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A zero-sum hybrid stochastic differential game with impulse controls

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Abstract In this paper, we study a zero-sum stochastic differential game with the following salient features: (i) the system state is dictated by a hybrid diffusion, (ii) both players use impulse controls, and (iii) the game takes place on an infinite time horizon. First, the dynamic programming principle for the problem is proven. Then, the lower and upper value functions of the game are characterized as the unique viscosity solution of the associated Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, which turns out to be a coupled system of variational inequalities with bilateral obstacles. Moreover, a verification theorem as a sufficient condition to identify a Nash equilibrium is established. The Nash equilibrium strategies for the two players, indicating when and how it is optimal to intervene, are given in terms of the obstacle part of the HJBI equation.

Keywords stochastic differential game, Markov chain, impulse control, HJBI equation, viscosity solution, verification theorem

1 Introduction

A zero-sum differential game is the problem where there are two players to control and influence the continuous-time dynamic system by making decisions non-cooperatively to achieve a Nash equilibrium. Research on the zero-sum deterministic differential game can be traced back to the Isaacs's pioneering work [1]. Later, Elliott and Kalton [2] introduced the concept of strategies for the two players and the definitions of lower and upper value functions for the game, based on which, Evans and Souganidis [3] proved that the two value functions are unique viscosity solutions to the associated Hamilton-Jacobi-Isaacs equations. Extension to the stochastic case was first carried out by Fleming and Souganidis [4] also by the viscosity solution theory. Then, with the joint effort of many researchers, now there have been enormously rich results on stochastic differential games; see, Tang and Hou [5] (switching control problem), Buckdahn and Li [6] (with recursive functional), Biswas [7] (driven by jump diffusion), Yu [8] (linear quadratic problem), and Tian et al. [9] (mean-field case).

A conventional assumption in the standard stochastic control theory is that control variables can be exerted with no direct costs, and thereby the controller can apply a control policy freely and continuously. However, it is frequently encountered in practical scenes that the control action causes a notable direct cost. Impulse control theory, instead, takes the non-negligible action costs into consideration and provides a natural formulation when the control and state processes are discontinuous. Typically, an impulse control is a sequence of pairs of action times and action magnitudes, and an optimal impulse control problem was often studied by the method of variational inequality, e.g., [10–22]. On the other hand, impulse control is commonly used in problems such as inventory and cash management (Constantinides and Richard [23]), exchange rate adjustment (Cadenillas and Zapatero [24]), dividend and reinsurance optimization (Wei et al. [25]), consumption utility maximization (Wu and Zhang [26]), and investment fund operation (Chang and Wu [27]).

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The hybrid diffusion, stemming from the need of more realistic models that better reflect a random environment, is a two-component process $(X(t), \theta(t))$ in which the first component $X(t)$ evolves according to a continuous diffusion process whose drift and diffusion coefficients depend on the regime of $\theta(t)$, where $\theta(t)$ is generally assumed to be a finite-state Markov chain. In recent years, the hybrid diffusion has received growing interest from the stochastic control community; see, Donnelly [28] and Lv and Wu [29] (maximum principle), Zhang et al. [30, 31] (dynamic programming), and Li and Zhou [32] and Li et al. [33] (linear quadratic problem). However, the study of game problems is relatively rare in the literature, especially for the case with impulse controls. In addition, the hybrid diffusion also has been employed extensively in various practical fields; we refer the readers to Zhou and Yin [34] for portfolio selection, Yao et al. [35] and Guo and Zhang [36] for European and American option pricing, respectively, Zhang and Zhou [37] for valuation of stock loans, Song et al. [38] for optimal harvesting, Zhu [39] for risk control, and the monographs by Yin and Zhu [40] and Yin and Zhang [41].

In this paper, we consider a zero-sum stochastic differential game with impulse controls under a hybrid diffusion model. Such kind of game problem, as described above, enjoys a wide applications in operations research and financial mathematics. We will follow the method of variational inequality and adopt the concept of strategies to study our zero-sum hybrid stochastic differential game with impulse controls. First, we state the corresponding dynamic programming principle (DPP) for the problem and provide a detailed proof, which is non-trivial with some technical issues to be overcome. In particular, a certain strategy (see (13)) is delicately designed to link the actions before and after the stopping time ν in the DPP via a time shifting approach (see (16)). Moreover, in contrast to Tang and Yong [11, Theorem 4.1] and Cosso [15, Lemma 4.3], where the stopping time ν in the DPP is restricted to taking at most countably many values, we further use an approximation approach to generalize the result to the situation where ν can take uncountably many values.

Then, based on the DPP, the lower and upper value functions of the game are proven to be the unique viscosity solution of the associated Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation (21). Note that the regime switching of the Markov chain leads to a coupling term $QW(x,\cdot)(i)$ (defined by (18)) in (21), so the current HJBI equation no longer satisfies the fundamental monotonicity condition in the user's guide by Crandall et al. [42]. This adds many mathematical challenges to the viscosity solution arguments. For example, in the proof of existence, a crucial auxiliary function $\psi(x, j)$ (defined by (28)) needs to be introduced, which serves as an augment of the test function $\varphi(x)$ by taking the Markov chain into account. Besides, the first jump time of the Markov chain and the first exit time of the state process from a neighborhood should be carefully discussed. On the other hand, in the proof of uniqueness, we also have to take efforts to deal with the coupling term $QW(x,\cdot)(i)$.

For a complete treatment of the problem, a verification theorem as a sufficient condition for Nash equilibriums is established. In this paper, the verification theorem is proven in a segment-by-segment way along the sequence of entire action times of the two players, where a segment means a period between two successive action times. Within each segment, we can apply Itô's formula and the continuation part of the HJBI equation works. If the intervention occurs, then the obstacle part of the HJBI equation takes over the game. Based on the obstacle part of the HJBI equation, we construct a pair of impulse controls for the two players, which is shown to be a Nash equilibrium. In the meantime, the solution of the HJBI equation turns out to be the value of the game.

The rest of this paper is organized as follows. Section 2 formulates the problem and gives some preliminary results. Section 3 states and proves the DPP. Section 4 is devoted to the viscosity solution characterization of the HJBI equation. Section 5 establishes a verification theorem for Nash equilibriums. Finally, Section 6 concludes the paper with some further remarks.

2 Problem formulation and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which a one-dimensional Brownian motion $B(t)$, $t \geqslant 0$, and a Markov chain $\theta(t)$, $t \geqslant 0$, are defined. Assume that $B(\cdot)$ and $\theta(\cdot)$ are independent. The Markov chain takes values in a finite state space $\mathcal{M} = \{1, \dots, M\}$. Let $Q = (q_{ij})_{i,j\in\mathcal{M}}$ denote the generator (i.e., the matrix of transition rates) of $\theta(\cdot)$ with $q_{ij} \geqslant 0$ for $i \neq j$ and $\sum_{j\in\mathcal{M}} q_{ij} = 0$ for each $i \in \mathcal{M}$. Let $\{\mathcal{F}_t\}_{t\geqslant 0}$ be the natural filtration of $B(\cdot)$ and $\theta(\cdot)$ augmented by all the null sets.

Let players I and II denote the two players in the game. In the following, we first present the definitions of admissible impulse controls for the two players.

Definition 1. An admissible impulse control for player I is a sequence of pairs $(\tau_m, \xi_m)_{m\geqslant 1}$, where $(\tau_m)_{m\geqslant 1}$ is an increasing sequence of stopping times with $\tau_m \to \infty$ as $m \to \infty$, representing the decisions on "when to intervene," and $(\xi_m)_{m\geqslant 1}$ is a sequence of $\mathcal{F}_{\tau_m}$-measurable random variables taking values in R, representing the decisions on "how to intervene." Let a process $u(t) \doteq \sum_{m\geqslant 1} \xi_m 1_{[\tau_m,\infty)}(t)$, $t \geqslant 0$ denote an admissible impulse control $(\tau_m, \xi_m)_{m\geqslant 1}$, where 1_A is the indicator function of a set A. The collection of all admissible impulse control processes $u(\cdot)$ for player I is defined as $\mathcal{U}$.

An admissible impulse control $(\rho_n, \eta_n)_{n\geqslant 1}$ for player II is defined similarly as a process $v(t) \doteq \sum_{n\geqslant 1} \eta_n 1_{[\rho_n,\infty)}(t)$, $t \geqslant 0$. The collection of all admissible impulse control processes $v(\cdot)$ for player II is defined as $\mathcal{V}$.

In our problem, the state equation is described by a hybrid diffusion $(X^{x,i}(\cdot), \theta^i(\cdot))$ with initial condition $(X^{x,i}(0-), \theta^i(0)) = (x, i) \in R \times \mathcal{M}$:

$$\begin{aligned} X^{x,i}(t) =& x + \int_0^t b(X^{x,i}(s), \theta^i(s))\mathrm{d}s + \int_0^t \sigma(X^{x,i}(s), \theta^i(s))\mathrm{d}B(s) \\ &+ \sum_{m\geqslant 1} \xi_m 1_{[\tau_m,\infty)}(t) \prod_{n\geqslant 1} 1_{\{\tau_m \neq \rho_n\}} + \sum_{n\geqslant 1} \eta_n 1_{[\rho_n,\infty)}(t), \quad t \geqslant 0, \end{aligned} \tag{1}$$

where $b, \sigma : R \times \mathcal{M} \mapsto R$, and $u(t) \doteq \sum_{m\geqslant 1} \xi_m 1_{[\tau_m,\infty)}(t)$ and $v(t) \doteq \sum_{n\geqslant 1} \eta_n 1_{[\rho_n,\infty)}(t)$ are admissible impulse control processes for players I and II, respectively. Note that (i) the sample paths of $X^{x,i}(\cdot)$ are right-continuous, and (ii) the infinite product in (1) means we take the convention that if both players want to intervene at the same time, only the action of player II will be taken into account.

Remark 1. We can also take the other convention that when the two players want to intervene at the same time, only the action of player I will be taken into account. In this case, the arguments are analogous to those presented in this paper; see Cosso [15] and Aïd et al. [18] for related discussion.

In a zero-sum game, besides the admissible impulse controls, we still need to define the following admissible impulse strategies for the two players; see also Cosso [15, Definition 2.7].

Definition 2. An admissible impulse strategy for player I is a mapping $\alpha[\cdot] : \mathcal{V} \mapsto \mathcal{U}$ such that for any stopping time τ and any $v_1, v_2 \in \mathcal{V}$ with $v_1(t) \equiv v_2(t)$ on $[0, \tau]$, it holds that $\alpha[v_1](t) \equiv \alpha[v_2](t)$ on $[0, \tau]$. An admissible impulse strategy $\beta[\cdot] : \mathcal{U} \mapsto \mathcal{V}$ for player II is defined similarly. The collections of all admissible impulse strategies for players I and II are defined as $\mathcal{A}$ and $\mathcal{B}$, respectively.

Remark 2. The most salient difference between the zero-sum game involving strategies with the so-called leader-follower game (see [43–45]) is that: the two players make decisions simultaneously in the zero-sum game, but hierarchically in the leader-follower game.

In the game, between players I and II there is an objective functional given as below, which is a reward for player I to maximize and a cost for player II to minimize:

$$J(x, i; u, v) = E\bigg[\int_0^\infty \mathrm{e}^{-rt} f(X^{x,i}(t), \theta^i(t))\mathrm{d}t - \sum_{m\geqslant 1} \mathrm{e}^{-r\tau_m} g(\xi_m) \prod_{n\geqslant 1} 1_{\{\tau_m \neq \rho_n\}} + \sum_{n\geqslant 1} \mathrm{e}^{-r\rho_n} h(\eta_n)\bigg], \tag{2}$$

where $f : R \times \mathcal{M} \mapsto R$ is the running reward or cost, $g : R \mapsto R$ is the intervention cost for player I, $h : R \mapsto R$ is the intervention cost for player II, and $r > 0$ is the discount factor.

The lower value function U and upper value function V of the game are defined as

$$U(x, i) = \inf_{\beta\in\mathcal{B}} \sup_{u\in\mathcal{U}} J(x, i; u, \beta[u]), \tag{3}$$

and

$$V(x, i) = \sup_{\alpha\in\mathcal{A}} \inf_{v\in\mathcal{V}} J(x, i; \alpha[v], v). \tag{4}$$

In the case $U = V$, we say the game admits a value.

Throughout the paper, we make the following assumptions.

(A1) For any $x, y \in R$ and $i \in \mathcal{M}$, there exist two positive constants C_1 and C_2 such that $(\phi = b, \sigma, f)$

$$|\phi(x, i)| \leqslant C_1, \quad |\phi(x, i) - \phi(y, i)| \leqslant C_2|x - y|.$$

(A2) $\inf_{\xi\in R} g(\xi) > 0$ and $\inf_{\eta\in R} h(\eta) > 0$. For any $\xi_1, \xi_2, \eta_1, \eta_2 \in R$, the following triangular inequalities hold:

$$g(\xi_1) + g(\xi_2) > g(\xi_1 + \xi_2), \quad h(\eta_1) + h(\eta_2) > h(\eta_1 + \eta_2).$$

(A3) The discount factor $r > C_2 + \frac{1}{2}C_2^2$.

Under Assumption (A1), for any $(x,i) \in R \times \mathcal{M}$ and $(u,v) \in \mathcal{U} \times \mathcal{V}$, the state equation (1) has a unique strong solution $X^{x,i}$ (see Yin and Zhu [40]). In the rest of this paper, let $X^{x,i;u,v}$ denote the solution if the dependence on the impulse controls needs to be emphasized.

Remark 3. In this paper, we have assumed that the state process, impulse control processes, and Brownian motion to be one-dimensional only for convenience of presentation. There is no essential difficulty to extend the results to the multi-dimensional case but with more complex notation.

Remark 4. Tang and Yong [11] and Cosso [15] first established the dynamic programming approach and viscosity solution characterization for impulse control and game problems (with no regime switching), respectively. In this sense, our paper can be regarded as a continuation and development of [11, 15]. On the one hand, involving the Markov chain in the problem adds much mathematical difficulty to the analysis, which has been mentioned in Section 1. On the other hand, in our paper a verification theorem as a sufficient condition for Nash equilibriums is also proven, which is absent in [11, 15].

Now let us give some preliminary results. First, one can readily obtain the following basic estimate of the state process with respect to its initial condition by Itô's formula and Gronwall's inequality.

Lemma 1. Under Assumption (A1), for any $x, y \in R$, $i \in \mathcal{M}$, and $u \in \mathcal{U}$, $v \in \mathcal{V}$, we have

$$E[|X^{x,i}(t) - X^{y,i}(t)|] \leqslant |x-y| \mathrm{e}^{(C_2 + \frac{1}{2}C_2^2)t}.$$

The following two propositions, which are concerned with the Lipschitz property of J, U, V and the boundedness of U, V, will play a key role in the subsequent analysis, whose proofs are similar to those of Propositions 3.2 and 3.6 of Cosso [15], respectively.

Proposition 1. Under Assumptions (A1)–(A3), the objective functional J defined by (2) as well as the lower value function U defined by (3) and the upper value function V defined by (4) are Lipschitz continuous with respect to x.

Proposition 2. Under Assumptions (A1)–(A3), the lower value function U defined by (3) and the upper value function V defined by (4) are bounded.

3 Dynamic programming principle

In this section, we state and prove the DPP for our zero-sum hybrid impulse game problem. At first, we present a lemma which is concerned with the path property of a Markov chain.

Lemma 2. A Markov chain is a right-continuous stochastic process with piecewise-constant sample paths.

Theorem 1. Under Assumptions (A1)–(A3), for any $x \in R$, $i \in \mathcal{M}$, and any stopping time ν, we have

$$\begin{aligned} U(x,i) = \inf_{\beta \in \mathcal{B}} \sup_{u \in \mathcal{U}} E\bigg[&\int_0^\nu \mathrm{e}^{-rt} f(X^{x,i;u,\beta[u]}(t), \theta^i(t)) \mathrm{d}t + \mathrm{e}^{-r\nu} U(X^{x,i;u,\beta[u]}(\nu), \theta^i(\nu)) \\ &- \sum_{\tau_m \leqslant \nu} \mathrm{e}^{-r\tau_m} g(\xi_m) \prod_{n \geqslant 1} 1_{\{\tau_m \neq \rho_n\}} + \sum_{\rho_n \leqslant \nu} \mathrm{e}^{-r\rho_n} h(\eta_n) \bigg], \end{aligned} \tag{5}$$

and

$$\begin{aligned} V(x,i) = \sup_{\alpha \in \mathcal{A}} \inf_{v \in \mathcal{V}} E\bigg[&\int_0^\nu \mathrm{e}^{-rt} f(X^{x,i;\alpha[v],v}(t), \theta^i(t)) \mathrm{d}t + \mathrm{e}^{-r\nu} V(X^{x,i;\alpha[v],v}(\nu), \theta^i(\nu)) \\ &- \sum_{\tau_m \leqslant \nu} \mathrm{e}^{-r\tau_m} g(\xi_m) \prod_{n \geqslant 1} 1_{\{\tau_m \neq \rho_n\}} + \sum_{\rho_n \leqslant \nu} \mathrm{e}^{-r\rho_n} h(\eta_n) \bigg]. \end{aligned} \tag{6}$$

Proof. We only prove (5) for the lower value function U; the other case (6) for the upper value function V is analogous. In the following, the proof is divided into two steps.

Step 1. In this step, we consider the case when ν takes at most countably many values $\{t_1, t_2, \dots\}$ with $t_{k+1} > t_k$ and $t_k \to \infty$ as $k \to \infty$. This step will be further divided into three sub-steps.

Sub-step 1. For convenience, let

$$\begin{aligned} G(u,\beta[u],\nu) =& E\bigg[\int_0^{\nu} \mathrm{e}^{-rt} f(X^{x,i;u,\beta[u]}(t),\theta^i(t))\mathrm{d}t + \mathrm{e}^{-r\nu}U(X^{x,i;u,\beta[u]}(\nu),\theta^i(\nu)) \\ & - \sum_{\tau_m\leqslant\nu} \mathrm{e}^{-r\tau_m} g(\xi_m)\prod_{n\geqslant 1} 1_{\{\tau_m\neq\rho_n\}} + \sum_{\rho_n\leqslant\nu} \mathrm{e}^{-r\rho_n} h(\eta_n)\bigg]. \end{aligned} \tag{7}$$

Given an $\varepsilon>0$, for any $u\in\mathcal{U}$ with $(\tau_m,\xi_m)_{m\geqslant1}$, there exists a strategy $\overline{\beta}\in\mathcal{B}$ with $(\overline{\rho}_n,\overline{\eta}_n)_{n\geqslant1}$ such that

$$G(u,\overline{\beta}[u],\nu) - \frac{\varepsilon}{2} < \inf_{\beta\in\mathcal{B}}\sup_{u\in\mathcal{U}} G(u,\beta[u],\nu). \tag{8}$$

For each fixed t_k, let $\mathcal{U}_{t_k}$ (respectively, $\mathcal{V}_{t_k}$) denote the set of admissible impulse controls for player I (respectively, player II) that start from t_k and are adapted to $\{\mathcal{F}_t^{t_k}\}_{t\geqslant0}$, where $\mathcal{F}_t^{t_k}\doteq\mathcal{F}_{t+t_k}$. Also, let $\mathcal{A}_{t_k}$ (respectively, $\mathcal{B}_{t_k}$) denote the corresponding set of admissible impulse strategies for player I (respectively, player II). Define $u_k(t)=u(t+t_k)$ for $t\geqslant0$, then $u_k\in\mathcal{U}_{t_k}$. Let $(\tau_m^k,\xi_m^k)_{m\geqslant1}$ denote the decision sequence associated with u_k.

From the Lipschitz property of J and U, we can partition R into intervals $\{I_d\}_{d\geqslant1}$ such that, given an $x_d\in I_d$, for any $y\in I_d$, $j\in\mathcal{M}$, and $\beta_k\in\mathcal{B}_{t_k}$,

$$U(y,j) > U(x_d,j) - \frac{\varepsilon}{6}, \tag{9}$$

and

$$J(x_d,j;u_k,\beta_k[u_k]) > J(y,j;u_k,\beta_k[u_k]) - \frac{\varepsilon}{6}. \tag{10}$$

In addition, for $x_d\in I_d$ and $j\in\mathcal{M}$, there exists a strategy $\widehat{\beta}_k\in\mathcal{B}_{t_k}$ with $(\widehat{\rho}_n^k,\widehat{\eta}_n^k)_{n\geqslant1}$ such that

$$U(x_d,j) > J(x_d,j;u_k,\widehat{\beta}_k[u_k]) - \frac{\varepsilon}{6}. \tag{11}$$

From (9)–(11), for any $y\in I_d$ and $j\in\mathcal{M}$, we have the following inequality:

$$U(y,j) > J(y,j;u_k,\widehat{\beta}_k[u_k]) - \frac{\varepsilon}{2}. \tag{12}$$

Let $A_{k,d,j}=\{\nu=t_k, X^{x,i;u,\overline{\beta}[u]}(t_k)\in I_d, \theta^i(t_k)=j\}$. Define a strategy $\widetilde{\beta}$ with $(\widetilde{\rho}_n,\widetilde{\eta}_n)_{n\geqslant1}$ as follows: for $u\in\mathcal{U}$,

$$\widetilde{\beta}[u(\cdot)](t) = \begin{cases} \overline{\beta}[u(\cdot)](t), & 0\leqslant t\leqslant\nu, \\ \displaystyle\sum_{k,d,j} 1_{A_{k,d,j}}\widehat{\beta}_k[u(\cdot+t_k)](t-t_k), & t>\nu. \end{cases} \tag{13}$$

Then, $\widetilde{\beta}\in\mathcal{B}$. Note that on $\{\nu=t_k\}$, for any $\widetilde{\rho}_n>t_k$, we have

$$\widetilde{\rho}_n = \widehat{\rho}_n^k + t_k, \quad \widetilde{\eta}_n = \widehat{\eta}_n^k.$$

Sub-step 2. On the one hand, it follows from (12) that

$$\begin{aligned} E[\mathrm{e}^{-r\nu}U(X^{x,i;u,\overline{\beta}[u]}(\nu),\theta^i(\nu))] &= \sum_{k,d,j} E[1_{A_{k,d,j}}\mathrm{e}^{-rt_k}U(X^{x,i;u,\overline{\beta}[u]}(t_k),j)] \\ &> \sum_{k,d,j} E[1_{A_{k,d,j}}\mathrm{e}^{-rt_k}J(X^{x,i;u,\overline{\beta}[u]}(t_k),j;u_k,\widehat{\beta}_k[u_k])] - \frac{\varepsilon}{2}. \end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned} & E\bigg[\int_{\nu}^{\infty} \mathrm{e}^{-rt} f(X^{x,i;u,\widetilde{\beta}[u]}(t),\theta^i(t))\mathrm{d}t \\ & \quad - \sum_{\tau_m>\nu} \mathrm{e}^{-r\tau_m} g(\xi_m)\prod_{n\geqslant1} 1_{\{\tau_m\neq\widetilde{\rho}_n\}} + \sum_{\widetilde{\rho}_n>\nu} \mathrm{e}^{-r\widetilde{\rho}_n} h(\widetilde{\eta}_n)\bigg] \\ & = \sum_{k,d,j} E\bigg[1_{A_{k,d,j}} E\bigg(\int_{t_k}^{\infty} \mathrm{e}^{-rt} f(X^{x,i;u,\widetilde{\beta}[u]}(t),\theta^i(t))\mathrm{d}t \\ & \quad - \sum_{\tau_m>t_k} \mathrm{e}^{-r\tau_m} g(\xi_m)\prod_{n\geqslant1} 1_{\{\tau_m\neq\widetilde{\rho}_n\}} + \sum_{\widetilde{\rho}_n>t_k} \mathrm{e}^{-r\widetilde{\rho}_n} h(\widetilde{\eta}_n)\bigg|\mathcal{F}_{t_k}\bigg)\bigg]. \end{aligned} \tag{15}$$

By changing $t \to t + t_k$, we get

$$\begin{aligned}
&E\bigg(\int_{t_k}^{\infty} \mathrm{e}^{-rt} f(X^{x,i;u,\widetilde{\beta}[u]}(t), \theta^i(t))\mathrm{d}t \\
&\quad - \sum_{\tau_m > t_k} \mathrm{e}^{-r\tau_m} g(\xi_m) \prod_{n \geqslant 1} 1_{\{\tau_m \neq \widetilde{\rho}_n\}} + \sum_{\widetilde{\rho}_n > t_k} \mathrm{e}^{-r\widetilde{\rho}_n} h(\widetilde{\eta}_n) \bigg| \mathcal{F}_{t_k}\bigg) \\
&= \mathrm{e}^{-rt_k} E\bigg(\int_0^{\infty} \mathrm{e}^{-rt} f(X^{x,i;u,\widetilde{\beta}[u]}(t + t_k), \theta^i(t + t_k))\mathrm{d}t \\
&\quad - \sum_{\tau_m^k > 0} \mathrm{e}^{-r\tau_m^k} g(\xi_m^k) \prod_{n \geqslant 1} 1_{\{\tau_m^k \neq \widehat{\rho}_n^k\}} + \sum_{\widehat{\rho}_n^k > 0} \mathrm{e}^{-r\widehat{\rho}_n^k} h(\widehat{\eta}_n^k) \bigg| \mathcal{F}_{t_k}\bigg) \\
&= \mathrm{e}^{-rt_k} J(X^{x,i;u,\overline{\beta}[u]}(t_k), j; u_k, \widehat{\beta}_k[u_k]).
\end{aligned} \tag{16}$$

From (14)–(16), we have

$$\begin{aligned}
E[\mathrm{e}^{-r\nu} U(X^{x,i;u,\overline{\beta}[u]}(\nu), \theta^i(\nu))] >& E\bigg[\int_{\nu}^{\infty} \mathrm{e}^{-rt} f(X^{x,i;u,\widetilde{\beta}[u]}(t), \theta^i(t))\mathrm{d}t \\
&- \sum_{\tau_m > \nu} \mathrm{e}^{-r\tau_m} g(\xi_m) \prod_{n \geqslant 1} 1_{\{\tau_m \neq \widetilde{\rho}_n\}} + \sum_{\widetilde{\rho}_n > \nu} \mathrm{e}^{-r\widetilde{\rho}_n} h(\widetilde{\eta}_n)\bigg] - \frac{\varepsilon}{2}.
\end{aligned}$$

Sub-step 3. Then, recalling the definition (7) of G, we obtain

$$J(x, i; u, \widetilde{\beta}[u]) - \frac{\varepsilon}{2} < G(u, \overline{\beta}[u], \nu). \tag{17}$$

Combining (8) and (17) gives

$$J(x, i; u, \widetilde{\beta}[u]) - \varepsilon < \inf_{\beta \in \mathcal{B}} \sup_{u \in \mathcal{U}} G(u, \beta[u], \nu).$$

As u is arbitrary in $\mathcal{U}$, we arrive at

$$U(x, i) \leqslant \inf_{\beta \in \mathcal{B}} \sup_{u \in \mathcal{U}} G(u, \beta[u], \nu).$$

Similarly, we can prove the reverse inequality, and thus the desired result follows.

Step 2. In this step, we consider a general stopping time ν. For $p = 1, 2, \ldots$, let

$$\nu_p = \sum_{l=1}^{\infty} \frac{l}{2^p} 1_{\{\frac{l-1}{2^p} < \nu \leqslant \frac{l}{2^p}\}}.$$

Then, ν_p is an $\mathcal{F}_t$-stopping time because $\{\nu_p \leqslant t\} = \{\nu \leqslant \frac{l_0}{2^p}\} \in \mathcal{F}_{\frac{l_0}{2^p}} \subset \mathcal{F}_t$, where $l_0 = \sup\{l : \frac{l}{2^p} \leqslant t\}$. Further, for each p, $\nu \leqslant \nu_p < \nu + \frac{1}{2^p}$. Note that

$$\left|\inf_{\beta \in \mathcal{B}} \sup_{u \in \mathcal{U}} G(u, \beta[u], \nu) - \inf_{\beta \in \mathcal{B}} \sup_{u \in \mathcal{U}} G(u, \beta[u], \nu_p)\right| \leqslant \sup_{\beta \in \mathcal{B}} \sup_{u \in \mathcal{U}} |G(u, \beta[u], \nu) - G(u, \beta[u], \nu_p)|.$$

So the whole task is to show that the right-hand side of the above inequality converges to 0 as $p \to \infty$.

In fact, it follows from the right-continuity of $X^{x,i;u,\beta[u]}(\cdot)$ and $\theta^i(\cdot)$, the Lipschitz continuity of U (see Proposition 1), and the dominated convergence theorem that

$$\begin{aligned}
|G(u, \beta[u], \nu) - G(u, \beta[u], \nu_p)| \leqslant& E\bigg[\int_{\nu}^{\nu_p} \mathrm{e}^{-rt} |f(X^{x,i;u,\beta[u]}(t), \theta^i(t))|\mathrm{d}t \\
&+ |\mathrm{e}^{-r\nu} U(X^{x,i;u,\beta[u]}(\nu), \theta^i(\nu)) - \mathrm{e}^{-r\nu_p} U(X^{x,i;u,\beta[u]}(\nu_p), \theta^i(\nu_p))| \\
&+ \sum_{\nu < \tau_m \leqslant \nu_p} \mathrm{e}^{-r\tau_m} g(\xi_m) \prod_{n \geqslant 1} 1_{\{\tau_m \neq \rho_n\}} + \sum_{\nu < \rho_n \leqslant \nu_p} \mathrm{e}^{-r\rho_n} h(\eta_n)\bigg]
\end{aligned}$$

converges to 0 uniformly in $u \in \mathcal{U}$ and $\beta \in \mathcal{B}$ as $p \to \infty$. We complete the proof.

Remark 5. It is worth mentioning that the key ingredients of the proof of Theorem 1 include the construction of the specially designed strategy (13), the time shift transformation (16), and the stopping time approximation scheme in Step 2, which are also the innovations and contributions of this paper.

4 Hamilton-Jacobi-Bellman-Isaacs equation

Let

$$\mathcal{L}W(x,i) = b(x,i)W'(x,i) + \frac{1}{2}\sigma^2(x,i)W''(x,i),$$

where $W'(x,i)$ and $W''(x,i)$ are the first-order and second-order derivatives of $W(x,i)$ with respect to x, respectively. Let

$$QW(x,\cdot)(i) = \sum_{j\neq i} q_{ij}[W(x,j) - W(x,i)], \tag{18}$$

which is the infinitesimal operator of the Markov chain. We also introduce the following two operators $\mathcal{G}$ and $\mathcal{H}$:

$$\mathcal{G}W(x,i) = \sup_{\xi\in R}\{W(x+\xi,i) - g(\xi)\}, \tag{19}$$

and

$$\mathcal{H}W(x,i) = \inf_{\eta\in R}\{W(x+\eta,i) + h(\eta)\}. \tag{20}$$

Remark 6. The definitions of $\mathcal{G}W(x,i)$ and $\mathcal{H}W(x,i)$ have an immediate and intuitive interpretation. If player I (respectively, player II) makes an impulse from x to $x+\xi$ (respectively, x to $x+\eta$), then the present Nash equilibrium payoff can be written as $W(x+\xi,i)-g(\xi)$ (respectively, $W(x+\eta,i)+h(\eta)$). We have considered the payoff in the present position and the intervention cost. Hence, $\mathcal{G}W(x,i)$ (respectively, $\mathcal{H}W(x,i)$) is actually the optimal impulse magnitude that player I (respectively, player II) would choose in case it wants to intervene.

In this section, based on the DPP, we show that the lower value function U and the upper value function V are the unique viscosity solution of the following HJBI equation, which consists of a coupled system of variational inequalities with bilateral obstacles:

$$\begin{aligned}\max\{\min\{&rW(x,i) - \mathcal{L}W(x,i) - f(x,i) - QW(x,\cdot)(i),\\ &W(x,i) - \mathcal{G}W(x,i)\}, W(x,i) - \mathcal{H}W(x,i)\} = 0.\end{aligned} \tag{21}$$

First, we give the definition of viscosity solution of (21), which is a natural generalization of the classical viscosity solution notion in Crandall et al. [42].

Definition 3. A continuous function $W(x,i)$, $x\in R$, and $i\in\mathcal{M}$, is said to be a viscosity subsolution (respectively, supersolution) of the HJBI equation (21) if, for any $i\in\mathcal{M}$,

$$\begin{aligned}\max\{\min\{&rW(\overline{x},i) - \mathcal{L}\varphi(\overline{x}) - f(\overline{x},i) - QW(\overline{x},\cdot)(i),\\ &W(\overline{x},i) - \mathcal{G}W(\overline{x},i)\}, W(\overline{x},i) - \mathcal{H}W(\overline{x},i)\} \leqslant 0 \quad (\text{respectively, } \geqslant 0)\end{aligned}$$

holds whenever $\varphi(x)\in C^2$ (i.e., twice continuously differentiable) and $W(x,i)-\varphi(x)$ has a local maximum (respectively, minimum) at $x=\overline{x}$. $W(x,i)$ is said to be a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

In what follows, we only prove the viscosity sub and supersolution properties for U, and those for V can be treated in a like manner. We begin with the following proposition of the operators $\mathcal{G}$ and $\mathcal{H}$, whose proof is similar to that of Cosso [15, Lemma 5.3].

Proposition 3. Under (A1)–(A3), the lower value function $U(x,i)$ defined by (3) satisfies

$$\max\{\min\{0, U(x,i) - \mathcal{G}U(x,i)\}, U(x,i) - \mathcal{H}U(x,i)\} = 0.$$

Theorem 2. Under Assumptions (A1)–(A3), the lower value function $U(x,i)$ defined by (3) is a viscosity solution of the HJBI equation (21).

Proof. We first consider the viscosity supersolution property, and the proof is divided into three steps.

Step 1. For any fixed $i\in\mathcal{M}$, let $\varphi(x)\in C^2$ be such that $U(x,i)-\varphi(x)$ attains its minimum at $x=\overline{x}$ in a neighbourhood $B_{\delta_1}(\overline{x}) \doteq (\overline{x}-\delta_1, \overline{x}+\delta_1)$ for some $\delta_1>0$. If $U(\overline{x},i)=\mathcal{H}U(\overline{x},i)$, then the proof is finished. Otherwise, from Proposition 3, we have $U(\overline{x},i)<\mathcal{H}U(\overline{x},i)$, or further $U(\overline{x},i)-\mathcal{H}U(\overline{x},i)\leqslant -2\gamma$ for some $\gamma>0$. Then,

$$U(x,i) - \mathcal{H}U(x,i) \leqslant -\gamma, \tag{22}$$

in $B_{\delta_2}(\overline{x})$ for some $\delta_2>0$ (noting the continuity of $U(\cdot,i)$ and $\mathcal{H}U(\cdot,i)$).

Step 2. Now, let $\delta=\delta_1\wedge\delta_2$. Let τ_θ be the first jump time of the Markov chain $\theta^i(\cdot)$ and τ_δ be the first exit time of $X^{\overline{x},i;u^0,\beta^0[u^0]}$ from $B_\delta(\overline{x})$, where u^0 and β^0 represent the no-impulse control for player I and no-impulse strategy for player II, respectively. Let $\nu=\tau_\theta\wedge\tau_\delta\wedge c$ for some constant $c>0$. From the DPP (5), for any $\varepsilon>0$ and the no-impulse control u^0, there exists a $\beta^\varepsilon\in\mathcal{B}$ with $(\rho_n^\varepsilon,\eta_n^\varepsilon)_{n\geqslant1}$ such that

$$\begin{aligned}U(\overline{x},i)>&E\bigg[\int_0^\nu \mathrm{e}^{-rt}f(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(t),i)\mathrm{d}t+\sum_{\rho_n^\varepsilon\leqslant\nu}\mathrm{e}^{-r\rho_n^\varepsilon}h(\eta_n^\varepsilon)\\&+\mathrm{e}^{-r\nu}U(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(\nu),\theta^i(\nu))\bigg]-\varepsilon.\end{aligned}\tag{23}$$

From the boundness of f and $\nu\leqslant c$, we have

$$E\bigg[\int_0^\nu \mathrm{e}^{-rt}\Big|f(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(t),i)-f(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)\Big|\mathrm{d}t\bigg]\leqslant K_1c.$$

It follows that

$$E\bigg[\int_0^\nu \mathrm{e}^{-rt}f(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(t),i)\mathrm{d}t\bigg]\geqslant E\bigg[\int_0^\nu \mathrm{e}^{-rt}f(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)\mathrm{d}t\bigg]-K_1c.\tag{24}$$

By the Lipschitz property of U, we have

$$\begin{aligned}&E\bigg[\Big|\mathrm{e}^{-r\nu}U(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(\nu),\theta^i(\nu))-\mathrm{e}^{-r\nu}U\Big(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu)+\sum_{\rho_n^\varepsilon\leqslant\nu}\eta_n^\varepsilon,\theta^i(\nu)\Big)\Big|\bigg]\\&\leqslant KE\bigg[\Big|X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(\nu)-\Big(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu)+\sum_{\rho_n^\varepsilon\leqslant\nu}\eta_n^\varepsilon\Big)\Big|\bigg].\end{aligned}$$

Note that

$$\begin{aligned}X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(\nu)=x&+\int_0^\nu b(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(s),\theta^i(s))\mathrm{d}s\\&+\int_0^\nu\sigma(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(s),\theta^i(s))\mathrm{d}B(s)+\sum_{\rho_n^\varepsilon\leqslant\nu}\eta_n^\varepsilon,\end{aligned}$$

and

$$\begin{aligned}X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu)+\sum_{\rho_n^\varepsilon\leqslant\nu}\eta_n^\varepsilon=x&+\int_0^\nu b(X^{\overline{x},i;u^0,\beta^0[u^0]}(s),\theta^i(s))\mathrm{d}s\\&+\int_0^\nu\sigma(X^{\overline{x},i;u^0,\beta^0[u^0]}(s),\theta^i(s))\mathrm{d}B(s)+\sum_{\rho_n^\varepsilon\leqslant\nu}\eta_n^\varepsilon.\end{aligned}$$

So we obtain

$$\begin{aligned}&E\bigg[\Big|X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(\nu)-\Big(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu)+\sum_{\rho_n^\varepsilon\leqslant\nu}\eta_n^\varepsilon\Big)\Big|\bigg]\\&\leqslant E\bigg[\int_0^\nu\Big|b(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(s),\theta^i(s))-b(X^{\overline{x},i;u^0,\beta^0[u^0]}(s),\theta^i(s))\Big|\mathrm{d}s\\&\quad+\Big|\int_0^\nu\Big(\sigma(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(s),\theta^i(s))-\sigma(X^{\overline{x},i;u^0,\beta^0[u^0]}(s),\theta^i(s))\Big)\mathrm{d}B(s)\Big|\bigg].\end{aligned}$$

Recall that $\nu\leqslant c$. By the boundedness of b, we have

$$E\bigg[\int_0^\nu\Big|b(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(t),\theta^i(s))-b(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),\theta^i(s))\Big|\mathrm{d}t\bigg]\leqslant Kc.$$

By the BDG inequality and the boundedness of σ, we have

$$\begin{aligned}&E\bigg[\Big|\int_0^\nu\Big(\sigma(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(s),\theta^i(s))-\sigma(X^{\overline{x},i;u^0,\beta^0[u^0]}(s),\theta^i(s))\Big)\mathrm{d}B(s)\Big|\bigg]\\&\leqslant E\bigg[\Big(\int_0^\nu\Big|\sigma(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(s),\theta^i(s))-\sigma(X^{\overline{x},i;u^0,\beta^0[u^0]}(s),\theta^i(s))\Big|^2\mathrm{d}s\Big)^{\frac{1}{2}}\bigg]\leqslant Kc^{\frac{1}{2}}.\end{aligned}$$

It follows that

$$E\Big[\mathrm{e}^{-r\nu}U(X^{\overline{x},i;u^0,\beta^\varepsilon[u^0]}(\nu),\theta^i(\nu))\Big] \geqslant E\Big[\mathrm{e}^{-r\nu}U\Big(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu)+\sum_{\rho_n^\varepsilon\leqslant\nu}\eta_n^\varepsilon,\theta^i(\nu)\Big)\Big]-K_2c^{\frac{1}{2}}. \tag{25}$$

Moreover,

$$\begin{aligned}
&E\Big[\mathrm{e}^{-r\nu}U\Big(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu)+\sum_{\rho_n^\varepsilon\leqslant\nu}\eta_n^\varepsilon,\theta^i(\nu)\Big)+\sum_{\rho_n^\varepsilon\leqslant\nu}\mathrm{e}^{-r\rho_n^\varepsilon}h(\eta_n^\varepsilon)-\mathrm{e}^{-r\nu}U(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu),\theta^i(\nu))\Big]\\
&\geqslant E\Big[\mathrm{e}^{-r\nu}\Big\{U\Big(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu)+\sum_{\rho_n^\varepsilon\leqslant\nu}\eta_n^\varepsilon,\theta^i(\nu)\Big)+h\Big(\sum_{\rho_n^\varepsilon\leqslant\nu}\eta_n^\varepsilon\Big)-U(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu),\theta^i(\nu))\Big\}\Big]\\
&\geqslant E\Big[\mathrm{e}^{-r\nu}1_{\{\tau_\alpha>(\tau_\delta\wedge c)\}}\Big\{U\Big(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu)+\sum_{\rho_n^\varepsilon\leqslant\nu}\eta_n^\varepsilon,i\Big)+h\Big(\sum_{\rho_n^\varepsilon\leqslant\nu}\eta_n^\varepsilon\Big)-U(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu),i)\Big\}\Big]\\
&\geqslant \gamma\mathrm{e}^{-rc}P(\tau_\alpha>(\tau_\delta\wedge c)),
\end{aligned} \tag{26}$$

where, in the above, the first inequality follows from the assumption (A2), the second inequality is owing to Proposition 3, and the third inequality is the result of (22).

In view of (24)–(26), Eq. (23) reduces to

$$\begin{aligned}
U(\overline{x},i)>&E\Big[\int_0^\nu \mathrm{e}^{-rt}f(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)\mathrm{d}t+\mathrm{e}^{-r\nu}U(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu),\theta^i(\nu))\Big]\\
&+\gamma\mathrm{e}^{-rc}P(\tau_\alpha>(\tau_\delta\wedge c))-K_1c-K_2c^{\frac{1}{2}}-\varepsilon.
\end{aligned} \tag{27}$$

For fixed γ and δ, since $P(\tau_\alpha>(\tau_\delta\wedge c))\uparrow 1$ as $c\downarrow 0$, we can select c and ε small enough such that the second line of (27) is greater than 0.

Step 3. Define a function $\psi: R\times\mathcal{M}\to R$ as follows:

$$\psi(x,j)=\begin{cases}\varphi(x)+U(\overline{x},i)-\varphi(\overline{x}), & j=i,\\ U(x,j), & j\neq i.\end{cases} \tag{28}$$

Applying Itô's formula to $\mathrm{e}^{-rt}\psi(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),\theta^i(t))$ between 0 and ν, we have

$$\begin{aligned}
&E[\mathrm{e}^{-r\nu}\psi(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu),\theta^i(\nu))]-\psi(\overline{x},i)\\
&=E\Big[\int_0^\nu \mathrm{e}^{-rt}\Big\{-r\psi(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)+\mathcal{L}\varphi(X^{\overline{x},i;u^0,\beta^0[u^0]}(t))\\
&\quad+\sum_{j\neq i}q_{ij}\Big(\psi(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),j)-\psi(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)\Big)\Big\}\mathrm{d}t\Big].
\end{aligned} \tag{29}$$

Note that $U(x,i)-\varphi(x)$ attains its minimum at $x=\overline{x}$ in $B_\delta(\overline{x})$, hence

$$U(\overline{x},i)-\varphi(\overline{x})\leqslant U(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)-\varphi(X^{\overline{x},i;u^0,\beta^0[u^0]}(t)),$$

i.e.,

$$\psi(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)\leqslant U(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i).$$

It follows from (28) that

$$\psi(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),j)=U(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),j).$$

In particular,

$$\begin{aligned}
&\sum_{j\neq i}q_{ij}\Big(\psi(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),j)-\psi(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)\Big)\\
&\geqslant\sum_{j\neq i}q_{ij}\Big(U(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),j)-U(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)\Big).
\end{aligned} \tag{30}$$

By (28)–(30), we obtain

$$\begin{aligned}&E\Big[\mathrm{e}^{-r\nu}U(X^{\overline{x},i;u^0,\beta^0[u^0]}(\nu),\theta^i(\nu))\Big]-U(\overline{x},i)\\&\quad\geqslant E\bigg[\int_0^{\nu}\mathrm{e}^{-rt}\bigg\{-rU(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)+\mathcal{L}\varphi(X^{\overline{x},i;u^0,\beta^0[u^0]}(t))\\&\qquad+\sum_{j\neq i}q_{ij}\Big(U(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),j)-U(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)\Big)\bigg\}\mathrm{d}t\bigg].\end{aligned}\tag{31}$$

It follows from (27) and (31) that

$$\begin{aligned}&E\bigg[\int_0^{\nu}\mathrm{e}^{-rt}\bigg\{rU(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)-\mathcal{L}\varphi(X^{\overline{x},i;u^0,\beta^0[u^0]}(t))\\&\qquad-f(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),i)-QU(X^{\overline{x},i;u^0,\beta^0[u^0]}(t),\cdot)(i)\bigg\}\mathrm{d}t\bigg]\\&\quad\geqslant\gamma\mathrm{e}^{-rc}P(\tau_\alpha>(\tau_\delta\wedge c))-K_1c-K_2c^{\frac{1}{2}}-\varepsilon>0.\end{aligned}$$

Dividing both sides of the above inequality by c and sending $c\to 0$, we have

$$rU(\overline{x},i)-\mathcal{L}\varphi(\overline{x})-f(\overline{x},i)-QU(\overline{x},\cdot)(i)\geqslant 0.$$

This gives the viscosity supersolution property.

The proof of subsolution property is parallel to that of supersolution property. Hence, we deduce that $U(x,i)$ is a viscosity solution of the HJBI equation (21).

In the next, we show the uniqueness of viscosity solution of the HJBI equation (21). This also indicates that the stochastic differential game admits a value. At first, we recall the definition of second order superdifferential of a function W at x:

$$J^{2,+}W(x)=\bigg\{(\zeta,\Lambda)\in R\times R:\limsup_{y\to x}\frac{W(y)-W(x)-(y-x)\zeta-\frac{1}{2}(y-x)^2\Lambda}{|y-x|^2}\leqslant 0\bigg\},$$

and the second order subdifferential of W at x is defined as $J^{2,-}W(x)=-J^{2,+}(-W)(x)$. Let $\overline{J}^{2,+}W(x)$ and $\overline{J}^{2,-}W(x)$ denote the closures of $J^{2,+}W(x)$ and $J^{2,-}W(x)$, respectively.

Now we introduce the following alternative definition of viscosity solution via the super- and subdifferentials, which is equivalent to Definition 3; see Crandall et al. [42, Section 2].

Definition 4. A continuous function $W(x,i)$, $x\in R$ and $i\in\mathcal{M}$, is said to be a viscosity subsolution (respectively, supersolution) of the HJBI equation (21) if, for any $x\in R$, $i\in\mathcal{M}$, and any $(\zeta,\Lambda)\in J^{2,+}W(x,i)$ (respectively, $J^{2,-}W(x,i)$), we have

$$\begin{aligned}\max\bigg\{\min\bigg\{&rW(x,i)-b(x,i)\zeta-\frac{1}{2}\sigma^2(x,i)\Lambda-f(x,i)-QW(x,\cdot)(i),\\&W(x,i)-\mathcal{G}W(x,i)\bigg\},W(x,i)-\mathcal{H}W(x,i)\bigg\}\leqslant 0\quad(\text{respectively, }\geqslant 0),\end{aligned}$$

where $W(x,i)$ is said to be a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

Remark 7. In view of the continuity, if $W(x,i)$ is a viscosity solution of the HJBI equation (21), then for any $(\zeta,\Lambda)\in\overline{J}^{2,+}W(x,i)$ (respectively, $\overline{J}^{2,-}W(x,i)$), the inequality in Definition 4 remains true; see Crandall et al. [42, Remark 2.4].

To proceed, we have to strengthen the assumption (A2) to the assumption (A4) as follows (see also Assumption ($\mathrm{H}_{c,\chi}$) in Cosso [15]):

(A4) $\inf_{\xi\in R}g(\xi)>0$ and $\inf_{\eta\in R}h(\eta)>0$. For any $\xi_1,\xi_2,\eta,\eta_1,\eta_2\in R$, there exists a positive constant c such that the following triangular inequalities hold:

$$g(\xi_1+\eta+\xi_2)\leqslant g(\xi_1)-h(\eta)+g(\xi_2)-c,$$

and

$$h(\eta_1+\eta_2) \leqslant h(\eta_1)+h(\eta_2)-c.$$

Then we have the following technical lemma that was borrowed from Cosso [15, Lemma 6.1] to handle the bilateral obstacles of the HJBI equation (21).

Lemma 3. Let assumptions (A1), (A3), and (A4) hold. Let $U(x,i)$ and $V(x,i)$ be a viscosity subsolution and a viscosity supersolution of the HJBI equation (21), respectively. Let x_1 be a point such that

$$V(x_1,i) \geqslant \mathcal{H}V(x_1,i),$$

or

$$V(x_1,i) < \mathcal{H}V(x_1,i), \quad U(x_1,i) \leqslant \mathcal{G}U(x_1,i).$$

Then, for any $\eta>0$, there exists an x_2 such that

$$U(x_2,i)-V(x_2,i)+\eta \geqslant U(x_1,i)-V(x_1,i),$$

and

$$V(x_2,i) < \mathcal{H}V(x_2,i), \quad U(x_2,i) > \mathcal{G}U(x_2,i).$$

Theorem 3. Let assumptions (A1), (A3), and (A4) hold. Let $U(x,i)$ and $V(x,i)$ be a viscosity subsolution and a viscosity supersolution of the HJBI equation (21), respectively. Suppose that $U(x,i)$ and $V(x,i)$ are bounded and Lipschitz continuous with respect to x. Then we have $U(x,i) \leqslant V(x,i)$ for any $(x,i) \in R\times\mathcal{M}$.

Proof. **Step 1.** It suffices to show that

$$\max_{i\in\mathcal{M}} \sup_{x\in R}\{U(x,i)-V(x,i)\} \leqslant 0.$$

We argue by contradiction. If it was not, in other words, if we have

$$d \doteq \max_{i\in\mathcal{M}} \sup_{x\in R}\{U(x,i)-V(x,i)\} > 0,$$

then there should exist x_1 and i_0 such that

$$U(x_1,i_0)-V(x_1,i_0) = d-\zeta_1 > 0,$$

for some $\zeta_1 \geqslant 0$ small enough.

From Lemma 3, for any $\zeta>0$, we can find an x_2 such that

$$U(x_2,i_0)-V(x_2,i_0) \doteq d-\zeta_2 \geqslant d-\zeta_1-\zeta,$$

for some $\zeta_2 \in [0,\zeta_1+\zeta]$, and

$$V(x_2,i_0) < \mathcal{H}V(x_2,i_0), \quad U(x_2,i_0) > \mathcal{G}U(x_2,i_0).$$

Let $x_0 \in \overline{B}_\delta(x_2)$, for some $\delta>0$, be a point such that

$$U(x_0,i_0)-V(x_0,i_0) = \sup_{x\in\overline{B}_\delta(x_2)}\{U(x,i_0)-V(x,i_0)\} \doteq d-\zeta_0 \doteq d_0, \tag{32}$$

for some $\zeta_0 \in [0,\zeta_2]$.

Step 2. For any $\varepsilon>0$, consider the following function:

$$\Phi_\varepsilon(x,y) = U(x,i_0)-V(y,i_0)-\frac{1}{2\varepsilon}|x-y|^2$$

in $\overline{B}_\delta(x_2)$. By Lemma 3.1 in [42], $\Phi_\varepsilon(x,y)$ attains its maximum d_ε in $\overline{B}_\delta(x_2)$ at some $(x_\varepsilon,y_\varepsilon)$ such that $d_\varepsilon \to d_0$ and $(x_\varepsilon,y_\varepsilon)\to(x_0,x_0)$, as $\varepsilon\to 0$. Moreover,

$$\lim_{\varepsilon\to 0}\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon} = 0.$$

In particular, it follows from the continuity of $U(\cdot, i_0)$, $\mathcal{G}U(\cdot, i_0)$, $V(\cdot, i_0)$, $\mathcal{H}V(\cdot, i_0)$ that, for ε sufficiently small,

$$V(y_\varepsilon, i_0) < \mathcal{H}V(y_\varepsilon, i_0), \quad U(x_\varepsilon, i_0) > \mathcal{G}U(x_\varepsilon, i_0). \tag{33}$$

Further, by Theorem 3.2 in [42], there exist $(p_\varepsilon, P_\varepsilon) \in \overline{J}^{2,+}U(x_\varepsilon, i_0)$ and $(q_\varepsilon, Q_\varepsilon) \in \overline{J}^{2,-}V(y_\varepsilon, i_0)$ such that $p_\varepsilon = q_\varepsilon = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}$ and

$$\begin{bmatrix} P_\varepsilon & 0 \\ 0 & -Q_\varepsilon \end{bmatrix} \leqslant \frac{1}{\varepsilon}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{\varepsilon}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}^2.$$

The last inequality implies that $\sigma^2(x_\varepsilon, i_0)P_\varepsilon - \sigma^2(y_\varepsilon, i_0)Q_\varepsilon \leqslant \frac{3}{\varepsilon}|\sigma(x_\varepsilon, i_0) - \sigma(y_\varepsilon, i_0)|^2$.

Step 3. From the viscosity subsolution property of U and the viscosity supersolution property of V and noting (33) and Remark 7, we have

$$rU(x_\varepsilon, i_0) - b(x_\varepsilon, i_0)\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \frac{1}{2}\sigma^2(x_\varepsilon, i_0)P_\varepsilon - f(x_\varepsilon, i_0) - QU(x_\varepsilon, \cdot)(i_0) \leqslant 0,$$

and

$$rV(y_\varepsilon, i_0) - b(y_\varepsilon, i_0)\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \frac{1}{2}\sigma^2(y_\varepsilon, i_0)Q_\varepsilon - f(y_\varepsilon, i_0) - QV(y_\varepsilon, \cdot)(i_0) \geqslant 0.$$

Combining the above two inequalities, we obtain

$$\begin{aligned} &r[U(x_\varepsilon, i_0) - V(y_\varepsilon, i_0)] - [b(x_\varepsilon, i_0) - b(y_\varepsilon, i_0)]\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \frac{1}{2}[\sigma^2(x_\varepsilon, i_0)P_\varepsilon - \sigma^2(y_\varepsilon, i_0)Q_\varepsilon] \\ &\quad - [f(x_\varepsilon, i_0) - f(y_\varepsilon, i_0)] - \sum_{j \neq i_0} q_{i_0,j}[U(x_\varepsilon, j) - U(x_\varepsilon, i_0)] \\ &\quad \leqslant -\sum_{j \neq i_0} q_{i_0,j}[V(y_\varepsilon, j) - V(y_\varepsilon, i_0)]. \end{aligned}$$

A re-arrangement of the coupling terms yields

$$\begin{aligned} &r[U(x_\varepsilon, i_0) - V(y_\varepsilon, i_0)] - [b(x_\varepsilon, i_0) - b(y_\varepsilon, i_0)]\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \frac{1}{2}[\sigma^2(x_\varepsilon, i_0)P_\varepsilon - \sigma^2(y_\varepsilon, i_0)Q_\varepsilon] \\ &\quad - [f(x_\varepsilon, i_0) - f(y_\varepsilon, i_0)] + \sum_{j \neq i_0} q_{i_0,j}[U(x_\varepsilon, i_0) - V(y_\varepsilon, i_0)] \\ &\quad \leqslant \sum_{j \neq i_0} q_{i_0,j}[U(x_\varepsilon, j) - V(y_\varepsilon, j)]. \end{aligned}$$

By sending ε to 0, we have

$$rd_0 + \sum_{j \neq i_0} q_{i_0,j}d_0 \leqslant \sum_{j \neq i_0} q_{i_0,j}[U(x_0, j) - V(x_0, j)] \leqslant \sum_{j \neq i_0} q_{i_0,j}d = \sum_{j \neq i_0} q_{i_0,j}(d_0 + \zeta_0),$$

i.e.,

$$d_0 \leqslant \frac{1}{r}\sum_{j \neq i_0} q_{i_0,j}\zeta_0.$$

This leads to a contraction to (32) for some ζ_0 sufficiently small.

Theorem 3 means that a viscosity subsolution will always be less than a viscosity supersolution. This leads to the uniqueness of the viscosity solution, since a viscosity solution is both a viscosity subsolution and a viscosity supersolution.

Proposition 4. Let assumptions (A1), (A3), and (A4) hold. Then, the viscosity solution of the HJBI equation (21) is unique in the space of bounded and Lipschitz continuous functions.

5 Verification theorem

This section establishes a verification theorem which can be used as a sufficient condition to find a Nash equilibrium and the value function.

For $i \in \mathcal{M}$, let

$$D_{1,i} = \{\phi(x,i) - \mathcal{G}\phi(x,i) > 0\},$$

and

$$D_{2,i} = \{\phi(x,i) - \mathcal{H}\phi(x,i) < 0\}.$$

Let $\partial D_{k,i}$ denote the boundary of $D_{k,i}$, $i \in \mathcal{M}$, $k = 1, 2$.

Remark 8. Actually, $D_{1,i}$ (respectively, $D_{2,i}$) is the so-called continuation region for player I (respectively, player II) in which it is better for player I (respectively, player II) to do nothing and let the state process continue than make an intervention and shift the state process to another position.

Theorem 4. Let $\phi(x,i)$, $x \in R$ and $i \in \mathcal{M}$, be a real-valued function satisfying:

(i) For $i \in \mathcal{M}$, $\phi(\cdot,i)$ belongs to class $C^2(R\backslash(\cup_{i\in\mathcal{M},k=1,2}\partial D_{k,i})) \cap C^1(R)$.

(ii) For $i \in \mathcal{M}$ and $x \in R$, $\mathcal{G}\phi(x,i) \leqslant \phi(x,i) \leqslant \mathcal{H}\phi(x,i)$.

(iii) For $i \in \mathcal{M}$ and $x \in D_{1,i}$, $\max\{r\phi(x,i) - \mathcal{L}\phi(x,i) - f(x,i) - Q\phi(x,\cdot)(i), \phi(x,i) - \mathcal{H}\phi(x,i)\} = 0$.

(iv) For $i \in \mathcal{M}$ and $x \in D_{2,i}$, $\min\{r\phi(x,i) - \mathcal{L}\phi(x,i) - f(x,i) - Q\phi(x,\cdot)(i), \phi(x,i) - \mathcal{G}\phi(x,i)\} = 0$.

Define u^* with $(\tau_m^*, \xi_m^*)_{m\geqslant 1}$ and v^* with $(\rho_n^*, \eta_n^*)_{n\geqslant 1}$ inductively as follows: $\tau_0^* = 0$, $\rho_0^* = 0$, and for $m \geqslant 1$, $n \geqslant 1$,

$$\begin{cases} \tau_m^* = \inf\limits_{t>\tau_{m-1}^*} \left\{ \phi(X^*(t), \theta^i(t)) = \sup\limits_{\xi\in R}\{\phi(X^*(t)+\xi, \theta^i(t)) - g(\xi)\} \right\}, \\ \xi_m^* = \arg\max\limits_{\xi\in R} \left\{ \phi(X^*(t)+\xi, \theta^i(t)) - g(\xi) \right\}, \end{cases} \tag{34}$$

and

$$\begin{cases} \rho_n^* = \inf\limits_{t>\rho_{n-1}^*} \left\{ \phi(X^*(t), \theta^i(t)) = \inf\limits_{\eta\in R}\{\phi(X^*(t)+\eta, \theta^i(t)) + h(\eta)\} \right\}, \\ \eta_n^* = \arg\min\limits_{\eta\in R} \left\{ \phi(X^*(t)+\eta, \theta^i(t)) + h(\eta) \right\}, \end{cases} \tag{35}$$

where $X^*(t) \doteq X^{x,i;u^*,v^*}(t)$. Then, (u^*, v^*) is a Nash equilibrium for players I and II and $\phi(x,i)$ is the value of the game.

Proof. It should be noticed that along the boundaries of the continuation regions $D_{1,i}$ and $D_{2,i}$, the function $\phi(\cdot,i)$, $i \in \mathcal{M}$, only belongs to C^1 but does not necessarily belong to C^2. In this situation, we can apply the smooth approximation argument introduced by Øksendal [46, Theorem 10.4.1 and Appendix D] to complement the smoothness needed for Itô's formula. Here in this proof, for convenience, we simply consider $\phi(\cdot,i)$, $i \in \mathcal{M}$, to be C^2 on the whole space; see also Guo and Zhang [36, Theorem 3.1] and Aïd et al. [18, Theorem 1].

First, we show that $\phi(x,i) \geqslant J(x,i;u,v^*)$, where u with $(\tau_m, \xi_m)_{m\geqslant 1}$ is arbitrary in $\mathcal{U}$ and v^* with $(\rho_n^*, \eta_n^*)_{n\geqslant 1}$ is given by (35) in which we use u instead of u^*. Let $(\nu_k)_{k\geqslant 1}$ be the entire intervention times of the two players no matter who intervenes. That is, if player I makes the intervention, then $\nu_k = \tau_m$ for some m otherwise if player II makes the intervention, then $\nu_k = \rho_n^*$ for some n. Using the sequence $(\nu_k)_{k\geqslant 1}$, the objective functional (2) becomes

$$J(x,i;u,v^*) = E\bigg[\int_0^\infty \mathrm{e}^{-rt} f(X(t),\theta^i(t))\mathrm{d}t + \sum_{k\geqslant 1} \mathrm{e}^{-r\nu_k}\{-g(\xi_m)1_{\{\nu_k=\tau_m\}} + h(\eta_n^*)1_{\{\nu_k=\rho_n^*\}}\}\bigg], \tag{36}$$

where $X(t) \doteq X^{x,i;u,v^*}(t)$.

Note that no impulse occurs between ν_{k-1} and ν_k, $k \geqslant 1$ (here, let $\nu_0 = 0$). Applying Itô's formula to $\mathrm{e}^{-rt}\phi(X(t),\theta^i(t))$ from ν_{k-1} to ν_k, we have

$$\begin{aligned} &E[\mathrm{e}^{-r\nu_k}\phi(X(\nu_k-),\theta^i(\nu_k)) - \mathrm{e}^{-r\nu_{k-1}}\phi(X(\nu_{k-1}),\theta^i(\nu_{k-1}))] \\ &= -E\bigg[\int_{\nu_{k-1}}^{\nu_k} \mathrm{e}^{-rt}\{(r-\mathcal{L})\phi(X(t),\theta^i(t)) - Q\phi(X(t),\cdot)(\theta^i(t))\}\mathrm{d}t\bigg] \\ &\leqslant -E\bigg[\int_{\nu_{k-1}}^{\nu_k} \mathrm{e}^{-rt} f(X(t),\theta^i(t))\mathrm{d}t\bigg], \end{aligned}$$

where the last inequality follows from $X(t) \in D_{2,\theta^i(t)}$ when $t \in (\nu_{k-1}, \nu_k)$.

Summing the indices $k \geqslant 1$, we obtain

$$\sum_{k\geqslant 1} E[\mathrm{e}^{-r\nu_k}\{\phi(X(\nu_k-),\theta^i(\nu_k)) - \phi(X(\nu_k),\theta^i(\nu_k))\}] \leqslant \phi(x,i) - E\left[\int_0^\infty \mathrm{e}^{-rt} f(X(t),\theta^i(t))\mathrm{d}t\right]. \tag{37}$$

Combining (36) and (37) yields

$$\begin{aligned} J(x,i;u,v^*) \leqslant \phi(x,i) + \sum_{k\geqslant 1} E[&\mathrm{e}^{-r\nu_k}\{-g(\xi_m)1_{\{\nu_k=\tau_m\}} + h(\eta_n^*)1_{\{\nu_k=\rho_n^*\}} \\ &+ \phi(X(\nu_k),\theta^i(\nu_k)) - \phi(X(\nu_k-),\theta^i(\nu_k))\}]. \end{aligned} \tag{38}$$

In the following, the analysis is divided into two cases:

(a) If $\nu_k = \tau_m$ for some m, it follows from condition (ii) that

$$\begin{aligned} \phi(X(\nu_k-),\theta^i(\nu_k)) &\geqslant \mathcal{G}\phi(X(\nu_k-),\theta^i(\nu_k)) \\ &= \sup_{\xi\in R}\{\phi(X(\nu_k-)+\xi,\theta^i(\nu_k)) - g(\xi)\} \\ &\geqslant \phi(X(\nu_k-)+\xi_m,\theta^i(\nu_k)) - g(\xi_m) \\ &= \phi(X(\nu_k),\theta^i(\nu_k)) - g(\xi_m). \end{aligned} \tag{39}$$

(b) If $\nu_k = \rho_n^*$ for some n, it follows from the definition (35) of $(\rho_n^*, \eta_n^*)_{n\geqslant 1}$ that

$$\begin{aligned} \phi(X(\nu_k-),\theta^i(\nu_k)) &= \mathcal{H}\phi(X(\nu_k-),\theta^i(\nu_k)) \\ &= \inf_{\eta\in R}\{\phi(X(\nu_k-)+\eta,\theta^i(\nu_k)) + h(\eta)\} \\ &= \phi(X(\nu_k-)+\eta_n^*,\theta^i(\nu_k)) + h(\eta_n^*) \\ &= \phi(X(\nu_k),\theta^i(\nu_k)) + h(\eta_n^*). \end{aligned} \tag{40}$$

From (38)–(40), we have $\phi(x,i) \geqslant J(x,i;u,v^*)$ for any $u \in \mathcal{U}$.

On the other hand, the proof of $\phi(x,i) \leqslant J(x,i;u^*,v)$ for any $v \in \mathcal{V}$ is symmetric, and the proof of $\phi(x,i) = J(x,i;u^*,v^*)$ is the same as above but with all inequalities becoming equalities. Hence, (u^*,v^*) defined by (34) and (35) is a Nash equilibrium for players I and II and $\phi(x,i)$ turns out to be the value of the game.

6 Concluding remarks

In the future, there are many interesting questions that deserve further investigation. In particular, to take into consideration that the Markov chain may have a large state space and exhibit a two-time-scale structure, solving the HJBI equation (21), either analytically or numerically, becomes a difficult task. In this case, an interesting and useful problem is to reduce the complexity and computational burden via a singular perturbation approach; in this connection, see Yin and Zhang [41] for more details.

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