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• **Supplementary File** •

Mean-square prescribed finite-time output consensus of high-order linear multi-agent systems

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Appendix A Proof of Lemma 1

Proof: The derivative of $a^{-k}(t)$ is $-ka^{-(k-1)}(t)$ for $k = 1, \cdots, n$. It is noted that $\dot{U}_i^{-1}(t) = a(t)U_i^{-1}(t)\bar{U}_i$, where $\bar{U}_i = \mathrm{diag}(0, -1, -2, \cdots, -(n-1))$. Then

$$\begin{aligned}&\dot{U}_i^{-1}(t)U_i(t) + \Lambda_i(t)\\ =&a(t)U_i^{-1}(t)\Big(\bar{U}_i + \frac{1}{a(t)}(A - \mu_1\lambda_i c(t)BK_1(t))\Big)U_i(t).\end{aligned} \tag{A1}$$

Let $k_{ih} = \mu_1\lambda_i\hat{b}_h$ for $h = 1, 2, \cdots, n$. The characteristic polynomial of the matrix $\bar{U}_i + \frac{1}{a(t)}(A - \mu_1\lambda_i c(t)BK_1(t))$ is $k_{i1} + \sum_{m=2}^{n-1} k_{im}\prod_{j=1}^{m}(\lambda + j - 1) + (\lambda + n - 1 + k_{in})\prod_{j=1}^{n-1}(\lambda + j - 1)$. It can be verified that the coefficients of the characteristic polynomial of the matrix $\bar{U}_i + \frac{1}{a(t)}(A - \mu_1\lambda_i c(t)BK_1(t))$ is related with the positive constants $k_{i1}, \cdots, k_{in}$, thus the eigenvalues of the matrix $\bar{U}_i + \frac{1}{a(t)}(A - \mu_1\lambda_i c(t)BK_1(t))$ can be negative by properly choosing the constants $k_{i1}, \cdots, k_{in}$. Next, we will show that $\hat{U}_i = U_i^{-1}(t)\left(\bar{U}_i + \frac{1}{a(t)}(A - \mu_1\lambda_i c(t)BK_1(t))\right)U_i(t) = U_i^{-1}(t)\bar{U}_iU_i(t) + \Lambda_{i1}$ is a constant matrix. Let $\hat{U}_i = (\hat{U}_{jh}^{[i]})_{n\times n}$, one has $\hat{U}_{jj}^{[i]} = \sum_{k=2}^{n}(-1)^k(k-1)u_{jk}^{[i]}(k_j^{[i]})^{k-1} - k_j^{[i]}$ and $\hat{U}_{jh}^{[i]} = \sum_{k=2}^{n}(-1)^k(k-1)u_{jk}^{[i]}(k_h^{[i]})^{k-1}$ for $j \neq h$. Every element of the matrix $\hat{U}_i$ is composed of the constants $u_{jk}^{[i]}$ and $k_h^{[i]}$ for $k = 2, \cdots, n$, $j, h \in \{1, \cdots, n\}$, thus $\hat{U}_i$ is a constant matrix. The proof is thus completed. □

Appendix B Proof of Theorem 1

We need to introduce the following lemma, which will be used in the proof.

Lemma 1. All the eigenvalues of the matrix $A - kBH(t)$ can be negative and different by properly choosing the positive constant k and the feedback gain matrix $H(t)$, where $H(t) = \begin{pmatrix} b_1a^n(t) & \cdots & b_na(t)\end{pmatrix}$, $a(t) > 0$ is a uniformly continuous time-varying gain.

Proof: The positive design parameters $b_i \in \mathbb{R}$ are chosen such that the polynomial $\lambda^n + b_na(t)\lambda^{n-1} + \cdots + b_2a^{n-1}(t)\lambda + b_1a^n(t)$ is equal to $\prod_{j=1}^{n}(\lambda + k_ja(t))$, k_j are positive constants for $i, j = 1, \cdots, n$, and $k_i \neq k_j$ for $i \neq j$. □

To better understand the Lemma 1, an example is presented here.

For the matrix $A \in \mathbb{R}^{3\times 3}$, the matrix $A - kBH(t)$ has the following form $\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -kb_1a^3(t) & -kb_2a^2(t) & -kb_3a(t)\end{pmatrix}$. Let $\bar{A} = A - kBH(t)$. Then, based on the traces of the matrices $I_3, \bar{A}, \bar{A}^2, \bar{A}^3, \bar{A}^4$, one can obtain the matrix $\hat{D} = \begin{pmatrix} \mathrm{tr}(I_3) & \mathrm{tr}(\bar{A}) & \mathrm{tr}(\bar{A}^2)\\ \mathrm{tr}(\bar{A}) & \mathrm{tr}(\bar{A}^2) & \mathrm{tr}(\bar{A}^3)\\ \mathrm{tr}(\bar{A}^2) & \mathrm{tr}(\bar{A}^3) & \mathrm{tr}(\bar{A}^4)\end{pmatrix}$,

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where $\mathrm{tr}(I_3) = 3$, $\mathrm{tr}(\bar{A}) = -kb_3a(t)$, $\mathrm{tr}(\bar{A}^2) = k^2b_3^2a^2(t) - 2kb_2a^2(t)$, $\mathrm{tr}(\bar{A}^3) = 3k^2b_2b_3a^3(t) - 3kb_1a^3(t) - k^3b_3^3a^3(t)$, $\mathrm{tr}(\bar{A}^4) = k^4b_3^4a^4(t) - 4k^3b_2b_3^2a^4(t) + 2k^2b_2^2a^4(t) + 4k^2b_1b_3a^4(t)$. Let $R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & kb_3a(t) & 1 \end{pmatrix}$, then $R_1\hat{D}R_1^T = \hat{D}_1$, where $\hat{D}_1 = \begin{pmatrix} 3 & -kb_3a(t) & -2kb_2a^2(t) \\ -kb_3a(t) & k^2b_3^2a^2(t) - 2kb_2a^2(t) & k^2b_2b_3a^3(t) - 3kb_1a^3(t) \\ -2kb_2a^2(t) & k^2b_2b_3a^3(t) - 3kb_1a^3(t) & -2k^2b_1b_3a^4(t) + 2k^2b_2^2a^4(t) \end{pmatrix}$. The first-order, second-order and third-order principal minor determinants of the matrix $\hat{D}_1$ are 3, $(3kb_3^2 - 6kb_2 - k^2b_3^2)a^2(t)$ and $(k^2(b_2^2b_3^2 - 4b_1b_3^3) + k(18b_1b_2b_3 - 4b_2^3) - 27b_1^2)a^6(t)$, respectively. The solution of the equation $3k_1b_3^2 - 6k_1b_2 - k_1^2b_3^2 = 0$ is $\bar{k}_{11} = \frac{3b_2}{b_3^2}$, the solutions of the equation $k_2^2(b_2^2b_3^2 - 4b_1b_3^3) + k_2(18b_1b_2b_3 - 4b_2^3) - 27b_1^2 = 0$ are $\bar{k}_{21} = \frac{2b_2^3 - 9b_1b_2b_3 + \sqrt{108b_1^2b_2^2b_3^2 - 38b_1b_2^4b_3 - 108b_1^3b_3^3 + 4b_2^6}}{b_2^2b_3^2 - 4b_1b_3^3}$ and $\bar{k}_{22} = \frac{2b_2^3 - 9b_1b_2b_3 - \sqrt{108b_1^2b_2^2b_3^2 - 38b_1b_2^4b_3 - 108b_1^3b_3^3 + 4b_2^6}}{b_2^2b_3^2 - 4b_1b_3^3}$. By properly choosing b_1, b_2, b_3 and k such that $b_2^2 \neq 4b_1b_3$ and $k \neq \bar{k}_{11}, \bar{k}_{21}, \bar{k}_{22}$, then the matrix $\bar{A} \in \mathbb{R}^{3\times 3}$ has three different eigenvalues. □

Let $\hat{e}_i(t) = y_i(t) - y_v(t)$ denote the output consensus error. In this paper, the purpose is to develop a distributed finite-time consensus protocol for the MAS to achieve the mean-square prescribed finite-time output consensus, i.e.,

$$\lim_{t\to T} E\Big\|\hat{e}_i(t)\Big\|^2 = 0 \tag{B1}$$

for $i = 0, 1, 2, \cdots, N$, where T denotes a-priori given and a user-defined finite time.

It is worth mentioning that, in the present setup, only the leader can have access to the information of the virtual leader and only a part of followers know the information of the leader. In this sense, we denote

$$\tilde{e}_0 = y_0 - y_v \tag{B2}$$

as the tracking error between the output of the leader $y_0(t)$ and the output of the virtual leader $y_v(t)$, and denote

$$\bar{e}_i = y_i - y_0 \tag{B3}$$

as the tracking error between the output of the follower nodes $y_i(t)$ and the output of the leader $y_0(t)$. From this definition, we obtain that

$$\hat{e}_i = \tilde{e}_0 + \bar{e}_i, \quad i = 0, 1, 2, \cdots, n, \tag{B4}$$

which means that a sufficient condition for obtaining $\lim_{t\to T} E\Big\|\hat{e}_i(t)\Big\|^2 = 0$ is $\lim_{t\to T} E\Big\|\tilde{e}_0(t)\Big\|^2 = 0$ and $\lim_{t\to T} E\Big\|\bar{e}_i(t)\Big\|^2 = 0$. In this sense, in the following, we focus on designing controllers for leader and followers such that $\lim_{t\to T} E\Big\|\tilde{e}_0(t)\Big\|^2 = 0$ and $\lim_{t\to T} E\Big\|\bar{e}_i(t)\Big\|^2 = 0$.

Define the consensus error as $e_i = x_i - x_0 \in \mathbb{R}^n$. Then, one has

$$\begin{aligned}
\dot{e}_i =& Ae_i + B(u_i - u_0) \\
=& Ae_i + B(K(t)\theta_0 - \mu_1c(t)z_i^{[1]}) \\
=& Ae_i + B\Big(K(t)\theta_0 - \mu_1c(t)\big(\sum_{j\in N_i}\mathcal{A}_{ij}(K_1(t)x_i - (K_1(t)x_j + \rho_{ij}n_{ij})) \\
& + \mathcal{A}_{i0}(K_1(t)x_i - (K_1(t)x_0 + \rho_{i0}n_{i0}))\big)\Big) \\
=& Ae_i - \mu_1c(t)BK_1(t)\Big(\sum_{j\in N_i}\mathcal{A}_{ij}(x_i - x_j) + \mathcal{A}_{i0}(x_i - x_0)\Big) \\
& + \mu_1c(t)B\Big(\sum_{j\in N_i}\mathcal{A}_{ij}\rho_{ij}n_{ij} + \mathcal{A}_{i0}\rho_{i0}n_{i0}\Big) + BK(t)\theta_0,
\end{aligned} \tag{B5}$$

and

$$\dot{\theta}_0 = \big(A - BK(t)\big)\theta_0 - BUv, \tag{B6}$$

for $i = 1, \cdots, N$.

Let $e = \mathrm{col}(e_1, \cdots, e_N) \in \mathbb{R}^{Nn}$ and $\bar{\theta}_0 = \mathrm{col}(\theta_0, \cdots, \theta_0) \in \mathbb{R}^{Nn}$. Rewrite (B5) in the following compact form

$$\begin{aligned}
\dot{e} =& (I_N \otimes A)e - \mu_1c(t)\Big((L + \mathcal{B}_r) \otimes (BK_1(t))\Big)e \\
& + (I_N \otimes BK(t))\bar{\theta}_0 + \mu_1c(t)\Big[\Big(I_N \otimes B\Big)D\Big]\eta, \\
\dot{\bar{\theta}}_0 =& \Big(I_N \otimes (A - BK(t))\Big)\bar{\theta}_0 - \Big(I_N \otimes (BU)\Big)\bar{v},
\end{aligned} \tag{B7}$$

where $D = \mathrm{diag}(d_1, \cdots, d_N) \in \mathbb{R}^{N\times N(N+1)}$, $d_i = (\mathcal{A}_{i0}, \mathcal{A}_{i1}, \cdots, \mathcal{A}_{iN}) \in \mathbb{R}^{1\times(N+1)}$, $\bar{v} = \mathrm{col}(v, \cdots, v)^{Nn_v}$, $n_i = \mathrm{col}(n_{i0}, n_{i1}, \cdots, n_{iN}) \in \mathbb{R}^{N+1}$, $\eta = \mathrm{col}(n_1, \cdots, n_N) \in \mathbb{R}^{N(N+1)}$.

We aim at making the consensus errors vanish in a-priori given, user-defined finite time T utilizing the time transformation method. First, we need to make some transformations for the system (B7) to realize time transformation.

Lemma 2. If Assumption 1 holds, then the matrix $L+\mathcal{B}_r$ is positive definite.

In virtue of Lemma 2, the matrix $L+\mathcal{B}_r$ is symmetric and positive definite, there must exist a unitary matrix $\hat{T}\in\mathbb{R}^{N\times N}$ satisfying $\hat{T}(L+\mathcal{B}_r)\hat{T}^T=\Omega$, where $\Omega=\mathrm{diag}(\lambda_1,\cdots,\lambda_N)$ and $\lambda_i>0$, $i=1,\cdots,N$, are the eigenvalues of the matrix $L+\mathcal{B}_r$. Define $\bar{\xi}=\mathrm{col}(\bar{\xi}_1,\cdots,\bar{\xi}_N)=(\hat{T}\otimes I_n)e$, then one has

$$\begin{aligned}\dot{\bar{\xi}}=&(I_N\otimes A)\bar{\xi}-\mu_1 c(t)\Big(\Omega\otimes(BK_1(t))\Big)\bar{\xi}\\&+(\hat{T}\otimes BK(t))\bar{\theta}_0+\mu_1 c(t)\Big[\Big(\hat{T}\otimes B\Big)D\Big]\eta.\end{aligned}\tag{B8}$$

Rewrite (B8) in the following form

$$\begin{aligned}\dot{\bar{\xi}}_i=&\Big(A-\mu_1\lambda_i c(t)BK_1(t)\Big)\bar{\xi}_i+(\hat{T}_i\otimes BK(t))\bar{\theta}_0\\&+\mu_1 c(t)\Big(\hat{T}_i\otimes B\Big)D\eta,\end{aligned}\tag{B9}$$

where $\hat{T}_i\in\mathbb{R}^{1\times N}$ is the ith row of the matrix $\hat{T}$.

According to Lemma 1, we obtain that all the eigenvalues of the matrices $A-BK(t)$ and $A-\mu_1\lambda_i c(t)BK_1(t)$ can be negative and have algebraic multiplicity one by properly selecting the feedback gain matrix $K(t)$, $K_1(t)$ and the parameters μ_1. Then the matrices $A-BK(t)$ and $A-\mu_1\lambda_i c(t)BK_1(t)$ can be diagonalized, on the time interval $[0,T)$, there exist nonsingular matrices $U_i(t)\in\mathbb{R}^{n\times n}$ such that $U_0^{-1}(t)(A-BK(t))U_0(t)=\Lambda_0(t)$ and $U_i^{-1}(t)(A-\mu_1\lambda_i c(t)BK_1(t))U_i(t)=\Lambda_i(t)$, where $\Lambda_i(t)=a(t)\Lambda_{i1}$, $\Lambda_{i1}=\mathrm{diag}(-k_1^{[i]},\cdots,-k_n^{[i]})$ and $k_j^{[i]}>0$ for $i=0,1,\cdots,N;j=1,\cdots,n$.

$U_i(t)=\begin{pmatrix}1&1&\cdots&1\\-k_1^{[i]}a(t)&-k_2^{[i]}a(t)&\cdots&-k_n^{[i]}a(t)\\\vdots&\vdots&\ddots&\vdots\\(-k_1^{[i]}a(t))^{n-1}&(-k_2^{[i]}a(t))^{n-1}&\cdots&(-k_n^{[i]}a(t))^{n-1}\end{pmatrix}$ for $i=0,1,\cdots,N$. On the time interval $[0,T)$, it is not difficult to verify that the matrix $U_i^{-1}(t)$ has the following form

$$U_i^{-1}(t)=\begin{pmatrix}u_{11}^{[i]}&u_{12}^{[i]}a^{-1}(t)&\cdots&u_{1n}^{[i]}a^{-(n-1)}(t)\\u_{21}^{[i]}&u_{22}^{[i]}a^{-1}(t)&\cdots&u_{2n}^{[i]}a^{-(n-1)}(t)\\\vdots&\vdots&\ddots&\vdots\\u_{n1}^{[i]}&u_{n2}^{[i]}a^{-1}(t)&\cdots&u_{nn}^{[i]}a^{-(n-1)}(t)\end{pmatrix},$$

where $u_{kj}^{[i]}$ are constants for $k,j\in\{1,\cdots,n\}$, $i=0,1,\cdots,N$.

Next, we continue to make some transformations to simplify the dynamics (B9). Let $\hat{\xi}_i=U_i^{-1}(t)\bar{\xi}_i$ and $\hat{\theta}_0=(I_N\otimes U_0^{-1}(t))\bar{\theta}_0$, then along the trajectory of (B9), we calculate the time derivative of $\hat{\xi}_i$, one has

$$\begin{aligned}\dot{\hat{\xi}}_i=&(\dot{U}_i^{-1}(t)U_i(t)+\Lambda_i(t))\hat{\xi}_i+\mu_1 c(t)U_i^{-1}(t)\Big(\hat{T}_i\otimes B\Big)D\eta\\&+U_i^{-1}(t)\Big(\hat{T}_i\otimes(BK(t)U_0(t))\Big)\hat{\theta}_0,\end{aligned}\tag{B10a}$$

$$\dot{\hat{\theta}}_0=I_N\otimes(\dot{U}_0^{-1}(t)U_0(t)+\Lambda_0(t))\hat{\theta}_0-I_N\otimes(U_0^{-1}(t)BU)\bar{v}.\tag{B10b}$$

It should be pointed out that the matrices $U_i^{-1}(t)\Big(\hat{T}_i\otimes(BK(t)U_0(t))\Big)$ and $U_0^{-1}(t)BU$ in the dynamics (B10a) and (B10b) can be written as $a(t)\Delta_i$ and $a^{-(n-1)}(t)\Pi$, respectively, where $\Delta_i\in\mathbb{R}^{n\times n}$ and $\Pi\in\mathbb{R}^{n\times n_v}$ are constant matrices. In addition, we obtain that

$$U_i^{-1}(t)\Big(\hat{T}_i\otimes B\Big)D=a(t)D_i(t),$$

where $D_i=a^{-n}(t)\Big(D_{i1}\ \cdots\ D_{i,N(N+1)}\Big)$ and $D_{ij}\in\mathbb{R}^n$ are constant vectors for $i=1,\cdots,N$, $j=1,\cdots,N(N+1)$.

Next, we introduce a time transformation method. First, we time transform the proposed smooth finite-time control structure possessing a dynamic equilibrium point into an infinite-time (that is, stretched) interval. The finite-time consensus problem can be converted to a stabilization problem defined over an infinite-time interval by introducing this time transformation. In this regard, the tools for analyzing the asymptotic convergence from standard Lyapunov stability theory can be utilized to address the finite-time consensus problem.

To begin with, define the time transformation $\theta(s)=T(1-e^{-s})$ and $t=\theta(s)$, which makes the finite-time controllers from the regular time interval $t\in[0,T)$ to the stretched time interval $s\in[0,\infty)$. Obviously, when $s\to\infty$, $t\to T$, which means that the methods for addressing the asymptotic convergence can result in the finite-time convergence. It should be pointed out that $\theta'(s)=\frac{d\theta(s)}{ds}=Te^{-s}=T-t=a^{-1}(t)$.

Let $\nu(t)$ represent a solution of the system described by (2). Let $\zeta_i(t)$ and $\zeta_0(t)$ represent the solution of the system described by (B10a) and (B10b), respectively. Define $v_s(s)=\nu(t)$, $\bar{\psi}_i(s)=\zeta_i(t)$ and $\psi_0(s)=\zeta_0(t)$, $\bar{v}_s(s)=\mathrm{col}(v_s(s),\cdots,v_s(s))$, $\bar{\psi}_0(s)=\mathrm{col}(\psi_0(s),\cdots,\psi_0(s))$. Then, one has

$$\begin{aligned}\bar{\psi}_i'(s)=&\bar{\Lambda}_i\bar{\psi}_i(s)+\Delta_i\bar{\psi}_0(s)+\mu_1 c(\theta(s))D_i(\theta(s))\eta(\theta(s)),\\\bar{\psi}_0'(s)=&I_N\otimes\bar{\Lambda}_0\bar{\psi}_0(s)-\theta'(s)a^{-(n-1)}(\theta(s))(I_N\otimes\Pi)\bar{v}_s(s),\\v_s'(s)=&Te^{-s}Sv_s(s),\end{aligned}\tag{B11}$$

where $c(\theta(s))=\frac{1}{s+\mu}$, $v_s^{'}(s)=\frac{dv_s(s)}{ds}$, $\bar{\psi}_i^{'}(s)=\frac{d\bar{\psi}_i(s)}{ds}$, $\bar{\psi}_0^{'}(s)=\frac{d\bar{\psi}_0(s)}{ds}$ and $\bar{v}_s(s)=\mathrm{col}(v_s(s),\cdots,v_s(s))\in\mathbb{R}^{Nn_v}$. It is noted that $a^{-1}(\theta(s))=\theta^{'}(s)=Te^{-s}$.

Let $\vartheta(s)=(Te^{-s})^n v_s(s)$, then one has

$$\vartheta^{'}(s)=-(nI_{n_v}-Te^{-s}S)\vartheta(s). \tag{B12}$$

Let $\bar{\psi}(s)=\mathrm{col}(\bar{\psi}_1(s),\cdots,\bar{\psi}_N(s))$, $\bar{\vartheta}(s)=\mathrm{col}(\vartheta(s),\cdots,\vartheta(s))\in\mathbb{R}^{Nn_v}$ and $\psi=\mathrm{col}(\bar{\psi}(s),\bar{\psi}_0,\bar{\vartheta}(s))$, it follows from (B11) that

$$\psi^{'}(s)=A_c\psi(s)+Te^{-s}A_{c1}\psi(s)+\mu_1c(\theta(s))B_c(s)\eta(\theta(s)), \tag{B13}$$

where $A_c=\begin{pmatrix}\bar{\Lambda}_c & \Delta & 0\\ 0 & I_N\otimes\bar{\Lambda}_0 & -I_N\otimes\Pi\\ 0 & 0 & -nI_{Nn_v}\end{pmatrix}$, $\Delta=\begin{pmatrix}\Delta_1\\ \vdots\\ \Delta_N\end{pmatrix}$, $A_{c1}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & I_N\otimes S\end{pmatrix}$, $\bar{\Lambda}_c=\mathrm{diag}(\bar{\Lambda}_1,\cdots,\bar{\Lambda}_N)$, $B_c=\begin{pmatrix}B_{c1}\\ 0\\ 0\end{pmatrix}$, $B_{c1}=\begin{pmatrix}D_1\\ \vdots\\ D_N\end{pmatrix}$. The matrices A_c and A_{c1} are constant matrices. It is noted that every element of the matrix $B_c(s)$ is bounded.

Rewrite the system (B13) in the following form, which is given by an Itô stochastic differential equation

$$d\psi=A_c\psi(s)ds+Te^{-s}A_{c1}\psi(s)ds+\mu_1c(\theta(s))B_c(s)dW, \tag{B14}$$

where $\int_0^s n_{ij}d\tau=w_{ij}$, $w_i=\mathrm{col}(w_{i0},w_{i1},w_{i2},\cdots,w_{iN})$, $W(t)=\mathrm{col}(w_1,w_2,\cdots,w_N)$ is an $N(N+1)$ dimensional standard Brownian motion.

Based on Lemma 1, we notice that by properly choosing the feedback gain matrix $K_1(t)$, A_c can be a Hurwitz matrix, then there is a positive definite matrix $P\in\mathbb{R}^{Nn\times Nn}$ satisfying the following Lyapunov equation

$$A_c^TP+PA_c=-Q, \tag{B15}$$

where $Q>0$ is a positive definite matrix.

Construct a Lyapunov function candidate as follows

$$V(s)=\psi^T(s)P\psi(s). \tag{B16}$$

Along the equation (B14), the derivative of V can be calculated as follows

$$\begin{aligned} dV=&-\psi^T(Q+Te^{-s}PA_{c1}+Te^{-s}A_{c1}^TP)\psi ds\\ &+2\mu_1c(\theta(s))\psi^TPB_cdW+\mu_1^2c^2(\theta(s))\mathrm{tr}(B_c^TPB_c)ds\\ \leqslant&-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)\bar{c}}c(\theta(s))Vds+\mu_1^2c^2(\theta(s))\mathrm{tr}(D_c)ds\\ &+2\mu_1c(\theta(s))\psi^TPB_cdW, \end{aligned} \tag{B17}$$

where $Q_1=Q+Te^{-s}PA_{c1}+Te^{-s}A_{c1}^TP$ is a positive definite matrix by properly choosing the feedback gain matrix $K_1(t)$ and Q, $D_c=B_c^TPB_c$.

Next, we will show

$$E\int_{s_0}^s c(\theta(\tau))\psi^TPB_cdW=0, \tag{B18}$$

for $s\geqslant s_0\geqslant 0$. Define a stopping time $\tau_{\bar{m}}^{s_0}\triangleq\inf\{s\in[s_0,T_1]:V(s)\geqslant\bar{m}\}$, where $s_0\geqslant 0, T_1\geqslant s_0, \bar{m}>0$. Then, by (B17), it follows that

$$\begin{aligned} &E[V(s\wedge\tau_{\bar{m}}^{s_0})\chi_{s\leqslant\tau_{\bar{m}}^{s_0}}]-E[V(t_0)]\\ \leqslant&-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)\bar{c}}\int_{s_0}^s c(\theta(\tau))V(\tau)d\tau+\mu_1^2\mathrm{tr}(D_c)\int_{s_0}^s c^2(\theta(\tau))d\tau\\ \leqslant&\mathrm{tr}(D_c)\int_{s_0}^s c^2(\theta(\tau))d\tau, \end{aligned}$$

which implies that $E[V(s\wedge\tau_{\bar{m}}^{s_0})\chi_{s\leqslant\tau_{\bar{m}}^{s_0}}]\leqslant M_1<\infty$, where the constant $M_1>0$ depends on s_0 and T_1. In addition, we have $\lim\limits_{M_1\to\infty}s\wedge\tau_{\bar{m}}^{s_0}=s$, then one has $\sup\limits_{s_0\leqslant s\leqslant T_1}E[V(s)]\leqslant M_1$ by Fatou lemma. Thus, $\forall s\in[s_0,T_1]$, one has

$$\begin{aligned} &E\int_{s_0}^s c^2(\theta(\tau))V(\tau)d\tau\\ \leqslant&\sup_{s_0\leqslant s\leqslant T_1}E[V(s)]\int_{s_0}^{T_1}c^2(\theta(\tau))d\tau<\infty, \end{aligned}$$

Since T_1 can be an arbitrarily positive constant, thus one has

$$E\int_{s_0}^s c^2(\theta(\tau))V(\tau)d\tau<\infty,\quad \forall s\geqslant s_0.$$

Furthermore, one has

$$\begin{aligned}&E\int_{s_0}^{s}c^2(\theta(\tau))\|\psi^TPB_c\|^2d\tau\\ \leqslant&\|P\|\|B_c\|^2E\int_{s_0}^{s}c^2(\theta(\tau))V(\tau)d\tau<\infty,\end{aligned}$$

by the property of Itô integral, we can obtain the equation (B18).

It follows from (B17)-(B18) that for any $s\geqslant 0$ and $h>0$, we have

$$\begin{aligned}&E[V(s+h)]-E[V(s)]\\ \leqslant&-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)\bar{c}}\int_{s}^{s+h}c(\theta(\tau))V(\tau)d\tau\\ &+\mu_1^2\mathrm{tr}(D_c)\int_{s_0}^{s}c^2(\theta(\tau))d\tau\\ \triangleq&V_1(s,s+h).\end{aligned}$$

Thus, it can be obtained that

$$\begin{aligned}&\limsup_{h\to 0^+}\frac{E[V(s+h)]-E[V(s)]}{h}\\ \leqslant&\limsup_{h\to 0^+}\frac{V_1(s,s+h)}{h}\\ =&-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)\bar{c}}c^2(\theta(s))E[V(s)]+\mu_1^2\mathrm{tr}(D_c)c^2(\theta(s)).\end{aligned}$$

By using the comparison lemma, for any $s\in[0,s+h]$, it yields

$$\begin{aligned}E[V(s)]\leqslant&\mu_1^2\mathrm{tr}(D_c)\int_0^s c^2(\theta(\tau))e^{-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)\bar{c}}\int_\tau^s c(\theta(h))dh}d\tau\\ &+E[V(0)]e^{-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)\bar{c}}\int_0^s c(\theta(\tau))d\tau}.\end{aligned}\tag{B19}$$

Since $\int_0^\infty c^2(\theta(\tau))d\tau<\infty$, it can be obtained that, for any $\epsilon>0$, there exists a positive constant $T^*>0$ such that $\int_{T^*}^\infty c^2(\theta(\tau))d\tau<\epsilon$. For the right-hand term in (B19), we can get

$$\begin{aligned}0\leqslant&\mathrm{tr}(D_c)\int_0^s c^2(\theta(\tau))e^{-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)\bar{c}}\int_\tau^s c(\theta(h))dh}d\tau\\ =&\mathrm{tr}(D_c)\int_0^{T^*} c^2(\theta(\tau))e^{-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)\bar{c}}\int_\tau^s c(\theta(h))dh}d\tau\\ &+\mathrm{tr}(D_c)\int_{T^*}^s c^2(\theta(\tau))e^{-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)\bar{c}}\int_\tau^s c(\theta(h))dh}d\tau\\ \leqslant&\mathrm{tr}(D_c)e^{-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)\bar{c}}\int_{T^*}^s c(\theta(h))dh}\int_0^{T^*}c^2(\theta(\tau))d\tau\\ &+\mathrm{tr}(D_c)\int_{T^*}^\infty c^2(\theta(\tau))d\tau\\ \leqslant&\mathrm{tr}(D_c)e^{-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)\bar{c}}\int_{T^*}^s c(\theta(h))dh}\int_0^{\infty}c^2(\theta(\tau))d\tau\\ &+\mathrm{tr}(D_c)\int_{T^*}^\infty c^2(\theta(\tau))d\tau\\ \leqslant&o(1)+\mathrm{tr}(D_c)\epsilon,\end{aligned}\tag{B20}$$

when $s\to\infty$. Since ϵ can be arbitrarily selected, it follows from (B20) that

$$\lim_{s\to\infty}\mathrm{tr}(D_c)\int_0^s c^2(\theta(\tau))e^{-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P)\bar{c}}\int_\tau^s c(\theta(h))dh}d\tau=0.\tag{B21}$$

Applying the limit (B21) to the system (B19) yields $\lim\limits_{s\to\infty}E[V(s)]=0$, furthermore, for all $\psi(0)$ on stretched time interval $s\in[0,\infty)$, $\lim\limits_{s\to\infty}E\|\psi(s)\|^2=0$ holds. As defined above, $\bar{\psi}_i(s)=\zeta_i(t)$ and $\psi_0(s)=\zeta_0(t)$ are the solutions of the systems described by (B10a) and (B10b), respectively. When $s\to\infty$, time t goes to the time T. Then, one has $\lim\limits_{t\to T}E\left\|e_i(t)\right\|^2=0$, $\lim\limits_{t\to T}E\left\|\theta_0(t)\right\|^2=0$.

Based on the regulator equation, one has

$$\begin{aligned}\tilde{e}_0=&Cx_0-Rv\\ =&C\theta_0.\end{aligned}\tag{B22}$$

As a result, we obtain $\lim\limits_{t\to T}E\left\|\tilde{e}_0(t)\right\|^2=0$. Thus, the mean-square prescribed finite-time state consensus is obtained for all the followers.

Then, one has $\lim_{t\to T} E\left\|\bar{e}_i(t)\right\|^2 = \lim_{t\to T} E\left\|Ce_i(t)\right\|^2 = 0$ and $\lim_{t\to T} E\left\|\tilde{e}_0(t)\right\|^2 = 0$, respectively. Thus, one has $\lim_{t\to T} E\left\|\hat{e}_i(t)\right\|^2 = 0$. As a result, all the followers can achieve the mean-square prescribed finite-time output consensus. The proof is thus completed. □

Appendix C A Numerical Example

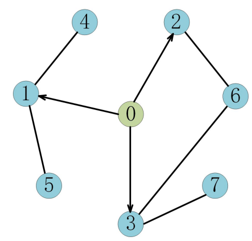

Figure C1 Communication network $\bar{\mathcal{G}}$

To validate our theoretical results, a numerical example is given in this section. Consider a MAS with one leader and seven followers, and let Figure C1 denote the communication graph among the eight agents. The dynamics of the follower agents are given by a third-order linear system. The system matrices of the virtual leader are given by $S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{36T^4} & 0 & -\frac{13}{36T^2} & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$. Then, the solution of the regulator equation is $X = \begin{pmatrix} I_3 & \mathbf{0} \end{pmatrix}$, and $U = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$.

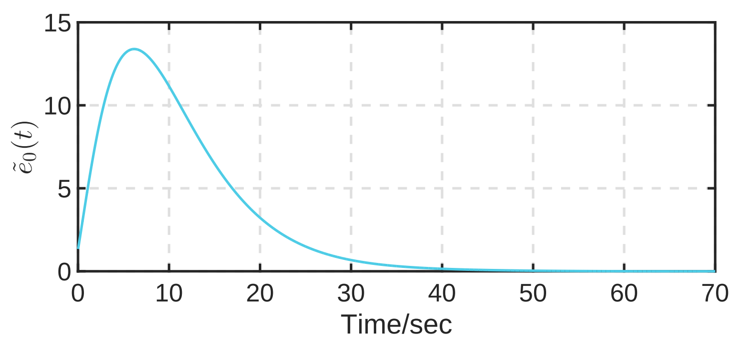

Figure C2 Evolution of $\tilde{e}_0(t)$

The eigenvalues of the matrix $L+\mathcal{B}_r$ are $\lambda_1(L+\mathcal{B}_r) = 0.2679$, $\lambda_2(L+\mathcal{B}_r) = 0.3820$, $\lambda_3(L+\mathcal{B}_r) = 1.0000$, $\lambda_4(L+\mathcal{B}_r) = 1.0000$, $\lambda_5(L+\mathcal{B}_r) = 2.6180$, $\lambda_6(L+\mathcal{B}_r) = 3.7321$, $\lambda_7(L+\mathcal{B}_r) = 4.0000$. Take $\mu_1 = 10$, $T = 70$, $a(t) = \frac{1}{T-t}$. The feedback gain matrix $K_1(t)$ is selected as $K_1(t) = \begin{pmatrix} 6000\frac{a^3(t)}{c(t)} & 1100\frac{a^2(t)}{c(t)} & 60\frac{a(t)}{c(t)} \end{pmatrix}$. $K(t) = \begin{pmatrix} 6000a^3(t) & 1100a^2(t) & 60a(t) \end{pmatrix}$. The communication noises $\rho_{ij} \in \mathbb{R}$ and $\varrho_{ij} \in \mathbb{R}$ are positive constants for $i = 1, 2, \cdots, 7$; $j = 0, 1, 2, \cdots, 7$. It can be verified that $\mu_1\lambda_i$ is not equal to k_{11}, k_{21} and k_{22}, thus, all the eigenvalues of the matrix $A - \mu_1\lambda_i c(t)BK_1(t)$ are negative and different. Under the proposed controllers, the state consensus errors $e_i = x_i - x_0$ converge in the mean-square sense, as shown in Figure C3. From Figure C3, it can be observed that mean-square prescribed finite-time state consensus is obtained for all the followers. The errors $\tilde{e}_0(t)$ and $\hat{e}_i(t)$ are depicted in Figures C2 and C4, respectively. As illustrated in Figures C3-C4, that is, $E\left\|y_i(t) - y_0(t)\right\|^2 \to 0$ and $E\left\|y_0(t) - y_v(t)\right\|^2 \to 0$, as time t goes to a prescribed finite time T, we notice that the mean-square prescribed finite-time output consensus problem is solved.

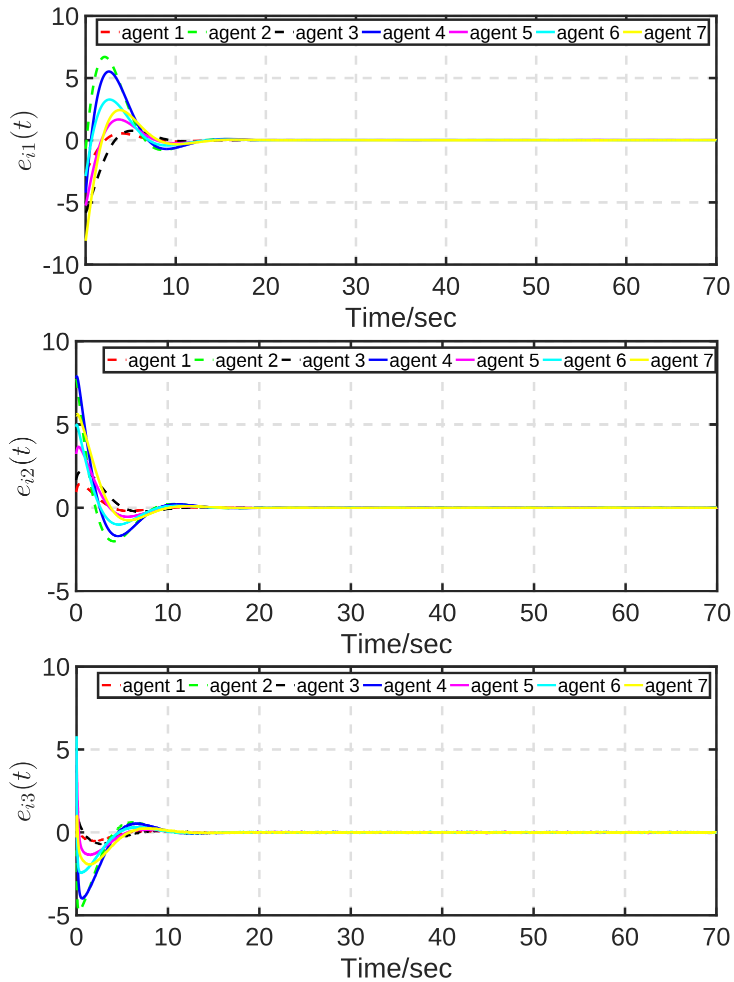

Figure C3 Evolution of the consensus errors $e_i(t) \in \mathbb{R}^3$ $(i = 1, \cdots, 7)$

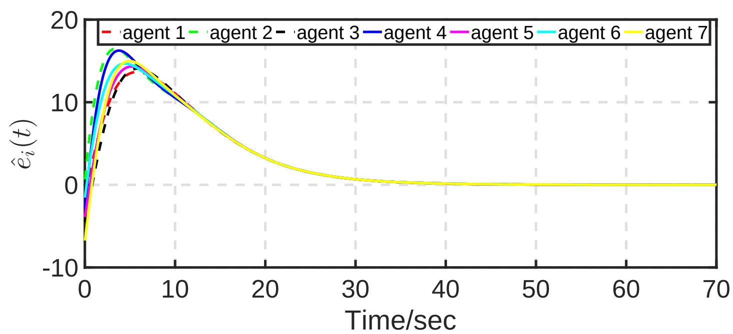

Figure C4 Evolution of $\hat{e}_i(t)$

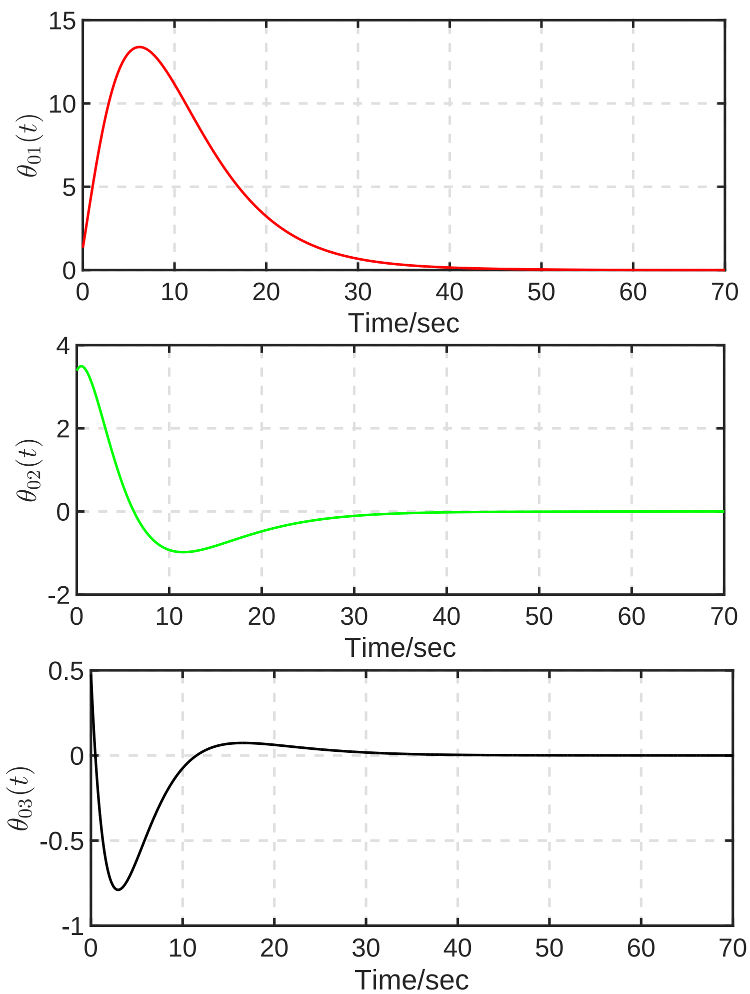

Figure C5 Evolution of $\theta_0(t) = \text{col}(\theta_{01}(t), \theta_{02}(t), \theta_{03}(t)) \in \mathbb{R}^3$ $(i = 1, \cdots, 7)$

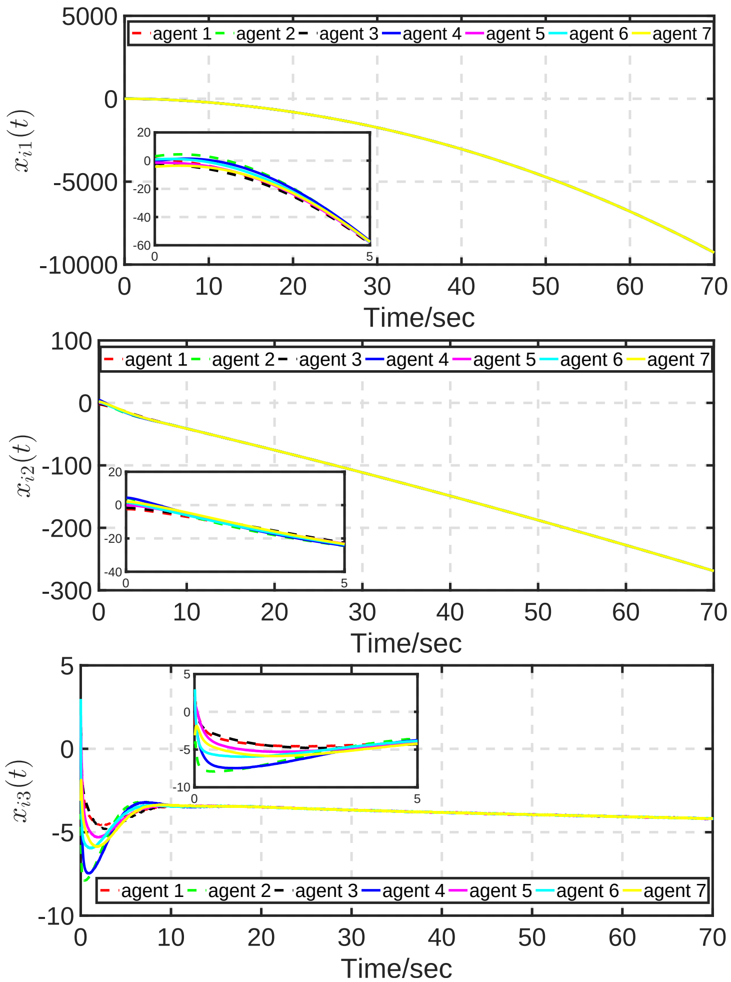

Figure C6 Evolution of the state $x_i(t) = \text{col}(x_{i1}(t), x_{i2}(t), x_{i3}(t)) \in \mathbb{R}^3$ $(i = 1, \cdots, 7)$