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Controllability Gramian-based measures of graph product networks

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Abstract This paper addresses the controllability measures of composite networks based on graph products, in which the graph product networks are either Kronecker product networks or Kronecker sum networks. The primary objective pursued here is to reveal the controllability Gramian-based measure links between the graph product composite network and its subnetworks. First, the analytical expression of the average controllability for the Kronecker product network is derived by using the corresponding controllability measures of its subnetworks. An upper bound of volumetric control energy is further obtained for the Kronecker product network. Then, the upper bounds are derived for the controllability measures of the Kronecker sum network, which provide a valuable reference for designing controllable networks in practical applications. Finally, the controllability of the Kronecker product network is compared with that of the Kronecker sum network, and it is found that when the subnetworks satisfy certain conditions, the Kronecker product network has better average controllability than the Kronecker sum network.

Keywords composite network, graph product, controllability Gramian-based measure, average controllability, control energy

1 Introduction

Several complex systems in the real world can be usefully modeled as complex networks, where the nodes represent elements of the systems and the edges denote the interactions between them [1]. With the development of graph theory and the widespread application of technologies such as the Internet, people are increasingly interested in the study of complex networks [2–7]. Nowadays, the general concept of networks has been applied to various fields, such as power grids, intelligent transportation systems, biological networks, economic networks, and social networks. The issue of network controllability has thus become an active area of research in network science and engineering [8, 9]. Numerous research results have focused on structural controllability [10–14]; exact controllability [15,16]; heterogeneous node dynamics [17]; edge dynamics [18–20]; structural robustness [21,22]; minimal input selection problems [23]; optimization [24–26]; and control energy [27–39]. Note that most of the previous results were obtained for a single (single-layer) network. Due to the complex structure and computational burdens of large-scale networks, the results obtained on the controllability of single networks cannot be directly applied to real-world networks, such as social networks, biological networks, and technological networks, which manifest highly intricate and interlinked structures.

Cybernetics has many tools for the efficient decomposition of large-scale systems. As discussed in the study by [40], a non-trivial graph can be decomposed into several factor systems, and every non-trivial graph has a prime factorization over the Kronecker product [41, 42]. Therefore, a large-scale complex system can be constructed and analyzed by virtue of its simple subsystems. In recent years, with the development of graph decomposition algorithms [43, 44], graph product networks have drawn increasing attention. The graph product network allows for the combination of network structures at different levels or components, providing a more effective means to model the structure and dynamics of complex systems. This has proved to be a powerful tool in studying various complex systems, including online

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social networks, biological networks, and power networks. In addition, graph product networks have played a pivotal role in broad applications in different areas of engineering. For instance, the stability of a composite feedback system can be analyzed by its subsystems [45]. In [46, 47], the controllability effect of subnetworks on an associated composite network was investigated. In the study by [48], the focus was on the switching behavior between subsystems in networked systems and how to coordinate and control these systems effectively. According to [44, 46, 49], the Cartesian graph product (Kronecker sum) and Kronecker product networks were explored. More general graph products, including the Kronecker sum, the Kronecker product, and strong products that illustrate the spectral and controllability properties of composite systems, can be found in [50].

Studying the controllability of graph product networks presents a significant challenge due to the increased computational complexity inevitably introduced by graph product operations. Therefore, a more efficient approach is needed to address this issue. In practical applications, consistent efforts have been made to study the controllability of complex networks from an energy perspective. Indeed, to quantify the degree of effort required to control the network, it has been necessary to consider control energy [51–53]. Several studies [28, 29, 33] have found that the largest and smallest eigenvalues, trace, inverse, and determinant of the controllability Gramian matrix have then been applied to quantify the control energy of a complex network. Meanwhile, the Gramian matrix can then be applied to the optimizations of the controller placement locations [34, 36] and the structure of a given network [37]. It should be pointed out, however, that the energy-related controllability of graph product networks is still rarely studied in the literature, except perhaps in the work by [53], which has revealed how the spectral properties of local factor systems can then be used to infer the energy-related controllability of composite networks. However, up to this point, it appears that there is no overarching theoretical framework for energy-related controllability of graph product networks. Therefore, it is important to begin to explore the controllability of graph product networks.

Inspired by the studies above, the controllability Gramian-based measures of graph product networks are investigated in this paper. Here, the graph product refers to the Kronecker product or the Kronecker sum. The controllability measures of the graph product network are then characterized by the trace and the determinant of the controllability Gramian, corresponding to average controllability and volumetric control energy, respectively. Unlike the findings of [53], in this work, the controllability of the composite network is herein characterized by the Gramian-based measures of its associated subnetworks, and the relationships between Gramian-based measures of the composite network and those of the associated subnetworks are then established. The main contributions made by this paper are summarized as follows.

(1) Firstly, the analytical expression of the average controllability of a Kronecker product network is derived by applying the corresponding controllability measures of its subnetworks. This means that instead of calculating the large-scale composite network directly, the average controllability of the resulting composite network can be calculated accurately from that of its smaller attendant subnetworks; therefore, it can greatly improve computational efficiency. In addition, an upper bound of the volumetric control energy for the Kronecker product network is then derived from the average controllability of its subnetworks. This implies that the volumetric control energy of the Kronecker product network can then be estimated by the average controllability measures of its smaller, attendant subnetworks.

(2) Secondly, the controllability measures of Kronecker sum networks are provided. The upper bounds of the attendant Gramian-based measures for the Kronecker sum network are then inferred from the average controllability measures of its subnetworks, taking advantage of the smaller size of these subnetworks, which implies that the Gramian-based measures of the Kronecker sum network can be estimated by using the candidate subnetworks before the design stage, which can guide us in then designing an appropriate Kronecker sum network.

(3) Finally, a comparison of Kronecker product networks and Kronecker sum networks is provided. Some sufficient conditions are given under which the composite network constructed by the Kronecker product has better average controllability than the composite network constructed by the Kronecker sum. Compared to the existing research, our approach offers a detailed comparison of the performance of two different network construction methods, aiding in a more comprehensive evaluation of their feasibility. This is crucial for selecting an appropriate network construction method to enhance network controllability.

In summary, this paper provides a theoretical framework for the energy-related controllability of graph product networks, and it reveals how the energy-related controllability of a composite network can be inferred from its attendant smaller subnetworks. The controllability Gramian-based measures of both

Kronecker product networks and Kronecker sum networks can be estimated well from the average controllability of the subnetworks in a computationally efficient manner and bypass calculating the large-scale composite network directly. The interesting results obtained provide practical ways to decompose or synthesize large-scale complex networks.

The rest of this paper is organized as follows. In Section 2, the notation, graph theory, and some preliminary results are introduced. In Section 3, energy-related controllability measures of graph product networks are then obtained. In Section 4, the controllability of Kronecker product networks is compared with that of Kronecker sum networks. Finally, Section 5 presents a conclusion.

2 Mathematical preliminaries and problem formulation

2.1 Notation

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ the set of the n-dimensional real vectors, and $\mathbb{R}^{m\times n}$ the set of $m\times n$ real matrices. Let I_n denote the $n\times n$ identity matrix and the basis vector e_i be the column vector with all zero entries except for the ith entry being one. Let M_{ij} denote the ith row and jth column entry of a matrix $M\in\mathbb{R}^{m\times n}$. Let $\mathrm{tr}(A)$ and $\det A$ denote the trace and determinant of a matrix $A\in\mathbb{R}^{n\times n}$, respectively. For the matrices $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{m\times m}$, let $A\otimes B$ and $A\oplus B$ denote the Kronecker product and the Kronecker sum, respectively, where $A\oplus B=A\otimes I_m+I_n\otimes B$.

2.2 Graphs

Consider a complex network characterized by an undirected signed graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{v_1,\dots,v_n\}$ is the node set and $\mathcal{E}=\{(v_i\sim v_j)\}$ is the edge set, in which $(v_i\sim v_j)$ denotes the edge between nodes v_i and v_j. Let $A=[a_{ij}]\in\mathbb{R}^{n\times n}$ denote the adjacency matrix of the undirected signed graph $\mathcal{G}$ with $a_{ij}\in\{-1,+1\}$ if $(v_i\sim v_j)\in\mathcal{E}$ and $a_{ij}=0$ otherwise. The Laplacian matrix is, thus, defined as $L=D-A$, where $D=\mathrm{diag}\{d_1,\dots,d_n\}$, $d_i=\sum_{j=1}^{n}|a_{ij}|$. Note that there are negative off-diagonal entries in the Laplacian matrix of the undirected signed graph. This means that zero is no longer the default eigenvalue of the signed Laplacian matrix. For a signed graph, if there exists a node bipartition such that the connections between nodes within the same group are positive and the connections between nodes in different groups are found to be negative, this is called a structurally balanced graph; otherwise, it is designated as a structurally unbalanced graph. All the eigenvalues of the Laplacian matrix L for a connected and structurally unbalanced signed graph $\mathcal{G}$ are then larger than zero [27, 54]. To ensure that the Laplacian matrix L is positive definite, this paper focuses on connected and structurally unbalanced signed graphs. However, it is important to note that, in a connected and structurally balanced signed graph, the Laplacian matrix L is semi-stable (having one zero eigenvalue), and the corresponding Gramian matrix is ill-defined. The standard Gramian matrix, therefore, cannot be used for structurally balanced signed graphs.

2.3 Generalized graph product networks

A graph product network can be regarded as a kind of composite network constructed by applying Kronecker product/sum operation from attendant smaller subgraphs [41, 46, 47]. Let $\mathcal{G}_1=(\mathcal{V}_1,\mathcal{E}_1)$ and $\mathcal{G}_2=(\mathcal{V}_2,\mathcal{E}_2)$ be two subnetworks. The generalized graph product is denoted by $\mathcal{G}=\mathcal{G}_1\triangle\mathcal{G}_2$, such as the Kronecker product $\mathcal{G}=\mathcal{G}_1\times\mathcal{G}_2$ and the Kronecker sum $\mathcal{G}=\mathcal{G}_1\square\mathcal{G}_2$. The generalized graph product $\mathcal{G}=\mathcal{G}_1\triangle\mathcal{G}_2$ has the node set $\mathcal{V}_1\times\mathcal{V}_2$ with nodes labeled as (i,p) for each $i\in\mathcal{V}_1$ and $p\in\mathcal{V}_2$. The directed product $\mathcal{G}=\mathcal{G}_1\times\mathcal{G}_2$ describes the edge relationship between node (i,p) and node (j,q) if and only if $(i\sim j)\in\mathcal{E}_1$ and $(p\sim q)\in\mathcal{E}_2$. The Cartesian product graph $\mathcal{G}=\mathcal{G}_1\square\mathcal{G}_2$ has an edge between node (i,p) and node (j,q) if and only if either $i=j$ and $(p\sim q)\in\mathcal{E}_2$, or $p=q$ and $(i\sim j)\in\mathcal{E}_1$. Examples of the Kronecker product and the Kronecker sum are illustrated in Figure 1.

2.4 Controllability and stability

Consider a continuous-time linear time-invariant networked system with Laplacian dynamics, described by

$$\dot{x}(t)=-Lx(t)+Bu(t), \tag{1}$$

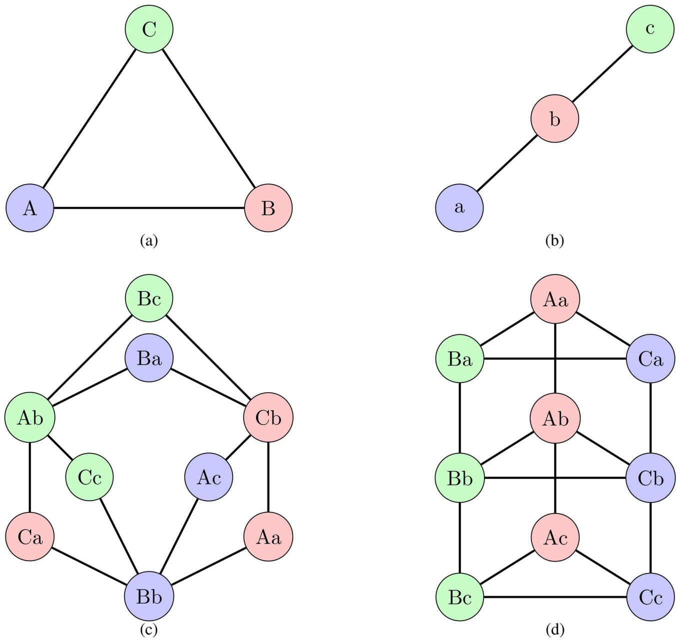

Figure 1 (Color online) Factor graphs and their product graphs. (a) $\mathcal{G}_1$; (b) $\mathcal{G}_2$; (c) $\mathcal{G}_1 \times \mathcal{G}_2$; (d) $\mathcal{G}_1 \square \mathcal{G}_2$.

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ indicate the state and input vectors, respectively, $L \in \mathbb{R}^{n\times n}$ is the Laplacian matrix and $B \in \mathbb{R}^{n\times m}$ is the input matrix. Assume that each external input controls one node, where the input matrix is $B = [e_{c_1}, \ldots, e_{c_m}] \in \mathbb{R}^{n\times m}$ with the basis vectors e_i, $i = c_1, \ldots, c_m$ indicating that the ith node, therefore, possesses external control.

Definition 1. The dynamical system (1) is Hurwitz stable (or simply stable) if all the eigenvalues of $-L$ have negative real parts.

Definition 2. The dynamical system (1) is said to be controllable over a finite time interval $[0, t_f]$ if there exists a control input $u(t) \in \mathbb{R}^m$, $t \in [0, t_f]$, that can drive the system from any given initial state $x(0) \in \mathbb{R}^n$ to any given target state $x_f \in \mathbb{R}^n$.

The system (1) is controllable if and only if the controllability Gramian

$$\mathcal{W} = \int_0^{t_f} \mathrm{e}^{-Lt} BB^{\mathrm{T}} \mathrm{e}^{-L^{\mathrm{T}}t} \mathrm{d}t \tag{2}$$

is positive definite. For a stable system, the controllability Gramian converges as $t_f \to \infty$. In this paper, an infinite-horizon Gramian is considered.

2.5 Controllability metrics of complex networks

The energy required to control a network can then describe how easy it is to control the network. Assuming that the initial state $x(0) = 0$ and the optimal control $u(t)$ is designed as is reported in [51], the minimum control energy required to steer the system from $x(0)$ into the desired target state x_f over the time interval $[0, t_f]$ is

$$E(t) = x_f^{\mathrm{T}} \mathcal{W}^{-1} x_f, \tag{3}$$

where $\mathcal{W} = \int_0^{t_f} \mathrm{e}^{-Lt} BB^{\mathrm{T}} \mathrm{e}^{-L^{\mathrm{T}}t} \mathrm{d}t$ is the controllability Gramian. It follows from (3) that the minimum energy required to control a network is then related to $\mathcal{W}^{-1}$.

If the eigenvalues of $\mathcal{W}$ are low in one direction, it will take a great deal of energy for the system to reach its final state in that direction [33]. Based on the eigenvalues of $\mathcal{W}$, many metrics for the size of network control energy have been described in [33]. In the present paper, the following two controllability Gramian-based metrics are used:

(1) $\mathrm{tr}(\mathcal{W})$: The average controllability $\mathrm{tr}(\mathcal{W})$ can be taken as the average of the control energy in all directions in the state space. The larger $\mathrm{tr}(\mathcal{W})$ means that the system is, therefore, easier to control. When $\mathrm{tr}(\mathcal{W}) \to 0$, this means that the system is uncontrollable.

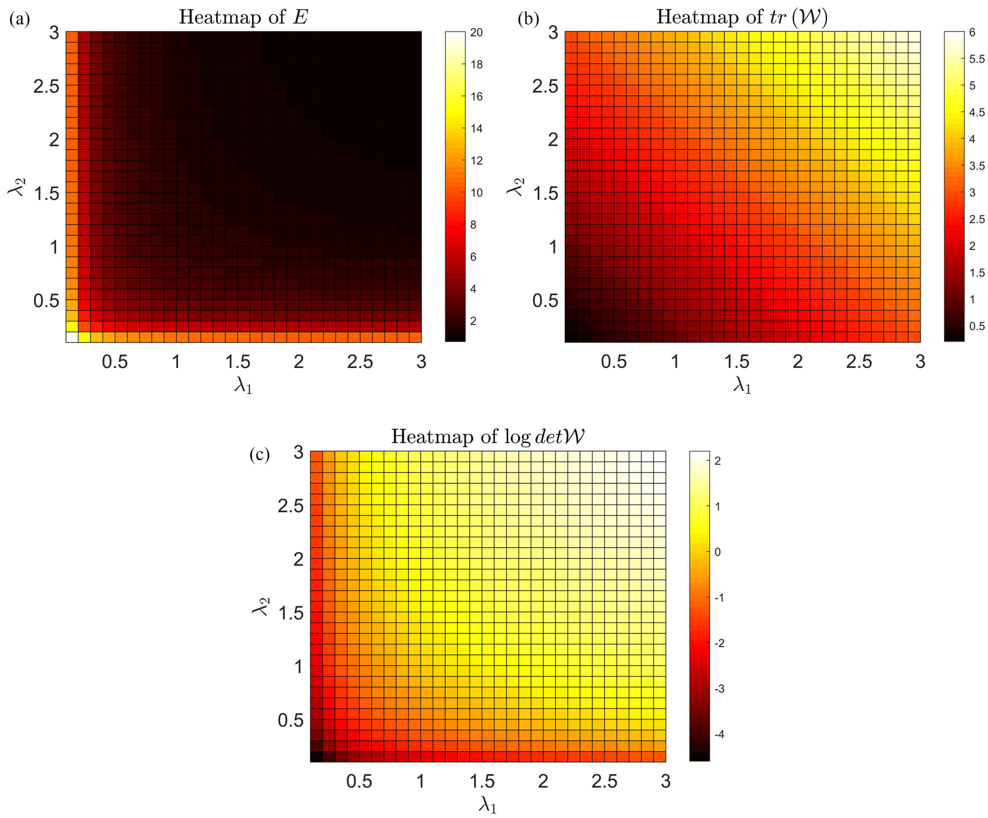

Figure 2 (Color online) Heat maps of the control energy and the controllability metrics. (a) Control energy E; (b) average controllability $\mathrm{tr}(\mathcal{W})$; (c) volumetric control energy $\log\det\mathcal{W}$.

(2) $\log\det\mathcal{W}$: The volumetric control energy $\log\det\mathcal{W}$ acts as a volumetric measure of the state set that can be reached with one unit or less of input energy. The term "volume" here refers to the magnitude of the set of accessible states in the state space rather than the physical volume per set. It denotes the extent or collection of all possible states a system can achieve, given an energy constraint. A large value of $\log\det\mathcal{W}$ indicates that a given system can undergo numerous state transitions with minimal energy input, while a small value of $\log\det\mathcal{W}$ implies significant limitations on the system's capacity for state changes. Note that for an uncontrollable system, the volumetric control energy is $\log\det\mathcal{W} \to -\infty$.

To illustrate the above two commonly used metrics, Figure 2 gives the heat maps for the control energy E, $\mathrm{tr}(\mathcal{W})$, and $\log\det\mathcal{W}$. Here, a 2-node connected network with $x_f^{\mathrm{T}} x_f = 1$ is then considered. Also, λ_1 and λ_2 are the eigenvalues of $\mathcal{W}$.

2.6 Composite complex networked systems

Analyzing a large-scale network directly is challenging, but one can start with the subnetworks that make up the network. Without any loss of generality, this paper considers the composite system consisting of two subsystems (subnetworks). The results obtained can be extended thereafter to multiple (more than two) subnetworks.

Considering two networked subsystems $(-L_1, B_1)$ and $(-L_2, B_2)$, where $L_1 \in \mathbb{R}^{n_1\times n_1}$ and $L_2 \in \mathbb{R}^{n_2\times n_2}$ are the Laplacian matrices of subnetworks $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, the input matrices are $B_1 = [e_{c_1^1}, \ldots, e_{c_{m_1}^1}] \in \mathbb{R}^{n_1\times m_1}$ with basis vectors $e_{c_i^1} \in \mathbb{R}^{n_1}$, $i = 1, \ldots, m_1$, indicating that the ith node of $\mathcal{G}_1$ possesses external control; and $B_2 = [e_{c_1^2}, \ldots, e_{c_{m_2}^2}] \in \mathbb{R}^{n_2\times m_2}$ with basis vectors $e_{c_j^2} \in \mathbb{R}^{n_2}$, $j = 1, \ldots, m_2$, indicating that the jth node of $\mathcal{G}_2$ possesses external control. The dynamics for the subsystems driven by their respective Laplacian matrices are

$$\dot{x}_1(t) = -L_1 x_1(t) + B_1 u_1(t), \tag{4}$$

$$\dot{x}_2(t) = -L_2 x_2(t) + B_2 u_2(t), \tag{5}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ denote the state vectors of the two subsystems. The dynamics for the composite system consisting of generalized graph product $\mathcal{G} = \mathcal{G}_1 \triangle \mathcal{G}_2$ can be written as

$$\dot{x}(t) = -Lx(t) + Bu(t), \tag{6}$$

where $L \in \mathbb{R}^{n\times n}$ is the Laplacian matrix of the composite network with

$$L = \begin{cases} L_1 \otimes L_2, & \text{if } \triangle = \times, \\ L_1 \oplus L_2, & \text{if } \triangle = \square, \end{cases} \tag{7}$$

and $B = B_1 \otimes B_2 \in \mathbb{R}^{n\times m}$. The notations $\triangle$, $\times$, and $\square$ denote the generalized graph product, the Kronecker product, and the Cartesian product (Kronecker sum), respectively.

In this paper, it is assumed that the two subsystems and the associated composite system are all Hurwitz stable and thereby controllable.

3 Controllability measures of graph product networks

This section discusses the controllability of graph product networks. Specifically, it investigates the average controllability and volumetric control energy. It characterizes the ultimate controllability of the composite network from the controllability Gramian-based measures of its subnetworks and thereby establishes new relationships between the controllability measures of the large-scale composite network and those of its smaller subnetworks.

The following results will be employed in the proof of the theorem.

Proposition 1. Consider two matrices $L_1 \in \mathbb{R}^{n_1\times n_1}$ and $L_2 \in \mathbb{R}^{n_2\times n_2}$. The eigenvalues and the corresponding eigenvectors of the Kronecker product $L_1 \otimes L_2$ are $\lambda_k = \mu_i \eta_j$ and $p_k = u_i \otimes v_j$, respectively; and the eigenvalues and the corresponding eigenvectors of the Kronecker sum $L_1 \oplus L_2$ are $\mu_i + \eta_j$ and $u_i \otimes v_j$, respectively; where μ_i and η_j are the eigenvalues of L_1 and L_2, respectively; and u_i and v_j are the corresponding eigenvectors of L_1 and L_2, respectively; $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$, and $k = 1, \ldots, n_1 n_2$.

Lemma 1 ([55])**.** Let $A \in \mathbb{R}^{n\times n}$ be positive definite (semi-positive definite) matrix, then

$$\det A \leqslant \prod_{i=1}^{n} A_{ii}, \tag{8}$$

where A_{ii} is the ith diagonal element of matrix A, $i = 1, \ldots, n$.

3.1 Controllability measures of Kronecker product networks

In this subsection, the controllability of Gramian-based measures of the Kronecker product networks is investigated.

Theorem 1. Considering two subsystems with Laplacian dynamics described by (4) and (5), the average controllability of the composite network generated by the Kronecker product is found to be

$$\operatorname{tr}(\mathcal{W}_\otimes) = 2\operatorname{tr}(\mathcal{W}_1)\operatorname{tr}(\mathcal{W}_2), \tag{9}$$

where $\mathcal{W}_\otimes$, $\mathcal{W}_1$, and $\mathcal{W}_2$ are the controllability Gramians of $(-L, B)$, $(-L_1, B_1)$, and $(-L_2, B_2)$, respectively, and $L_1 \in \mathbb{R}^{n_1\times n_1}$, $L_2 \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{n_1\times m_1}$, $B_2 \in \mathbb{R}^{n_2\times m_2}$, $L = L_1 \otimes L_2$, $B = B_1 \otimes B_2$.

Proof. Since L is a real-symmetric matrix, there is a similarity transformation matrix $P \in \mathbb{R}^{n\times n}$, satisfying $L = P\Lambda P^{\mathrm{T}}$, where $PP^{\mathrm{T}} = I_n$, $n = n_1 n_2$, and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1, \ldots, \lambda_n$ being the eigenvalues of the Laplacian matrix L. Hence, $\mathrm{e}^{Lt} = P\mathrm{e}^{\Lambda t}P^{\mathrm{T}}$. Then, the controllability Gramian of the Kronecker product network is given by

$$\mathcal{W}_\otimes = \int_0^\infty \mathrm{e}^{-Lt}BB^{\mathrm{T}}\mathrm{e}^{-L^{\mathrm{T}}t}\mathrm{d}t = P\Phi P^{\mathrm{T}}, \tag{10}$$

where

$$\Phi = \int_0^\infty \mathrm{e}^{-\Lambda t}P^{\mathrm{T}}BB^{\mathrm{T}}P\mathrm{e}^{-\Lambda^{\mathrm{T}}t}\mathrm{d}t. \tag{11}$$

Therefore,

$$\mathrm{tr}\left(\mathcal{W}_{\otimes}\right)=\mathrm{tr}\left(P\Phi P^{\mathrm{T}}\right)=\mathrm{tr}\left(\Phi\right). \tag{12}$$

Note that both B_1 and B_2 are composed of basis vectors, and $B=B_1\otimes B_2$, one has $B=[e_{c_1},\ldots,e_{c_m}]\in\mathbb{R}^{n\times m}$ with basis vectors $e_{c_k}\in\mathbb{R}^n$, $k=1,\ldots,m$, where $m=m_1m_2$. Then $BB^{\mathrm{T}}=\sum_{k=1}^{m}e_{c_k}e_{c_k}^{\mathrm{T}}$, and the ith diagonal element of Φ is given by

$$\begin{aligned}
\Phi_{ii}&=e_i^{\mathrm{T}}\Phi e_i\\
&=\sum_{k=1}^{m}e_i^{\mathrm{T}}\int_0^\infty \mathrm{e}^{-\Lambda t}P^{\mathrm{T}}e_{c_k}e_{c_k}^{\mathrm{T}}P\mathrm{e}^{-\Lambda^{\mathrm{T}}t}\mathrm{d}te_i\\
&=\sum_{k=1}^{m}\int_0^\infty e_i^{\mathrm{T}}\mathrm{e}^{-\Lambda t}P^{\mathrm{T}}e_{c_k}e_{c_k}^{\mathrm{T}}P\mathrm{e}^{-\Lambda^{\mathrm{T}}t}e_i\mathrm{d}t\\
&=\sum_{k=1}^{m}\int_0^\infty \left(e_{c_k}^{\mathrm{T}}P\mathrm{e}^{-\Lambda^{\mathrm{T}}t}e_i\right)^{\mathrm{T}}\left(e_{c_k}^{\mathrm{T}}P\mathrm{e}^{-\Lambda^{\mathrm{T}}t}e_i\right)\mathrm{d}t\\
&=\sum_{k=1}^{m}\int_0^\infty \left(e_{c_k}^{\mathrm{T}}Pe_i\mathrm{e}^{-\lambda_i t}\right)^{\mathrm{T}}\left(e_{c_k}^{\mathrm{T}}Pe_i\mathrm{e}^{-\lambda_i t}\right)\mathrm{d}t\\
&=\sum_{k=1}^{m}\int_0^\infty \left(P_{c_k i}\mathrm{e}^{-\lambda_i t}\right)^2\mathrm{d}t\\
&=\int_0^\infty \mathrm{e}^{-2\lambda_i t}\mathrm{d}t\sum_{k=1}^{m}P_{c_k i}^2\\
&=\frac{1}{2\lambda_i}\sum_{k=1}^{m}P_{c_k i}^2,
\end{aligned} \tag{13}$$

where $P_{c_k i}$ denotes the c_kth row and ith column entry of the matrix P.

Similarly, one has $L_1=U\Lambda_1U^{\mathrm{T}}$ and $L_2=V\Lambda_2V^{\mathrm{T}}$, where $UU^{\mathrm{T}}=I_{n_1}$, $VV^{\mathrm{T}}=I_{n_2}$, $\Lambda_1=\mathrm{diag}(\mu_1,\ldots,\mu_{n_1})$, and $\Lambda_2=\mathrm{diag}\left(\eta_1,\ldots,\eta_{n_2}\right)$.

Let the controllability Gramians of $(-L_1,B_1)$ and $(-L_2,B_2)$ be $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively, one has

$$\mathcal{W}_1=U\Phi_1U^{\mathrm{T}},\ \mathcal{W}_2=V\Phi_2V^{\mathrm{T}},$$

and

$$\mathrm{tr}\left(\mathcal{W}_1\right)=\mathrm{tr}\left(\Phi_1\right),\ \mathrm{tr}\left(\mathcal{W}_2\right)=\mathrm{tr}\left(\Phi_2\right),$$

where

$$\Phi_1=\int_0^\infty \mathrm{e}^{-\Lambda_1 t}U^{\mathrm{T}}B_1B_1^{\mathrm{T}}U\mathrm{e}^{-\Lambda_1^{\mathrm{T}}t}\mathrm{d}t, \tag{14}$$

$$\Phi_2=\int_0^\infty \mathrm{e}^{-\Lambda_2 t}V^{\mathrm{T}}B_2B_2^{\mathrm{T}}V\mathrm{e}^{-\Lambda_2^{\mathrm{T}}t}\mathrm{d}t. \tag{15}$$

Then the ith diagonal elements of Φ_1 and Φ_2 are $[\Phi_1]_{ii}=\frac{1}{2\mu_i}\sum_{k=1}^{m_1}U_{c_k^1 i}^2$ and $[\Phi_2]_{ii}=\frac{1}{2\eta_i}\sum_{k=1}^{m_2}V_{c_k^2 i}^2$, respectively. Therefore, one has

$$\mathrm{tr}\left(\mathcal{W}_1\right)=\sum_{i=1}^{n_1}\frac{1}{2\mu_i}\sum_{k=1}^{m_1}U_{c_k^1 i}^2,\ \mathrm{tr}\left(\mathcal{W}_2\right)=\sum_{i=1}^{n_2}\frac{1}{2\eta_i}\sum_{k=1}^{m_2}V_{c_k^2 i}^2. \tag{16}$$

Substituting (13) into (12) and by Proposition 1, one has

$$\begin{aligned}
\mathrm{tr}\left(\mathcal{W}_{\otimes}\right)&=\sum_{i=1}^{n}\frac{1}{2\lambda_i}\sum_{k=1}^{m}P_{c_k i}^2\\
&=\frac{1}{2}\sum_{p=1}^{n_1}\sum_{q=1}^{n_2}\left(\frac{1}{\mu_p\eta_q}\sum_{u=1}^{m_1}\sum_{v=1}^{m_2}U_{c_u^1 p}^2V_{c_v^2 q}^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{p=1}^{n_1}\sum_{q=1}^{n_2}\left(\sum_{u=1}^{m_1}\frac{U_{c_u^1 p}^2}{\mu_p}\sum_{v=1}^{m_2}\frac{V_{c_v^2 q}^2}{\eta_q}\right)\\
&= \frac{1}{2}\left(\sum_{p=1}^{n_1}\sum_{u=1}^{m_1}\frac{U_{c_u^1 p}^2}{\mu_p}\right)\left(\sum_{q=1}^{n_2}\sum_{v=1}^{m_2}\frac{V_{c_v^2 q}^2}{\eta_q}\right)\\
&= \frac{1}{2}\left(\sum_{p=1}^{n_1}\frac{1}{\mu_p}\sum_{u=1}^{m_1}U_{c_u^1 p}^2\right)\left(\sum_{q=1}^{n_2}\frac{1}{\eta_q}\sum_{v=1}^{m_2}V_{c_v^2 q}^2\right)\\
&= \frac{1}{2}\times 2\sum_{p=1}^{n_1}[\Phi_1]_{pp}\times 2\sum_{q=1}^{n_2}[\Phi_2]_{qq}\\
&= 2\mathrm{tr}\left(\Phi_1\right)\mathrm{tr}\left(\Phi_2\right)\\
&= 2\mathrm{tr}\left(\mathcal{W}_1\right)\mathrm{tr}\left(\mathcal{W}_2\right),
\end{aligned}
$$

which is equivalent to (9). The proof is, thus, finished.

Theorem 1 implies that the average controllability of the Kronecker product network can be calculated accurately by using the average controllability of its subnetworks. For a large-scale composite network generated by the Kronecker product, Theorem 1 can improve the computational efficiency of the average controllability considerably. Furthermore, from (9), one has $\mathrm{tr}\left(\mathcal{W}_\otimes\right) \neq 0$ if and only if neither $\mathrm{tr}\left(\mathcal{W}_1\right)$ nor $\mathrm{tr}\left(\mathcal{W}_2\right)$ is 0. This implies that a composite network generated by the Kronecker product is found to be controllable only if all the average controllability measures of its subnetworks are greater than 0.

Example 1. Consider two subsystems with

$$
L_1 = \begin{bmatrix} 2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}, L_2 = \begin{bmatrix} 2 & -1 & -1\\ -1 & 2 & 1\\ -1 & 1 & 2\end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}.
$$

The resulting Kronecker product network can be obtained with

$$
L = L_1\otimes L_2 = \begin{bmatrix}
4 & -2 & -2 & 2 & -1 & -1 & 2 & -1 & -1\\
-2 & 4 & 2 & -1 & 2 & 1 & -1 & 2 & 1\\
-2 & 2 & 4 & -1 & 1 & 2 & -1 & 1 & 2\\
2 & -1 & -1 & 4 & -2 & -2 & 2 & -1 & -1\\
-1 & 2 & 1 & -2 & 4 & 2 & -1 & 2 & 1\\
-1 & 1 & 2 & -2 & 2 & 4 & -1 & 1 & 2\\
2 & -1 & -1 & 2 & -1 & -1 & 4 & -2 & -2\\
-1 & 2 & 1 & -1 & 2 & 1 & -2 & 4 & 2\\
-1 & 1 & 2 & -1 & 1 & 2 & -2 & 2 & 4
\end{bmatrix}, \quad B = B_1\otimes B_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}^{\mathrm{T}}.
$$

It can then be calculated that $\mathrm{tr}\left(\mathcal{W}_1\right) = \frac{3}{4}$, $\mathrm{tr}\left(\mathcal{W}_2\right) = \frac{3}{4}$, and $\mathrm{tr}\left(\mathcal{W}_\otimes\right) = \frac{9}{8}$, where $\mathcal{W}_\otimes$, $\mathcal{W}_1$, and $\mathcal{W}_2$ are the controllability Gramians of $(-L, B)$, $(-L_1, B_1)$, and $(-L_2, B_2)$, respectively. This verifies Theorem 1.

Corollary 1. Consider N subsystems with Laplacian dynamics. The average controllability of the Kronecker product network with multiple (more than two) subnetworks is

$$
\mathrm{tr}\left(\mathcal{W}_\otimes\right) = 2^{N-1}\prod_{k=1}^{N}\mathrm{tr}\left(\mathcal{W}_k\right), \tag{17}
$$

where $\mathcal{W}_\otimes$ and $\mathcal{W}_k$ are the controllability Gramians of $(-L, B)$ and $(-L_k, B_k)$, respectively, for $k = 1,\ldots,N$, and $L = L_1\otimes\cdots\otimes L_N$, $B = B_1\otimes\cdots\otimes B_N$.

Proof. Corollary 1 can be obtained directly from Theorem 1. Thus, the proof is omitted here.

Corollary 1 shows that the average controllability of a large-scale composite network can be obtained by computing its attendant small subnetworks. In other words, the average controllability of a composite system with more than two subnetworks can then be obtained quickly by applying (17).

Theorem 2. Consider two subsystems with Laplacian dynamics described by (4) and (5). The volumetric control energy of the composite network generated by the Kronecker product then satisfies

$$\log \det \mathcal{W}_{\otimes} \leqslant n_1 n_2 \log \frac{2\mathrm{tr}\left(\mathcal{W}_1\right) \mathrm{tr}\left(\mathcal{W}_2\right)}{n_1 n_2}, \tag{18}$$

where $\mathcal{W}_{\otimes}$, $\mathcal{W}_1$, and $\mathcal{W}_2$ are the controllability Gramians of $(-L, B)$, $(-L_1, B_1)$, and $(-L_2, B_2)$, respectively, and $L_1 \in \mathbb{R}^{n_1 \times n_1}$, $L_2 \in \mathbb{R}^{n_2 \times n_2}$, $L = L_1 \otimes L_2$, $B = B_1 \otimes B_2$.

Proof. By (10), one has

$$\begin{aligned} \log \det \mathcal{W}_{\otimes} &= \log \det \left(P \Phi P^{\mathrm{T}}\right) \\ &= \log \left(\det P \det \Phi \det P^{\mathrm{T}}\right) \\ &= \log \det \Phi. \end{aligned} \tag{19}$$

Substituting (13) into (19) and using Proposition 1 and Lemma 1, one has

$$\begin{aligned} \log \det \Phi &\leqslant \log \prod_{i=1}^{n} \left(\frac{1}{2\lambda_i} \sum_{k=1}^{m} P_{c_k i}^2 \right) \\ &= \sum_{i=1}^{n} \log \left(\frac{1}{2\lambda_i} \sum_{k=1}^{m} P_{c_k i}^2 \right) \\ &= \sum_{p=1}^{n_1} \sum_{q=1}^{n_2} \log \left(\sum_{u=1}^{m_1} \sum_{v=1}^{m_2} \frac{1}{2\mu_p \eta_q} U_{c_u^1 p}^2 V_{c_v^2 q}^2 \right) \\ &= \sum_{p=1}^{n_1} \sum_{q=1}^{n_2} \log \left(\frac{1}{2} \sum_{u=1}^{m_1} \frac{1}{\mu_p} U_{c_u^1 p}^2 \sum_{v=1}^{m_2} \frac{1}{\eta_q} V_{c_v^2 q}^2 \right) \\ &= \sum_{p=1}^{n_1} \sum_{q=1}^{n_2} \left(\log \left(\sum_{u=1}^{m_1} \frac{1}{\mu_p} U_{c_u^1 p}^2 \right) + \log \left(\sum_{v=1}^{m_2} \frac{1}{\eta_q} V_{c_v^2 q}^2 \right) - \log 2 \right) \\ &= n_2 \sum_{p=1}^{n_1} \log \left(\sum_{u=1}^{m_1} \frac{1}{\mu_p} U_{c_u^1 p}^2 \right) + n_1 \sum_{q=1}^{n_2} \log \left(\sum_{v=1}^{m_2} \frac{1}{\eta_q} V_{c_v^2 q}^2 \right) - n_1 n_2 \log 2. \end{aligned} \tag{20}$$

Note that

$$[\Phi_1]_{pp} = \frac{1}{2\mu_p} \sum_{u=1}^{m_1} U_{c_u^1 p}^2, \; [\Phi_2]_{qq} = \frac{1}{2\eta_q} \sum_{v=1}^{m_2} V_{c_v^2 i}^2, \tag{21}$$

where Φ_1 and Φ_2 are defined by (14) and (15), respectively. By arithmetic-geometric mean inequality, one has

$$\left(\prod_{p=1}^{n_1} [\Phi_1]_{pp} \right)^{\frac{1}{n_1}} \leqslant \frac{\sum_{p=1}^{n_1} [\Phi_1]_{pp}}{n_1} = \frac{\mathrm{tr}\left(\mathcal{W}_1\right)}{n_1}, \tag{22}$$

$$\left(\prod_{q=1}^{n_2} [\Phi_2]_{qq} \right)^{\frac{1}{n_2}} \leqslant \frac{\sum_{q=1}^{n_2} [\Phi_2]_{qq}}{n_2} = \frac{\mathrm{tr}\left(\mathcal{W}_2\right)}{n_2}. \tag{23}$$

Substituting (21) into (20), and using (22) and (23) gives us

$$\begin{aligned} \log \det \mathcal{W}_{\otimes} &= \log \det \Phi \\ &\leqslant n_2 \sum_{p=1}^{n_1} \log \left(2 \left[\Phi_1\right]_{pp} \right) + n_1 \sum_{q=1}^{n_2} \log \left(2 \left[\Phi_2\right]_{qq} \right) - n_1 n_2 \log 2 \end{aligned}$$

$$= n_2 \log \left(\prod_{p=1}^{n_1} [\Phi_1]_{pp} \right) + n_1 \log \left(\prod_{q=1}^{n_2} [\Phi_2]_{qq} \right) + n_1 n_2 \log 2$$

$$\leqslant n_1 n_2 \log \frac{\operatorname{tr}(\mathcal{W}_1)}{n_1} + n_1 n_2 \log \frac{\operatorname{tr}(\mathcal{W}_2)}{n_2} + n_1 n_2 \log 2, \tag{24}$$

which is equivalent to (18). The proof is, thus, finished.

Theorem 2 demonstrates that the volumetric control energy of a Kronecker product network can be estimated by the average controllability measures of its subnetworks.

3.2 Controllability measures of Kronecker sum networks

In [53], the controllability measures of a composite network can be obtained from the spectral properties of the subnetworks. However, for a large-scale network, the eigenvalues of the Laplacian matrix are not easily obtained. To estimate the controllability measures of a large-scale composite network, its controllability measures must be bounded. In this subsection, the relationships between the controllability measures of a Kronecker sum network and those of its subnetworks are to be established.

Lemma 2 ([53])**.** Consider two subsystems with Laplacian dynamics as described by (4) and (5). The average controllability of a composite network generated by the Kronecker sum is

$$\operatorname{tr}(\mathcal{W}_\oplus) = \frac{1}{2} \sum_{u=1}^{m_1} \sum_{v=1}^{m_2} \sum_{p=1}^{n_1} \sum_{q=1}^{n_2} \left(\frac{1}{\mu_p + \eta_q} U_{c_u^1 p}^2 V_{c_v^2 q}^2 \right), \tag{25}$$

where $\mathcal{W}_\oplus$ is the controllability Gramian of $(-L, B)$, $L = L_1 \oplus L_2$ and $B = B_1 \otimes B_2$. Also, $U \in \mathbb{R}^{n_1 \times n_1}$ and $V \in \mathbb{R}^{n_2 \times n_2}$ are the orthogonal eigenvector matrices of L_1 and L_2, respectively, thereby satisfying $UU^{\mathrm{T}} = I_{n_1}$, $VV^{\mathrm{T}} = I_{n_2}$. Moreover, $U_{c_u^1 p}$ and $V_{c_v^2 q}$ are the (c_u^1, p)th entry and (c_v^2, q)th entry of U and V, respectively.

Theorem 3. Consider two subsystems with Laplacian dynamics as described by (4) and (5). The average controllability of the Kronecker sum network thereby satisfies

$$\operatorname{tr}(\mathcal{W}_\oplus) \leqslant \frac{m_2}{4} \operatorname{tr}(\mathcal{W}_1) + \frac{m_1}{4} \operatorname{tr}(\mathcal{W}_2), \tag{26}$$

where $\mathcal{W}_\oplus$, $\mathcal{W}_1$, and $\mathcal{W}_2$ are the controllability Gramians of $(-L, B)$, $(-L_1, B_1)$, and $(-L_2, B_2)$, respectively, and $L_1 \in \mathbb{R}^{n_1 \times n_1}$, $L_2 \in \mathbb{R}^{n_2 \times n_2}$, $B_1 \in \mathbb{R}^{n_1 \times m_1}$, $B_2 \in \mathbb{R}^{n_2 \times m_2}$, $L = L_1 \oplus L_2$, $B = B_1 \otimes B_2$.

Proof. According to the Cauchy-Schwarz inequality, one has

$$\left(\frac{1}{\mu_p} + \frac{1}{\eta_q} \right) (\mu_p + \eta_q) \geqslant 4, \tag{27}$$

$$\frac{1}{\mu_p + \eta_q} \leqslant \frac{1}{4} \left(\frac{1}{\mu_p} + \frac{1}{\eta_q} \right). \tag{28}$$

Substituting (28) into (25) and by (16), one has

$$\begin{aligned}
\operatorname{tr}(\mathcal{W}_\oplus) &= \frac{1}{2} \sum_{u=1}^{m_1} \sum_{v=1}^{m_2} \sum_{p=1}^{n_1} \sum_{q=1}^{n_2} \left(\frac{1}{\mu_p + \eta_q} U_{c_u^1 p}^2 V_{c_v^2 q}^2 \right) \\
&\leqslant \frac{1}{8} \sum_{u=1}^{m_1} \sum_{v=1}^{m_2} \sum_{p=1}^{n_1} \sum_{q=1}^{n_2} \left(\left(\frac{1}{\mu_p} + \frac{1}{\eta_q} \right) U_{c_u^1 p}^2 V_{c_v^2 q}^2 \right) \\
&= \frac{1}{8} \sum_{u=1}^{m_1} \sum_{v=1}^{m_2} \left(\sum_{p=1}^{n_1} \frac{1}{\mu_p} U_{c_u^1 p}^2 \right) \left(\sum_{q=1}^{n_2} V_{c_v^2 q}^2 \right) + \frac{1}{8} \sum_{u=1}^{m_1} \sum_{v=1}^{m_2} \left(\sum_{p=1}^{n_1} U_{c_u^1 p}^2 \right) \left(\sum_{q=1}^{n_2} \frac{1}{\eta_q} V_{c_v^2 q}^2 \right) \\
&= \frac{1}{8} \left(\sum_{u=1}^{m_1} \sum_{p=1}^{n_1} \frac{1}{\mu_p} U_{c_u^1 p}^2 \right) \left(\sum_{v=1}^{m_2} \sum_{q=1}^{n_2} V_{c_v^2 q}^2 \right) + \frac{1}{8} \left(\sum_{u=1}^{m_1} \sum_{p=1}^{n_1} U_{c_u^1 p}^2 \right) \left(\sum_{v=1}^{m_2} \sum_{q=1}^{n_2} \frac{1}{\eta_q} V_{c_v^2 q}^2 \right) \\
&= \frac{m_2}{8} \left(\sum_{u=1}^{m_1} \sum_{p=1}^{n_1} \frac{1}{\mu_p} U_{c_u^1 p}^2 \right) + \frac{m_1}{8} \left(\sum_{v=1}^{m_2} \sum_{q=1}^{n_2} \frac{1}{\eta_q} V_{c_v^2 q}^2 \right)
\end{aligned}$$

$$= \frac{m_2}{4}\mathrm{tr}\left(\mathcal{W}_1\right) + \frac{m_1}{4}\mathrm{tr}\left(\mathcal{W}_2\right), \tag{29}$$

where $\sum_{p=1}^{n_1} U_{c_u^1 p}^2 = 1$ and $\sum_{q=1}^{n_2} V_{c_v^2 q}^2 = 1$ are used. The proof is, thus, finished.

According to (26), the average controllability of the Kronecker sum network can be estimated well by using the candidate subnetworks before design. The result guides the design of a desired controllable Kronecker sum network.

Theorem 4. Consider two subsystems with Laplacian dynamics as described by (4) and (5). The volumetric control energy of the Kronecker sum network thereby satisfies

$$\log\det\mathcal{W}_{\oplus} \leqslant \frac{n_1 n_2}{2}\log\frac{\mathrm{tr}\left(\mathcal{W}_1\right)\mathrm{tr}\left(\mathcal{W}_2\right)}{n_1 n_2}, \tag{30}$$

where $\mathcal{W}_{\oplus}$, $\mathcal{W}_1$, and $\mathcal{W}_2$ are the controllability Gramians of $(-L_{\oplus}, B)$, $(-L_1, B_1)$, and $(-L_2, B_2)$, respectively, and $L_1 \in \mathbb{R}^{n_1\times n_1}$, $L_2 \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{n_1\times m_1}$, $B_2 \in \mathbb{R}^{n_2\times m_2}$, $L = L_1 \oplus L_2$, $B = B_1 \otimes B_2$.

Proof. Similar to (20), one has

$$\log\det\mathcal{W}_{\oplus} \leqslant \sum_{p=1}^{n_1}\sum_{q=1}^{n_2}\log\left(\frac{1}{2}\sum_{u=1}^{m_1}\sum_{v=1}^{m_2}\frac{1}{\mu_p+\eta_q}U_{c_u^1 p}^2 V_{c_v^2 q}^2\right), \tag{31}$$

where $U \in \mathbb{R}^{n_1\times n_1}$ and $V \in \mathbb{R}^{n_2\times n_2}$ are the orthogonal eigenvector matrices of L_1 and L_2, respectively, and μ_p and η_q are the eigenvalues of U and V, respectively, $p = 1,\ldots,n_1$, $q = 1,\ldots,n_2$, and $U_{c_u^1 p}$ and $V_{c_v^2 q}$ are the $\left(c_u^1, p\right)$th entry and $\left(c_v^2, q\right)$th entry of U and V, respectively.

Note that

$$\frac{1}{\mu_p+\eta_q} \leqslant \min\left\{\frac{1}{\mu_p},\ \frac{1}{\eta_q}\right\}, \tag{32}$$

one has

$$\begin{aligned}
\frac{1}{2}\sum_{u=1}^{m_1}\sum_{v=1}^{m_2}\frac{1}{\mu_p+\eta_q}U_{c_u^1 p}^2 V_{c_v^2 q}^2 &\leqslant \min\left\{\frac{1}{2}\sum_{u=1}^{m_1}\sum_{v=1}^{m_2}\frac{1}{\mu_p}U_{c_u^1 p}^2 V_{c_v^2 q}^2,\ \frac{1}{2}\sum_{u=1}^{m_1}\sum_{v=1}^{m_2}\frac{1}{\eta_q}U_{c_u^1 p}^2 V_{c_v^2 q}^2\right\}\\
&= \min\left\{\frac{1}{2}\sum_{u=1}^{m_1}\frac{1}{\mu_p}U_{c_u^1 p}^2\sum_{v=1}^{m_2}V_{c_v^2 q}^2,\ \frac{1}{2}\sum_{u=1}^{m_1}U_{c_u^1 p}^2\sum_{v=1}^{m_2}\frac{1}{\eta_q}V_{c_v^2 q}^2\right\}\\
&\leqslant \min\left\{\frac{1}{2}\sum_{u=1}^{m_1}\frac{1}{\mu_p}U_{c_u^1 p}^2,\ \frac{1}{2}\sum_{v=1}^{m_2}\frac{1}{\eta_q}V_{c_v^2 q}^2\right\}\\
&= \min\left\{\left[\Phi_1\right]_{pp},\ \left[\Phi_2\right]_{qq}\right\},
\end{aligned} \tag{33}$$

where $\sum_{u=1}^{m_1} U_{c_u^1 p}^2 \leqslant 1$, $\sum_{v=1}^{m_2} V_{c_v^2 q}^2 \leqslant 1$ (which are obtained from the properties of the row/column vectors of the orthogonal matrices [55]), and Eq. (21) are used.

Substituting (33) into (31), and by (22) and (23), one has

$$\begin{aligned}
\log\det\mathcal{W}_{\oplus} &\leqslant \min\left\{\sum_{p=1}^{n_1}\sum_{q=1}^{n_2}\log\left[\Phi_1\right]_{pp},\ \sum_{p=1}^{n_1}\sum_{q=1}^{n_2}\log\left[\Phi_2\right]_{qq}\right\}\\
&= \min\left\{n_2\log\prod_{p=1}^{n_1}\left[\Phi_1\right]_{pp},\ n_1\log\prod_{q=1}^{n_2}\left[\Phi_2\right]_{qq}\right\}\\
&\leqslant \min\left\{n_1 n_2\log\frac{\mathrm{tr}\left(\mathcal{W}_1\right)}{n_1},\ n_1 n_2\log\frac{\mathrm{tr}\left(\mathcal{W}_2\right)}{n_2}\right\}\\
&\leqslant \frac{n_1 n_2}{2}\log\frac{\mathrm{tr}\left(\mathcal{W}_1\right)}{n_1} + \frac{n_1 n_2}{2}\log\frac{\mathrm{tr}\left(\mathcal{W}_2\right)}{n_2},
\end{aligned} \tag{34}$$

which is equivalent to (30). The proof is, thus, finished.

Remark 1. Both Theorems 3 and 4 can be generalized to Kronecker sum networks with multiple (i.e., more than two) subnetworks.

4 Comparison of Kronecker product networks with Kronecker sum networks

In this section, the average controllability relationship between Kronecker product networks and Kronecker sum networks is examined in some detail. The subnetworks considered here are characterized by structurally unbalanced signed graphs.

Theorem 5. Consider two connected and structurally unbalanced subnetworks $\mathcal{G}_1$ and $\mathcal{G}_2$ with the Laplacian dynamics as described by (4) and (5), respectively, then

$$\frac{\operatorname{tr}(\mathcal{W}_\otimes)}{\operatorname{tr}(\mathcal{W}_\oplus)} \geqslant \frac{8}{m_1+m_2}\min\{\operatorname{tr}(\mathcal{W}_1), \operatorname{tr}(\mathcal{W}_2)\}, \tag{35}$$

where $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_\otimes$, and $\mathcal{W}_\oplus$ are the controllability Gramians of $(-L_1, B_1)$, $(-L_2, B_2)$, $(-L_\otimes, B)$, and $(-L_\oplus, B)$, respectively, and $L_\otimes = L_1 \otimes L_2$, $L_\oplus = L_1 \oplus L_2$, $B = B_1 \otimes B_2$, $L_1 \in \mathbb{R}^{n_1\times n_1}$, $L_2 \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{n_1\times m_1}$, $B_2 \in \mathbb{R}^{n_2\times m_2}$; m_1 and m_2 are the numbers of controlled nodes of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

Proof. By Theorems 1 and 3, one has

$$\begin{aligned}\frac{\operatorname{tr}(\mathcal{W}_\otimes)}{\operatorname{tr}(\mathcal{W}_\oplus)} &\geqslant \frac{2\operatorname{tr}(\mathcal{W}_1)\operatorname{tr}(\mathcal{W}_2)}{\frac{m_2}{4}\operatorname{tr}(\mathcal{W}_1)+\frac{m_1}{4}\operatorname{tr}(\mathcal{W}_2)}\\ &= \frac{8\operatorname{tr}(\mathcal{W}_1)\operatorname{tr}(\mathcal{W}_2)}{m_2\operatorname{tr}(\mathcal{W}_1)+m_1\operatorname{tr}(\mathcal{W}_2)}.\end{aligned} \tag{36}$$

Since

$$m_2\operatorname{tr}(\mathcal{W}_1)+m_1\operatorname{tr}(\mathcal{W}_2) \leqslant (m_1+m_2)\max\{\operatorname{tr}(\mathcal{W}_1), \operatorname{tr}(\mathcal{W}_2)\}, \tag{37}$$

substituting (37) into (36), one has

$$\begin{aligned}\frac{\operatorname{tr}(\mathcal{W}_\otimes)}{\operatorname{tr}(\mathcal{W}_\oplus)} &\geqslant \frac{8\operatorname{tr}(\mathcal{W}_1)\operatorname{tr}(\mathcal{W}_2)}{m_2\operatorname{tr}(\mathcal{W}_1)+m_1\operatorname{tr}(\mathcal{W}_2)}\\ &\geqslant \frac{8}{m_1+m_2}\frac{\operatorname{tr}(\mathcal{W}_1)\operatorname{tr}(\mathcal{W}_2)}{\max\{\operatorname{tr}(\mathcal{W}_1), \operatorname{tr}(\mathcal{W}_2)\}}\\ &= \frac{8}{m_1+m_2}\min\{\operatorname{tr}(\mathcal{W}_1), \operatorname{tr}(\mathcal{W}_2)\},\end{aligned} \tag{38}$$

which is equivalent to (35). The proof is, thus, finished.

According to Theorem 5, as the average controllability of the subnetwork increases, the ratio of the controllability of the Kronecker product network to that of the Kronecker sum network will also increase.

Theorem 6. Consider two connected and structurally unbalanced subnetworks $\mathcal{G}_1$ and $\mathcal{G}_2$ with the Laplacian dynamics as described by (4) and (5), respectively. If $\operatorname{tr}(\mathcal{W}_1) \geqslant \frac{m_1}{4}$ and $\operatorname{tr}(\mathcal{W}_2) \geqslant \frac{m_2}{4}$, then

$$\operatorname{tr}(\mathcal{W}_\otimes) \geqslant \operatorname{tr}(\mathcal{W}_\oplus), \tag{39}$$

where $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_\otimes$, and $\mathcal{W}_\oplus$ are the controllability Gramians of $(-L_1, B_1)$, $(-L_2, B_2)$, $(-L_\otimes, B)$, and $(-L_\oplus, B)$, respectively, and $L_\otimes = L_1 \otimes L_2$, $L_\oplus = L_1 \oplus L_2$, $B = B_1 \otimes B_2$, $L_1 \in \mathbb{R}^{n_1\times n_1}$, $L_2 \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{n_1\times m_1}$, $B_2 \in \mathbb{R}^{n_2\times m_2}$; m_1 and m_2 are the numbers of controlled nodes of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

Proof. Since

$$\operatorname{tr}(\mathcal{W}_1) \geqslant \frac{m_1}{4}, \quad \operatorname{tr}(\mathcal{W}_2) \geqslant \frac{m_2}{4}, \tag{40}$$

by Theorems 1 and 3, one has

$$\begin{aligned}\operatorname{tr}(\mathcal{W}_\otimes) &= 2\operatorname{tr}(\mathcal{W}_1)\operatorname{tr}(\mathcal{W}_2)\\ &= \operatorname{tr}(\mathcal{W}_1)\operatorname{tr}(\mathcal{W}_2)+\operatorname{tr}(\mathcal{W}_1)\operatorname{tr}(\mathcal{W}_2)\\ &\geqslant \frac{m_1}{4}\operatorname{tr}(\mathcal{W}_2)+\frac{m_2}{4}\operatorname{tr}(\mathcal{W}_1)\\ &\geqslant \operatorname{tr}(\mathcal{W}_\oplus),\end{aligned} \tag{41}$$

which is equivalent to (39). The proof is, thus, finished.

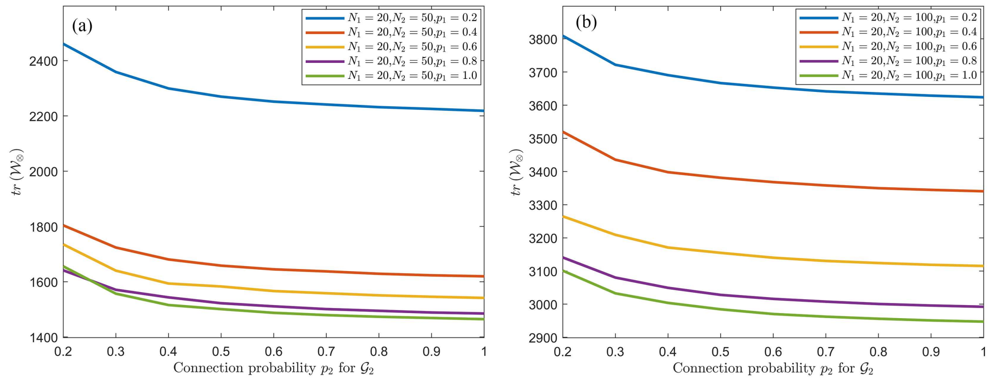

Figure 3 (Color online) Relationship between $\mathrm{tr}\,(\mathcal{W}_\otimes)$ and connection probabilities p_1 and p_2. The experiment is repeated 10 times. (a) $N_1 = 20$ and $N_2 = 50$; (b) $N_1 = 20$ and $N_2 = 100$.

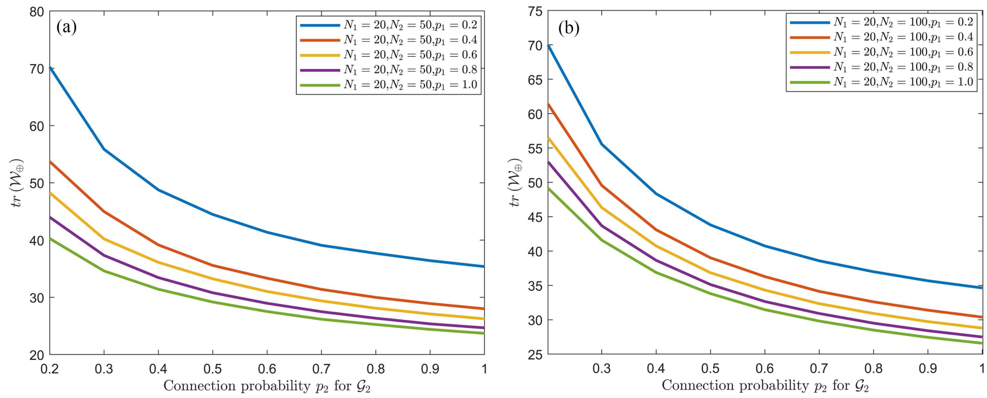

Figure 4 (Color online) Relationship between $\mathrm{tr}\,(\mathcal{W}_\oplus)$ and connection probabilities p_1 and p_2. The experiment is repeated 10 times. (a) $N_1 = 20$ and $N_2 = 50$; (b) $N_1 = 20$ and $N_2 = 100$.

Theorem 6 gives a sufficient condition under which the composite network constructed by the Kronecker product is found to have better average controllability than the composite network constructed by the Kronecker sum.

To illustrate the above theoretical results, the relationship of average controllability between Kronecker product networks and Kronecker sum networks is examined using comprehensive numerical simulations. The subnetworks are first generated by the well-known Erdös-Rényi (ER) random networks. To then assess the effect of subnetwork structures on the composite network controllability, control input is injected into each node of the network. To then ensure that the subnetworks are structurally unbalanced, a negative cycle is added to each subnetwork.

In Figures 3(a), 4(a), and 5(a), the number of nodes of the subnetwork $\mathcal{G}_1$ is set to $N_1 = 20$, the connection probabilities p_1 are set to 0.2, 0.4, 0.6, 0.8, and 1.0, respectively, and the number of nodes of the subnetwork $\mathcal{G}_2$ is set to $N_2 = 50$, the connection probabilities p_2 are set to 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1.0, respectively. Figure 3(a) shows the average controllability of the Kronecker product network constructed by $\mathcal{G}_1$ and $\mathcal{G}_2$. It can then be observed that when the number of nodes is fixed, the average controllability decreases with an increase in the connection probability of the subnetworks. A similar observation can be made for the Kronecker sum network, as shown in Figure 4(a). This indicates that the graph product network composed of sparse subnetworks has better average controllability than the one composed of denser subnetworks. In Figure 5(a), $\mathrm{tr}\,(\mathcal{W}_\otimes)$ is much larger than $\mathrm{tr}\,(\mathcal{W}_\oplus)$, and their ratio $\frac{\mathrm{tr}(\mathcal{W}_\otimes)}{\mathrm{tr}(\mathcal{W}_\oplus)}$ increases with the increase of the connection probability of the subnetworks. This then suggests that the Kronecker product network has better average controllability than the Kronecker sum network.

Additionally, graph product networks with $N_1 = 20$ and $N_2 = 100$ are tested, as shown in Figures 3(b), 4(b), and 5(b). The similar results observed in Figures 3(a), 4(a), and 5(a) can be also observed in Figures 3(b), 4(b), and 5(b). Indeed, the theoretical results agree well with the numerical simulations.

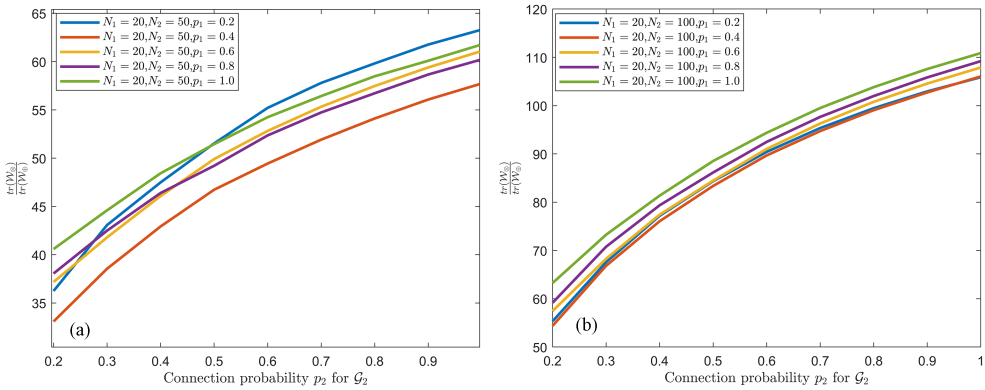

Figure 5 (Color online) Relationship between $\frac{\mathrm{tr}(\mathcal{W}_\otimes)}{\mathrm{tr}(\mathcal{W}_\oplus)}$ and connection probabilities p_1 and p_2. The experiment is repeated 10 times. (a) $N_1 = 20$ and $N_2 = 50$; (b) $N_1 = 20$ and $N_2 = 100$.

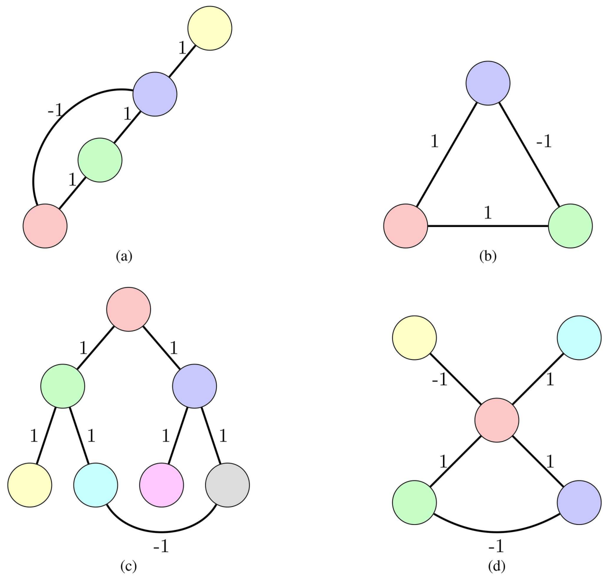

Figure 6 (Color online) Examples of widely used special subgraphs where the number of nodes equals the number of edges. (a) A signed chain with a cycle; (b) a signed cycle graph; (c) a signed tree with a cycle; (d) a signed star with a cycle.

Next, a corollary is obtained for some sparse yet widely used subnetworks. The subnetworks considered here are characterized by structurally unbalanced signed graphs in which the number of nodes is found to be equal to the number of edges. Figure 6 illustrates the examples of the topologies of these subnetworks.

Lemma 3 is given for the proof of Corollary 2.

Lemma 3. Consider a connected and structurally unbalanced signed network $\mathcal{G}$ with the Laplacian dynamics as described by (1), with $m = n$ and $B = I_n$. Then,

$$\mathrm{tr}\left(\mathcal{W}\right) \geqslant \frac{n^2}{4\hat{e}}, \tag{42}$$

where $\mathcal{W}$ is the controllability Gramian of $(-L, B)$, n and $\hat{e}$ are the numbers of nodes and edges of $\mathcal{G}$, respectively, and m is the number of control inputs.

Proof. Note that for $n = m$, by (13), one has

$$\Phi_{ii} = \frac{1}{2\lambda_i}\sum_{k=1}^{n} P_{c_k i}^2 = \frac{1}{2\lambda_i}, \tag{43}$$

then

$$\mathrm{tr}\left(\mathcal{W}\right) = \sum_{i=1}^{n}\Phi_{ii} = \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\lambda_i}, \tag{44}$$

where $\sum_{k=1}^{n} P_{c_k i}^2 = 1$ is used and λ_i is the ith eigenvalues of L, $i = 1, \ldots, n$.

According to Cauchy-Schwarz inequality, one has

$$\left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\right)\left(\sum_{i=1}^{n} \lambda_i\right) \geqslant \left(\sum_{i=1}^{n} \frac{1}{\sqrt{\lambda_i}}\sqrt{\lambda_i}\right)^2, \tag{45}$$

$$\sum_{i=1}^{n} \frac{1}{\lambda_i} \geqslant \frac{n^2}{\sum_{i=1}^{n} \lambda_i} = \frac{n^2}{\sum_{i=1}^{n} \hat{d}_i}, \tag{46}$$

where $\hat{d}_i$ is the degree of node i and $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \hat{d}_i$ is employed [56]. Then,

$$\operatorname{tr}(\mathcal{W}) = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{\lambda_i} \geqslant \frac{n^2}{2\sum_{i=1}^{n} \hat{d}_i} = \frac{n^2}{4\hat{e}}, \tag{47}$$

where $\hat{e}$ is the number of edges.

Corollary 2. Consider two connected and structurally unbalanced subnetworks $\mathcal{G}_1$ and $\mathcal{G}_2$ with the Laplacian dynamics as described by (4) and (5), respectively. If the number of edges in each subnetwork is equal to the number of nodes and all the nodes of $\mathcal{G}_i$ are under control, i.e., $n_i = \hat{e}_i$, $m_i = n_i$, and $B_i = I_{n_i}$, $i = 1, 2$. Then,

$$\operatorname{tr}(\mathcal{W}_{\otimes}) \geqslant \operatorname{tr}(\mathcal{W}_{\oplus}), \tag{48}$$

where $\mathcal{W}_{\otimes}$ and $\mathcal{W}_{\oplus}$ are the controllability Gramians of $(-L_{\otimes}, B)$ and $(-L_{\oplus}, B)$, respectively, and $L_{\otimes} = L_1 \otimes L_2$, $L_{\oplus} = L_1 \oplus L_2$, $B = B_1 \otimes B_2$, $L_1 \in \mathbb{R}^{n_1 \times n_1}$, $L_2 \in \mathbb{R}^{n_2 \times n_2}$, $B_1 \in \mathbb{R}^{n_1 \times m_1}$, $B_2 \in \mathbb{R}^{n_2 \times m_2}$; n_i, $\hat{e}_i$, and m_i are the numbers of nodes, edges, and controlled nodes of the subnetwork $\mathcal{G}_i$, respectively, $i = 1, 2$.

Proof. Corollary 2 can be directly obtained from Theorem 6 and Lemma 3, so the proof is, therefore, omitted here.

5 Conclusion

This paper investigates the controllability Gramian-based measures of graph product networks and reveals novel relationships of the controllability measures between the subnetworks and the resulting graph product networks. First, the analytical expression for the average controllability of the Kronecker product network is derived. This allows for the subsequent accurate calculation of the average controllability of the composite network from its relatively smaller subnetworks, offering a more efficient approach to large-scale networks. Then, the upper bound of the volumetric control energy of the Kronecker product network is estimated. Furthermore, the upper bounds of the controllability Gramian-based measures of the Kronecker sum network are also estimated. Finally, the comparison of the Kronecker product networks with the Kronecker sum networks is provided. It is demonstrated that the Kronecker product network has better average controllability than the Kronecker sum network when the average controllability of the subnetwork satisfies certain conditions. These findings, therefore, provide valuable references for designing controllable networks in practical applications. In summary, this work paves the way for new research directions in an attempt to properly control graph product networks, even multiplex networks (or multilayer networks) [57, 58] and higher-order networks [59].

In future research, there is potential to explore other types of composite networks, such as tensor product composite networks. Additionally, studying the stability/observability of composite networks is an important avenue in further research, which includes exploring the stability/observability measures of composite networks and examining the relationship between the stability/observability of the attendant subsystems and the associated composite network. Moreover, robustness and optimization [60–63] of network-controllability-based graph products, and even the controllability of networks with higher-order interactions [64] are also worthy of in-depth investigation.

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