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Mean-square exponential stability of stochastic Volterra systems in infinite dimensions

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Abstract Over the past decade, the study of stability theory in integro-differential systems has grown significantly owing to their relevance in solving physical and engineering problems, such as viscoelasticity and thermo-viscoelasticity in materials with memory properties. This paper concentrates on a class of infinite-dimensional stochastic integro-differential systems. We establish the well-posedness of the system and identify mild solutions to the system and an abstract stochastic Cauchy problem. This identification is identified by employing a semigroup approach combined with Yosida approximation. We derive sufficient conditions that ensure the mean-square exponential stability of mild solutions to the system boils down to the boundedness of a certain function and a norm estimate for the stochastic part. These conditions are implemented through the semigroup approach and the composition operator method. Illustrative examples are provided and the obtained theoretical results are validated by numerical simulations.

Keywords stochastic integro-differential equations, mean-square exponential stability, stochastic distributed parameter systems, Hardy space, composition operators

1 Introduction

In this paper, we are concerned with the following infinite-dimensional linear stochastic integro-differential equation system:

$$\begin{cases} \mathrm{d}x(t) = \left[Ax(t) + \int_0^t a(t-s)Ax(s)\,\mathrm{d}s\right]\mathrm{d}t + \left[Bx(t) + \int_0^t b(t-s)B_1x(s)\,\mathrm{d}s\right]\mathrm{d}W(t), \\ x(0) = x_0. \end{cases} \tag{1}$$

Here, $x(t) \in L^2_{\mathcal{F}_t}(\Omega; X)$ indicates the system state at time t; $x_0 \in L^2_{\mathcal{F}_0}(\Omega; X)$ represents the initial state; A is the generator of strongly continuous semigroup (simply, C_0-semigroup or operator semigroup; see [1,2] for more information) $\{T(t) : t \geqslant 0\}$ on X; $a(\cdot)$, $b(\cdot) \in \mathcal{H}^1(\mathbb{R}_+)$ are kernel functions; $W(\cdot)$ is a Q-Brownian motion (called also Q-Wiener process; see [3,4] for more details) on V with the strictly positive covariance operator $Q \in \mathcal{L}_1(V)$, more explicitly, Q is self-adjoint and $\langle Qz, z\rangle_V \geqslant m\|z\|^2_V$ for any $z \in V$ for some constant $m > 0$; the operators B, $B_1 \colon X \to \mathcal{L}^0_2$ are bounded. Obviously, Eq. (1) is a stochastic version of the following Volterra system:

$$\begin{cases} \dot{x}(t) = Ax(t) + \int_0^t a(t-s)Ax(s)\,\mathrm{d}s, \\ x(0) = x_0. \end{cases} \tag{2}$$

Here, $x(t) \in X$ indicates the system state at time t; as usual, the dot stands for the derivative with regard to time; $x_0 \in X$ represents the initial state; $a(\cdot)$ and A are defined as in (1).

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The Volterra system (2) models some problems related to viscoelasticity and thermo-viscoelasticity, taking into account the memory behavior of the materials. This is applicable in various areas, such as continuum mechanism (e.g., simple shear motion, rod torsion and simple tension) and transient velocity fields in an isotropic viscoelastic fluid [5, Chapter I.5], Timoshenko beam [6] and in particular, in heat conduction with memory [7, 8]. The convolution term in (2) appears also in the theory of fractional calculus and fractional differential equations. Fractional differential equations strive to describe the dynamic evolution of physical models that incorporate memory effects more realistically [9–11], which are essentially forming a class of integro-differential equations with specific convolution terms. The fractional derivatives and integrals can be roughly regarded as interpolations between common derivatives and integrals. The most popular fractional integral is the Riemann-Liouville fractional integral owing to its power kernel function. However, the power law distribution presents certain challenges in practical applications, leading to the development of the α-order Caputo derivative was developed:

$$(D^{\alpha} f)(t) := \int_0^t K_{\alpha}(t-s) f'(s)\,\mathrm{d}s, \quad t \geqslant 0,$$

where $0 \leqslant \alpha < 1$. In this type of fractional derivatives, different kernel functions $K_{\alpha}(\cdot)$ results in different fractional derivatives, including the well-known Caputo-Fabrizio derivative [12], Atangana-Baleanu derivative [13] and Atangana-Gómez derivative [14].

The study into evolutionary integral equations, focusing on their well-posedness, was extensively discussed by Prüss [5] in 1993. As outlined in [5, Chapter I] (see also [15]), Eq. (2) is well-posed. This means that there exists a strong solution $x(\cdot, x_0)$ on $\mathbb{R}_+$ to (2) for any $x_0 \in D(A)$; moreover, for any sequence $\{x_n\}_{n=1}^{\infty} \subset D(A)$ with $x_n \to 0$ it follows that $x(\cdot, x_n) \to 0$ in X uniformly on each compact interval. This implies the existence of what is termed the resolvent family $\{S(t) : t \geqslant 0\}$ for (2) exists, meaning a family $\{S(t) : t \geqslant 0\}$ of linear bounded operators in X that satisfies the following conditions: (i) $S(0) = I$ and $S(\cdot)$ is strongly continuous on $\mathbb{R}_+$; (ii) $S(t)$ commutes with A for every $t \geqslant 0$, meaning that $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for every $t \geqslant 0$ and $x \in D(A)$; (iii) $S(\cdot)x_0$ is a strong solution to (2) for each $x_0 \in D(A)$. It is obvious that the resolvent family generalizes the concept of the C_0-semigroup, maintaining strong continuity without adhering to the semigroup property. Further examination reveals that the mild solution to (2) with the initial value $x_0 \in X$ is given by $S(\cdot)x_0$, and $\{S(t) : t \geqslant 0\}$ is exponentially bounded.

However, it is challenging to represent strong and mild solutions to the stochastic Volterra system (1) or even general abstract stochastic differential equations in a similar manner as "$S(t)x_0$". In Section 3, the well-posedness of (1), more precisely, the existence and uniqueness of the mild solution, along with the continuous dependence of the mild solution on initial data, will be explored by applying the Banach fixed point theorem. Additionally, a correspondence between the strong solutions of (1) and those of an abstract stochastic Cauchy problem shall be established by employing embedding methods and a semigroup approach. These techniques have also been adopted in [2, Section VI.7] and [16, 17] to establish such correspondences for abstract deterministic integro-differential equations. In the deterministic context, the density of $D(A)$ in X implies that the strong solutions are dense in the set of all mild solutions for (2). In the stochastic context, nevertheless, the density of $D(A)$ in X is not sufficient to derive the density of strong solutions in the set of all mild solutions for (1) or general abstract stochastic differential equations. This implies that this correspondence cannot be naturally extended from strong solutions to mild solutions for (1). For this reason, the Yosida approximation of (1) must be introduced, which is usually employed to extend some properties from strong solutions to mild solutions for abstract stochastic differential equations, as seen in [3, 4, 18, 19]. Recent literature on abstract stochastic integro-differential equations includes works by [19–22], with the exponential stability being investigated in [20, 21] therein.

It is widely recognized from the celebrated Paley-Wiener theorem (see [23]; for its vector-valued version [24, Proposition 12.5.4]) that the Laplace transformation

$$\mathscr{L}\colon L^2(\mathbb{R}_+; X) \to H^2(\mathbb{C}_+; X)$$

is an isometric isomorphism. Following this, the issue of estimating composition operators on Hardy spaces arises naturally when needing to ascertain the L^2 norm of the solutions on $\mathbb{R}_+$ has to be required for abstract integro-differential equations. Therefore, the theory of composition operators on Hardy spaces (see [25–27] for more detailed information) plays an important role in abstract (deterministic or stochastic) integro-differential equations, which has been utilized to study infinite-time admissibility and infinite-time exact observability of abstract Volterra systems in [28, 29]. The mean-square exponential stability of the mild solution to (1), the main result of this paper, will be established in Section 4 by employing a semigroup approach and applying a composition operator theory.

2 Preliminaries

Let X, V be two separable Hilbert spaces. Let $(\Omega, \mathcal{F}, \boldsymbol{F}, \mathbb{P})$ be a complete probability space with the natural filtration $\boldsymbol{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). $L^2_{\mathcal{F}_t}(\Omega; X)$ denotes the Hilbert space consisting of all $\mathcal{F}_t$-measurable random variables $\xi\colon \Omega \to X$ with

$$\|\xi\|^2_{L^2_{\mathcal{F}_t}(\Omega;X)} := \mathbb{E}\|\xi\|^2_X := \int_\Omega \|\xi\|^2_X \,\mathrm{d}\mathbb{P} < \infty.$$

$\mathbb{R}_+ := [0, \infty)$ and $\mathbb{C}_+ := \{\, \lambda \in \mathbb{C} \mid \mathrm{Re}\lambda > 0 \,\}$. $\mathcal{H}^1(\mathbb{R}_+; Y)$ represents the vector-valued Sobolev space of all squarely integrable functions with first order derivatives being still squarely integrable for some Banach space Y, and $\mathcal{H}^1(\mathbb{R}_+) := \mathcal{H}^1(\mathbb{R}_+; \mathbb{R})$. $\mathcal{L}_1(V)$ denotes the set of all trace-class operators (called also nuclear operators; see, for instance, [30, Section VI.2] and [31, Chapter III]) on V, which is a Banach space endowed with the trace norm

$$\|P\|_{\mathcal{L}_1(V)} := \mathrm{tr}\big((P^*P)^{\frac{1}{2}}\big) := \sum_{i=1}^{\infty} \big\langle (P^*P)^{\frac{1}{2}} e_i, e_i \big\rangle_V, \quad \forall\, P \in \mathcal{L}_1(V)$$

for an orthonormal basis $\{e_i\}_{i=1}^{\infty} \subset V$. $V_0 := Q^{\frac{1}{2}}V$ stands for the image of V under the operator $Q^{\frac{1}{2}}$ which is a separable Hilbert space equipped with the inner product

$$\langle u, v\rangle_{V_0} := \langle Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v\rangle_V, \quad \forall\, u, v \in V_0.$$

$\mathcal{L}_2(V_0, Y) = \big\{\, F \in \mathcal{L}(V_0, Y) \mid \mathrm{tr}[(FQ^{\frac{1}{2}})(FQ^{\frac{1}{2}})^*] < \infty \,\big\}$ is the set of Hilbert-Schmidt operators from V_0 into some separable Hilbert space Y, which is a separable Hilbert space endowed with the inner product

$$\langle F, G\rangle_{\mathcal{L}_2(V_0,Y)} := \mathrm{tr}[(FQ^{\frac{1}{2}})(GQ^{\frac{1}{2}})^*], \quad \forall\, F, G \in \mathcal{L}_2(V_0, Y),$$

and $\mathcal{L}^0_2 := \mathcal{L}_2(V_0, X)$. We refer to [32, Section I.2] and [3, Section IV.2] for V_0 and $\mathcal{L}^0_2$, and [30, Section VI.2] for Hilbert-Schmidt operators. $H^2(\mathbb{C}_+; X)$ stands for the Hardy space on the right half plane (see [26, 33] for more information) consisting of all vector-valued holomorphic functions on $\mathbb{C}_+$ with

$$\|f\|^2_{H^2(\mathbb{C}_+;X)} := \sup_{x>0} \int_{-\infty}^{\infty} \|f(x+iy)\|^2_X \,\frac{\mathrm{d}y}{2\pi} < \infty.$$

Introduce the following important solution space for abstract stochastic differential equations. For any Banach space Y and any $T_0 > 0$, denote

$$C_{\mathbb{F}}([0,T_0]; L^2(\Omega; Y)) := \big\{\, \psi\colon [0,T_0]\times\Omega \to Y \,\big|\, \psi(\cdot)\colon [0,T_0] \to L^2_{\mathcal{F}_{T_0}}(\Omega; Y) \text{ is } \boldsymbol{F}\text{-adapted and continuous} \,\big\},$$

which forms a Banach space equipped with the norm

$$\|\psi(\cdot)\|_{C_{\mathbb{F}}([0,T_0];L^2(\Omega;Y))} := \sup_{t\in[0,T_0]} \big(\mathbb{E}\|\psi(t)\|^2_Y\big)^{\frac{1}{2}}, \quad \forall\, \psi \in C_{\mathbb{F}}([0,T_0]; L^2(\Omega; Y)).$$

We refer to [20, 22] for the notions of strong and mild solutions of stochastic Volterra equations, and consider the following inhomogeneous form of (1):

$$\begin{cases} \mathrm{d}x(t) = \left[Ax(t) + \displaystyle\int_0^t a(t-s)Ax(s)\,\mathrm{d}s + f_0(t)\right]\mathrm{d}t + \left[Bx(t) + \displaystyle\int_0^t b(t-s)B_1x(s)\,\mathrm{d}s\right]\mathrm{d}W(t), \\ x(0) = x_0, \end{cases} \tag{3}$$

where $f_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(\mathbb{R}_+; X))$. Let $T_0 > 0$.

Definition 1. An X-valued, $\boldsymbol{F}$-adapted, continuous stochastic process $x(\cdot)$ is called a strong solution to (3) on time interval $[0,T_0]$ if

(a) $x(t)\in D(A)$ for almost all $(t,\omega)\in[0,T_0]\times\Omega$, and

$$\begin{aligned} Ax(\cdot)+\int_0^{\cdot} a(\cdot-s)Ax(s)\,\mathrm{d}s+f_0(\cdot)\in L^1(0,T_0;X), \qquad \text{a.s.},\\ Bx(\cdot)+\int_0^{\cdot} b(\cdot-s)B_1x(s)\,\mathrm{d}s\in L^2(0,T_0;\mathcal{L}_2^0), \qquad \text{a.s.}; \end{aligned} \tag{4}$$

(b) for any $t\in[0,T_0]$,

$$\begin{aligned} x(t)=x_0&+\int_0^t Ax(s)\,\mathrm{d}s+\int_0^t\int_0^r a(r-s)Ax(s)\,\mathrm{d}s\,\mathrm{d}r+\int_0^t f_0(s)\,\mathrm{d}s\\ &+\int_0^t B(x(s))\,\mathrm{d}W(s)+\int_0^t\int_0^r b(r-s)B_1x(s)\,\mathrm{d}s\,\mathrm{d}W(r), \qquad \text{a.s.} \end{aligned} \tag{5}$$

Definition 2. An X-valued, $\boldsymbol{F}$-adapted, continuous stochastic process $x(\cdot)$ is called a mild solution to (3) on time interval $[0,T_0]$ if $f_0\in L^1(0,T_0;X)$ almost surely, Eq. (4) holds, and

$$\begin{aligned} x(t)=S(t)x_0&+\int_0^t S(t-s)f_0(s)\,\mathrm{d}s+\int_0^t S(t-s)B(x(s))\,\mathrm{d}W(s)\\ &+\int_0^t S(t-r)\int_0^r b(r-s)B_1x(s)\,\mathrm{d}s\,\mathrm{d}W(r), \qquad \text{a.s.} \end{aligned}$$

for any $t\in[0,T_0]$, where $\{S(t):t\geqslant 0\}$ is the resolvent family for (2).

Definition 3. Stochastic Volterra system (1) is said to be mean-square exponentially stable if there exist two constants $M\geqslant 1$ and $\omega<0$ such that for every mild solution $x(\cdot,x_0)$ to (1) with the initial datum $x_0\in L^2_{\mathcal{F}_0}(\Omega;X)$,

$$\mathbb{E}\|x(t,x_0)\|_X^2\leqslant M\mathrm{e}^{\omega t}\mathbb{E}\|x_0\|_X^2, \quad \forall\, t\geqslant 0.$$

We conclude this section with an example of a finite-dimensional system that is mean-square exponentially stable to provide deeper insight into (1). This example will be revisited in Example 2, where we will apply our main results.

Example 1. Let us take

$$X=\mathbb{R}, \qquad V=\mathbb{R}, \qquad Q=I, \qquad A=-2I; \qquad a(t)=-\mathrm{e}^{-2t}, \qquad b(t)=0, \qquad t\geqslant 0;$$

$$B(x)v:=B_1(x)v:=\frac{1}{3}xv, \qquad v\in V, \quad \forall\, x\in X$$

in (1) with initial $x(0)=x_0\in X$ almost surely. Then Eq. (1) becomes the following stochastic ordinary integro-differential equation system:

$$\begin{cases} \mathrm{d}x(t)=\left(-2x(t)+2\displaystyle\int_0^t \mathrm{e}^{-2(t-s)}x(s)\,\mathrm{d}s\right)\mathrm{d}t+\dfrac{1}{3}x(t)\mathrm{d}W(t),\\ x(0)=x_0, \qquad \text{a.s.} \end{cases} \tag{6}$$

According to Itô's formula (see, for instance, [34, Theorem 6.2]), the solution to (6) is given by

$$x(t)=\frac{1}{2}\left(\mathrm{e}^{(\sqrt{2}-\frac{37}{18})t}+\mathrm{e}^{(-\sqrt{2}-\frac{37}{18})t}\right)\mathrm{e}^{\frac{1}{3}W(t)}x_0, \qquad t\geqslant 0, \qquad \text{a.s.}$$

It is not hard from the standard exponential martingale properties of Brownian motions to get

$$\mathbb{E}|x(t)|^2=\frac{1}{4}\left(\mathrm{e}^{(\sqrt{2}-\frac{35}{18})t}+\mathrm{e}^{(-\sqrt{2}-\frac{35}{18})t}\right)^2x_0^2, \quad \forall\, t\geqslant 0,$$

which guarantees the mean-square exponential stability of (6).

3 Well-posedness

In this section, we concentrate on the well-posedness (existence and uniqueness of the mild solution, continuous dependence of the mild solution on the initial value) of the stochastic Volterra equation (1), and identify mild solutions of (1) with those of an abstract stochastic Cauchy problem.

The well-posedness of (1) is established as demonstrated in Theorem 1.

Theorem 1. Let $f_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(\mathbb{R}_+; X))$ and $T_0 > 0$. Then Eq. (3) with the initial $x_0 \in L^2_{\mathcal{F}_0}(\Omega; X)$ admits a unique mild solution $x(\cdot\,, x_0)$ in $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$. Moreover, there exists a constant $K > 0$ such that

$$\|x(\cdot\,, x_0)\|_{C_{\mathbb{F}}([0,T_0];L^2(\Omega;X))} \leqslant K\big(\|x_0\|_{L^2_{\mathcal{F}_0}(\Omega;X)} + \sqrt{T_0}\|f_0\|_{L^2_{\mathcal{F}_0}(\Omega;L^2(0,T_0;X))}\big), \quad \forall\, x_0 \in L^2_{\mathcal{F}_0}(\Omega; X). \tag{7}$$

In particular, the stochastic Volterra equation (1) is well-posed which is the special case when $f_0 \equiv 0$ almost surely.

Proof. Let $x_0 \in L^2_{\mathcal{F}_0}(\Omega; X)$. Define the map Γ: for all $\psi \in C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$,

$$\begin{aligned}(\Gamma\psi)(t) = S(t)x_0 &+ \int_0^t S(t-s)f_0(s)\,\mathrm{d}s + \int_0^t S(t-s)B(\psi(s))\,\mathrm{d}W(s)\\ &+ \int_0^t S(t-r)\int_0^r b(r-s)B_1\psi(s)\,\mathrm{d}s\,\mathrm{d}W(r), \qquad t \in [0, T_0]\end{aligned}$$

in that $\{S(t) : t \geqslant 0\}$ is the resolvent family of (2). Let $\psi \in C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$, $t_1 \in [0, T_0]$ and $|\tau|$ be sufficiently small. Then

$$\mathbb{E}\|(\Gamma\psi)(t_1+\tau) - (\Gamma\psi)(t_1)\|^2_X \leqslant 4\sum_{i=1}^4 \mathbb{E}\|F_i(t_1+\tau) - F_i(t_1)\|^2_X.$$

According to [35, Proposition 1.4], the Hölder inequality and the strong continuity of resolvent family, we can obtain that as $\tau \to 0$,

$$\begin{aligned}\mathbb{E}\|F_4(t_1+\tau) - F_4(t_1)\|^2_X \leqslant 2\int_0^{t_1} &\mathbb{E}\big\|[S(t_1+\tau-s) - S(t_1-s)]B\psi(s)\big\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s\\ &+ 2\int_{t_1}^{t_1+\tau} \mathbb{E}\big\|S(t_1+\tau-s)B\psi(s)\big\|^2_{\mathcal{L}_2^0}\,\mathrm{d}s \to 0.\end{aligned}$$

Similarly, from the Fubini theorem, we have that as $\tau \to 0$, $\mathbb{E}\|F_3(t_1+\tau) - F_3(t_1)\|^2_X \to 0$. It is not hard to derive that $\mathbb{E}\|F_i(t_1+\tau) - F_i(t_1)\|^2_X \to 0$, $i = 1, 2$, as $\tau \to 0$. Thus, $\Gamma\psi$ is mean-square continuous on $[0, T_0]$ and Γ maps $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$ into itself. Introduce on $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$ the norm

$$\|\psi(\cdot)\|_{\exp} := \sup_{t\in[0,T_0]} (\mathrm{e}^{-\theta t}\mathbb{E}\|\psi(t)\|^2_X)^{\frac{1}{2}}$$

for some $\theta > 0$. It is easy to verify that $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$ forms also a Banach space under the norm $\|\cdot\|_{\exp}$, and $\|\cdot\|_{C_{\mathbb{F}}([0,T_0];L^2(\Omega;X))}$ is equivalent to $\|\cdot\|_{\exp}$ on $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$. Now we prove that Γ is contractive for $\theta > 0$ sufficiently large. Let $\varphi \in C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$. Invoking [35, Proposition 1.4], the Hölder inequality and the Fubini theorem again, we have

$$\begin{aligned}\mathbb{E}\|(\Gamma\psi)(t) - (\Gamma\varphi)(t)\|^2_X \leqslant 2\sup_{s\in[0,t]}\|S(s)\|^2_{\mathcal{L}(X)} &\sup_{s\in[0,t]} \mathrm{e}^{-\theta s}\mathbb{E}\|\psi(s) - \varphi(s)\|^2_X \int_0^t \mathrm{e}^{\theta s}\,\mathrm{d}s\\ &\cdot \big(t\,\|B_1\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)}\|b\|^2_{L^2[0,t]} + \|B\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)}\big), \quad \forall\, t \in [0, T_0].\end{aligned}$$

Hence, the above yields

$$\|\Gamma\psi - \Gamma\varphi\|^2_{\exp} \leqslant \frac{1 - \mathrm{e}^{-\theta T_0}}{\theta}\kappa\|\psi - \varphi\|^2_{\exp},$$

where

$$\kappa := 2\sup_{t\in[0,T_0]}\|S(t)\|^2_{\mathcal{L}(X)}\big(\|B\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)} + T_0\|B_1\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)}\|b\|^2_{L^2[0,T_0]}\big). \tag{8}$$

Then Γ is contractive for $\theta > 0$ sufficiently large. In light of the Banach fixed point theorem, Γ enjoys a unique fixed point $x(\cdot)$ in $C_{\mathbb{F}}([0,T_0];L^2(\Omega;X))$ which is the unique mild solution to (3) with the initial value x_0. Further, by virtue of [35, Proposition 1.4], the Hölder inequality and the Fubini theorem, we can derive

$$\begin{aligned}\mathbb{E}\|x(t)\|_X^2 \leqslant 4\sup_{t\in[0,T_0]}\|S(t)\|^2_{\mathcal{L}(X)}\Bigg(&\mathbb{E}\|x_0\|_X^2+T_0\|f_0\|^2_{L^2_{\mathcal{F}_0}(\Omega;L^2(0,T_0;X))}+\|B\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)}\int_0^t\mathbb{E}\big\|x(s)\big\|_X^2\,\mathrm{d}s\\&+T_0\,\|B_1\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)}\|b\|^2_{L^2[0,T_0]}\int_0^t\mathbb{E}\big\|x(s)\big\|_X^2\mathrm{d}s\Bigg),\quad\forall\, t\in[0,T_0].\end{aligned}\tag{9}$$

Denote $K := 2\sup_{t\in[0,T_0]}\|S(t)\|_{\mathcal{L}(X)}\max\{1,\mathrm{e}^{\kappa T_0}\}$ where κ is defined as (8). By employing the well-known Gronwall inequality (see, e.g., [18, Section II.4]), Eq. (9) becomes (7). We now claim that the assertion holds.

Remark 1. As done in [36], it seems to be possible that introduce the Banach space

$$\mathcal{S}_\theta := \Big\{\psi\in C_{\mathbb{F}}([0,\tau];L^2(\Omega;X))\mid \tau>0 \text{ and } \lim_{t\to\infty}\mathrm{e}^{\theta t}\|\psi(t)\|_{L^2_{\mathcal{F}_t}(\Omega;X)}=0\Big\}$$

equipped with the norm

$$\|\psi(\cdot)\|_{\mathcal{S}_\theta} := \sup_{t\in\mathbb{R}_+}\left(\mathrm{e}^{\theta t}\mathbb{E}\|\psi(t)\|_X^2\right)^{\frac{1}{2}},\quad\forall\,\psi\in\mathcal{S}_\theta$$

for some $\theta\geqslant 0$ and apply the Banach fixed point theorem to achieve the mean-square asymptotic or exponential stability of mild solutions to (1). In this way, the parameters $M\geqslant 1$ and $\omega<0$ need to be explicitly estimated for which $\|S(t)\|_X\leqslant M\mathrm{e}^{\omega t}$, $t\geqslant 0$ holds. However, to our knowledge, there is not yet a satisfactory result that can explicitly estimate both M and ω for general Volterra systems (2). It should be mentioned that the result established in Theorem 4 does not explicitly depend on M or ω.

Introduce the product Hilbert space $\mathcal{X} := X\times L^2(\mathbb{R}_+;X)$ endowed with the inner product

$$\left\langle\begin{bmatrix}x_1\\f_1\end{bmatrix},\begin{bmatrix}x_2\\f_2\end{bmatrix}\right\rangle_{\mathcal{X}} := \langle x_1,x_2\rangle_X+\langle f_1,f_2\rangle_{L^2(\mathbb{R}_+;X)},\quad\forall\begin{bmatrix}x_1\\f_1\end{bmatrix},\begin{bmatrix}x_2\\f_2\end{bmatrix}\in\mathcal{X}.\tag{10}$$

It is obvious from separability of X that $L^2(\mathbb{R}_+;X)$ is separable and so is $\mathcal{X}$. Define the operator

$$\mathcal{A}_0 := \begin{bmatrix}A & \delta_0\\ 0 & \dfrac{\mathrm{d}}{\mathrm{d}s}\end{bmatrix},\qquad D(\mathcal{A}_0) := D(A)\times\mathcal{H}^1(\mathbb{R}_+;X)\subset\mathcal{X}.\tag{11}$$

Here, δ_0 is the Dirac measure in 0, namely, $\delta_0(f)=f(0)$ for any $f\in\mathcal{H}^1(\mathbb{R}_+;X)$; $\frac{\mathrm{d}}{\mathrm{d}s}$ denotes the first order derivative operator, i.e., $\frac{\mathrm{d}}{\mathrm{d}s}f := f'$ for all $f\in\mathcal{H}^1(\mathbb{R}_+;X)$. Define

$$\mathbb{A}x := a(\cdot)Ax,\quad\forall\, x\in D(\mathbb{A})=D(A),\tag{12}$$

$$\mathbb{B}x := b(\cdot)B_1x,\quad\forall\, x\in X.\tag{13}$$

Denote

$$\begin{gathered}\mathcal{A}_1 := \begin{bmatrix}0&0\\ \mathbb{A}&0\end{bmatrix},\qquad \mathcal{A} := \mathcal{A}_0+\mathcal{A}_1=\begin{bmatrix}A&\delta_0\\0&\dfrac{\mathrm{d}}{\mathrm{d}s}\end{bmatrix}+\begin{bmatrix}0&0\\ \mathbb{A}&0\end{bmatrix},\\ D(\mathcal{A})=D(\mathcal{A}_1)=D(A)\times\mathcal{H}^1(\mathbb{R}_+;X)\subset\mathcal{X},\end{gathered}\tag{14}$$

and

$$\mathcal{B} := \begin{bmatrix}B&0\\ \mathbb{B}&0\end{bmatrix}.\tag{15}$$

As done in [17] and [2, Sections VI.3 and VI.7], by embedding X into the product Hilbert space $\mathcal{X}$ and adopting semigroup approach, we expect the strong solution of (3) to correspond with the first coordinate of the strong solution of the following abstract stochastic Cauchy problem:

$$\begin{cases}\mathrm{d}z(t)=\mathcal{A}z(t)\,\mathrm{d}t+\mathcal{B}(z(t))\,\mathrm{d}W(t),\\ z(0)=z_0 := \begin{bmatrix}x_0\\f_0\end{bmatrix}.\end{cases}\tag{16}$$

Let us start with the well-posedness of (16).

Proposition 1. Let separable product Hilbert space $\mathcal{X} = X \times L^2(\mathbb{R}_+; X)$ with the inner product as (10) and $T_0 > 0$. Then $\mathcal{A}$ defined as in (14) generates a C_0-semigroup on $\mathcal{X}$, and the abstract stochastic Cauchy problem (16) with the initial value $z_0 \in L^2_{\mathcal{F}_0}(\Omega; \mathcal{X})$ admits a unique mild solution $z(\cdot\,, z_0)$ satisfying $z(\cdot\,, z_0) \in C_{\mathbb{F}}([0, T_0]; L^2(\Omega; \mathcal{X}))$. Moreover, there exists a constant $K \geqslant 0$ such that

$$\|z(\cdot\,, z_0)\|_{C_{\mathbb{F}}([0,T_0];L^2(\Omega;\mathcal{X}))} \leqslant K\|z_0\|_{L^2_{\mathcal{F}_0}(\Omega;\mathcal{X})}, \quad \forall\, z_0 \in L^2_{\mathcal{F}_0}(\Omega; \mathcal{X}).$$

Proof. Define on $\mathcal{X}$ the operators

$$\mathcal{T}_0(t) := \begin{bmatrix} T(t) & R(t) \\ 0 & S_l(t) \end{bmatrix}, \qquad t \geqslant 0, \tag{17}$$

where $S_l(t)$ (form the left shift semigroup $\{S_l(t) : t \geqslant 0\}$ on $L^2(\mathbb{R}_+; X)$, see [37, Example 2.3.2 (ii)]) and $R(t)$ are defined on $L^2(\mathbb{R}_+; X)$:

$$(S_l(t)f)(\cdot) := f(\cdot + t), \qquad R(t)f := \int_0^t T(t-s)f(s)\,\mathrm{d}s, \quad \forall\, f \in L^2(\mathbb{R}_+; X). \tag{18}$$

Introduce the Banach space $D(\mathcal{A}_0, \|\cdot\|_{\mathcal{A}_0})$ equipped with the graph norm

$$\left\|\begin{bmatrix} x \\ f \end{bmatrix}\right\|_{\mathcal{A}_0} := \left(\left\|\begin{bmatrix} x \\ f \end{bmatrix}\right\|_{\mathcal{X}}^2 + \left\|\mathcal{A}_0\begin{bmatrix} x \\ f \end{bmatrix}\right\|_{\mathcal{X}}^2\right)^{\frac{1}{2}}, \quad \forall \begin{bmatrix} x \\ f \end{bmatrix} \in D(\mathcal{A}_0).$$

We have proven in [15, Theorem 2.1] that $\mathcal{A}$ generates a C_0-semigroup $\{\mathcal{T}(t) : t \geqslant 0\}$ on $\mathcal{X}$. This is achieved via a relatively bounded perturbation theorem [38, Corollary III.1.5] for C_0-semigroup and by showing that $\mathcal{A}_1$ is bounded on $(D(\mathcal{A}_0), \|\cdot\|_{\mathcal{A}_0})$.

According to [4, Theorem 3.14], well-posedness of the mild solution to (16) in $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$ are guaranteed by Hilbert-Schmidt boundedness of $\mathcal{B}$ defined as in (15). Indeed, it holds that

$$\begin{aligned}\left\|\mathcal{B}\begin{bmatrix} x \\ f \end{bmatrix}\right\|_{\mathcal{L}_2(V_0,\mathcal{X})}^2 = \left\|\begin{bmatrix} Bx \\ \mathbb{B}x \end{bmatrix}\right\|_{\mathcal{L}_2(V_0,\mathcal{X})}^2 &\leqslant \|Bx\|_{\mathcal{L}_2^0}^2 + \|b(\cdot)B_1x\|_{\mathcal{L}_2(V_0,L^2(\mathbb{R}_+;X))}^2 \\ &\leqslant \left(\|B\|_{\mathcal{L}(X,\mathcal{L}_2^0)}^2 + \|b\|_{L^2(\mathbb{R}_+)}^2\|B_1\|_{\mathcal{L}(X,\mathcal{L}_2^0)}^2\right)\left\|\begin{bmatrix} x \\ f \end{bmatrix}\right\|_{\mathcal{X}}^2\end{aligned}$$

for each $x \in X$ and each $f \in L^2(\mathbb{R}_+; X)$. This proves the assertion.

The following result shows the one-to-one relationship between the strong solution to (3) and the first coordinate of the strong solution to (16).

Theorem 2. Let $T_0 > 0$, $f_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(\mathbb{R}_+; X))$ and $x_0 \in L^2_{\mathcal{F}_0}(\Omega; X)$. Assume that $f_0 \in \mathcal{H}^1(\mathbb{R}_+; X)$ and $x_0 \in D(A)$ almost surely. Then Eq. (3) admits a strong solution $x(\cdot)$ on $[0, T_0]$ if and only if (16) admits a strong solution $z(\cdot)$ on $[0, T_0]$. In this case, $x(\cdot)$ coincides with the first coordinate of $z(\cdot)$.

Proof. (a) The "if" part. Let $z(\cdot) := \left[\begin{smallmatrix} z_1(\cdot) \\ z_2(\cdot) \end{smallmatrix}\right]$ be a strong solution to (16) on $[0, T_0]$ with the initial value $z_0 := \left[\begin{smallmatrix} x_0 \\ f_0 \end{smallmatrix}\right]$ and $t \in [0, T_0]$. Then it follows that

$$\begin{aligned} z(t) &= z_0 + \int_0^t \mathcal{A}z(s)\,\mathrm{d}s + \int_0^t \mathcal{B}(z(s))\,\mathrm{d}W(s) \\ &= \begin{bmatrix} x_0 \\ f_0 \end{bmatrix} + \int_0^t \begin{bmatrix} A & \delta_0 \\ \mathbb{A} & \frac{\mathrm{d}}{\mathrm{d}s} \end{bmatrix}\begin{bmatrix} z_1(s) \\ z_2(s) \end{bmatrix}\mathrm{d}s + \int_0^t \begin{bmatrix} B & 0 \\ \mathbb{B} & 0 \end{bmatrix}\begin{bmatrix} z_1(s) \\ z_2(s) \end{bmatrix}\mathrm{d}W(s), \qquad \text{a.s.} \end{aligned} \tag{19}$$

Focusing on the first coordinate of the above (19), one has

$$z_1(t) = x_0 + \int_0^t Az_1(s) + \delta_0 z_2(s)\,\mathrm{d}s + \int_0^t B(z_1(s))\,\mathrm{d}W(s), \qquad \text{a.s.} \tag{20}$$

It is useful to note that the variation of the parameters formula

$$\mathcal{T}(s)z = \mathcal{T}_0(s)z + \int_0^s \mathcal{T}_0(s-r)\mathcal{A}_1\mathcal{T}(r)z\,\mathrm{d}r, \quad \forall\, s \geqslant 0,\, z \in D(\mathcal{A})$$

holds by means of [2, Proposition VI.7.21] and a strong solution to (16) is also a mild solution to (16) by virtue of [3, Theorem 6.5]. These, together with the stochastic Fubini theorem [4, Theorem 2.141], imply

$$\begin{aligned} z(t) &= \mathcal{T}_0(t)z_0 + \int_0^t \mathcal{T}_0(t-s)\mathcal{A}_1 z(s)\,\mathrm{d}s + \int_0^t \mathcal{T}_0(t-s)\mathcal{B}(z(s))\,\mathrm{d}W(s) \\ &= \begin{bmatrix} T(t)x_0 + R(t)f_0 \\ S_l(t)f_0 \end{bmatrix} + \int_0^t \begin{bmatrix} T(t-s) & R(t-s) \\ 0 & S_l(t-s) \end{bmatrix} \begin{bmatrix} 0 \\ \mathbb{A}z_1(s) \end{bmatrix} \mathrm{d}s \\ &\quad + \int_0^t \begin{bmatrix} T(t-s) & R(t-s) \\ 0 & S_l(t-s) \end{bmatrix} \begin{bmatrix} Bz_1(s) \\ \mathbb{B}z_1(s) \end{bmatrix} \mathrm{d}W(s), \qquad \text{a.s.} \end{aligned} \tag{21}$$

Hence, it is remarkable by focusing on the second coordinate of (21) that

$$z_2(t) = S_l(t)f_0 + \int_0^t S_l(t-s)\mathbb{A}z_1(s)\,\mathrm{d}s + \int_0^t S_l(t-s)\mathbb{B}(z_1(s))\,\mathrm{d}W(s), \qquad \text{a.s.} \tag{22}$$

Combining (20) with (22), we deduce (5).

(b) The "only if" part. Let $x(\cdot)$ be a strong solution on $[0, T_0]$ to (3) with the initial value x_0 and $t \in [0, T_0]$. Let us take $z_1(\cdot) = x(\cdot)$ and $z_2(\cdot)$ as (22). We see readily that (20) holds. It is sufficient to verify that $z_2(\cdot)$ satisfies

$$z_2(t) = f_0 + \int_0^t \left(\mathbb{A}z_1(s) + \frac{\mathrm{d}}{\mathrm{d}s}z_2(s)\right) \mathrm{d}s + \int_0^t \mathbb{B}(z_1(s))\,\mathrm{d}W(s), \qquad \text{a.s.} \tag{23}$$

To that purpose, we note that for all $0 \leqslant s \leqslant t \leqslant T$ and $v \in V$,

$$S_l(t-s)\mathbb{A}z_1(s) = a(\cdot + t - s)Ax(s) \in \mathcal{H}^1(\mathbb{R}_+; X), \qquad \text{a.s.,} \tag{24}$$

$$S_l(t-s)\mathbb{B}(z_1(s))v = b(\cdot + t - s)B_1(x(s))v \in \mathcal{H}^1(\mathbb{R}_+; X), \qquad \text{a.s.} \tag{25}$$

In addition, we have

$$\int_0^{T_0}\int_0^t \left\|\frac{\mathrm{d}}{\mathrm{d}s}S_l(t-r)\mathbb{A}z_1(r)\right\|_{L^2(\mathbb{R}_+;X)} \mathrm{d}r\,\mathrm{d}t \leqslant \int_0^{T_0}\int_0^t \left\|a'(\cdot + t - r)Ax(r)\right\|_{L^2(\mathbb{R}_+;X)} \mathrm{d}r\,\mathrm{d}t < \infty \tag{26}$$

almost surely and

$$\int_0^{T_0}\int_0^t \left\|\frac{\mathrm{d}}{\mathrm{d}s}S_l(t-r)\mathbb{B}z_1(r)\right\|^2_{\mathcal{L}_2(V_0, L^2(\mathbb{R}_+;X))} \mathrm{d}r\,\mathrm{d}t = \int_0^{T_0}\int_0^t \left\|b'(\cdot + t - r)B_1x(r)\right\|^2_{\mathcal{L}_2(V_0, L^2(\mathbb{R}_+;X))} \mathrm{d}r\,\mathrm{d}t < \infty \tag{27}$$

almost surely. In light of [32, Proposition 1.3.5], Eqs. (24)–(27), together with $f_0 \in \mathcal{H}^1(\mathbb{R}_+; X)$ almost surely, ensure that Eq. (23) is satisfied. Therefore, $z(\cdot) := \left[\begin{smallmatrix} z_1(\cdot) \\ z_2(\cdot) \end{smallmatrix}\right]$ is a strong solution on $[0, T_0]$ to (16) with the initial state $z_0 := \left[\begin{smallmatrix} x_0 \\ f_0 \end{smallmatrix}\right]$. The proof is complete.

In order to extend Theorem 2 to the case of the mild solutions, we introduce the Yosida approximating systems of (3) as follows:

$$\begin{cases} \mathrm{d}x_\lambda(t) = \left[Ax_\lambda(t) + \displaystyle\int_0^t a(t-s)Ax_\lambda(s)\,\mathrm{d}s + I_\lambda(J_\lambda f_0)(t)\right]\mathrm{d}t \\ \qquad\qquad + I_\lambda\left[Bx_\lambda(t) + \displaystyle\int_0^t (J_\lambda b)(t-s)B_1x_\lambda(s)\,\mathrm{d}s\right]\mathrm{d}W(t), \\ x_\lambda(0) = I_\lambda x_0, \end{cases} \tag{28}$$

and the Yosida approximating systems of (16) as follows:

$$\begin{cases} \mathrm{d}z_\lambda(t) = \mathcal{A}z_\lambda(t)\,\mathrm{d}t + \mathcal{I}_\lambda \mathcal{B}(z_\lambda(t))\,\mathrm{d}W(t), \\ z_\lambda(0) = \mathcal{I}_\lambda z_0. \end{cases} \tag{29}$$

Here, $\lambda \in \rho(A) \cap \rho(\frac{\mathrm{d}}{\mathrm{d}s}) = \rho(A) \cap \mathbb{C}_+$ (see [37, Example 3.3.1 (ii)] for spectrum of $\frac{\mathrm{d}}{\mathrm{d}s}$); $I_\lambda := \lambda R(\lambda, A)$ and $J_\lambda := \lambda R(\lambda, \frac{\mathrm{d}}{\mathrm{d}s})$, namely (recall [37, Example 3.3.2 (ii)] for resolvent of $\frac{\mathrm{d}}{\mathrm{d}s}$),

$$(J_\lambda f)(s) = \lambda \int_s^\infty \mathrm{e}^{\lambda(s-\tau)} f(\tau)\,\mathrm{d}\tau, \quad \forall\, f \in L^2(\mathbb{R}_+; X), \tag{30}$$

the operator $\mathcal{I}_\lambda$ is defined on $\mathcal{X}$ by

$$\mathcal{I}_\lambda := \begin{bmatrix} I_\lambda & 0 \\ 0 & I_\lambda J_\lambda \end{bmatrix}.$$

Denote

$$L_\lambda(t) := Bx_\lambda(t) + \int_0^t (J_\lambda b)(t-s) B_1 x_\lambda(s)\,\mathrm{d}s, \quad \forall\, t \geqslant 0,\, w \in \Omega.$$

Proposition 2. Let $\lambda \in \rho(A) \cap \mathbb{C}_+$ and $T_0 > 0$. If $x_\lambda(\cdot)$ is a mild solution on $[0, T_0]$ to (28), then $x_\lambda(\cdot)$ is also a strong solution on $[0, T_0]$ to (28).

Proof. Let $t \in [0, T_0]$. We begin with the resolvent family $\{S(t) : t \geqslant 0\}$ for (2). It is known from Section 1 and [15] (see also originally [5, Chapter I]) that the strong solution to (2) with the initial value $x_0 \in D(A)$ is given by $S(\cdot)x_0$, i.e.,

$$S(t)x = x + \int_0^t AS(s)x\,\mathrm{d}s + \int_0^t \int_0^r a(r-s)AS(s)x\,\mathrm{d}s\,\mathrm{d}r, \quad \forall\, x \in D(A). \tag{31}$$

Let $x_\lambda(\cdot)$ be a mild solution on $[0, T_0]$ to (28). Then $x_\lambda(\cdot)$ can be rewritten as

$$x_\lambda(t) = S(t)I_\lambda x_0 + \int_0^t S(t-s)I_\lambda(J_\lambda f_0)(s)\,\mathrm{d}s + \int_0^t S(t-s)I_\lambda L_\lambda(s)\,\mathrm{d}W(s), \qquad \text{a.s.}$$

According to the Fubini theorem and (31), we can calculate

$$\begin{aligned} &\int_0^t \int_0^r a(r-s)A \int_0^s S(s-\xi)I_\lambda(J_\lambda f_0)(\xi)\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}r + \int_0^t A \int_0^r S(r-\xi)I_\lambda(J_\lambda f_0)(\xi)\,\mathrm{d}\xi\,\mathrm{d}r \\ &= \int_0^t S(t-\xi)I_\lambda(J_\lambda f_0)(\xi)\,\mathrm{d}\xi - \int_0^t I_\lambda(J_\lambda f_0)(\xi)\,\mathrm{d}\xi, \qquad \text{a.s.} \end{aligned} \tag{32}$$

Analogously, it follows by virtues of the stochastic Fubini theorem [4, Theorem 2.141] and (31) that

$$\begin{aligned} &\int_0^t \int_0^r a(r-s)A \int_0^s S(s-\xi)I_\lambda L_\lambda(\xi)\,\mathrm{d}W(\xi)\,\mathrm{d}s\,\mathrm{d}r + \int_0^t A \int_0^r S(r-\xi)I_\lambda L_\lambda(\xi)\,\mathrm{d}W(\xi)\,\mathrm{d}r \\ &= \int_0^t S(t-\xi)I_\lambda L_\lambda(\xi)\,\mathrm{d}W(\xi) - \int_0^t I_\lambda L_\lambda(\xi)\,\mathrm{d}W(\xi), \qquad \text{a.s.} \end{aligned} \tag{33}$$

Here, all integrals in (32) and (33) are well-defined due to the fact that $S(t)$ commutes with A on $D(A)$ and $AS(t)I_\lambda = S(t)AI_\lambda$ is bounded on X for any $t \geqslant 0$. Therefore, combining (31)–(33), we obtain

$$\int_0^t \int_0^r a(r-s)Ax_\lambda(s)\,\mathrm{d}s\,\mathrm{d}r + \int_0^t Ax_\lambda(r)\,\mathrm{d}r = x_\lambda(t) - I_\lambda x_0 - \int_0^t I_\lambda(J_\lambda f_0)(s)\,\mathrm{d}s - \int_0^t I_\lambda L_\lambda(s)\,\mathrm{d}W(s)$$

almost surely as claimed.

Lemma 1 ([17])**.** Let Y be a Banach space. Then δ_0 defined as (11) is bounded on $\mathcal{H}^1(\mathbb{R}_+; Y)$ and $\|\delta_0\|_{\mathcal{L}(\mathcal{H}^1(\mathbb{R}_+;Y),Y)} \leqslant \sqrt{2}$, that is, for each $f \in \mathcal{H}^1(\mathbb{R}_+; Y)$,

$$\|f(0)\|_Y^2 \leqslant 2\big(\|f\|^2_{L^2(\mathbb{R}_+;Y)} + \|f'\|^2_{L^2(\mathbb{R}_+;Y)}\big).$$

The following is the main result in this section which associates the mild solution to (3) with the first coordinate of the mild solution to (16).

Theorem 3. Let $T_0 > 0$, $f_0 \in L^2_{\mathcal{F}_0}(\Omega; L^2(\mathbb{R}_+; X))$ and $x_0 \in L^2_{\mathcal{F}_0}(\Omega; X)$. Then in $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$, the unique mild solution to (3) coincides with the first coordinate of the unique mild solution to the abstract stochastic Cauchy problem (16); in particular, the unique mild solution to (1) coincides with the first coordinate of the unique mild solution to (16) with the initial value $z_0 := [\begin{smallmatrix} x_0 \\ 0 \end{smallmatrix}]$, that is, the case of $f_0 \equiv 0$ almost surely.

Proof. Let $\lambda \in \rho(A) \cap \rho(\frac{\mathrm{d}}{\mathrm{d}s}) = \rho(A) \cap \mathbb{C}_+$. From [39, Proposition 1.2] (see also [38, Lemma II.3.4]) we know

$$\lim_{\lambda\to\infty} I_\lambda x = x, \qquad \lim_{\lambda\to\infty} J_\lambda f = f, \qquad \lim_{\lambda\to\infty} \mathcal{I}_\lambda \begin{bmatrix} x \\ f \end{bmatrix} = \begin{bmatrix} x \\ f \end{bmatrix}, \quad \forall\, x \in X,\, f \in L^2(\mathbb{R}_+; X). \tag{34}$$

In other words, $I_\lambda \to I$ strongly on X, $J_\lambda \to J$ strongly on $L^2(\mathbb{R}_+; X)$ and $\mathcal{I}_\lambda \to \mathcal{I}$ strongly on $\mathcal{X}$ as $\lambda \to \infty$, where I, J and $\mathcal{I}$ represent the identity operators on X, $L^2(\mathbb{R}_+; X)$ and $\mathcal{X}$ separately. Replacing respectively (3) and (16) by (28) and (29), one can establish the analogues of Theorem 1, Proposition 1 and Theorem 2. Briefly, Eqs. (28) and (29) are well-posed in $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$ and $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; \mathcal{X}))$, respectively; the unique strong solution to (28) coincides with the first coordinate of the unique strong solution to (29). It is worth recalling [32, Proposition 1.3.5]. Each mild solution to (29) is also its strong solution since $\mathcal{A}\mathcal{I}_\lambda$ is bounded on $\mathcal{X}$ in which $\mathcal{A}$ is as (14). Indeed, because of the facts that J_λ commutes with I_λ in terms of (30), $AI_\lambda = \lambda^2 R(\lambda, A) - \lambda I$ is bounded on X, $\frac{\mathrm{d}}{\mathrm{d}s} J_\lambda = \lambda^2 R(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}) - \lambda J$ is bounded on $L^2(\mathbb{R}_+; X)$ and δ_0 is bounded on $\mathcal{H}^1(\mathbb{R}_+; X)$ from Lemma 1, we can compute that

$$\begin{aligned}
\|\mathcal{A}\mathcal{I}_\lambda z\|^2_{\mathcal{X}} &\leqslant 2\|AI_\lambda x\|^2_X + 2\|\delta_0 J_\lambda I_\lambda f\|^2_X + 2\|a(\cdot)AI_\lambda x\|^2_{L^2(\mathbb{R}_+;X)} + 2\left\|\frac{\mathrm{d}}{\mathrm{d}s} J_\lambda I_\lambda f\right\|^2_{L^2(\mathbb{R}_+;X)} \\
&\leqslant 2\big(1 + \|a\|^2_{L^2(\mathbb{R}_+)}\big)\|AI_\lambda\|^2_{\mathcal{L}(X)}\|x\|^2_X + \Bigg(4\|J_\lambda\|^2_{\mathcal{L}(L^2(\mathbb{R}_+;X))} \\
&\quad + 6\left\|\frac{\mathrm{d}}{\mathrm{d}s} J_\lambda\right\|^2_{\mathcal{L}(L^2(\mathbb{R}_+;X))}\Bigg)\|I_\lambda\|^2_{\mathcal{L}(X)}\|f\|^2_{L^2(\mathbb{R}_+;X)}, \quad \forall\, z := \begin{bmatrix} x \\ f \end{bmatrix} \in \mathcal{X}.
\end{aligned}$$

Further, the strong solution $z_\lambda(\cdot)$ to (29) converges to $z(\cdot)$ in $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; \mathcal{X}))$ due to [4, Theorem 3.22], where $z(\cdot)$ is the unique mild solution to (16). All the above, together with Proposition 2, conclude that the mild solution $x_\lambda(\cdot)$ to (28) converges to $z_1(\cdot)$ in $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$ as $\lambda \to \infty$, where $z_1(\cdot)$ indicates the first coordinate of the unique mild solution to (16).

Let $x_\lambda(\cdot)$ be the unique mild solution to (28) in $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$ and $x(\cdot)$ the unique mild solution to (3) in $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$. Clearly, It is enough to prove that $x_\lambda(\cdot)$ converges to $x(\cdot)$ in $C_{\mathbb{F}}([0, T_0]; L^2(\Omega; X))$ as $\lambda \to \infty$. Denote

$$L(t) := Bx(t) + \int_0^t b(t-s)B_1 x(s)\,\mathrm{d}s, \quad \forall\, t \geqslant 0,\, w \in \Omega.$$

Let $t \in [0, T_0]$. Then we have

$$\begin{aligned}
x_\lambda(t) - x(t) = S(t)(I_\lambda x_0 - x_0) &+ \int_0^t S(t-s)\big[I_\lambda(J_\lambda f_0)(s) - f_0(s)\big]\,\mathrm{d}s \\
&+ \int_0^t S(t-s)\big[I_\lambda L_\lambda(s) - L(s)\big]\,\mathrm{d}W(s), \qquad \text{a.s.}
\end{aligned}$$

Consequently, we can estimate

$$\mathbb{E}\|x_\lambda(t) - x(t)\|^2_X \leqslant 7\big(R_1(t) + R_2(t) + R_3(t)\big). \tag{35}$$

Here,

$$\begin{aligned}
R_1(t) := \mathbb{E}&\left\|\int_0^t S(t-s)I_\lambda \int_0^s (J_\lambda b)(s-r)B_1(x_\lambda(r) - x(r))\,\mathrm{d}r\,\mathrm{d}W(s)\right\|^2_X \\
&+ \mathbb{E}\left\|\int_0^t S(t-s)I_\lambda B(x_\lambda(s) - x(s))\,\mathrm{d}W(s)\right\|^2_X,
\end{aligned}$$

$$R_2(t) := \mathbb{E}\big\|S(t)(I_\lambda - I)x_0\big\|_X^2 + \mathbb{E}\left\|\int_0^t S(t-s)(I_\lambda - I)f_0(s)\,\mathrm{d}s\right\|_X^2 + \mathbb{E}\left\|\int_0^t S(t-s)(I_\lambda - I)L(s)\,\mathrm{d}W(s)\right\|_X^2,$$

$$R_3(t) := \mathbb{E}\left\|\int_0^t S(t-s)I_\lambda \int_0^s (J_\lambda b - b)(s-r)B_1(x(r))\,\mathrm{d}r\,\mathrm{d}W(s)\right\|_X^2 + \mathbb{E}\left\|\int_0^t S(t-s)I_\lambda[(J_\lambda - J)f_0](s)\,\mathrm{d}s\right\|_X^2.$$

By making use of [35, Proposition 1.4], the Hölder inequality and the Fubini theorem, one obtains

$$R_1(t) \leqslant \|I_\lambda\|_{\mathcal{L}(X)}^2 \sup_{t\in[0,T_0]} \|S(t)\|_{\mathcal{L}(X)}^2 \Big(T_0\|B_1\|_{\mathcal{L}(X,\mathcal{L}_2^0)}^2 \|b\|_{L^2[0,T_0]}^2 \|J_\lambda\|_{\mathcal{L}(L^2(\mathbb{R}_+;X))}^2 + \|B\|_{\mathcal{L}(X,\mathcal{L}_2^0)}^2\Big) \int_0^t \mathbb{E}\|x_\lambda(s) - x(s)\|_X^2\,\mathrm{d}s; \tag{36}$$

analogously,

$$R_2(t) \leqslant \left(\mathbb{E}\|(I_\lambda - I)x_0\|_X^2 + \int_0^{T_0} \mathbb{E}\big\|(I_\lambda - I)f_0(s)\big\|_X^2\,\mathrm{d}s + T_0\int_0^{T_0} \mathbb{E}\big\|(I_\lambda - I)L(s)\big\|_{\mathcal{L}_2^0}^2\,\mathrm{d}s\right) \sup_{t\in[0,T_0]} \|S(t)\|_X^2; \tag{37}$$

$$R_3(t) \leqslant \Big(T_0^2\,\|B_1\|_{\mathcal{L}(X,\mathcal{L}_2^0)}^2 \|(J_\lambda - J)b\|_{L^2[0,T_0]}^2 \|x(\cdot)\|_{C_{\mathbb{F}}([0,T_0];L^2(\Omega;X))}^2 + T_0\|(J_\lambda - J)f_0\|_{L_{\mathcal{F}_0}^2(\Omega;L^2(0,T_0;X))}^2\Big) \|I_\lambda\|_{\mathcal{L}(X)}^2 \sup_{t\in[0,T_0]} \|S(t)\|_X^2. \tag{38}$$

Denote

$$\kappa := 4 \sup_{t\in[0,T_0]} \|S(t)\|_{\mathcal{L}(X)}^2 \big(4T_0\|B_1\|_{\mathcal{L}(X,\mathcal{L}_2^0)}^2 \|b\|_{L^2[0,T_0]}^2 + \|B\|_{\mathcal{L}(X,\mathcal{L}_2^0)}^2\big),$$

and

$$K_\lambda := \sup_{t\in[0,T_0]} \|S(t)\|_X^2 \Bigg(4T_0\|(J_\lambda - J)f_0\|_{L_{\mathcal{F}_0}^2(\Omega;L^2(0,T_0;X))}^2 + 4T_0^2\,\|B_1\|_{\mathcal{L}(X,\mathcal{L}_2^0)}^2 \|(J_\lambda - J)b\|_{L^2[0,T_0]}^2 \|x(\cdot)\|_{C_{\mathbb{F}}([0,T_0];L^2(\Omega;X))}^2 + \int_0^{T_0} \mathbb{E}\big\|(I_\lambda - I)f_0(s)\big\|_X^2\,\mathrm{d}s + T_0\int_0^{T_0} \mathbb{E}\big\|(I_\lambda - I)L(s)\big\|_{\mathcal{L}_2^0}^2\,\mathrm{d}s + \mathbb{E}\|(I_\lambda - I)x_0\|_X^2\Bigg).$$

For $\lambda > 0$ large enough, due to $\|J_\lambda\|_{\mathcal{L}(L^2(\mathbb{R}_+;X))} \leqslant 2$ and $\|I_\lambda\|_{\mathcal{L}(X)} \leqslant 2$, it is very simple to check by noticing (36)–(38) that

$$R_1(t) \leqslant \kappa \int_0^t \mathbb{E}\|x_\lambda(s) - x(s)\|_X^2\,\mathrm{d}s \qquad \text{and} \qquad R_2(t) + R_3(t) \leqslant K_\lambda. \tag{39}$$

Plugging (39) into (35) and utilizing the well-known Gronwall inequality (see, e.g., [18, Section II.4]), we deduce

$$\|x_\lambda(\cdot) - x(\cdot)\|_{C_{\mathbb{F}}([0,T_0];L^2(\Omega;X))}^2 = \sup_{t\in[0,T_0]} \mathbb{E}\|x_\lambda(t) - x(t)\|_X^2 \leqslant 7K_\lambda \max\{\mathrm{e}^{7\kappa T_0}, 1\}. \tag{40}$$

From the Lebesgue dominated convergence theorem, Eq. (34) implies $K_\lambda \to 0$. This, together with (40), means that $x_\lambda(\cdot)$ converges to $x(\cdot)$ in $C_{\mathbb{F}}([0,T_0];L^2(\Omega;X))$. The assertion follows from the uniqueness of the limit.

Remark 2. We emphasize that since the C_0-semigroup $\{\mathcal{T}(t) : t \geqslant 0\}$ generated by $\mathcal{A}$ is very difficult to calculate explicitly, mild solutions of (3) and (16) are identified by Yosida approximation rather than the direct method.

4 Mean-square exponential stability

In this section, we focus on the mean-square exponential stability of the mild solution to the stochastic Volterra system (1). By recalling Theorem 3, this can be translated into analyzing the mean-square exponential stability of the mild solution to the abstract stochastic Cauchy problem (16). To make our findings applicable, we have to restrict $L^2(\mathbb{R}_+;X)$ to a certain closed subspace. This further limits the kernel functions $a(\cdot)$ and $b(\cdot)$, but despite these constraints, our conclusions remain applicable to (1) when kernel functions exponentially decay. As highlighted in Section 1, this scenario encompasses the use of the Caputo-Fabrizio fractional derivative.

Let $a(\cdot)$ be as in (1) and $f(\cdot)\in L^2(\mathbb{R}_+;X)$. For all $\lambda\in\rho(\frac{\mathrm{d}}{\mathrm{d}s})=\mathbb{C}_+$, we define

$$\begin{aligned}\tilde{a}_\lambda(\tau)&:=\left(R\left(\lambda,\frac{\mathrm{d}}{\mathrm{d}s}\right)a\right)(\tau)=\int_\tau^\infty \mathrm{e}^{\lambda(\tau-t)}a(t)\,\mathrm{d}t, \qquad \tau\geqslant 0;\\ \tilde{f}_\lambda(\tau)&:=\left(R\left(\lambda,\frac{\mathrm{d}}{\mathrm{d}s}\right)f\right)(\tau)=\int_\tau^\infty \mathrm{e}^{\lambda(\tau-t)}f(t)\,\mathrm{d}t, \qquad \tau\geqslant 0,\end{aligned}\tag{41}$$

where $\rho(\frac{\mathrm{d}}{\mathrm{d}s})$ and $R(\lambda,\frac{\mathrm{d}}{\mathrm{d}s})$ stand for the resolvent set and the resolvent of $\frac{\mathrm{d}}{\mathrm{d}s}$, respectively. The Laplace transforms of $a(\cdot)$ and $f(\cdot)$ are denoted by

$$\hat{a}(\lambda):=\delta_0\tilde{a}_\lambda, \qquad \hat{f}(\lambda):=\delta_0\tilde{f}_\lambda, \qquad \lambda\in\mathbb{C}_+,\tag{42}$$

respectively, where δ_0 defined as near (11) means the Dirac measure in 0. Let us note that the integral representation of $\delta_0 R(\lambda,\frac{\mathrm{d}}{\mathrm{d}s})$ and the Laplace transform are identified. We refer to [40, Chapter V] for more details on Laplace transform and [41] for more information on vector-valued Laplace transform. Let $\hat{a}(\lambda)\neq -1$ for every $\lambda\in\mathbb{C}_+$. Define

$$h(\lambda):=\frac{1}{\hat{a}(\lambda)+1}, \qquad \varphi(\lambda):=\frac{\lambda}{\hat{a}(\lambda)+1}, \qquad \lambda\in\mathbb{C}_+.\tag{43}$$

For any $\lambda\in\mathbb{C}_+$, define on $L^2(\mathbb{R}_+;X)$ the operator

$$H(\lambda):=h(\lambda)R\left(\lambda,\frac{\mathrm{d}}{\mathrm{d}s}\right)\mathbb{A}R(\varphi(\lambda),A)\delta_0R\left(\lambda,\frac{\mathrm{d}}{\mathrm{d}s}\right)+R\left(\lambda,\frac{\mathrm{d}}{\mathrm{d}s}\right),$$

where A and $\mathbb{A}$ are given as in (1) and (12) separately. Then as shown in [15, Lemma 3.2], although we cannot get the C_0-semigroup generated by $\mathcal{A}$ explicitly, the resolvent of $\mathcal{A}$ can be calculated as the following, where $\mathcal{A}$ is as in (14).

Lemma 2 ([15])**.** Let X be a Hilbert space, A generate a C_0-semigroup on X and $a(\cdot)\in\mathcal{H}^1(\mathbb{R}_+)$. If $\hat{a}(\lambda)\neq-1$ and $\varphi(\lambda)\in\rho(A)$ for any $\lambda\in\mathbb{C}_+$, then $\mathbb{C}_+\subset\rho(A)$ implies that $\mathbb{C}_+\subset\rho(\mathcal{A})$ and

$$R(\lambda,\mathcal{A})=\begin{bmatrix}h(\lambda)R(\varphi(\lambda),A) & h(\lambda)R(\varphi(\lambda),A)\delta_0R(\lambda,\frac{\mathrm{d}}{\mathrm{d}s})\\ h(\lambda)R(\lambda,\frac{\mathrm{d}}{\mathrm{d}s})\mathbb{A}R(\varphi(\lambda),A) & H(\lambda)\end{bmatrix}, \quad \forall\,\lambda\in\mathbb{C}_+,$$

here, $\mathbb{A}$ and $\mathcal{A}$ are defined as in (12) and (14), respectively.

Theorem 4. Assume that

(i) A generates an exponentially stable C_0-semigroup $\{T(t):t\geqslant 0\}$ on separable Hilbert space X, i.e., there exist two constants $M_1\geqslant 1$, $\omega_1<0$ such that $\|T(t)\|_{\mathcal{L}(X)}\leqslant M_1\mathrm{e}^{\omega_1 t}$ for all $t\geqslant 0$;

(ii) $a(\cdot)\in\mathcal{H}^1(\mathbb{R}_+)$ satisfy that $\hat{a}(\lambda)\neq-1$ and $\varphi(\lambda)\in\rho(A)$ for all $\lambda\in\mathbb{C}_+$, where $\hat{a}(\cdot)$ is as in (42) and $\varphi(\cdot)$ is defined as in (43);

(iii) $\mathfrak{M}$ is a closed subspace of $L^2(\mathbb{R}_+;X)$ such that $S_l(t)\mathfrak{M}\subset\mathfrak{M}$ for every $t\geqslant 0$, that is, $\mathfrak{M}$ is $\{S_l(t):t\geqslant 0\}$-invariant (see [38, Paragraphs I.1.11 and II.2.3]), where $S_l(t)$ are defined as (18). Moreover, it follows that $\mathbb{B}\in\mathcal{L}(X,\mathcal{L}_2(V_0,\mathfrak{M}))$ given as (13) and $a(\cdot)Ax\in\mathfrak{M}$ for all $x\in D(A)$;

(iv) There exist two real-valued functions $p(\cdot)$, $q(\cdot)$ such that for all $\lambda\in\mathbb{C}_+$, $f\in\mathfrak{M}$,

$$\left\|\hat{f}(\lambda)\right\|_X^2\leqslant p(\lambda)\|f\|^2_{L^2(\mathbb{R}_+;X)} \qquad \text{and} \qquad \left\|\tilde{f}_\lambda(\cdot)\right\|^2_{L^2(\mathbb{R}_+;X)}\leqslant q(\lambda)\|f\|^2_{L^2(\mathbb{R}_+;X)}.\tag{44}$$

If

$$\sup_{\mathrm{Re}\lambda>0}\xi(\lambda)<\infty \qquad \text{and} \qquad \|B\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)}+\|b\|^2_{L^2(\mathbb{R}_+)}\|B_1\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)}<1/\zeta,\tag{45}$$

then stochastic Volterra system (1) is mean-square exponentially stable. Here,

$$\xi(\lambda) := |h(\lambda)|^2 \Big[4\big\|\tilde{a}_\lambda(\cdot)\varphi(\lambda)R(\varphi(\lambda), A) - \tilde{a}_\lambda(\cdot)I\big\|^2_{\mathcal{L}(X,L^2(\mathbb{R}_+;X))} + 2\|R(\varphi(\lambda), A)\|^2_{\mathcal{L}(X)}\Big] \big(1 + p(\lambda)\big) + 2q(\lambda), \qquad \lambda \in \mathbb{C}_+,$$

$$\zeta := \frac{M_1^2}{-\omega_1} \sup_{\mathrm{Re}\lambda>0} |h(\lambda)|^2 \sup_{\mathrm{Re}\lambda>0} (1 + p(\lambda)) \sup_{\mathrm{Re}\lambda>0} \frac{\mathrm{Re}\lambda}{\mathrm{Re}\,\varphi(\lambda)}, \tag{46}$$

$h(\cdot)$, $\varphi(\cdot)$ are as (43) and $\tilde{a}_\lambda(\cdot)$ is defined as (41).

Proof. To begin with, we validate that Proposition 1, Theorems 2 and 3 still follow if $L^2(\mathbb{R}_+; X)$ is replacing by $\mathfrak{M}$. Denote $\mathcal{M} := X \times \mathfrak{M}$. Immediately, $\mathcal{M}$ is a closed subspace of $\mathcal{X} := X \times L^2(\mathbb{R}_+; X)$. Let $x \in X$ and $f \in \mathfrak{M}$. Then the $\{S_l(t) : t \geqslant 0\}$-invariance of $\mathfrak{M}$ implies

$$\mathcal{T}_0(t)\begin{bmatrix} x \\ f \end{bmatrix} = \begin{bmatrix} T(t) & R(t) \\ 0 & S_l(t) \end{bmatrix}\begin{bmatrix} x \\ f \end{bmatrix} = \begin{bmatrix} T(t)x + R(t)f \\ S_l(t)f \end{bmatrix} \in \mathcal{M}, \qquad t \geqslant 0,$$

namely, $\mathcal{M}$ is $\{\mathcal{T}_0(t) : t \geqslant 0\}$-invariant, where $\mathcal{T}_0(t)$ are defined by (17) and $R(\cdot)$ is given by (18). According to [38, Paragraph I.1.11], the operators $\mathcal{T}_0|_{\mathcal{M}}(t)$ defined by

$$\mathcal{T}_0|_{\mathcal{M}}(t)\begin{bmatrix} x \\ f \end{bmatrix} := \mathcal{T}_0(t)\begin{bmatrix} x \\ f \end{bmatrix}, \quad \forall \begin{bmatrix} x \\ f \end{bmatrix} \in \mathcal{M}$$

form a C_0-semigroup $\{\mathcal{T}_0|_{\mathcal{M}}(t) : t \geqslant 0\}$ on $\mathcal{M}$. By making use of [38, Paragraph II.2.3], the generator of the C_0-semigroup $\{\mathcal{T}_0|_{\mathcal{M}}(t) : t \geqslant 0\}$ on $\mathcal{M}$ is derived by $\mathcal{A}_0|_{\mathcal{M}} = \mathcal{A}_0$ with the domain

$$D(\mathcal{A}_0|_{\mathcal{M}}) := D(\mathcal{A}_0) \cap \mathcal{M} = D(A) \times (\mathcal{H}^1(\mathbb{R}_+; X) \cap \mathfrak{M}),$$

where $\mathcal{A}_0$ is as (11). Define the operator $\mathcal{A}_1|_{\mathcal{M}} := \mathcal{A}_1$ mapping $D(\mathcal{A}_0|_{\mathcal{M}})$ into $\mathcal{M}$ and the operator $\mathcal{B}|_{\mathcal{M}} := \mathcal{B}$ mapping $\mathcal{M}$ into $\mathcal{L}_2(V_0, \mathcal{M})$, where $\mathcal{A}_1$ and $\mathcal{B}$ are given as in (14) and (15) separately. Similar to the proof of Proposition 1, it is simple to verify that $\mathcal{A}_1|_{\mathcal{M}}$ is bounded on the Banach space $(D(\mathcal{A}_0|_{\mathcal{M}}), \|\cdot\|_{\mathcal{A}_0|_{\mathcal{M}}})$ equipped with the graph norm

$$\left\|\begin{bmatrix} x \\ f \end{bmatrix}\right\|^2_{\mathcal{A}_0|_{\mathcal{M}}} := \left(\left\|\begin{bmatrix} x \\ f \end{bmatrix}\right\|^2_{\mathcal{M}} + \left\|\mathcal{A}_0|_{\mathcal{M}}\begin{bmatrix} x \\ f \end{bmatrix}\right\|^2_{\mathcal{M}}\right)^{\frac{1}{2}}, \quad \forall \begin{bmatrix} x \\ f \end{bmatrix} \in \mathcal{M},$$

and hence $\mathcal{A}|_{\mathcal{M}} := \mathcal{A}_0|_{\mathcal{M}} + \mathcal{A}_1|_{\mathcal{M}}$ generates a C_0-semigroup on $\mathcal{M}$, denoted $\{\mathcal{T}|_{\mathcal{M}}(t) : t \geqslant 0\}$. $\mathcal{B}|_{\mathcal{M}}$ is clearly bounded on $\mathcal{M}$. Therefore, in the case of replacing $L^2(\mathbb{R}_+; X)$ by $\mathfrak{M}$, Eq. (16) can be reformulated as

$$\begin{cases} \mathrm{d}z(t) = \mathcal{A}|_{\mathcal{M}} z(t)\,\mathrm{d}t + \mathcal{B}|_{\mathcal{M}}(z(t))\,\mathrm{d}W(t), \\ z(0) = z_0 := \begin{bmatrix} x_0 \\ f_0 \end{bmatrix}, \end{cases}$$

where $f_0 \in \mathfrak{M}$. Analogously, Theorems 2 and 3 also hold with $L^2(\mathbb{R}_+; X)$ is replacing by $\mathfrak{M}$.

Next, we demonstrate that the operator norm estimate

$$\sup_{\|\eta\|_{\mathcal{M}}=1} \int_0^\infty \|\mathcal{B}|_{\mathcal{M}}\mathcal{T}|_{\mathcal{M}}(t)\eta\|^2_{\mathcal{L}_2(V_0,\mathcal{M})}\,\mathrm{d}t < 1$$

holds. Equivalently, there exists a constant $K < 1$ such that

$$\|\mathcal{B}|_{\mathcal{M}}\mathcal{T}|_{\mathcal{M}}(\cdot)\eta\|^2_{L^2(\mathbb{R}_+;\mathcal{L}_2(V_0,\mathcal{M}))} \leqslant K\|\eta\|^2_{\mathcal{M}}, \quad \forall\, \eta \in \mathcal{M}. \tag{47}$$

Let $\eta := [\begin{smallmatrix} x \\ f \end{smallmatrix}] \in \mathcal{M}$. By virtues of the vector-valued version of Paley-Wiener theorem [24, Proposition 12.5.4], the integral representation of resolvent [38, Theorem II.1.10] and Lemma 2, we have

$$\begin{aligned} \|\mathcal{B}|_{\mathcal{M}}\mathcal{T}|_{\mathcal{M}}(\cdot)\eta\|^2_{L^2(\mathbb{R}_+;\mathcal{L}_2(V_0,\mathcal{M}))} &= \|\mathcal{B}|_{\mathcal{M}}R(\cdot, \mathcal{A}|_{\mathcal{M}})\eta\|^2_{H^2(\mathbb{C}_+;\mathcal{L}_2(V_0,\mathcal{M}))} \\ &\leqslant \big(\|B\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)} + \|b\|^2_{L^2(\mathbb{R}_+)}\|B_1\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)}\big) \sup_{\mathrm{Re}\lambda>0} |h(\lambda)|^2 \\ &\quad \cdot \|R(\varphi(\cdot), A)\|^2_{\mathcal{L}(X,H^2(\mathbb{C}_+;X))} \sup_{\mathrm{Re}\lambda>0} (\|x + \hat{f}(\lambda)\|^2_X). \end{aligned} \tag{48}$$

Define on Hardy space $H^2(\mathbb{C}_+;X)$ the composition operator C_φ by $(C_\varphi f)(\lambda) := f(\varphi(\lambda))$ for all $f \in H^2(\mathbb{C}_+;X)$. With the aids of [26, Theorem 4 and Proposition 1], we know that the composition operator C_φ is bounded on $H^2(\mathbb{C}_+;X)$ if and only if φ has a finite angular derivative at infinity, or equivalently,

$$\kappa := \sup_{\mathrm{Re}\lambda>0} \frac{\mathrm{Re}\lambda}{\mathrm{Re}\,\varphi(\lambda)} < \infty,$$

in this case, $\|C_\varphi\|^2_{\mathcal{L}(H^2(\mathbb{C}_+;X))} = \kappa$. Hence, Eq. (48) becomes

$$\begin{aligned}\|\mathcal{B}|_{\mathcal{M}}\mathcal{T}|_{\mathcal{M}}(\cdot)\eta\|^2_{L^2(\mathbb{R}_+;\mathcal{L}_2(V_0,\mathcal{M}))} \leqslant{} & \left(\|B\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)} + \|b\|^2_{L^2(\mathbb{R}_+)}\|B_1\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)}\right) \sup_{\mathrm{Re}\lambda>0} |h(\lambda)|^2 \\ & \cdot \|C_\varphi\|^2_{\mathcal{L}(H^2(\mathbb{C}_+;X))}\|R(\cdot,A)\|^2_{\mathcal{L}(X,H^2(\mathbb{C}_+;X))} \\ & \cdot \sup_{\mathrm{Re}\lambda>0} (2\|x\|^2_X + 2\|\hat{f}(\lambda)\|^2_X).\end{aligned} \tag{49}$$

From Theorem 4 (i) and using the Paley-Wiener theorem again, we obtain

$$\|R(\cdot,A)\|^2_{\mathcal{L}(X,H^2(\mathbb{C}_+;X))} = \|T(\cdot)\|^2_{\mathcal{L}(X,L^2(\mathbb{R}_+;X))} \leqslant \frac{M_1^2}{-2\omega_1}. \tag{50}$$

Combining (44)–(46), (49) and (50), we claim that Eq. (47) follows with

$$K = \left(\|B\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)} + \|b\|^2_{L^2(\mathbb{R}_+)}\|B_1\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)}\right)\zeta < 1.$$

Finally, as shown in the proof of [15, Theorem 3.3], the C_0-semigroup $\{\mathcal{T}|_{\mathcal{M}}(t) : t \geqslant 0\}$ is exponentially stable, since Theorem 4 (i) implies $\{\, \lambda \in \mathbb{C} \mid \mathrm{Re}\lambda > 0 \,\} \subset \rho(A)$. Therefore, in the light of [32, Theorem 2.2.2] (or originally, see [42, Theorem 1]), the mild solution $z(\cdot, z_0)$ to the abstract stochastic Cauchy problem (16) with the initial datum $z_0 := \left[\begin{smallmatrix} x_0 \\ f_0 \end{smallmatrix}\right]$ is mean-square exponentially stable, where $f_0 \in L^2_{\mathcal{F}_0}(\Omega; M)$. Fixing $f_0 \equiv 0$ almost surely and utilizing Theorem 3, one deduces

$$\mathbb{E}\|x(t,x_0)\|^2_X \leqslant \mathbb{E}\left\| z\left(t, \begin{bmatrix} x_0 \\ 0 \end{bmatrix}\right)\right\|^2_{\mathcal{M}} \leqslant M\mathrm{e}^{\omega t}\mathbb{E}\left\|\begin{bmatrix} x_0 \\ 0\end{bmatrix}\right\|^2_{\mathcal{M}} = M\mathrm{e}^{\omega t}\mathbb{E}\|x_0\|^2_X$$

for some constants $M \geqslant 1$, $\omega < 0$. This concludes the proof.

From Lemma 2 we learn that the norm of resolvent operator needs to be estimated. However, in general, the resolvent operator $R(\lambda, A)$ of a closed linear operator A is difficult to calculate on an infinite-dimensional space. It is easier to explore the special case of exponential-decay kernel functions

$$a(t) = \alpha_1 \mathrm{e}^{-\beta t}, \qquad b(t) = \alpha_2\mathrm{e}^{-\beta t}, \qquad t \geqslant 0, \tag{51}$$

where α_1, $\alpha_2 \in \mathbb{R}$ and $\beta > 0$. In this case, the notations $\hat{a}(\cdot), h(\cdot), \varphi(\cdot)$ and $\tilde{a}_\lambda(\cdot)$ become

$$\hat{a}(\lambda) = \frac{\alpha_1}{\lambda+\beta}, \qquad h(\lambda) = \frac{\lambda+\beta}{\lambda+\beta+\alpha_1}, \qquad \varphi(\lambda) = \frac{\lambda(\lambda+\beta)}{\lambda+\beta+\alpha_1}, \qquad \lambda \in \mathbb{C}_+, \tag{52}$$

where $\beta + \alpha_1 \geqslant 0$, and

$$\tilde{a}_\lambda(\tau) = \mathrm{e}^{-\beta\tau}\frac{\alpha_1}{\lambda+\beta}, \qquad \tau \geqslant 0.$$

Based on Theorem 4, taking the subspace $\mathfrak{M} = \{\, \mathrm{e}^{-\beta\cdot}x \mid x \in X \,\}$ of $L^2(\mathbb{R}_+;X)$, we can obtain Corollary 1.

Corollary 1. Assume that

(i) A generates an exponentially stable C_0-semigroup $\{T(t) : t \geqslant 0\}$ on separable Hilbert space X, i.e., there exist two constants $M_1 \geqslant 1$, $\omega_1 < 0$ such that $\|T(t)\|_{\mathcal{L}(X)} \leqslant M_1\mathrm{e}^{\omega_1 t}$ for all $t \geqslant 0$;

(ii) $a(\cdot)$, $b(\cdot) \in \mathcal{H}^1(\mathbb{R}_+)$ are of the form as (51) and satisfy that $\alpha_1 < 0$, $\beta > -\alpha_1$ and $\varphi(\lambda) \in \rho(A)$ for all $\lambda \in \mathbb{C}_+$, where $\varphi(\cdot)$ is defined as in (52).

If

$$\sup_{\mathrm{Re}\lambda>0} \|R(\varphi(\lambda),A)\|_{\mathcal{L}(X)} < K \qquad \text{and} \qquad \|B\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)} + \frac{\alpha_2^2}{2\beta}\|B_1\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)} < \frac{1}{\zeta},$$

then stochastic Volterra system (1) is mean-square exponentially stable. Here,

$$\zeta := \frac{M_1^2}{-\omega_1}\left(2 + \frac{2\alpha_1^2}{(\beta+\alpha_1)^2}\right)\left(1+\frac{2}{\beta}\right)\sup_{\mathrm{Re}\lambda>0}\frac{\mathrm{Re}\lambda}{\mathrm{Re}\,\varphi(\lambda)}.$$

5 Simulation results

In this section, we provide two examples to illustrate the result obtained in Section 4 with numerical simulations. The first example is finite dimensional, for which mean-square exponential stability has been established in Example 1. The deterministic counterpart of the second example corresponds to Coleman and Gurtin's model [7], utilizing exponential-decay kernel functions. As highlighted in Section 1, the convolution terms with this type of kernel functions can be equated to the Caputo-Fabrizio fractional derivative [12].

Example 2. Reconsider Example 1. It is easy to verify that $|T(t)| = \mathrm{e}^{-2t}$ and

$$\varphi(\lambda) = \frac{\lambda(\lambda+\beta)}{\lambda+\beta+\alpha_1} = \lambda + 1 - \frac{1}{\lambda+1} \in \mathbb{C}\backslash\{-2\} = \rho(A), \quad \forall\, \lambda \in \mathbb{C}_+.$$

Thus, Corollary 1 (i) and (ii) are fulfilled. A direct computation yields

$$\sup_{\mathrm{Re}\lambda>0} |R(\varphi(\lambda), A)| = \frac{1}{\inf_{\mathrm{Re}\lambda>0} |\lambda + 3 - \frac{1}{\lambda+1}|} = \frac{1}{2}.$$

Furthermore, it is clear from $V = \mathbb{R}$ and $Q = I$ that $\|B\|_{\mathcal{L}(X,\mathcal{L}_0^2)} = \|B\|_{\mathcal{L}(\mathbb{R},\mathcal{L}(\mathbb{R}))} = 1/3$, so we get

$$\zeta = \frac{M_1^2}{-\omega_1}\left(2 + \frac{2\alpha_1^2}{(\beta+\alpha_1)^2}\right)\left(1 + \frac{2}{\beta}\right) \sup_{\mathrm{Re}\lambda>0} \frac{\mathrm{Re}\lambda}{\mathrm{Re}\,\varphi(\lambda)} = 4 \cdot \sup_{\mathrm{Re}\lambda>0} \frac{\mathrm{Re}\lambda}{\mathrm{Re}\lambda + 1 - \frac{\mathrm{Re}\lambda+1}{|\lambda+1|^2}} = 4,$$

$$\|B\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)} + \frac{\alpha_2^2}{2\beta}\|B_1\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)} = \frac{1}{9} < \frac{1}{4} = \frac{1}{\zeta}.$$

With the aid of Corollary 1, we claim that Eq. (6) is mean-square exponentially stable.

Example 3. Consider the following stochastic partial integro-differential system based on Coleman-Gurtin model [8, Eq. (3)] of heat conduction with memory

$$\begin{cases} \mathrm{d}z(w,t) = \left[z_{ww}(w,t) - \int_0^t \mathrm{e}^{-2(t-s)} z_{ww}(w,s)\,\mathrm{d}s - 4z(w,t) + 4\int_0^t \mathrm{e}^{-2(t-s)} z(w,s)\,\mathrm{d}s\right]\mathrm{d}t \\ \qquad\qquad + \left[\frac{1}{2}z(w,t) + \int_0^t \mathrm{e}^{-2(t-s)} z(w,s)\,\mathrm{d}s\right]\mathrm{d}W(t), \\ z_w(0,t) = z(\pi,t) = 0, \qquad t \geqslant 0, \\ z(w,0) = z_0(w), \qquad w \in [0,\pi), \end{cases} \tag{53}$$

where $W(\cdot)$ is a one dimensional standard Brownian motion. Obviously, it is based on the one dimensional stochastic heat equation without memory effect (see [18, Example 3.1], [32, Example 2.4.1], [3, Example 5.7] for more examples of one dimensional stochastic heat equation)

$$\begin{cases} \mathrm{d}z(w,t) = \big(z_{ww}(w,t) - 4z(w,t)\big)\mathrm{d}t + \frac{1}{2}z(w,t)\,\mathrm{d}W(t), \\ z_w(0,t) = z(\pi,t) = 0, \qquad t \geqslant 0, \\ z(w,0) = z_0(w), \qquad w \in [0,\pi). \end{cases} \tag{54}$$

Intuitively, Eq. (53) describes the phenomenon of heat conduction on a rod with length $l = \pi$ under stochastic perturbation; $z(w,t)$ represents the temperature of the rod at position w at time t, the temperature at the extreme right end of the rod is constant at 0 while the heat flux density at the extreme left end is 0 (i.e., the left end is adiabatic); further, the rod is composed of a hereditary material, and the kernel functions $a(\cdot)$ and $b(\cdot)$ describing its memory characteristics are as follows:

$$a(t) = -\mathrm{e}^{-2t}, \qquad b(t) = 2\mathrm{e}^{-2t}, \qquad t \geqslant 0. \tag{55}$$

To restate (53) into the form of stochastic Volterra system (1), we take (all the function spaces are taken to be real)

$$X = L^2(0,\pi), \qquad V = \mathbb{R}, \qquad Q = I; \qquad \hat{A} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad D(\hat{A}) = \{\, f \in \mathcal{H}^2(0,\pi) \mid f'(0) = f(\pi) = 0 \,\};$$

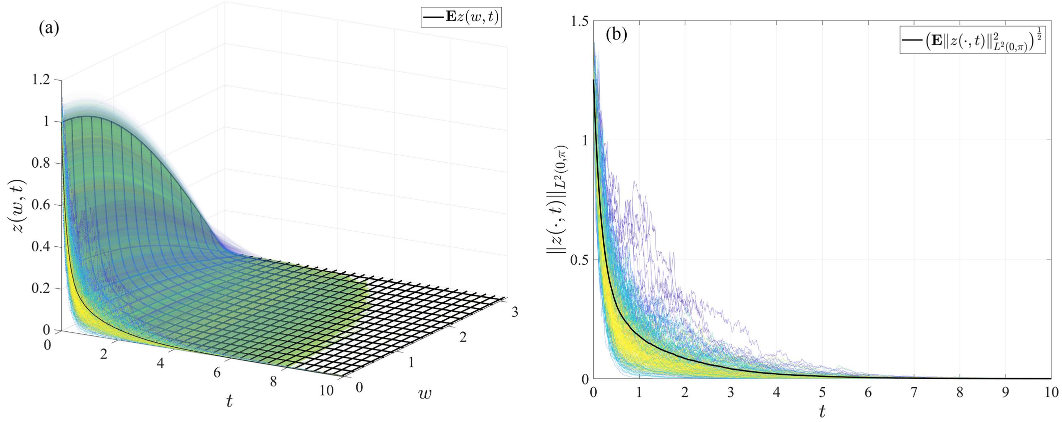

Figure 1 (Color online) Numerical results of (53) with (56). (a) 300 samples and expectation of 1000 samples; (b) $L^2(0,\pi)$ norm of 300 samples and mean square of 1000 samples.

$$A = \hat{A} - 4I, \qquad D(A) = D(\hat{A}); \qquad B(f)v := B_1(f)v := \frac{1}{2}f(\cdot)v, \qquad v \in V, \quad \forall\, f \in X,$$

and the kernel functions $a(\cdot)$, $b(\cdot)$ are as shown in (55). For the sake of completeness, we reaffirm some properties of $\hat{A}$ (see, e.g., [24, Example 2.6.10]). Actually, $\hat{A}$ is a self-adjoint operator on X with the spectrum

$$\sigma(\hat{A}) = \sigma_p(\hat{A}) = \left\{ \mu_n := -\left(n - \frac{1}{2}\right)^2 \,\middle|\, n = 1, 2, \dots \right\}.$$

In addition, $\hat{A}$ generates a contraction exponentially stable C_0-semigroup $\{\hat{T}(t) : t \geqslant 0\}$ on X meeting $\|\hat{T}(t)\|_{\mathcal{L}(X)} \leqslant \mathrm{e}^{-\frac{1}{4}t}$ for all $t \geqslant 0$. It is simple to see that $\|T(t)\|_{\mathcal{L}(X)} \leqslant \mathrm{e}^{-\frac{17}{4}t}$ and

$$\varphi(\lambda) = \frac{\lambda(\lambda+\beta)}{\lambda+\beta+\alpha_1} = \lambda + 1 - \frac{1}{\lambda+1} \in \rho(A), \quad \forall\, \lambda \in \mathbb{C}_+,$$

so Corollary 1 (i) and (ii) are satisfied. Based on [24, Proposition 3.2.8], the self-adjointness of A leads to

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} = \left(\inf_{s\in\sigma(A)} |\lambda - s|\right)^{-1}, \quad \forall\, \lambda \in \rho(A).$$

Then we immediately deduce

$$\sup_{\mathrm{Re}\lambda>0} \|R(\varphi(\lambda), A)\|_{\mathcal{L}(X)} = \sup_{\mathrm{Re}\lambda>0} \frac{1}{\inf_{s\in\sigma(A)} |\varphi(\lambda) - s|} = \frac{1}{\inf_{\mathrm{Re}\lambda>0} \left|\lambda + \frac{21}{4} - \frac{1}{\lambda+1}\right|} = \frac{17}{4}.$$

Moreover, it is evident from $V = \mathbb{R}$ and $Q = I$ that $\|B\|_{\mathcal{L}(X,\mathcal{L}_2^0)} = \|B\|_{\mathcal{L}(L^2(0,\pi),\mathcal{L}(\mathbb{R},L^2(0,\pi)))} = 1/2$. Similarly, $\|B_1\|_{\mathcal{L}(X,\mathcal{L}_2^0)} = 1/2$. Thus, we have the calculations

$$\zeta = \frac{M_1^2}{-\omega_1}\left(2 + \frac{2\alpha_1^2}{(\beta+\alpha_1)^2}\right)\left(1 + \frac{2}{\beta}\right) \sup_{\mathrm{Re}\lambda>0} \frac{\mathrm{Re}\lambda}{\mathrm{Re}\,\varphi(\lambda)} = \frac{32}{17} \cdot \sup_{\mathrm{Re}\lambda>0} \frac{\mathrm{Re}\lambda}{\mathrm{Re}\lambda + 1 - \frac{\mathrm{Re}\lambda+1}{|\lambda+1|^2}} = \frac{32}{17},$$

$$\|B\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)} + \frac{\alpha_2^2}{2\beta}\|B_1\|^2_{\mathcal{L}(X,\mathcal{L}_2^0)} = \frac{1}{2} < \frac{17}{32} = \frac{1}{\zeta}.$$

According to Corollary 1, we infer that Eq. (53) is mean-square exponentially stable.

On the other hand, in order to text the mean-square exponential stability of (53), we compute numerically 1000 samples of solutions to (53) with the initial condition

$$z_0(w) = \cos\frac{w}{2}, \qquad w \in [0,\pi), \qquad \text{a.s.} \tag{56}$$

300 samples and expectation $\mathbb{E}\,z(w,t)$ of them with (56) are shown in Figure 1(a). $L^2(0,\pi)$ norm of 300 samples and mean square $\left(\mathbb{E}\|z(\cdot,t)\|^2_{L^2(0,\pi)}\right)^{\frac{1}{2}}$ of 1000 samples with (56) are plotted in Figure 1(b). It is now straight forward from Figure 1(b) that the solution of (53) with the initial condition (56) is mean-square exponentially stable.

6 Conclusion

For a class of infinite-dimensional stochastic integro-differential equation systems, this study has provided the pathway to connect mild solutions of such equations with those of an abstract Cauchy problem by employing a semigroup approach and the Yosida approximation. Furthermore, we have established sufficient conditions that ensure the mean-square exponential stability of these mild solutions boils down to the boundedness of a function and a norm estimate for the stochastic part. Our result works well in scenarios where the system's convolution terms carry exponential-decay kernel functions as in (51), which correspond to the Caputo-Fabrizio fractional derivative. Therefore, it will be interesting to explore the sufficient conditions across more general cases in future work. In addition, our results operate under the assumption that B and B_1 are bounded on X. It will be interesting to expect the mean-square exponential stability of strong solutions to the equation if B and B_1 with $D(A) \subset D(B) \subset X$ and $D(A) \subset D(B_1) \subset X$ are bounded on $D(A)$.

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