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Controllability of descriptor multi-agent systems with signed networks

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Abstract This paper studies the controllability of descriptor multi-agent systems (DMASs) with signed networks, where the networks are signed, and agents have descriptor linear dynamics. First, the basis for the eigenspace of the system matrix is derived completely and qualitatively by leveraging the maximum generalized left Jordan chain defined in this paper. Taking advantage of the explicit form of the basis, necessary and sufficient conditions for the controllability of DMASs are established. Furthermore, a necessary and sufficient condition is provided to ensure the controllability of DMASs with heterogeneous dynamics. Particularly, for some special cases, controllability conditions are expressed more precisely in terms of eigenvectors. Then, the relationship between the controllability of DMASs with signed and unsigned networks is investigated. It is shown that the controllability of DMASs under structurally balanced signed networks is equivalent to that under the associated underlying unsigned networks whether the individual dynamics is homogeneous or heterogeneous. Finally, theoretical results are applied to the multi-agent supporting systems.

Keywords controllability, signed networks, descriptor multi-agent systems, heterogeneous dynamics

1 Introduction

Controllability is an important concept in modern control theory, and it plays a fundamental role in the analysis and synthesis of linear control systems. Roughly speaking, controllability quantifies the ability to steer the state of the dynamical system from any initial condition to any desired final state within finite time. For multi-agent systems (MASs), the significance of controllability is driving all agents to achieve desirable configurations from any initial states by directly controlling a few agents. Tanner [1] first formulated the controllability of MASs under a leader-follower framework. Subsequently, most researchers investigated controllability in terms of graphical tools and algebraic criteria [2–8].

The aforementioned results on controllability focus only on MASs with unsigned networks, where the interactions between agents are all cooperative. However, antagonistic interactions are ubiquitous in reality, such as social networks [9], where individual relationships can be both friendly and hostile. In practice, the cooperative-antagonistic interactions among agents are described by signed graphs [10], where the positive and negative edges represent the cooperative and antagonistic interactions, respectively. Recently, complex dynamic behaviors on signed networks have been extensively investigated, including bipartite consensus, bipartite tracking, and bipartite containment [10–13]. Additionally, the controllability of signed networks has received widespread attention in the control community [14–16]. In [14], a graph-theoretic characterization of an upper bound on the controllable subspace was proposed. The structural controllability of multi-agent networks defined over directed signed graphs was considered in [15]. Ref. [15] proved that a multi-agent network is structurally controllable if and only if the communication digraph is leader-follower connected. Recent advances in this regard include [16], in which the lower and upper bounds for the dimension of edge controllable subspace of signed networks are derived.

At present, there have been some exciting studies devoted to the controllability of higher-dimensional systems, where agents have general linear dynamics, and the system dynamics is generally multi-input/

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multi-output [17–20]. However, in natural and artificial MASs, there usually exist algebraic constraints among state variables of agents, in this case, each agent in MASs should be modeled as a descriptor system. It is well-recognized that descriptor systems offer a more general framework for modeling physical systems than traditional state-space representations. Different from the general linear system case, for descriptor linear systems, there are four types of controllability, namely, C-controllability, R-controllability, I-controllability, and S-controllability. It is noted that R-controllability guarantees the realizability for the system from any initial condition to any reachable state, and understanding R-controllability can often serve as a stepping stone to exploring other types of controllability. In view of this, many researchers have investigated R-controllability. Early work by Yip and Sincovec [21] laid the theoretical foundation for the analysis of R-controllability in descriptor systems. Subsequent studies have further extended the R-controllability criterion [22–24]. In practical applications, the analysis of R-controllability provides theoretical support for modeling and simulation, and lays the foundation for optimizing control strategies and improving system performance in various applications. For example, in mechanical systems, it enhances operability and safety by ensuring the system can reach target positions [25]. In economic dynamic models, it helps predict behavior and design macroeconomic policies [26]. Considering the theoretical value and practical significance of studying R-controllability, this paper is particularly motivated to investigate the R-controllability.

In practice, many MASs inherently exhibit descriptor characteristics, further highlighting the importance of studying their controllability. One typical example is multi-agent supporting systems used in earthquake damage-prevention buildings [27], which are singular when each agent is supported by multiple pillars. Recently, the coordination control of descriptor multi-agent systems (DMASs) has been drawing increasing attention, including consensus [28], containment [29], and others. In addition, it is worth noting that all the aforementioned methods and results on the controllability of MASs no longer apply to the controllability of DMASs, the interplay between the differential and algebraic parts of the system necessitates new analytical tools and control strategies. Hence, it is of both theoretical and practical importance to address the controllability of DMASs.

Furthermore, while significant efforts have been made to study the controllability and observability of DMASs [30], existing criteria often rely on solving complex matrix equations. However, due to the high computational complexity of solving matrix equations, how to get more concise and practical controllability criteria for DMASs with antagonistic interactions is a challenging problem. On one hand, the system matrix of DMASs consists of the Laplacian matrix and agents' system matrices, which increases the complexity of analyzing the eigenvector of the system matrix. On the other hand, the Laplacian matrix under directed signed networks is asymmetric, and its row sum is nonzero, thereby most existing tools and methods for undirected unweighted unsigned networks are no longer applicable to signed networks. To date, a comprehensive investigation into the controllability of DMASs with signed networks remains largely unexplored. Addressing this gap is not only of theoretical interest but also holds practical significance for designing effective control strategies in DMASs. Therefore, the motivation behind this research lies in developing concise and practical controllability criteria for DMASs with signed networks, addressing both theoretical challenges and practical needs in the control of distributed systems.

In this paper, the controllability of DMASs with signed networks is investigated, in which each agent is modeled as a descriptor linear system, and the communication digraph admits both positively and negatively weighted edges to represent the intuitive cooperative and antagonistic interactions between agents, respectively. It is noticed that results in this paper can be directly extended to the observability of DMASs due to the duality between controllability and observability of descriptor systems. The main contributions of this paper are threefold:

(1) The controllability of DMASs with signed networks is studied in the scenario where the Laplacian matrix is nondiagonalizable and diagonalizable, and the explicit form of eigenvectors of the system matrix is qualitatively presented, respectively. Particularly, the maximum generalized left Jordan chain is defined in this paper, based on this, the basis for the eigenspace of the system matrix with nondiagonalizable Laplacian matrix is derived, which is, to the best of our knowledge, first explicitly and completely characterized. Taking advantage of the explicit form for the basis, necessary and sufficient conditions on the controllability of DMASs are established. These conditions reveal how the controllability of DMASs is affected by the network topology, the subsystem dynamics, the external control inputs, and the inner coupling. Compared with the recent results in [30], conditions in this paper do not require to solve matrix equations. Thus the computational complexity is greatly reduced.

(2) The controllability of DMASs with heterogeneous dynamics is investigated, where heterogeneous

dynamics means that different agents in the DMASs may have different parameter matrices. A necessary and sufficient condition is established, which can be checked independently for every individual subsystem, and is also applicable to the DMASs with homogeneous dynamics. Moreover, for some specific constraints, the corresponding controllability criteria are expressed more precisely in terms of eigenvectors.

(3) The relationship between the controllability of DMASs with signed networks and with unsigned networks is investigated. It is shown that the controllability of DMASs with homogeneous dynamics under structurally balanced networks is equivalent to that under unsigned networks whose edge weights are all nonnegative from the perspective of eigenvectors. For heterogeneous dynamics, the controllability is also equivalent under structurally balanced networks and unsigned networks. Based on these arguments, the property of some special networks can be used to effectively identify the uncontrollability of the whole system.

The rest of this paper is organized as follows. Section 2 introduces some preliminaries and formulates the model. Section 3 presents the results on controllability of DMASs with homogeneous dynamics; heterogeneous dynamics case is considered in Section 4. In Section 5, the relationship between the controllability of DMASs under signed and unsigned networks is studied. Theoretical results are applied to the multi-agent supporting systems in Section 6. The conclusion is drawn in Section 7.

Throughout this paper, let $\mathbb{R}$ ($\mathbb{C}$), $\mathbb{R}^n$ ($\mathbb{C}^n$), and $\mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$) be the sets of real (complex) numbers, real (complex) n-vectors, and real (complex) matrices of size $n\times m$, respectively. Let $\mathbb{C}_{\rm p}^k$ be the set of all k times piecewise continuously differentiable functions. Set $Z^q=\{1,\dots,q\}$ collects the positive integers no larger than q. Let $I_n\in\mathbb{R}^{n\times n}$ be the identity matrix, $e_i^{(n)}$ be the i-th canonical basis of $\mathbb{R}^n$, $\mathbf{0}$ be an all-zero matrix with compatible dimension. For matrices $A_1,\dots,A_t\in\mathbb{R}^{n\times n}$, $\mathrm{diag}(A_1,\dots,A_t)$ denotes a block diagonal matrix with diagonal elements $A_1,\dots,A_t$, particularly, when $n=1$, $\mathrm{diag}(A_1,\dots,A_t)$ is a diagonal matrix. Let $\mathrm{span}\{v_1,\dots,v_t\}$ be the vector space spanned by the vectors $v_1,\dots,v_t$. Moreover, $\sigma(E,A)=\{s\in\mathbb{C}\mid |s|<\infty,\ \det(sE-A)=0\}$ represents the set of finite eigenvalues of the matrix pair (E,A). Let $A\otimes B$ be the Kronecker product of matrices A and B. For matrices $M\in\mathbb{R}^{m\times r}$ and $V\in\mathbb{R}^{m\times s}$, $M\perp V$ means that $v^{\rm T}M=0$ for some nonzero v in the column space of V, and similarly $M\not\perp V$ indicates that no such v exists. Let $V_1\odot V_2$ be the direct sum of spaces V_1 and V_2. The cardinality of a finite set S is defined as $|S|$. If $S\subset R$, then the complement of S in R is defined as $R\backslash S$. For matrix $A=[a_{ij}]$, $|A|=[|a_{ij}|]$. The signature matrix set is defined as $\mathcal{D}=\{\mathrm{diag}(d_1,\dots,d_n)\mid d_i\in\{\pm1\}\}$. Two matrices $M_1,M_2\in\mathbb{R}^{n\times n}$ are said to be signature similar if $\exists D\in\mathcal{D}$ such that $M_2=DM_1D$.

2 Preliminaries and model formulation

2.1 Signed graph

Let $\mathcal{G}=\{\mathcal{V},\mathcal{E},W\}$ be a directed signed graph, where $\mathcal{V}=\{1,\dots,N\}$ is the set of vertices, $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the set of edges, and $W=[w_{ij}]\in\mathbb{R}^{N\times N}$ is the signed weight matrix of $\mathcal{G}$, $w_{ij}\neq0$ if and only if the edge $(j,i)\in\mathcal{E}$, and w_{ij} can be positive or negative. In addition, we assume that $w_{ii}=0$ for all $i\in\mathbb{Z}^N$. Let $\mathcal{G}_+=\{\mathcal{V},\mathcal{E},|W|\}$ and $\mathcal{G}_-=\{\mathcal{V},\mathcal{E},-|W|\}$ be the all-positive graph and all-negative graph corresponding to $\mathcal{G}$, respectively. Usually, $\mathcal{G}_+$ can also be regarded as the underlying unsigned graph of the signed graph $\mathcal{G}$. The neighbor of vertex i is defined as $\mathcal{N}_i=\{j\mid(j,i)\in\mathcal{E}\}$. A path from vertexes i_1 to i_k is a sequence of edges $\{(i_1,i_2),(i_2,i_3),\dots,(i_{k-1},i_k)\}$, in which all vertices are distinct. A cycle is a path beginning and ending with the same vertex, i.e., $i_1=i_k$. A cycle is positive (negative) if $w_{i_2i_1}w_{i_3i_2}\cdots w_{i_1i_k}>0$ (<0). A graph is a star if it has a center vertex i, and every vertex except i belongs to $\{j\mid(i,j)\in\mathcal{E}\}$. The Laplacian matrix associated with the signed graph $\mathcal{G}$ is $L=[l_{ij}]=\Delta-W$, where $\Delta=\mathrm{diag}(\sum_{j\in\mathcal{N}_1}|w_{1j}|,\dots,\sum_{j\in\mathcal{N}_N}|w_{Nj}|)$.

Definition 1 ([10,31])**.** Let $\mathcal{G}$ be a signed graph with the vertex set $\mathcal{V}$.

(1) $\mathcal{G}$ is structurally balanced if it has a bipartition of two nonempty subsets $\mathcal{V}_1,\mathcal{V}_2$, with $\mathcal{V}_1\bigcup\mathcal{V}_2=\mathcal{V}$ and $\mathcal{V}_1\bigcap\mathcal{V}_2=\emptyset$, such that $w_{ij}\geqslant0$ if $i,j\in\mathcal{V}_s$ ($s\in\{1,2\}$) and $w_{ij}\leqslant0$ if $i\in\mathcal{V}_s,j\in\mathcal{V}_t,\ s\neq t$ ($s,t\in\{1,2\}$);

(2) $\mathcal{G}$ is structurally anti-balanced if it has a bipartition of two nonempty subsets $\mathcal{V}_1,\mathcal{V}_2$, with $\mathcal{V}_1\bigcup\mathcal{V}_2=\mathcal{V}$ and $\mathcal{V}_1\bigcap\mathcal{V}_2=\emptyset$, such that $w_{ij}\leqslant0$ if $i,j\in\mathcal{V}_s$ ($s\in\{1,2\}$) and $w_{ij}\geqslant0$ if $i\in\mathcal{V}_s,j\in\mathcal{V}_t,\ s\neq t$ ($s,t\in\{1,2\}$);

(3) $\mathcal{G}$ is strictly structurally unbalanced if $\mathcal{G}$ is neither balanced nor anti-balanced.

2.2 Descriptor linear systems

Consider a descriptor linear system

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times q}$ are constant matrices, and E may be singular, if E is nonsingular, system (1) is called a normal linear system. In the sequel, system (1) is frequently defined as (E, A, B) for simplicity.

Definition 2 ([32])**.** The matrix pair (E, A) is called regular if $\det(sE - A)$ is not identically zero for some $s \in \mathbb{C}$.

It follows from the standard decomposition[1)] of descriptor linear systems that there exist two nonsingular matrices Q and P such that under the transformation (P, Q) system (1) becomes the following equivalent standard decomposition form:

$$\begin{cases} \dot{x}_1(t) = A_1x_1(t) + B_1u(t), \\ N\dot{x}_2(t) = x_2(t) + B_2u(t), \end{cases} \tag{2}$$

where $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, the matrix $N \in \mathbb{R}^{n_2\times n_2}$ is nilpotent.

Definition 3 ([32])**.** Given the regular descriptor linear system (2), a vector $\omega \in \mathbb{R}^n$ is said to be a reachable vector of system (2), if there exists an initial condition $x_1(0) = x_{10}$, an admissible control input $u(t) \in \mathbb{C}_{\mathrm{p}}^{h-1}$, and some $t_1 > 0$ such that the response of the system (2) satisfies

$$x(t_1, u, x_{10}) = \begin{bmatrix} x_1(t_1) \\ x_2(t_1) \end{bmatrix} = \omega.$$

Note that in the above definition the admissible control input is confined to $u(t) \in \mathbb{C}_{\mathrm{p}}^{h-1}$, which represents the set of $h - 1$ times piecewise continuously differentiable functions. Let

$$\mathscr{R}_t[x_{10}] = \left\{\omega \mid \exists u(t) \in \mathbb{C}_{\mathrm{p}}^{h-1} \text{ s.t. } x(t, u, x_{10}) = \omega \in \mathbb{R}^n\right\};$$

then $\mathscr{R}_t[x_{10}]$ is the set of reachable states at time t from the initial condition $x_1(0) = x_{10}$. Furthermore, let

$$\mathscr{R}_t = \bigcup_{x_{10}\in\mathbb{R}^{n_1}} \mathscr{R}_t[x_{10}],$$

and

$$\mathscr{H}_t = \left\{x \mid x = \begin{bmatrix} \mathrm{e}^{A_1t}x_{10} \\ 0 \end{bmatrix}, x_{10} \in \mathbb{R}^{n_1}\right\};$$

then $\mathscr{R}_t$ is the state reachable set at time t for system (2) from all possible initial condition $x_1(0) = x_{10} \in \mathbb{R}^{n_1}$, and $\mathscr{H}_t$ is the set of free reachable state at time t starting from all possible initial condition x_{10}.

Definition 4 ([32])**.** The regular descriptor system (2) is called R-controllable, if it is controllable in the reachable subspace $\mathscr{R}_t$, or more precisely, for any prescribed $t_1 > 0$, $x_{10} \in \mathbb{R}^{n_1}$, and $\omega \in \mathscr{R}_{t_1}$, there always exists an admissible control input $u(t) \in \mathbb{C}_{\mathrm{p}}^{h-1}$ such that the response of system (2) starting from the initial value $x_1(0) = x_{10}$ satisfies $x(t_1, u, x_{10}) = \omega$.

In this paper, the controllability of the DMASs refers to the R-controllability.

Lemma 1 ([32])**.** A regular system (E, A, B) is R-controllable if and only if $\mathrm{rank}[sE - A \quad B] = n, \forall s \in \mathbb{C}, |s| < \infty$.

Consider a descriptor linear system

$$\begin{cases} E\dot{x}(t) = Ax(t), \\ y = Hx(t), \end{cases} \tag{3}$$

where $H \in \mathbb{R}^{l\times n}$, $y(t) \in \mathbb{R}^l$ is the output of the system. In the sequel, system (3) is frequently defined as (E, A, H) for simplicity.

1) Given the descriptor system (1) with (E, A) regular, there exist two nonsingular matrices Q and P such that $\tilde{E} = QEP = \mathrm{diag}(I_{n_1}, N)$, $\tilde{A} = QAP = \mathrm{diag}(A_1, I_{n_2})$, $\tilde{B} = QB = \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$, where $n_1 + n_2 = n$, the matrix $N \in \mathbb{R}^{n_2\times n_2}$ is nilpotent, and the involved partitions are compatible.

Definition 5 ([32])**.** The regular system (3) is R-observable if for arbitrarily given $x(t) \in \mathscr{H}_t$, there exists a time point $t_f \geqslant t$, such that the state $x(t)$ can be uniquely determined by the system output $y(\tau)$, $t \leqslant \tau \leqslant t_f$.

Lemma 2 ([32])**.** A regular system (E, A, H) is R-observable if and only if

$$\operatorname{rank}\begin{bmatrix} sE - A \\ H \end{bmatrix} = n, \ \forall s \in \mathbb{C}, \ |s| < \infty.$$

2.3 Model formulation

Consider a DMAS consisting of N agents labeled $\mathcal{V} = \{1, \ldots, N\}$. Without loss of generality, $\mathcal{V}_{\mathrm{L}} = \{1, \ldots, M\}$ denotes the set of the leaders with $M \leqslant N$, and $\mathcal{V}_{\mathrm{F}} = \mathcal{V} \backslash \mathcal{V}_{\mathrm{L}}$ denotes the set of the followers. For each follower $i \in \mathcal{V}_{\mathrm{F}}$, the dynamics is governed by

$$E\dot{x}_i(t) = Ax_i(t) + Cz_i(t), \tag{4}$$

where $x_i(t) \in \mathbb{R}^n$ and $z_i(t) \in \mathbb{R}^s$ are the state and coupling variable of the agent i, $E \in \mathbb{R}^{n\times n}$ may be singular, and $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n\times s}$ are system matrix and inner coupling matrix. For each leader $i \in \mathcal{V}_{\mathrm{L}}$, the dynamics is

$$E\dot{x}_i(t) = Ax_i(t) + Cz_i(t) + Bu_i(t), \tag{5}$$

where $u_i(t) \in \mathbb{R}^q$ is the control input applied to agent i, and $B \in \mathbb{R}^{n\times q}$ is input matrix. The coupling variable $z_i(t)$ for each agent $i \in \mathcal{V}$ is determined by the so-called diffusive coupling rule based on the neighbor relations as follows [6]:

$$z_i(t) = K \sum_{j\in\mathcal{N}_i} w_{ij}\left(x_j(t) - \operatorname{sign}\left(w_{ij}\right) x_i(t)\right), \tag{6}$$

where $K \in \mathbb{R}^{s\times n}$ is the feedback gain matrix. Let $\Omega = \operatorname{diag}(\delta_1, \ldots, \delta_N)$, where $\delta_i = 1$ indicates that agent i is a leader, and $\delta_i = 0$ otherwise. Let $X = \left[x_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}\right]^{\mathrm{T}}$ and $U = \left[u_1^{\mathrm{T}}, \ldots, u_N^{\mathrm{T}}\right]^{\mathrm{T}}$. Then, the DMAS can be rewritten in a compact form as follows:

$$\Xi\dot{X} = \Phi X + \Psi U, \tag{7}$$

where $\Xi = I_N \otimes E \in \mathbb{R}^{Nn\times Nn}$, $\Phi = I_N \otimes A - L \otimes CK \in \mathbb{R}^{Nn\times Nn}$, and $\Psi = \Omega \otimes B \in \mathbb{R}^{Nn\times Nq}$. Additionally, assume that the DMAS is regular, namely, $\det(s\Xi - \Phi) \not\equiv 0$ for some $s \in \mathbb{C}$.

Remark 1. This paper focuses on the controllability of DMASs. However, observability is also a fundamental and important topic in the study of distributed cooperative control of DMASs. Consider a DMAS in the form of

$$\begin{cases} \Xi\dot{X} = \Phi X + \Psi U, \\ Y = \Xi X, \end{cases} \tag{8}$$

where $\Xi = I_N \otimes H \in \mathbb{R}^{Nl\times Nn}$, $Y = \left[y_1^{\mathrm{T}}, \ldots, y_N^{\mathrm{T}}\right]^{\mathrm{T}}$, and $y_i(t) \in \mathbb{R}^l$ is the output of agent i. Then the following system:

$$\begin{cases} \Xi^{\mathrm{T}}\dot{Z} = \Phi^{\mathrm{T}} Z + \Xi^{\mathrm{T}} V, \\ W = \Psi^{\mathrm{T}} Z \end{cases} \tag{9}$$

is called the dual system of system (8). Due to the dual principle between the controllability (observability) of system (8) and the observability (controllability) of system (9), system (8) is R-controllable (R-observable) if and only if its dual system (9) is R-observable (R-controllable) [32].

3 Controllability of DMASs with homogeneous dynamics

In this section, we focus on the controllability of DMASs with homogeneous dynamics, to facilitate the discussion, first introducing the following definitions.

Definition 6 ([32])**.** Let p^1 be the left eigenvector of (E, A) associated with eigenvalue λ; then a sequence of nonzero row vectors $p^1, \ldots, p^g$ is called the left Jordan chain of (E, A) corresponding to p^1 if $p^{i+1}(A - \lambda E) = p^i E$ for $i \in \mathbb{Z}^{g-1}$ and there does not exist $p \in \mathbb{C}^{1\times n}$ such that $p(A - \lambda E) = p^g E$, where g is the length of the Jordan chain.

Definition 7. Let $A,\ T \in \mathbb{R}^{n\times n}$, γ^1 be the left eigenvector of (E, A) associated with eigenvalue λ, then a sequence of nonzero row vectors $\gamma^1, \gamma^2, \dots, \gamma^u$ is called the generalized left Jordan chain of (E, A) about T corresponding to γ^1 if $\gamma^{i+1}(\lambda E - A) = \gamma^i T$ for $i \in \mathbb{Z}^{u-1}$ and there does not exist $\gamma \in \mathbb{C}^{1\times n}$ such that $\gamma(\lambda E - A) = \gamma^u T$, where u is the length. Particularly, if for any other generalized left Jordan chain $\gamma^1, \tilde{\gamma}^2, \dots, \tilde{\gamma}^{\tilde{u}}$ corresponding to γ^1, there holds $u \geqslant \tilde{u}$, then $\gamma^1, \gamma^2, \dots, \gamma^u$ is called the maximum generalized left Jordan chain of (E, A) about T corresponding to γ^1.

Note that the maximum generalized left Jordan chain of (E, A) about T corresponding to γ^1 is not unique. Next, we provide a lemma about the maximum generalized left Jordan chain, which can contribute to capturing a basis for the left eigenspace of the system matrix.

Lemma 3. Let γ_l^1 with $l \in \mathbb{Z}^\eta$ be a basis for the left eigenspace of (E, A) associated with eigenvalue λ, where $\eta \geqslant 1$ is the geometric multiplicity of λ. Assume that $\gamma_l^1, \gamma_l^2, \dots, \gamma_l^{u_l}$ is a maximum generalized left Jordan chain of (E, A) about T corresponding to γ_l^1, where u_l is the length. Then for any generalized left Jordan chain $\gamma_s^1, \tilde{\gamma}_s^2, \dots, \tilde{\gamma}_s^{\tilde{u}_s}$ corresponding to γ_s^1, there holds $\tilde{\gamma}_s^i = \sum_{r=1}^{i-1}\sum_{l=1}^{\eta} \alpha_{rl}\gamma_l^{i-r} + \gamma_s^i$ for $s \in \mathbb{Z}^\eta$, $i \in \{2, \dots, \tilde{u}_s\}$, where $\alpha_{rl} \in \mathbb{R}$ are not all zero.

Proof. Using induction, if $i = 2$, since $\tilde{\gamma}_s^2(\lambda E - A) = \gamma_s^1 T = \gamma_s^2(\lambda E - A)$, and $\gamma_l^1(\lambda E - A) = 0$ for all $l \in \mathbb{Z}^\eta$, then $\tilde{\gamma}_s^2 = \sum_{l=1}^{\eta} \alpha_{1l}\gamma_l^1 + \gamma_s^2$ holds.

Assume that $\tilde{\gamma}_s^{i-1} = \sum_{r=1}^{i-1-1}\sum_{l=1}^{\eta} \alpha_{rl}\gamma_l^{i-1-r} + \gamma_s^{i-1}$ holds. Next we prove that $\tilde{\gamma}_s^i = \sum_{r=1}^{i-1}\sum_{l=1}^{\eta} \alpha_{rl}\gamma_l^{i-r} + \gamma_s^i$ holds. Let $\hat{\gamma}_s^j$ be a vector which satisfies $\hat{\gamma}_s^j(\lambda E - A) = \gamma_s^{j-1}T$ for all $j \in \{2, \dots, i-1\}$. Then we have $(\hat{\gamma}_s^j - \gamma_s^j)(\lambda E - A) = 0$. Note that γ_l^1 with $l \in \mathbb{Z}^\eta$ are linear independent left eigenvectors of (E, A) corresponding to λ. Thus there exist $k_l \in \mathbb{R}$ such that $\hat{\gamma}_s^j = \sum_{l=1}^{\eta} k_l\gamma_l^1 + \gamma_s^j$, which are not all zero. By $\tilde{\gamma}_s^i(\lambda E - A) = \tilde{\gamma}_s^{i-1}T$, $\gamma_s^i(\lambda E - A) = \gamma_s^{i-1}T$, then

$$
\begin{aligned}
(\tilde{\gamma}_s^i - \gamma_s^i)(\lambda E - A) &= \sum_{r=1}^{i-1-1}\sum_{l=1}^{\eta} \alpha_{rl}\gamma_l^{i-1-r}T \\
&= \sum_{r=1}^{i-1-1}\sum_{l=1}^{\eta} \alpha_{rl}\hat{\gamma}_l^{i-r}(\lambda E - A) \\
&= \left(\sum_{r=1}^{i-1-1}\sum_{l=1}^{\eta} \alpha_{rl}\left(\sum_{l=1}^{\eta} k_l\gamma_l^1\right) + \gamma_l^{i-r}\right)(\lambda E - A).
\end{aligned}
$$

Since there exists $h_l \in \mathbb{R}$ such that $\tilde{\gamma}_s^i - \gamma_s^i - (\sum_{r=1}^{i-1-1}\sum_{l=1}^{\eta} \alpha_{rl}(\sum_{l=1}^{\eta} k_l\gamma_l^1) + \gamma_l^{i-r}) = \sum_{l=1}^{\eta} h_l\gamma_l^1$, we have $\tilde{\gamma}_s^i = \sum_{r=1}^{i-1-1}\sum_{l=1}^{\eta} \alpha_{rl}\gamma_l^{i-r} + \sum_{l=1}^{\eta} \alpha_{i-1,l}\gamma_l^1 + \gamma_s^i$, where $\alpha_{i-1,l} = k_l(\sum_{r=1}^{i-1-1}\sum_{l=1}^{\eta} \alpha_{rl}) + h_l$. Therefore, $\tilde{\gamma}_s^i = \sum_{r=1}^{i-1}\sum_{l=1}^{\eta} \alpha_{rl}\gamma_l^{i-r} + \gamma_s^i$ holds. In particular, if γ_l^{i-r} does not exist, then $\alpha_{rl} = \alpha_{r-1,l} = \cdots = \alpha_{1l} = 0$.

Proposition 1. Assume that the topological matrix L is nondiagonalizable with Jordan canonical form $J = \mathrm{diag}(J_1, \dots, J_s)$, where $J_i = \lambda_i I_{a_i} + N_{a_i} \in \mathbb{C}^{a_i\times a_i}$, λ_i is its eigenvalue with corresponding left eigenvector h_i^1. Let $h_i^1, \dots, h_i^{a_i}$ be the left Jordan chain of L corresponding to h_i^1 for all $i \in \mathbb{Z}^s$, $\sigma(E, A - \lambda_i CK) = \{\omega_i^1, \dots, \omega_i^{q_i}\}$ be the set of finite eigenvalues of $(E, A - \lambda_i CK)$, β_{ijl}^1 with $l \in \mathbb{Z}^{\eta_{ij}}$ be a basis for the left eigenspace corresponding to ω_i^j, where $\eta_{ij} \geqslant 1$ is the geometric multiplicity of ω_i^j. Let $\beta_{ijl}^1, \dots, \beta_{ijl}^{u_{ijl}}$ denote a maximum generalized left Jordan chain of $(E, A - \lambda_i CK)$ about $-CK$ corresponding to β_{ijl}^1 for all $l \in \mathbb{Z}^{\eta_{ij}}$. Then the following statements hold:

(i) The set of finite eigenvalues of (Ξ, Φ) is $\bigcup_{i=1}^{s} \sigma(E, A - \lambda_i CK)$;

(ii) $\sum_{t=1}^{r} h_i^{r-t+1} \otimes \beta_{ijl}^t$ with $r \in \mathbb{Z}^{\theta_{ijl}}$, $l \in \mathbb{Z}^{\eta_{ij}}$ is a basis for the left eigenspace of (Ξ, Φ) associated with ω_i^j, where $\theta_{ijl} = \min\{a_i, u_{ijl}\}$.

Proof. Regarding (i), let H be a nonsingular matrix such that $HLH^{-1} = J$ and $J = \mathrm{diag}(J_1, \dots, J_s)$ is the Jordan canonical form of L, where $J_i = \lambda_i I_{a_i} + N_{a_i} \in \mathbb{C}^{a_i\times a_i}$, N_{a_i} is a matrix with ones on the subdiagonal, and zeros elsewhere. Then, we have $(H \otimes I_n)(I_N \otimes E)(H^{-1} \otimes I_n) = (I_N \otimes E)$.

Let $\bar{\Phi} = (H \otimes I_n)\Phi(H^{-1} \otimes I_n)$, then

$$
\begin{aligned}
\bar{\Phi} &= (H \otimes I_n)(I_N \otimes A - L \otimes CK)(H^{-1} \otimes I_n) \\
&= I_N \otimes A - J \otimes CK \\
&= I_N \otimes A - \mathrm{diag}(J_1, \dots, J_s) \otimes CK
\end{aligned}
$$

$$= \mathrm{diag}\left(I_{a_1} \otimes A - J_1 \otimes CK, \ldots, I_{a_s} \otimes A - J_s \otimes CK\right)$$
$$= \mathrm{diag}\left(I_{a_1} \otimes (A - \lambda_1 CK) - N_{a_1} \otimes CK, \ldots, I_{a_s} \otimes (A - \lambda_s CK) - N_{a_s} \otimes CK\right).$$

Since (Ξ, Φ) and $(\Xi, \bar{\Phi})$ have the same eigenvalues and $\bar{\Phi}$ is a block-diagonal matrix, one has $\sigma(\Xi, \Phi) = \bigcup_{i=1}^{s} \sigma(E, A - \lambda_i CK)$.

Regarding (ii), let $H_i = [h_i^{a_i \mathrm{T}}, \ldots, h_i^{1\mathrm{T}}]^{\mathrm{T}}$, then $(H_i \otimes I_n)(I_N \otimes A - L \otimes CK) = (I_{a_i} \otimes (A - \lambda_i CK) - N_{a_i} \otimes CK)(H_i \otimes I_n)$. Thus, $(\omega_i^j, \tau(H_i \otimes I_n))$ is a left eigenpair of (Ξ, Φ) if and only if (ω_i^j, τ) is a left eigenpair of $(I_{a_i} \otimes E,\ I_{a_i} \otimes (A - \lambda_i CK) - N_{a_i} \otimes CK)$ for $\tau \in \mathbb{C}^{1 \times (n a_i)}$. Letting $\tau = [\tau_1, \ldots, \tau_{a_i}]$, then $\tau_1, \ldots, \tau_{a_i}$ satisfy

$$\begin{cases} \tau_1(\omega_i^j E - (A - \lambda_i CK)) = 0, \\ \tau_{g+1}(\omega_i^j E - (A - \lambda_i CK)) = \tau_g(-CK) \end{cases}$$

for all $g \in \mathbb{Z}^{a_i - 1}$. By Definition 7, there exists $k \in \mathbb{Z}^{a_i}$ such that $\tau_1 = \cdots = \tau_{k-1} = 0$ and $\tau_k, \ldots, \tau_{a_i}$ are the first $a_i - k + 1$ vectors of $\beta_{ijs}^1, \tilde{\beta}_{ijs}^2, \ldots, \tilde{\beta}_{ijs}^{\tilde{u}_{ijs}}$ for all $s \in \mathbb{Z}^{\eta_{ij}}$, where $\beta_{ijs}^1, \tilde{\beta}_{ijs}^2, \ldots, \tilde{\beta}_{ijs}^{\tilde{u}_{ijs}}$ can be any generalized left Jordan chain of $(E, A - \lambda_i CK)$ about $-CK$ corresponding to β_{ijs}^1. Let $\mathcal{B}$ be the set of generalized left Jordan chains of $(E, A - \lambda_i CK)$ about $-CK$ corresponding to β_{ijs}^1 for all $s \in \mathbb{Z}^{\eta_{ij}}$. Thus the left eigenspace corresponding to ω_i^j is $\mathcal{T} = \{\tau \mid \tau = \sum_{t=1}^{r} e_{a_i - r + t}^{(a_i)\mathrm{T}} \otimes \tilde{\beta}_{ijs}^t$ with $\beta_{ijs}^1, \tilde{\beta}_{ijs}^2, \ldots, \tilde{\beta}_{ijs}^{\tilde{u}_{ijs}} \in \mathcal{B}, r \in \mathbb{Z}^{\tilde{\theta}_{ijs}}, s \in \mathbb{Z}^{\eta_{ij}}\}$, where $\tilde{\theta}_{ijs} = \min\{a_i, \tilde{u}_{ijs}\}$ and $\tilde{\beta}_{ijs}^1 = \beta_{ijs}^1$. Next, we prove that $\sum_{t=1}^{r} e_{a_i - r + t}^{(a_i)\mathrm{T}} \otimes \beta_{ijl}^t$ with $r \in \mathbb{Z}^{\theta_{ijl}}$, $l \in \mathbb{Z}^{\eta_{ij}}$ is a maximal linearly independent group of $\mathcal{T}$.

Since β_{ijl}^1 with $l \in \mathbb{Z}^{\eta_{ij}}$ are linearly independent, obviously, $\sum_{t=1}^{r} e_{a_i - r + t}^{(a_i)\mathrm{T}} \otimes \beta_{ijl}^t$ with $r \in \mathbb{Z}^{\theta_{ijl}}$, $l \in \mathbb{Z}^{\eta_{ij}}$ are linearly independent. In addition, for any $\sum_{t=1}^{r} e_{a_i - r + t}^{(a_i)\mathrm{T}} \otimes \tilde{\beta}_{ijs}^t \in \mathcal{T}$, by Lemma 3, we have $\tilde{\beta}_{ijs}^t = \sum_{a=1}^{t-1} \sum_{l=1}^{\eta_{ij}} \alpha_{al} \beta_{ijl}^{t-a} + \beta_{ijs}^t$, then $\sum_{t=1}^{r} e_{a_i - r + t}^{(a_i)\mathrm{T}} \otimes \tilde{\beta}_{ijs}^t = \sum_{t=1}^{r} (e_{a_i - r + t}^{(a_i)\mathrm{T}} \otimes \beta_{ijs}^t) + \sum_{a=1}^{r-1} \sum_{l=1}^{\eta_{ij}} \alpha_{al} (\sum_{t=a+1}^{r} e_{a_i - r + t}^{(a_i)\mathrm{T}} \otimes \beta_{ijl}^{t-a})$ for all $r \in \mathbb{Z}^{\tilde{\theta}_{ijl}}$, $l \in \mathbb{Z}^{\eta_{ij}}$; thus $\sum_{t=1}^{r} e_{a_i - r + t}^{(a_i)\mathrm{T}} \otimes \beta_{ijl}^t$ with $l \in \mathbb{Z}^{\eta_{ij}}$, $r \in \mathbb{Z}^{\theta_{ijl}}$ is a maximal linearly independent group.

Therefore, we obtain that $\sum_{t=1}^{r} e_{a_i - r + t}^{(a_i)\mathrm{T}} \otimes \beta_{ijl}^t$ with $r \in \mathbb{Z}^{\theta_{ijl}}$, $l \in \mathbb{Z}^{\eta_{ij}}$ is a basis for the left eigenspace of $(I_{a_i} \otimes E,\ I_{a_i} \otimes (A - \lambda_i CK) - N_{a_i} \otimes CK)$ associated with ω_i^j, where $\theta_{ijl} = \min\{a_i, u_{ijl}\}$. Then, $\sum_{t=1}^{r} h_i^{r-t+1} \otimes \beta_{ijl}^t$ with $l \in \mathbb{Z}^{\eta_{ij}}$, $r \in \mathbb{Z}^{\theta_{ijl}}$ is a basis for the left eigenspace of (Ξ, Φ) associated with ω_i^j, where $\theta_{ijl} = \min\{a_i, u_{ijl}\}$.

Proposition 1 provides a complete set of basis for the eigenspace of the system matrix leveraging the maximum generalized left Jordan chain defined in Definition 7. Moreover, Proposition 1 fully characterizes the relationship between the eigenvector of the system matrix and the eigenvector of L and $(E, A - \lambda_i CK)$. Next, taking advantage of the explicit form for the basis, a necessary and sufficient condition for the R-controllability of system (7) is established.

Theorem 1. Assume that the topological matrix L is nondiagonalizable with Jordan canonical form $J = \mathrm{diag}(J_1, \ldots, J_s)$, where $J_i = \lambda_i I_{a_i} + N_{a_i} \in \mathbb{C}^{a_i \times a_i}$, λ_i is its eigenvalue with corresponding left eigenvector h_i^1. Let $h_i^1, \ldots, h_i^{a_i}$ be the left Jordan chain of L corresponding to h_i^1 for all $i \in \mathbb{Z}^s$, $\sigma(E, A - \lambda_i CK) = \{\omega_i^1, \ldots, \omega_i^{q_i}\}$ be the set of finite eigenvalues of $(E, A - \lambda_i CK)$, β_{ijl}^1 with $l \in \mathbb{Z}^{\eta_{ij}}$ be a basis for the left eigenspace corresponding to ω_i^j, where $\eta_{ij} \geqslant 1$ is the geometric multiplicity of ω_i^j. Let $\beta_{ijl}^1, \ldots, \beta_{ijl}^{u_{ijl}}$ denote a maximum generalized left Jordan chain of $(E, A - \lambda_i CK)$ about $-CK$ corresponding to β_{ijl}^1 for all $l \in \mathbb{Z}^{\eta_{ij}}$. Then DMASs (7) is R-controllable if and only if the following holds:

(i) $\nu(\Omega \otimes B) \neq 0$ for all $\nu \in \mathcal{U}_{ij} \backslash \{\mathbf{0}\}$, $j \in \mathbb{Z}^{q_i}$, $i \in \mathbb{Z}^s$, where $\mathcal{U}_{ij} = \mathrm{span}\{\nu_{ijl}^r \mid \nu_{ijl}^r = \sum_{t=1}^{r} h_i^{r-t+1} \otimes \beta_{ijl}^t,\ l \in \mathbb{Z}^{\eta_{ij}},\ r \in Z^{\theta_{ijl}}\}$, $\theta_{ijl} = \min\{a_i, u_{ijl}\}$;

(ii) if $\omega_{i_1}^{j_1} = \cdots = \omega_{i_t}^{j_t} = \mu$, where $\omega_{i_k}^{j_k} \in \sigma(E, A - \lambda_{i_k} CK)$, and $\mu \notin \sigma(E, A - \lambda_r CK)$ for all $r \in \mathbb{Z}^s \backslash \{i_1, \ldots, i_t\}$, then $\nu(\Omega \otimes B) \neq 0$ for all $\nu \in \odot_{k=1}^{t} \mathcal{U}_{i_k j_k} \backslash \{\mathbf{0}\}$.

Proof. Necessity. By Proposition 1, $\mathcal{U}_{ij}$ is the left eigenspace of (Ξ, Φ) associated with the eigenvalue ω_i^j. If system (7) is R-controllable, then $\nu(\Omega \otimes B) \neq 0$ for all $\nu \in \mathcal{U}_{ij} \backslash \{\mathbf{0}\}$, $j \in \mathbb{Z}^{q_i}, i \in \mathbb{Z}^s$. Condition (i) is proven.

For condition (ii), assume that $\omega_{i_1}^{j_1} = \cdots = \omega_{i_t}^{j_t} = \mu$ and $\mu \notin \sigma(E, A - \lambda_r CK)$ for all $r \in \mathbb{Z}^s \backslash \{i_1, \ldots, i_t\}$, then the left eigenspace of (Ξ, Φ) corresponding to μ is $\odot_{k=1}^{t} \mathcal{U}_{i_k j_k}$. Therefore, if system (7) is R-controllable, then $\nu(\Omega \otimes B) \neq 0$ for all $\nu \in \odot_{k=1}^{t} \mathcal{U}_{i_k j_k} \backslash \{\mathbf{0}\}$.

Sufficiency. If system (7) is R-uncontrollable, then there exist an eigenvalue $\bar{\omega}$ of (Ξ, Φ) and the corresponding left eigenvector $\bar{\xi}$, such that $\bar{\xi}\Psi = 0$, which includes two cases.

Case 1. If $\bar{\omega} \in \sigma(E, A - \lambda_{i_0} CK)$ and $\bar{\omega} \notin \sigma(E, A - \lambda_r CK)$ for all $r \in \mathbb{Z}^s$, $r \neq i_0$, it follows from Proposition 1 that $\mathcal{U}_{i_0 j_0} = \text{span}\{\nu^r_{i_0 j_0 l_0} \mid \nu^r_{i_0 j_0 l_0} = \sum_{t=1}^{r} h_{i_0}^{r-t+1} \otimes \beta^t_{i_0 j_0 l_0},\ l_0 \in \mathbb{Z}^{\eta_{i_0 j_0}},\ r \in Z^{\theta_{i_0 j_0 l_0}}\}$ is the left eigenspace of (Ξ, Φ) corresponding to $\bar{\omega}$. Therefore, if system (7) is R-uncontrollable, there exists a left eigenvector $\bar{\xi} \in \mathcal{U}_{i_0 j_0}$ such that $\bar{\xi}(\Omega \otimes B) = 0$, which contradicts condition (i).

Case 2. If $\omega^{j_1}_{i_1} = \cdots = \omega^{j_t}_{i_t} = \bar{\omega}$, and $\bar{\omega} \notin \sigma(E, A - \lambda_r CK)$ for all $r \in \mathbb{Z}^s \backslash \{i_1, \ldots, i_t\}$, then there exists a left eigenvector $\bar{\xi} \in \odot_{k=1}^{t} \mathcal{U}_{i_k j_k}$ such that $\bar{\xi}(\Omega \otimes B) = 0$, which contradicts condition (ii).

Therefore, if system (7) is R-uncontrollable, then at least one condition in Theorem 1 does not hold, which completes the proof.

Theorem 1 presents a precise and efficient condition for determining the R-controllability of DMASs with higher-dimensional states. As we all know, the R-controllability of descriptor systems is equivalent to the controllability of standard state-space systems. Here, the controllability conditions in Theorem 1 are obtained by taking advantage of the structural characteristics of the descriptor system captured in Proposition 1, which is more clear than those obtained by transforming descriptor systems into the standard state-space system. Therefore, on one hand, the conditions in Theorem 1 provide more precise and efficient criteria for determining the controllability of large-scale DMASs. On the other hand, they also more intuitively reveal how the R-controllability of the whole system is affected by the network topology, the subsystem dynamics, the external control inputs, and the inner coupling.

Actually, when the agents are governed by single-integrator dynamics, system (7) becomes $\dot{x} = -Lx + \Omega u$, then the controllability of the system is completely determined by the underlying topology given by the pair of matrices L and Ω. Therefore, (L, Ω) being controllable means that a system with single-integrator agent dynamics is controllable. The controllability of (L, Ω) reveals the effect of underlying topology on the controllability of system (7). Note that the computational complexity of solving matrix equations in [30] is at least $O(N^4 n^3)$, whereas checking the new criterion established here incurs no more than $O(N^4 + Nn^4 + N^3 n^3)$ operations; the computational complexity can greatly be reduced. The following illustrates the efficiency of Theorem 1.

Example 1. Consider a DMAS consisting of three agents, and

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ CK = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ \Omega = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $l_{21} = -1$ represents that agent 1 is cooperative to agent 2, $l_{32} = 1$ represents that agent 2 is antagonistic to agent 3. The Jordan form of L is $J = [1, 1, 0; 0, 1, 0; 0, 0, 0]$ with corresponding left eigenvectors $h^1_1 = [1, -1, 0], h^2_1 = [-1, 0, -1]$, and $h^1_2 = [1, 0, 0]$, respectively. The eigenvalue of $(E, A - CK)$ is $\omega^1_1 = 0$ with the left eigenvector $\beta^1_{111} = [1, 0]$, and the eigenvalue of (E, A) is $\omega^1_2 = 1$ with the left eigenvector $\beta^1_{211} = [1, 0]$. Thus, $\nu^1_{111}(\Omega \otimes B) \neq 0$ and $\nu^1_{211}(\Omega \otimes B) \neq 0$. Therefore, this DMAS is R-controllable.

Remark 2. It can be seen that if system (7) is R-controllable, $\nu^1_{ijl}(\Omega \otimes B) \neq 0$ for all $l \in \mathbb{Z}^{\eta_{ij}}$, $j \in \mathbb{Z}^{q_i}, i \in \mathbb{Z}^s$. It follows that $(h^1_i \Omega) \otimes (\beta^1_{ijl} B) \neq 0$, where h^1_i is an arbitrary left eigenvector of L corresponding to λ_i, and β^1_{ijl} is an arbitrary left eigenvector of $(E, A - \lambda_i CK)$ corresponding to ω^j_i. Therefore, (L, Ω) is controllable and $(E, A - \lambda_i CK, B)$ is R-controllable for all $i \in \mathbb{Z}^s$. Moreover, if the matrix L has an eigenvalue 0 (e.g., $\mathcal{G}$ has a directed spanning tree), the R-controllability of the subsystem (E, A, B) is necessary for the controllability of the whole networked system. Hence one can see that the controllability of underlying network structure and the R-controllability of subsystem are the key factors that determine the controllability of the whole system when the matrix L has an eigenvalue 0, the internal interaction is also one of the factors that affect the R-controllability of the system when $\lambda_i \neq 0$.

Next, we consider a special case that L is diagonalizable, for which more intuitive and concise controllability conditions are obtained. In contrast to the case of nondiagonalizable L, the left eigenvectors of (Ξ, Φ) can be directly captured by the left eigenvectors of L and $(E, A - \lambda_i CK)$.

Proposition 2. Assume that the topological matrix L is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_N$, and p_i is the corresponding left eigenvector. Let $\sigma(E, A - \lambda_i CK) = \{\omega^1_i, \ldots, \omega^{q_i}_i\}$ be the set of finite eigenvalues of $(E, A - \lambda_i CK)$ for all $i \in \mathbb{Z}^N$, γ^l_{ij} with $l \in \mathbb{Z}^{\eta_{ij}}$ be a basis for the left eigenspace corresponding to ω^j_i for all $j \in \mathbb{Z}^{q_i}$, where $\eta_{ij} \geqslant 1$ is the geometric multiplicity of ω^j_i. Then the following statements hold:

(i) the set of finite eigenvalues of (Ξ, Φ) is $\bigcup_{i=1}^{N} \sigma(E, A-\lambda_i CK)$;

(ii) $p_i \otimes \gamma_{ij}^1, \ldots, p_i \otimes \gamma_{ij}^{\eta_{ij}}$ is a basis for the left eigenspace of (Ξ, Φ) corresponding to ω_i^j for all $j \in \mathbb{Z}^{q_i}, i \in \mathbb{Z}^N$.

Proof. Regarding (i), let P be a nonsingular matrix satisfying $PLP^{-1} = Q$, where $Q = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$; then we have $(P \otimes I_n)(I_N \otimes E)\left(P^{-1} \otimes I_n\right) = (I_N \otimes E)$.

Let $\widetilde{\Phi} = (P \otimes I_n)\Phi\left(P^{-1} \otimes I_n\right)$, then

$$\begin{aligned}\widetilde{\Phi} &= (P \otimes I_n)(I_N \otimes A - L \otimes CK)\left(P^{-1} \otimes I_n\right)\\ &= I_N \otimes A - Q \otimes CK\\ &= I_N \otimes A - \operatorname{diag}(\lambda_1, \ldots, \lambda_N) \otimes CK\\ &= \operatorname{diag}(A-\lambda_1 CK, \ldots, A-\lambda_N CK).\end{aligned}$$

Since $(\Xi, \widetilde{\Phi})$ and (Ξ, Φ) have the same eigenvalues, $\sigma(\Xi, \Phi) = \bigcup_{i=1}^{N} \sigma(E, A-\lambda_i CK)$.

Regarding (ii), let $P = \left[p_1^{\mathrm{T}}, \ldots, p_N^{\mathrm{T}}\right]^{\mathrm{T}}$, obviously, $(P \otimes I_n)\Phi = \widetilde{\Phi}(P \otimes I_n)$. Thus, $(\omega_i^j, \tau(P \otimes I_n))$ is a left eigenpair of (Ξ, Φ) if and only if (ω_i^j, τ) is a left eigenpair of $(\Xi, \widetilde{\Phi})$ for $\tau \in \mathbb{C}^{1\times(Nn)}$. Since γ_{ij}^l with $l \in \mathbb{Z}^{\eta_{ij}}$ is a basis for the left eigenspace of $(E, A-\lambda_i CK)$ associated with ω_i^j, then $e_i^{(N)\mathrm{T}} \otimes \gamma_{ij}^l$ with $l \in \mathbb{Z}^{\eta_{ij}}$ is a basis for the left eigenspace of $(\Xi, \widetilde{\Phi})$ associated with ω_i^j. Therefore, $p_i \otimes \gamma_{ij}^1, \ldots, p_i \otimes \gamma_{ij}^{\eta_{ij}}$ is a basis for the left eigenspace of (Ξ, Φ) corresponding to ω_i^j for all $j \in \mathbb{Z}^{q_i}, i \in \mathbb{Z}^N$.

In what follows, we establish a necessary and sufficient condition for the R-controllability of system (7).

Theorem 2. Assume that the topological matrix L can be diagonalized with eigenvalues $\lambda_1, \ldots, \lambda_N$; p_i is the left eigenvector corresponding to λ_i for all $i \in \mathbb{Z}^N$. Let $\sigma(E, A-\lambda_i CK) = \{\omega_i^1, \ldots, \omega_i^{q_i}\}$ be the set of finite eigenvalues of $(E, A-\lambda_i CK)$, γ_{ij}^l with $l \in \mathbb{Z}^{\eta_{ij}}$ be a basis for the left eigenspace corresponding to ω_i^j for all $j \in \mathbb{Z}^{q_i}$, where $\eta_{ij} \geqslant 1$ is the geometric multiplicity of ω_i^j. Then system (7) is R-controllable if and only if the following holds:

(i) (L, Ω) is controllable;

(ii) $(E, A-\lambda_i CK, B)$ is R-controllable for all $i \in \mathbb{Z}^N$;

(iii) if $\omega_{i_1}^{j_1} = \cdots = \omega_{i_t}^{j_t} = \mu$, where $\omega_{i_k}^{j_k} \in \sigma(E, A-\lambda_{i_k} CK)$, and $\mu \notin \sigma(E, A-\lambda_r CK)$ for all $r \in \mathbb{Z}^N \backslash \{i_1, \ldots, i_t\}$, then

$$\left[\sum_{k=1}^{t} \sum_{l=1}^{\eta_{i_k j_k}} \alpha_{kl}\left(p_{i_k} \otimes \gamma_{i_k j_k}^l\right)\right](\Omega \otimes B) \neq 0,$$

where $\alpha_{kl} \in \mathbb{R}$ are not all zero for all $k \in Z^t$, $l \in Z^{\eta_{i_k j_k}}$.

Proof. Necessity. If system (7) is R-controllable, then $\left(p_i \otimes \gamma_{ij}^l\right)(\Omega \otimes B) \neq 0$ for all $l \in \mathbb{Z}^{\eta_{ij}}, j \in \mathbb{Z}^{q_i}, i \in \mathbb{Z}^N$. Thus $(p_i\Omega) \otimes \left(\gamma_{ij}^l B\right) \neq 0$, which yields

$$p_i \Omega \neq 0 \text{ for all } i \in \mathbb{Z}^N, \tag{10}$$

$$\gamma_{ij}^l B \neq 0 \text{ for all } l \in \mathbb{Z}^{\eta_{ij}},\ j \in \mathbb{Z}^{q_i},\ i \in \mathbb{Z}^N. \tag{11}$$

Since p_i and γ_{ij}^l are arbitrary left eigenvectors of L and $(E, A-\lambda_i CK)$, respectively, then it follows from (10) that (L, Ω) is controllable. Similarly, Eq. (11) implies that $(E, A-\lambda_i CK, B)$ is R-controllable for all $i \in \mathbb{Z}^N$. Conditions (i) and (ii) are proven.

For condition (iii), suppose that $\omega_{i_1}^{j_1} = \cdots = \omega_{i_t}^{j_t} = \mu$ and $\mu \notin \sigma(E, A-\lambda_r CK)$ for all $r \in \mathbb{Z}^N \backslash \{i_1, \ldots, i_t\}$. Then, all the left eigenvectors of (Ξ, Φ) corresponding to μ can be expressed as $\sum_{k=1}^{t} \sum_{l=1}^{\eta_{i_k}} \alpha_{kl}(p_{i_k} \otimes \gamma_{i_k j_k}^l)$, where p_{i_k} is the left eigenvector of L corresponding to λ_{i_k}, $\gamma_{i_k j_k}^l$ is a basis for the left eigenspace of $(E, A-\lambda_{i_k} CK)$ associated with μ, $\eta_{i_k j_k}$ denotes the corresponding geometric multiplicity, and $\alpha_{kl} \in \mathbb{R}$ are not all zero for all $k \in \mathbb{Z}^t$, $l \in \mathbb{Z}^{\eta_{i_k}}$. If system (7) is R-controllable, then

$$\left[\sum_{k=1}^{t} \sum_{l=1}^{\eta_{i_k j_k}} \alpha_{kl}\left(p_{i_k} \otimes \gamma_{i_k j_k}^l\right)\right](\Omega \otimes B) \neq 0, \tag{12}$$

for any scalars α_{kl}, which are not all zero.

Sufficiency. According to Lemma 1, if system (7) is R-uncontrollable, then there exists a left eigenpair $(\hat{\omega}, \hat{\xi})$ of (Ξ, Φ) satisfying

$$\hat{\xi}\Psi = 0, \tag{13}$$

which includes two cases as follows.

Case 1. If $\hat{\omega} \in \sigma(E, A-\lambda_{i_c}CK)$ and $\hat{\omega} \notin \sigma(E, A-\lambda_r CK)$ for all $r \in \mathbb{Z}^N,\ r \neq i_c$, then $\hat{\xi}$ can be expressed as $\sum_{l=1}^{\eta_{i_c j_c}} \alpha_l \left(p_{i_c} \otimes \gamma_{i_c j_c}^l\right)$, where p_{i_c} is the left eigenvector of L corresponding to λ_{i_c}, $\gamma_{i_c j_c}^1, \ldots, \gamma_{i_c j_c}^{\eta_{i_c j_c}}$ is a basis for the left eigenspace of $(E, A - \lambda_{i_c}CK)$ corresponding to $\hat{\omega}$, and $[\alpha_1, \ldots, \alpha_{\eta_{i_c j_c}}]$ is a nonzero vector. Thus, Eq. (13) implies

$$\left[\sum_{l=1}^{\eta_{i_c j_c}} \alpha_l \left(p_{i_c} \otimes \gamma_{i_c j_c}^l\right)\right](\Omega \otimes B) = \sum_{l=1}^{\eta_{i_c j_c}} \alpha_l \left(p_{i_c}\Omega\right) \otimes \left(\gamma_{i_c j_c}^l B\right) = \left(p_{i_c}\Omega\right) \otimes \left(\sum_{l=1}^{\eta_{i_c j_c}} \alpha_l \gamma_{i_c j_c}^l B\right) = 0. \tag{14}$$

Then, we obtain

$$p_{i_c}\Omega = 0 \tag{15}$$

or

$$\sum_{l=1}^{\eta_{i_c j_c}} \alpha_l \gamma_{i_c j_c}^l B = 0. \tag{16}$$

We have p_{i_c} and $\sum_{l=1}^{\eta_{i_c j_c}} \alpha_l \gamma_{i_c j_c}^l$ are arbitrary left eigenvectors of L and $(E, A - \lambda_{i_c}CK)$, respectively. In this case, if system (7) is R-uncontrollable, then it follows from (15) that (L, Ω) is uncontrollable or there exists λ_{i_c} such that $(E, A - \lambda_{i_c}CK, B)$ is R-uncontrollable.

Case 2. If $\omega_{i_1}^{j_1} = \cdots = \omega_{i_t}^{j_t} = \hat{\omega}$ and $\hat{\omega} \notin \sigma(E, A - \lambda_r CK)$ for all $r \in \mathbb{Z}^N \backslash \{i_1, \ldots, i_t\}$, then $\hat{\xi}$ can be expressed as $\sum_{k=1}^{t} \sum_{l=1}^{\eta_{i_k j_k}} \alpha_{kl} \left(p_{i_k} \otimes \gamma_{i_k j_k}^l\right)$, where p_{i_k} is the left eigenvector of L corresponding to λ_{i_k} for all $k \in \mathbb{Z}^t$, $\gamma_{i_k j_k}^l$ with $l \in Z^{\eta_{i_k j_k}}$ is a basis for the left eigenspace of $(E, A - \lambda_{i_k}CK)$ associated with $\hat{\omega}$ for all $k \in \mathbb{Z}^t$, $\eta_{i_k j_k}$ denotes the corresponding geometric multiplicity, and α_{kl} are not all zero for all $k \in \mathbb{Z}^t,\ l \in Z^{\eta_{i_k j_k}}$. By (13), there exist α_{kl} for all $k \in \mathbb{Z}^t$ and $l \in Z^{\eta_{i_k}}$ such that

$$\left[\sum_{k=1}^{t} \sum_{l=1}^{\eta_{i_k j_k}} \alpha_{kl} \left(p_{i_k} \otimes \gamma_{i_k j_k}^l\right)\right](\Omega \otimes B) = 0, \tag{17}$$

which contradicts condition (iii).

Therefore, if system (7) is R-uncontrollable, then at least one condition in Theorem 2 does not hold, which completes the proof.

The following example illustrates the efficiency of Theorem 2.

Example 2. Consider a DMAS consisting of two agents with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ CK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\ L = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\ \Omega = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $l_{12} = -1$ represents that agent 2 is cooperative to agent 1, $l_{21} = 1$ represents that agent 1 is antagonistic to agent 2. The eigenvalues of L are $1 \pm i \in \mathbb{C}$, with the corresponding left eigenvectors $p_1 = [1, i],\ p_2 = [1, -i]$. We derive that (L, Ω) is controllable, and $(E, A-(1+i)CK, B),\ (E, A-(1-i)CK, B)$ are R-controllable. Moreover, $(E, A - (1 + i)CK)$ and $(E, A - (1 - i)CK)$ have a common eigenvalue 1, with the corresponding left eigenvectors $\gamma_{11} = \gamma_{21} = [1, 0]$. Thus, $\alpha_{11}(p_1\Omega \otimes \gamma_{11}B) + \alpha_{21}(p_2\Omega \otimes \gamma_{21}B) \neq 0$ for any nonzero $[\alpha_{11}, \alpha_{21}]$. Therefore, this DMAS is R-controllable.

4 Controllability of DMASs with heterogeneous dynamics

In the previous subsections, we assume that all agents in the DMASs have identical descriptor linear dynamics, that is $E_1 = \cdots = E_N,\ A_1 = \cdots = A_N,\ B_1 = \cdots = B_N,\ C_1 = \cdots = C_N$. However, homogeneity is only an ideal assumption because of various restrictions in the practical systems. Therefore, this subsection focuses on the controllability of DMASs with heterogeneous dynamics, which means that different agents in the DMASs may have different parameter matrices, i.e., E_i, A_i, B_i, and C_i may be different for different agents.

Consider a DMAS consisting of N agents with heterogeneous descriptor linear dynamics, described by

$$E_i\dot{x}_i(t) = A_ix_i(t) + C_iz_i(t) + \delta_iB_iu_i(t), \tag{18}$$

where $x_i(t) \in \mathbb{R}^n$, $z_i(t) \in \mathbb{R}^s$, $u_i(t) \in \mathbb{R}^q$, $E_i, A_i \in \mathbb{R}^{n\times n}$, $C_i \in \mathbb{R}^{n\times s}$, $B_i \in \mathbb{R}^{n\times q}$. δ_i indicates that whether agent $i \in \mathcal{V}_\mathrm{L}$ or $i \in \mathcal{V}_\mathrm{F}$, i.e., $\delta_i = 1$ if agent $i \in \mathcal{V}_\mathrm{L}$, and $\delta_i = 0$ otherwise. And

$$z_i(t) = \sum_{j\in\mathcal{N}_i} w_{ij}\left(K_jx_j(t) - \mathrm{sign}\left(w_{ij}\right)K_ix_i(t)\right), \tag{19}$$

where $K_i \in \mathbb{R}^{s\times n}$ is the feedback gain matrix. Let $X = \left[x_1^\mathrm{T}, \ldots, x_N^\mathrm{T}\right]^\mathrm{T}$ and $U = \left[u_1^\mathrm{T}, \ldots, u_N^\mathrm{T}\right]^\mathrm{T}$. Additionally, define the matrix $\Upsilon = \Lambda - \Gamma$, where $\Lambda = \mathrm{diag}(A_1, \ldots, A_N) \in \mathbb{R}^{Nn\times Nn}$ and $\Gamma = [\Gamma_{ij}] \in \mathbb{R}^{Nn\times Nn}$ with $\Gamma_{ij} = l_{ij}C_iK_j$, where $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ is the Laplacian matrix. Let $\Sigma = \mathrm{diag}(\delta_1B_1, \ldots, \delta_NB_N) \in \mathbb{R}^{Nn\times Nq}$ and $\Theta = \mathrm{diag}(E_1, \ldots, E_N) \in \mathbb{R}^{Nn\times Nn}$. One can rewrite the system (18) in a compact form as follows:

$$\Theta\dot{X} = \Upsilon X + \Sigma U. \tag{20}$$

The previous studies show that when the agents have identical dynamics, the system is R-controllable only if the network topology is controllable. For the DMASs with heterogeneous dynamics, interconnections between each pair of subsystems are different matrix-weighted edges; then controllability of the network topology is no longer a decisive factor as will be seen in the following example.

Example 3. Consider a DMAS consisting of three agents with

$$E_1 = E_2 = E_3 = \begin{bmatrix}1&0\\0&0\end{bmatrix},\ A_1 = \begin{bmatrix}1&0\\0&2\end{bmatrix},\ A_2 = \begin{bmatrix}1&0\\0&1\end{bmatrix},\ A_3 = \begin{bmatrix}2&0\\0&2\end{bmatrix},\ B_1 = B_2 = B_3 = \begin{bmatrix}1&0\\0&1\end{bmatrix},$$

$$C_1 = C_2 = C_3 = \begin{bmatrix}0&1\\1&0\end{bmatrix},\ K_1 = \begin{bmatrix}0&1\\1&0\end{bmatrix},\ K_2 = \begin{bmatrix}2&1\\1&3\end{bmatrix},\ K_3 = \begin{bmatrix}3&1\\1&0\end{bmatrix},\ L = \begin{bmatrix}0&0&0\\-1&1&0\\1&0&1\end{bmatrix},\ \Omega = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

It is straightforward to verify that (L, Ω) is uncontrollable (i.e., $\mathrm{rank}[\Omega \quad L\Omega \quad L^2\Omega] = 2 < 3$). By calculating, we obtain $\mathrm{rank}[s\Xi - \Phi \quad \Psi] = 6$ for all $s \in \mathbb{C}$, $|s| < \infty$, thus the DMASs is R-controllable.

The above-mentioned example reveals that the heterogeneity of agent dynamics will enhance the R-controllability of the DMASs in the case that the network topology is uncontrollable, the controllability of the network topology is no longer necessary for the R-controllability of the heterogeneous DMASs. However, to guarantee R-controllability of the whole system, it is still necessary for the network topology to be leader-follower connected.

Definition 8. A directed signed graph $\mathcal{G}$ is leader-follower connected if for each $i \in \mathcal{V}_\mathrm{F}$, there exists $j \in \mathcal{V}_\mathrm{L}$ so that there is a path from j to i in $\mathcal{G}$.

Theorem 3. System (20) is R-controllable only if the network topology is leader-follower connected.

Proof. By contradiction, suppose that there exists $i \in \mathcal{V}_\mathrm{F}$ which has not a path from $j \in \mathcal{V}_\mathrm{L}$ to i in $\mathcal{G}$, that is, there are some agents having interconnections among them without external control and interconnections from other agents. Then, there exists a permutation matrix P, such that

$$P^\mathrm{T}LP = \begin{bmatrix}L_{11} & \mathbf{0}\\ L_{21} & L_{22}\end{bmatrix}, P^\mathrm{T}\Omega = \begin{bmatrix}\mathbf{0}\\ \Omega_2\end{bmatrix},$$

where $\Omega = \mathrm{diag}(\delta_1, \ldots, \delta_N)$. Construct $\bar{P}$ by replacing each nonzero element in P with I_n, and replacing zero elements with all-zero matrices of appropriate dimensions. Then, one has

$$\bar{P}^\mathrm{T}\Upsilon\bar{P} = \begin{bmatrix}\Upsilon_{11} & \mathbf{0}\\ \Upsilon_{21} & \Upsilon_{22}\end{bmatrix}, \bar{P}^\mathrm{T}\Theta\overline{P} = \begin{bmatrix}\Theta_1 & \\ & \Theta_2\end{bmatrix}, \bar{P}^\mathrm{T}\Sigma = \begin{bmatrix}\mathbf{0}\\ \Sigma_2\end{bmatrix}.$$

Assume that v is a left eigenvector of $(\Theta_1, \Upsilon_{11})$ corresponding to an eigenvalue λ, that is, $v\Upsilon_{11} = \lambda v\Theta_1$. Construct a new vector $\bar{v} = [v \ \mathbf{0}]$. Then one has

$$\bar{v}\begin{bmatrix}\Upsilon_{11} & \mathbf{0}\\ \Upsilon_{21} & \Upsilon_{22}\end{bmatrix} = \lambda[v \ \mathbf{0}]\begin{bmatrix}\Theta_1 & \\ & \Theta_2\end{bmatrix} = \lambda\bar{v}\begin{bmatrix}\Theta_1 & \\ & \Theta_2\end{bmatrix}, \bar{v}\begin{bmatrix}\mathbf{0}\\ \Sigma_2\end{bmatrix} = \mathbf{0}.$$

Therefore, system (20) is R-uncontrollable. This is a contradiction.

In what follows, a necessary and sufficient condition is presented for the R-controllability of system (20).

Theorem 4. System (20) is R-controllable if and only if the only solution to

$$\begin{cases} t_i\left(sE_i - A_i + l_{ii}C_iK_i\right) + \left(\sum_{j=1,j\neq i}^{N} l_{ji}t_j\right)C_jK_i = 0, \\ \delta_i t_i B_i = 0, \quad i = 1,\ldots,N \end{cases} \tag{21}$$

is $t_i = 0$ for all $s \in \mathbb{C}$ and $i \in \mathbb{Z}^N$.

Proof. According to Lemma 1, system (20) is R-controllable if and only if

$$\begin{cases} t\Upsilon = st\Theta, \\ t\Sigma = 0 \end{cases} \tag{22}$$

has a unique solution $t = 0$ for all $s \in \mathbb{C}$. Let $t = [t_1,\ldots,t_N] \in \mathbb{C}^{1\times Nn}$ with $t_i = [t_{i1},\ldots,t_{in}] \in \mathbb{C}^{1\times n}$. Then, we have

$$\begin{cases} [t_1,\ldots,t_N]\left(s\Theta - \Lambda + \Gamma\right) = 0, \\ [t_1,\ldots,t_N]\,\Sigma = 0, \end{cases} \tag{23}$$

which is equivalent to (21).

Theorem 4 only needs to check for every individual subsystem, which makes it easier to judge the R-controllability of (20). Clearly, the condition in (21) is also efficient for DMASs with homogeneous dynamics. Hereafter, several R-controllability conditions can be derived for special cases.

Theorem 5. If L is nondiagonalizable with Jordan canonical form $J = \mathrm{diag}(J_1,\ldots,J_s)$, where $J_i = \lambda_i I_{a_i} + N_{a_i} \in \mathbb{C}^{a_i\times a_i}$, λ_i is its eigenvalue with corresponding left eigenvector h_i^1. Let $h_i^1,\ldots,h_i^{a_i}$ be the left Jordan chain of L corresponding to h_i^1 for all $i \in \mathbb{Z}^s$. Assume that $E_i = E$, $C_iK_j = CK$, and $A_i - \lambda CK = A - \lambda CK$ for all $\lambda \in \sigma(L)$, $i,j \in Z^N$. System (20) is R-controllable if and only if the following holds:

(i) $\nu\Sigma \neq 0$ for all $\nu \in \mathcal{U}_{ij}\backslash\{\mathbf{0}\}$, $j \in \mathbb{Z}^{q_i}$, $i \in \mathbb{Z}^s$, where $\mathcal{U}_{ij} = \mathrm{span}\{\nu_{ijl}^r \mid \nu_{ijl}^r = \sum_{t=1}^{r} h_i^{r-t+1} \otimes \beta_{ijl}^t,\ l \in \mathbb{Z}^{\eta_{ij}},\ r \in Z^{\theta_{ijl}}\}$, $\theta_{ijl} = \min\{a_i, u_{ijl}\}$;

(ii) if $\omega_{i_1}^{j_1} = \cdots = \omega_{i_t}^{j_t} = \mu$, where $\omega_{i_k}^{j_k} \in \sigma(E, A - \lambda_{i_k}CK)$, and $\mu \notin \sigma(E, A - \lambda_r CK)$ for all $r \in \mathbb{Z}^s\backslash\{i_1,\ldots,i_t\}$, then $\nu\Sigma \neq 0$ for all $\nu \in \odot_{k=1}^{t}\mathcal{U}_{i_kj_k}\backslash\{\mathbf{0}\}$, where β_{ijl}^t, u_{ijl}, $\omega_{i_k}^{j_k}$ are defined as in Theorem 2.

Proof. Let $H = [h_1^{a_1\mathrm{T}},\ldots,h_1^{1\mathrm{T}},\ldots,h_s^{a_s\mathrm{T}},\ldots,h_s^{1\mathrm{T}}]^{\mathrm{T}}$ be a nonsingular matrix satisfying $HLH^{-1} = J = \mathrm{diag}(J_1,\ldots,J_s)$, where $J_i = \lambda_i I_{a_i} + N_{a_i} \in \mathbb{C}^{a_i\times a_i}$, λ_i is its eigenvalue with corresponding left eigenvector h_i^1, and $h_i^1,\ldots,h_i^{a_i}$ is the left Jordan chain of L corresponding to h_i^1, $h_i^{a_i} = [h_{i1}^{a_i},\ldots,h_{iN}^{a_i}]$. Then we have

$$(H\otimes I_n)\Upsilon = \left[\overline{\Upsilon}_1^{\mathrm{T}},\ldots,\overline{\Upsilon}_s^{\mathrm{T}}\right]^{\mathrm{T}},$$

where

$$\overline{\Upsilon}_i = \begin{bmatrix} h_{i1}^{a_i}(A_1 - \lambda_i CK) - h_{i1}^{a_i-1}CK & \cdots & h_{iN}^{a_i}(A_N - \lambda_i CK) - h_{iN}^{a_i-1}CK \\ \vdots & & \vdots \\ h_{i1}^{2}(A_1 - \lambda_i CK) - h_{i1}^{1}CK & \cdots & h_{iN}^{2}(A_N - \lambda_i CK) - h_{iN}^{1}CK \\ h_{i1}^{1}(A_1 - \lambda_i CK) & \cdots & h_{iN}^{1}(A_N - \lambda_i CK) \end{bmatrix} \in \mathbb{C}^{na_i\times Nn}.$$

Since $C_iK_j = CK$ and $A_i - \lambda CK = A - \lambda CK$ for all $\lambda \in \sigma(L)$, one has

$$\begin{aligned} &(H\otimes I_n)\Upsilon(H^{-1}\otimes I_n) \\ &\quad = \mathrm{diag}\left(I_{a_1}\otimes A - J_1\otimes CK,\ldots,I_{a_s}\otimes A - J_s\otimes CK\right) \\ &\quad = \mathrm{diag}\left(I_{a_1}\otimes(A - \lambda_1 CK) - N_{a_1}\otimes CK,\ldots,I_{a_s}\otimes(A - \lambda_s CK) - N_{a_s}\otimes CK\right). \end{aligned}$$

The following proof is similar to Theorem 1 and hence omitted.

Theorem 6. If L is diagonalizable, $E_i = E$, $C_i = C$, and $A_i - \lambda CK_i = A - \lambda CK$ for all $\lambda \in \sigma(L)$, $i \in Z^N$, system (20) is R-controllable if and only if the following holds:

(i) $\left(p_i \otimes \gamma_{ij}^l\right) \Sigma \neq 0$ for all $l \in \mathbb{Z}^{\eta_{ij}},\ j \in \mathbb{Z}^{q_i},\ i \in \mathbb{Z}^N$;

(ii) if $\omega_{i_1}^{j_1} = \cdots = \omega_{i_t}^{j_t} = \mu$, where $\omega_{i_k}^{j_k} \in \sigma(E, A - \lambda_{i_k} CK)$, and $\mu \notin \sigma(E, A - \lambda_r CK)$ for all $r \in \mathbb{Z}^N \backslash \{i_1, \dots, i_t\}$, then

$$\left[\sum_{k=1}^{t} \sum_{l=1}^{\eta_{i_k j_k}} \alpha_{kl} \left(p_{i_k} \otimes \gamma_{i_k j_k}^l\right)\right] \Sigma \neq 0,$$

where $\alpha_{kl} \in \mathbb{R}$ are not all zero for all $k \in Z^t$, $l \in Z^{\eta_{i_k j_k}}$, p_i, γ_{ij}^l, ω_i^j are defined as in Theorem 1.

Proof. Let $P = \left[p_1^{\mathrm{T}}, \dots, p_N^{\mathrm{T}}\right]^{\mathrm{T}}$ be a nonsingular matrix satisfying $PLP^{-1} = Q$, where $p_i = [p_{i1}, \dots, p_{iN}]$, $Q = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$. Then we have

$$(P \otimes I_n)\Upsilon = \begin{bmatrix} p_{11}(A_1 - \lambda_1 CK_1) & \cdots & p_{1N}(A_N - \lambda_1 CK_N) \\ \vdots & & \vdots \\ p_{N1}(A_1 - \lambda_N CK_1) & \cdots & p_{NN}(A_N - \lambda_N CK_N) \end{bmatrix}.$$

Since $A_1 - \lambda CK_1 = \cdots = A_N - \lambda CK_N$ for all $\lambda \in \sigma(L)$, one has $(P \otimes I_n)\Upsilon(P^{-1} \otimes I_n) = \mathrm{diag}(A - \lambda_1 CK, \dots, A - \lambda_N CK)$.

The following proof is similar to Theorem 2 and hence omitted.

Theorems 5 and 6 suggest only in some specific cases. We can clearly and completely characterize the basis for the eigenspace of (Θ, Υ), which means that in heterogeneous case, some properties of a few matrices with small dimensions can also be verified to provide precise and efficient criteria for determining the controllability of large-scale DMASs. The characterization for the general case remains a topic for further research. In addition, it is noted that only in these special cases, the controllability of (L, Ω) is necessary to ensure the R-controllability of heterogeneous DMASs.

Consider the case of the communication between agents is performed through one-dimensional connections, i.e., the input matrix is $B_i \in \mathbb{R}^{n\times 1}$, the inner coupling matrix is $C_i \in \mathbb{R}^{n\times 1}$, and the feedback gain matrix is $K_i \in \mathbb{R}^{1\times n}$. The following theorem provides necessary conditions for the R-controllability of system (20). Before proceeding, we propose the following notations.

Let

$$\mathcal{U} = \{i \mid \delta_i \neq 0, i = 1, \dots, N\}$$

denote the index set of the controlled agent.

Assume that $\sigma(E_1, A_1) = \cdots = \sigma(E_N, A_N)$, and for any $s \in \sigma(E_i, A_i)$, define a matrix set

$$\mathcal{P}(s) = \left\{ \left[\eta_1^{\mathrm{T}}, \dots, \eta_N^{\mathrm{T}}\right] \,\middle|\, \begin{matrix} \eta_i \in \mathcal{P}_1(s), i \notin \mathcal{U} \\ \eta_i \in \mathcal{P}_2(s), i \in \mathcal{U} \end{matrix} \right\},$$

where $\mathcal{P}_1(s) = \left\{v_i \in \mathbb{C}^{1\times n} \mid v_i(sE_i - A_i) = 0\right\}$ and $\mathcal{P}_2(s) = \left\{v_i \in \mathbb{C}^{1\times n} \mid v_i B_i = 0, v_i \in \mathcal{P}_1(s)\right\}$.

Theorem 7. Suppose that $\sigma(E_1, A_1) = \cdots = \sigma(E_N, A_N)$, $|\mathcal{U}| < N$, $B_i \in \mathbb{R}^{n\times 1}$, $C_i = C \in \mathbb{R}^{n\times 1}$, $K_i = K \in \mathbb{R}^{1\times n}$. Then system (20) is R-controllable only if the following holds:

(i) for any $s \in \sigma(E_i, A_i)$ and $\alpha \in \mathcal{P}(s), \alpha L \neq 0$ if $\alpha \neq 0$;

(ii) for any $s \notin \sigma(E_i, A_i)$, $\mathrm{rank}(I_N + WL, M) = N$, where $W = \mathrm{diag}\{\gamma_1, \dots, \gamma_N\}$, $M = \mathrm{diag}\{\theta_1, \dots, \theta_N\}$, $\gamma_i = K(sE_i - A_i)^{-1} C$, and $\theta_i = \delta_i K(sE_i - A_i)^{-1} B_i$.

Proof. Suppose that condition (i) is not necessary. Then, there exist an $s_0 \in \sigma(E_i, A_i)$ and a nonzero matrix $\alpha \in \mathcal{P}(s_0)$ such that $\alpha L = 0$ with $\alpha = \left[\alpha_1^{\mathrm{T}}, \dots, \alpha_N^{\mathrm{T}}\right] \in \mathbb{C}^{n\times N}$ and $\alpha_i \in \mathbb{C}^{1\times n}$. Let $\bar{\alpha} = [\alpha_1, \dots, \alpha_N]$, then $\bar{\alpha}\Sigma = 0$ and

$$\bar{\alpha}(s_0\Theta - \Upsilon) = \bar{\alpha}(s_0\Theta - \Lambda + \Gamma) = \left[\left(\sum_{j=1}^{N} l_{j1}\alpha_j\right) CK, \dots, \left(\sum_{j=1}^{N} l_{jN}\alpha_j\right) CK\right] = 0,$$

which contradicts the R-controllability of the system.

Suppose that condition (ii) is not necessary, there exists an $s_0 \notin \sigma(E_i, A_i)$, satisfying rank$(I_N + W_0L, M_0) < N$, where $W_0 = \mathrm{diag}\{\gamma_{10}, \ldots, \gamma_{N0}\}, M = \mathrm{diag}\{\theta_{10}, \ldots, \theta_{N0}\}$, $\gamma_{i0} = K(s_0E_i - A_i)^{-1}C$, and $\theta_{i0} = \delta_i K(s_0E_i - A_i)^{-1}B_i$.

Thus, there exists a nonzero vector $v = [v_1, \ldots, v_N] \in \mathbb{C}^{1\times N}$ such that

$$v\left(I_N + W_0L\right) = 0 \quad \text{and} \quad vM_0 = 0.$$

Let $\xi = [\xi_1, \ldots, \xi_N]$ with $\xi_i = v_iK\left(s_0E_i - A_i\right)^{-1}$. By $v \neq 0$, one has $\xi \neq 0$. Then one has

$$\xi\Sigma = \mathrm{diag}(\delta_1 v_1 K\left(s_0E_1 - A_1\right)^{-1} B_1, \ldots, \delta_N v_N K\left(s_0E_N - A_N\right)^{-1} B_N) = vM_0 = 0$$

and

$$\begin{aligned}
&\xi\left(s_0\Theta - \Upsilon\right)\\
&= (v_1K\left(s_0E_1 - A_1\right)^{-1}, \ldots, v_NK\left(s_0E_N - A_N\right)^{-1}) \times (\mathrm{diag}\left(s_0E_1 - A_1, \ldots, s_0E_N - A_N\right) + L \otimes CK)\\
&= v \otimes K + ((v_1, \ldots, v_N)\,\mathrm{diag}(K\left(s_0E_1 - A_1\right)^{-1} C, \ldots, K\left(s_0E_N - A_N\right)^{-1} C)L) \otimes K\\
&= v \otimes K + ((v_1, \ldots, v_N)\,\mathrm{diag}\left(\gamma_{10}, \ldots, \gamma_{N0}\right) L) \otimes K\\
&= v\left(I_N + W_0L\right) \otimes K = 0,
\end{aligned}$$

which contradicts the R-controllability of the system.

Theorem 8. Suppose that $\sigma(E_1, A_1) = \cdots = \sigma(E_N, A_N)$, $|\mathcal{U}| < N$, $B_i \in \mathbb{R}^{n\times 1}$, $C_i \in \mathbb{R}^{n\times 1}$, $K_i \in \mathbb{R}^{1\times n}$. Then system (20) is R-controllable if the following holds:

(i) (E_i, A_i, C_i) is R-controllable for all $i \in \mathbb{Z}^N$;

(ii) (E_i, A_i, K_i) is R-observable for all $i \in \mathbb{Z}^N$;

(iii) for any $s \in \sigma(E_i, A_i)$ and $\alpha \in \mathcal{P}(s)$, $\alpha L \neq 0$ if $\alpha \neq 0$;

(iv) for any $s \notin \sigma(E_i, A_i)$, rank$(I_N + WL, M) = N$, where $W = \mathrm{diag}\{\gamma_1, \ldots, \gamma_N\}$, $M = \mathrm{diag}\{\theta_1, \ldots, \theta_N\}$, $\gamma_i = K_i\left(sE_i - A_i\right)^{-1} C_i$, and $\theta_i = \delta_i K_i\left(sE_i - A_i\right)^{-1} B_i$.

Proof. For $s \in \mathbb{C}$, suppose that there exists a vector $\xi = [\xi_1, \ldots, \xi_N], \xi_i \in \mathbb{C}^{1\times n}$ such that $\xi(s\Theta - \Upsilon) = 0$ and $\xi\Sigma = 0$, which implies

$$\xi_i\left(sE_i - A_i\right) + \sum_{j=1}^{N} l_{ji}\xi_j C_j K_i = 0, i = 1, \ldots, N \tag{24}$$

and

$$\xi_i B_i = 0, i \in \mathcal{U}. \tag{25}$$

If $s \in \sigma(E_i, A_i)$ then rank$\left(sE_i - A_i\right) < n$ for all $i \in \mathbb{Z}^N$, thus

$$\sum_{j=1}^{N} l_{ji}\xi_j C_j = 0 \tag{26}$$

for all $i \in \mathbb{Z}^N$. Otherwise, K_i can be linearly represented by the row vectors of $sE_i - A_i$, therefore

$$\mathrm{rank}\begin{bmatrix} sE_i - A_i \\ K_i \end{bmatrix} = \mathrm{rank}\left(sE_i - A_i\right) < n,$$

which contradicts the R-observability of (A_i, K_i). Substitute (26) into (24), then

$$\xi_i\left(sE_i - A_i\right) = 0, i = 1, \ldots, N. \tag{27}$$

Therefore,

$$\sum_{j=1}^{N} l_{ji}\xi_j\left(sE_j - A_j\right) = 0, i = 1, \ldots, N. \tag{28}$$

In light of (26) and condition (i), it follows that

$$\sum_{j=1}^{N} l_{ji}\xi_j = 0, i = 1, \ldots, N. \tag{29}$$

Let $\alpha = \left[\xi_1^{\mathrm{T}}, \ldots, \xi_N^{\mathrm{T}}\right]$. From (25) and (27), it can be verified $\alpha \in \mathcal{P}(s)$. Eq. (29) is equivalent to $\alpha L = 0$. Therefore, by condition (iii), we have $\alpha = 0$.

If $s \notin \sigma\left(E_i, A_i\right)$, then $sE_i - A_i$ is invertible for all $i \in \mathbb{Z}^N$. It follows from (24) that

$$\xi_i = -\sum_{j=1}^{N} l_{ji}\xi_j C_j K_i \left(sE_i - A_i\right)^{-1}, i = 1, \ldots, N. \tag{30}$$

Let $v_i = \sum_{j=1}^{N} l_{ji}\xi_j C_j$. Then

$$\xi_i = -v_i K_i \left(sE_i - A_i\right)^{-1}, i = 1, \ldots, N \tag{31}$$

and

$$v_i = \sum_{j=1}^{N} l_{ji}\xi_j C_j = -\sum_{j=1}^{N} l_{ji} v_j K_j \left(sE_j - A_j\right)^{-1} C_j = -\sum_{j=1}^{N} l_{ji} v_j \gamma_j, \tag{32}$$

where $\gamma_i = K_i \left(sE_i - A_i\right)^{-1} C_i$. Let $v = [v_1, \ldots, v_N]$, and rewrite (32) as

$$v(I_N + WL) = 0. \tag{33}$$

Combining (25) and (31), it is easy to verify that $v_i K_i \left(sE_i - A_i\right)^{-1} B_i = 0$ for $i \in \mathcal{U}$, which is equivalent to $vM = 0$, where M is defined in (iv) of Theorem 8. From (33) and condition (iv), we have $v = 0$, thus $\xi_i = 0$ for all $i \in \mathbb{Z}^N$. Therefore, Eq. (20) is R-controllable.

Theorems 7 and 8 give the necessary and sufficient conditions for the controllability of the heterogeneous DMASs with one-dimensional communication respectively, which only need to be verified properties of subsystem and network topology. It is worth noting that the condition obtained in Theorem 7 is still a necessary condition for the homogeneous DMASs to be R-controllable, and the condition established in Theorem 8 is a necessary and sufficient condition to ensure the R-controllability of homogeneous DMASs. This observation reveals that some criteria derived for DMASs with homogeneous dynamics can be applicable to DMASs with heterogeneous dynamics only in some special cases.

5 Relationship between signed networks and unsigned networks

In this subsection, we investigate the relationship between the controllability of DMASs under signed networks and under unsigned networks. Specifically, we explore the influence of antagonistic interactions on controllability by studying the controllability of structurally balanced and structurally anti-balanced networks.

Definition 9 ([31])**.** Let $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}, W_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}, W_2)$ be two signed graphs with Laplacian matrices L_1 and L_2, respectively. We say $\mathcal{G}_1$ and $\mathcal{G}_2$ are switching equivalent, denoted as $\mathcal{G}_1 \sim \mathcal{G}_2$, if and only if L_1 and L_2 are signature similar, i.e., $L_2 = DL_1D, D \in \mathcal{D}$.

Lemma 4 ([10])**.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a signed graph; then it has $2^{|\mathcal{E}|-|\mathcal{V}|+1}$ switching equivalent classes.

Lemma 5 ([31])**.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$ be a signed graph; then $\mathcal{G} \sim \mathcal{G}_+$ if $\mathcal{G}$ is structurally balanced, and $\mathcal{G} \sim \mathcal{G}_-$ if $\mathcal{G}$ is structurally anti-balanced.

In the following, the relationship among the R-controllability of the DMASs under different signed networks is discussed. For brevity, we use the R-controllability of $\mathcal{G}$ to represent the R-controllability of system (7) with interaction network $\mathcal{G}$.

Proposition 3. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two interaction networks of system (7), L_1 and L_2 be, respectively, the Laplacian matrices of $\mathcal{G}_1$ and $\mathcal{G}_2$, if $\mathcal{G}_1 \sim \mathcal{G}_2$, that is, $L_1 = DL_2D, D \in \mathcal{D}$, then the following conditions hold:

(i) If L_1 is nondiagonalizable, $\sum_{t=1}^{r} h_i^{r-t+1} \otimes \beta_{ijl}^t$ with $r \in \mathbb{Z}^{\theta_{ijl}}$, and $l \in \mathbb{Z}^{\eta_{ij}}$ is a basis for the left eigenspace of (Ξ, Φ) with L_1 corresponding to the eigenvalue ω_i^j, then $\sum_{t=1}^{r} h_i^{r-t+1} D \otimes \beta_{ijl}^t$ with $r \in \mathbb{Z}^{\theta_{ijl}}$, $l \in \mathbb{Z}^{\eta_{ij}}$ is a basis for the left eigenspace of (Ξ, Φ) with L_2 corresponding to ω_i^j, where $\theta_{ijl} = \min\{a_i, u_{ijl}\}$, h_i^{r-t+1}, β_{ijl}^t are defined as in Proposition 1;

(ii) If L_1 is diagonalizable, and $p_i \otimes \gamma_{ij}^1, \ldots, p_i \otimes \gamma_{ij}^{\eta_{ij}}$ is a basis for the left eigenspace of (Ξ, Φ) with L_1 corresponding to the eigenvalue ω_i^j for all $j \in \mathbb{Z}^{q_i}, i \in \mathbb{Z}^N$, then $p_i D \otimes \gamma_{ij}^1, \ldots, p_i D \otimes \gamma_{ij}^{\eta_{ij}}$ is a basis for the left eigenspace of (Ξ, Φ) with L_2 corresponding to ω_i^j for all $j \in \mathbb{Z}^{q_i}, i \in \mathbb{Z}^N$, where p_i, γ_{ij} are defined as in Proposition 2.

Proof. (i) If L_1 is nondiagonalizable, then there exists a nonsingular matrix H such that $HL_1H^{-1} = J$, where J is the Jordan canonical form of L_1. Then

$$HDL_2DH^{-1} = J \quad \Rightarrow \quad HDL_2(HD)^{-1} = J.$$

That is, h_i is the eigenvector of L_1 if and only if h_iD is the eigenvector of L_2. The following proof is similar to Proposition 1.

(ii) The analysis is similar to condition (i).

Theorem 9. For system (7) with $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}, W)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}, W)$, if $\mathcal{G}_1 \sim \mathcal{G}_2$, then the R-controllability of $\mathcal{G}_1$ is equivalent to the R-controllability of $\mathcal{G}_2$.

Proof. Suppose that L_1 and L_2 are, respectively, the Laplacian matrices of the interaction networks $\mathcal{G}_1$ and $\mathcal{G}_2$. For simplicity, here we only prove the case of diagonzaliable L_1, and the case of nondiagonzaliable L_1 is similar. Then condition (i) of Theorem 2 with L_2 is that (L_2, Ω) is controllable, which implies $p_iD\Omega \neq 0$ for all $i \in \mathbb{Z}^N$. Since D and Ω are diagonal matrix, then

$$p_iD\Omega \neq 0 \Leftrightarrow p_i\Omega D \neq 0 \Leftrightarrow p_i\Omega \neq 0 \quad \text{for all } i \in \mathbb{Z}^N,$$

which means that condition (i) is equivalent for L_1 and L_2. Since L_1 and L_2 have the same eigenvalues, condition (ii) remains the same. And condition (iii) of Theorem 2 with L_2 is $[\sum_{k=1}^{t} \sum_{l=1}^{\eta_{i_kj_k}} \alpha_{kl} (p_{i_k} D \otimes \gamma_{i_kj_k}^l)]\ (\Omega \otimes B) \neq 0$, which is equivalent to $[\sum_{k=1}^{t} \sum_{l=1}^{\eta_{i_kj_k}} \alpha_{kl} \left(p_{i_k} \otimes \gamma_{i_kj_k}^l\right)](\Omega \otimes B)(D \otimes I_q) \neq 0$, and also equivalent for L_1. Therefore, the R-controllability of $\mathcal{G}_1$ is equivalent to that of $\mathcal{G}_2$.

Remark 3. Combining Lemma 5 and the above results, we can prove that the R-controllability of $\mathcal{G}$ is equivalent to the R-controllability of $\mathcal{G}_+$ ($\mathcal{G}_-$) when $\mathcal{G}$ is structurally balanced (anti-balanced). Additionally, if $\mathcal{G}$ is a path, star or tree, $\mathcal{G}$ has only one switching equivalent class by Lemma 4, thus $\mathcal{G}$ is structurally balanced, then the R-controllability of $\mathcal{G}$ is equivalent to $\mathcal{G}_+$. As we see, positive cycle is also structurally balanced. From Lemma 4, cycles can be divided into two different switching equivalence classes: one contains the positive cycle and the other contains the negative cycle. Hence, the R-controllability of all the positive cycles is equivalent, and the R-controllability of all the negative cycles is also equivalent.

In Theorem 9, we find that the R-controllability of DMASs with some special networks is not affected by edge sign, which implies that we can use the associated unsigned networks to study the R-controllability of DMASs. Particularly, for some special networks such as paths, stars, and trees, whether the interactions between agents are cooperative or antagonistic is often unknown precisely. Thus, we investigate the R-controllability of $\mathcal{G}$ only by analyzing the R-controllability of $\mathcal{G}_+$. In addition, the property of these special networks can be used to effectively identify the R-uncontrollability of the whole system sometimes, which is demonstrated by the following example.

Example 4. Considering system (7) with star network given in Figure 1, the external control input applies to the central agent. Due to interactions among agents uncertainty, the edge sign might be hard to know, while the absolute value of each edge weight is known to be equal. However, this does not affect our judgment of the R-controllability of the system. From Theorem 9, we can use the associated unsigned star network to study the R-controllability of DMASs. As we all know, (L, Ω) corresponding to an unsigned star network with the center agent as the single leader is controllable if and only if each edge has a different weight [33], thus (L, Ω) is uncontrollable, and the DMASs is R-uncontrollable by Theorem 2. In this case, the R-uncontrollability of the whole system can be diagnosed even if the agent dynamics and the interactions among agents are unknown.

The following theorem provides the relationship between the R-controllability of different signed graph $\mathcal{G}$ for the case of heterogeneous dynamics.

Theorem 10. Let $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}, W_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}, W_2)$ be two interaction networks of system (20). If $\mathcal{G}_1 \sim \mathcal{G}_2$, then the R-controllability of $\mathcal{G}_1$ is equivalent to that of $\mathcal{G}_2$.

Proof. Suppose that L_1 and L_2 are, respectively, the Laplacian matrices of the interaction networks $\mathcal{G}_1$ and $\mathcal{G}_2$. If $\mathcal{G}_1 \sim \mathcal{G}_2$, then $\exists D \in \mathcal{D}$ such that $L_2 = DL_1D$. Then, we have

$$\begin{aligned}
&\operatorname{rank}\left[s\Theta_1 - \Upsilon_1 \quad \Sigma_1\right] \\
&= \operatorname{rank}\left[s\Theta_1 - \Lambda_1 + \Gamma_1 \quad \Sigma_1\right] \\
&= \operatorname{rank}[D \otimes I_n]\left[s\Theta_1 - \Lambda_1 + \Gamma_1 \quad \Sigma_1\right] \begin{bmatrix} D \otimes I_n & \\ & D \otimes I_q \end{bmatrix} \\
&= \operatorname{rank}\left[(D \otimes I_n)(s\Theta_1 - \Lambda_1 + \Gamma_1)(D \otimes I_n) \quad (D \otimes I_n)\Sigma_1(D \otimes I_q)\right] \\
&= \operatorname{rank}\left[s\Theta_1 - \Lambda_1 + \overline{\Gamma} \quad \Sigma_1\right] \\
&= \operatorname{rank}\left[s\Theta_2 - \Upsilon_2 \quad \Sigma_2\right],
\end{aligned}$$

where $\Upsilon_2 = \Lambda_1 - \overline{\Gamma}$, $\overline{\Gamma_{ij}} = \overline{l_{ij}} C_i K_j$ with $L_2 = [\overline{l_{ij}}]$. Therefore, the R-controllability of $\mathcal{G}_1$ is equivalent to that of $\mathcal{G}_2$.

Remark 4. For the case of heterogeneous dynamics, even if we cannot characterize the relationship of eigenvectors under different signed graphs, we can also obtain the above results through switching equivalence. It is not difficult to find whether system dynamics is homogeneous or heterogeneous, controllability of system under structurally balanced network is not affected by the sign of interconnections between agents.

6 Applications in multi-agent supporting systems

A multi-agent supporting system (MASS) is a supporting system, whose configuration is formed by separate agents and it has potential applications in earthquake damage prevention in buildings, water-floating plants, and large-diameter parabolic antennae or telescopes [34]. When an MASS consists of many independent blocks and each block is supported by several pillars, each agent in this MASS becomes a singular system [27]. Consider an MASS of three agents with a directed path network. Consider the case when each agent is supported by two pillars called Unit I and Unit II, respectively, as shown in [27], where $\bar{m}$ is the mass, $\bar{d}$ is the damping coefficient, and $\bar{k}$ is the stiffness coefficient. The dynamics of each agent is described by (1), where

$$x_i = \begin{bmatrix} h_{i1} \\ v_{i1} \\ h_{i2} \\ v_{i2} \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{\bar{k}}{\bar{m}} & -\frac{\bar{d}}{\bar{m}} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

$B^{\mathrm{T}} = [0, 1, 0, 0]$, h_{i1}, v_{i1}, h_{i2}, and v_{i2} represent the heights and velocities of Unit I and Unit II, respectively. Without loss of generality, let the parameters of each agent be $\bar{m} = \bar{d} = \bar{k} = 10$. The network topology among the agents is given as in Figure 2, where positive (negative) edges represent that the direction of the force of the agents is consistent (opposite), thus the interaction between the agents is cooperative (antagonistic). Agent 1 is subjected to external inputs, and the corresponding matrix L is $[0, 0, 0; -1, 1, 0; 0, -1, 1]$. Let $C^{\mathrm{T}} = [0, -1, -1, 0]$, $K = [1, 0, 0, 0]$. For an earthquake damage-preventing building, when an earthquake occurs, each agent in this MASS tries to keep the building horizontal, which means that the height and velocity of each agent should reach the desired states. Obviously the system matrix satisfies the condition of Theorem 1. By calculating, we obtain $\operatorname{rank}[s\Xi - \Phi \quad \Psi] = 12$ for all $s \in \mathbb{C}$, $|s| < \infty$. This means that system (7) is R-controllable, which is consistent with Theorem 1. The trajectories of agents are depicted in Figure 3[2)], which implies that we can effectively control the

2) It follows from the standard decomposition of descriptor linear systems that there exist two nonsingular matrices Q and P such that under the transformation (P, Q) system (1) is decomposed into the slow subsystem $\dot{x_1}(t) = A_1x_1(t) + B_1u(t)$ and fast subsystem $N\dot{x_2}(t) = x_2(t) + B_2u(t)$, where $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, the matrix $N \in \mathbb{R}^{n_2 \times n_2}$ is nilpotent. System (1) is R-controllable if and only if the slow subsystem is controllable. Therefore, the simulation diagram only gives the state trajectory of the slow subsystem, where $n_1 = 6$.

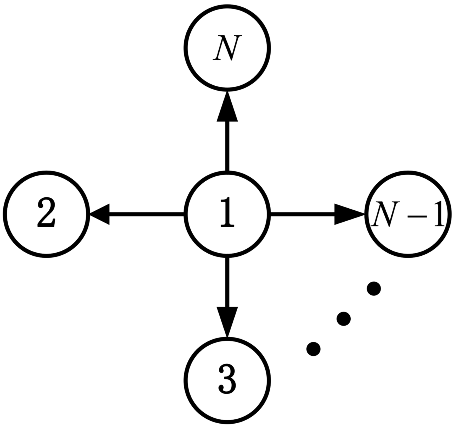

Figure 1 Directed star network with $N > 2$.

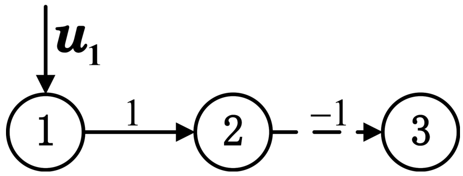

Figure 2 Interaction topologies of MASS. The solid and dashed lines represent the positive and negative edges, respectively.

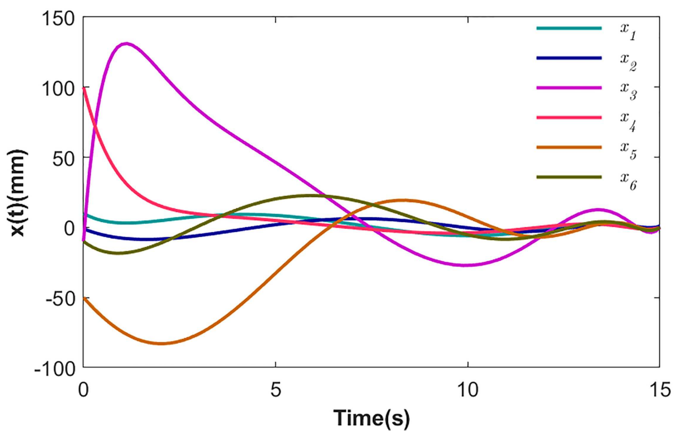

Figure 3 (Color online) State trajectories of agents from $x_0 = [10; -1; -10; 100; -50; -10]^{\mathrm{T}}$ at time 0 to $x_1 = [0; 0; 0; 0; 0; 0]^{\mathrm{T}}$ at time 15.

MASS through external control input to keep the building stable after the earthquake.

7 Conclusion

In this paper, we investigated the controllability of DMASs with directed signed networks, where the DMASs consist of descriptor linear dynamical subsystems. We completely characterized a set of basis for the eigenspace of the system matrix, which is related to our definition of the maximum generalized left Jordan chain. Based on this result, we derived several necessary and sufficient conditions for the controllability of DMASs in the cases of nondiagonalizable and diagonalizable topological matrices. Moreover, for the DMASs with heterogeneous dynamics, we established some controllability conditions of general and special cases. Furthermore, we presented the relationship between the controllability of DMASs with signed networks and with unsigned networks. It is shown that the controllability of DMASs with homogeneous (heterogeneous) dynamics under structurally balanced networks is equivalent to that under unsigned networks whose edge weights are all nonnegative. Theoretical results were used to deal with controllability problems of multi-agent supporting systems. Our results can be further extended to more realistic communication networks such as time-varying or switching topologies.

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