}=\emptyset, V_{2_2}=\emptyset$, $V_{1_1}\cap V_3=\emptyset$, and $(\mathcal{L}_2,\mathcal{B}_2)$ is controllable (since $\mathrm{rank}(\mathcal{L}_2,\mathcal{B}_2)=2$), which satisfy the conditions of Corollary 2, therefore, this NCPN is controllable.

Fig. F8 represents the all agents' movement trajectories from the arbitrary initial state to the desired state, where agents are denoted as $\bigstar$ in $\mathcal{G}_1$ and $\circ$ in $\mathcal{G}_2$, respectively, and a 'rectangle' shows the final configuration (the desired state, denoted as $\triangleright$).

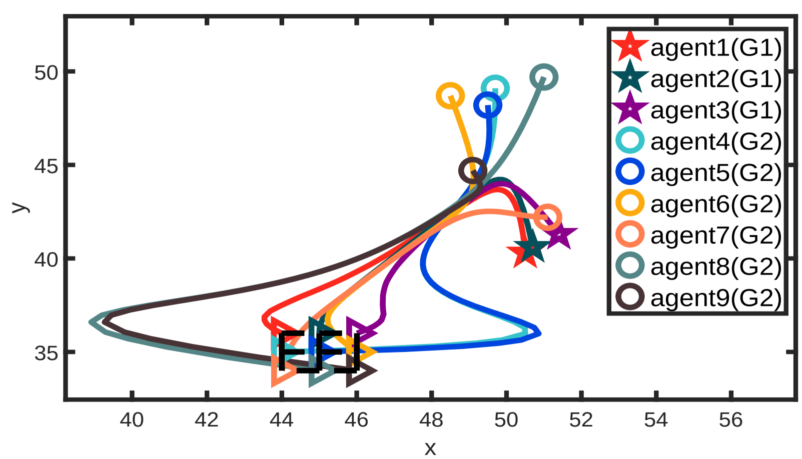

Figure F8 A rectangle configuration for $\mathcal{G}$ in Example 4.

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