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Design of zero-determinant strategies and its application to networked repeated games

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Abstract Using the semi-tensor product (STP) of matrices, the profile evolutionary equation (PEE) for repeated finite games is obtained. By PEE, the zero-determinant (ZD) strategies are developed for general finite games. A formula is then obtained to design ZD strategies for general finite games with multiple players and asymmetric strategies. A necessary and sufficient condition is obtained to ensure the availability of the designed ZD strategies. It follows that player i can unilaterally design $k_i - 1$ (one less than the number of its strategies) dominating linear relations about the expected payoffs of all players. Finally, the fictitious opponent player is proposed for networked repeated games (NRGs). A technique is proposed to simplify the model by reducing the number of frontier strategies.

Keywords finite repeated game, profile evolutionary equation, ZD strategy, networked repeated games, semi-tensor product of matrices

1 Introduction

In 2012, the zero-determinant (ZD) strategy was first proposed by Press and Dyson [1], which shows that in an iterated prisoner's dilemma there exist strategies that dominate any evolutionary opponent. Since then it has attracted considerable attention from the game theoretic community as well as computer, information, systems, and control communities. Ref. [2] called it "an underway revolution in game theory", because it reveals that in a repeated game, a player can unilaterally control its opponent's payoff.

A significant development in the following research is the so-called "Akin's Lemma" [3], which is a generalization of Press-Dyson's pioneering work without using the determinant form. Akin's original work is about a two-player two-strategy game (prisoner's dilemma). Then various extensions have been done. In [4], the ZD strategies of two-player two-strategy discounted games were discussed. Ref. [5] considered two-player continuous strategy discounted games. A surprising fact is that as a player adopts the ZD strategy, its actions restricting to two discrete levels of cooperation are enough to enforce a linear relationship between the payoffs of two players even if the opponent has infinitely many donation levels to choose. Multiplayer ZD strategies in games with two actions have been discussed both for undiscounted payoffs [6, 7] and for discounted payoffs [8]. The most general case with multiple players and an arbitrary number of strategies is also investigated by [9, 10].

Meanwhile, the characteristics of ZD strategies have also been investigated widely. Particularly, the stability of ZD strategy was analyzed in [11]; the robustness of ZD strategies has been investigated in [12]; the ZD strategies of noisy repeated games were investigated by [13]; the influence of misperception on ZD strategies was discussed in [14], and the evolutionary stability of ZD strategies has also been investigated widely [15–17]. ZD strategy technique has also been used for some particular kinds of games, such as application to public goods games [7], mining pole games [18], and snowdrift game [19].

The early studies concern more about the ZD strategy design [1, 6, 20]. Most later studies focus on general properties of linear relation for average payoffs of players. For instance, Ref. [10] proved the

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existence of the solution of a set of linear relations of average payoffs enforced by ZD strategies and the independence of the relations. Ref. [9] is also concentrated on the existence theorem.

When the applications of ZD strategies to real game problems are concerned, design formulas or numerical algorithms are necessary. Ref. [21] provided a method to design ZD strategies for 2×2 asymmetric games. When asymmetric games are considered, the shortage of similar studies in previous studies has been claimed in [21].

Another promising development of ZD strategy is its application to networked evolutionary games. For instance, Ref. [11] considered the extortion strategy under myopic best response arrangement over networks. Several kinds of evolutions of ZD strategy on networked evolutionary games are revealed by [22, 23]. Ref. [24] considered the cooperative mining ZD strategy in block-chain networks.

Recently, a new matrix product called the semi-tensor product (STP) of matrices, was proposed [25,26], and it has been applied to solve some problems in game theory, including the modeling and analysis of networked repeated games (NRGs) [27], providing a formula to verify whether a finite game is potential [28,29], investigating the vector space structure of finite games and its orthogonal decompositions [30–32], application to traffic congestion games [33], diffusion games [34], and logical dynamic games [35], just to mention a few. Readers who are interested in the STP approach to finite games are referred to a survey paper [36].

Using STP, this paper presents a profile evolutionary equation (PEE) for general finite repeated games, essentially the same as the Markov matrix for the memory-one game in [1]. Then a detailed design technique and rigorous proofs are presented for this general case, which are generalizations of those proposed firstly by [1]. A necessary and sufficient condition is obtained for the availability of ZD strategies. As a by-product, we also prove that if a player has k_i strategies it can provide unilaterally $k_i - 1$ linear payoff relations using ZD strategies.

Finally, the ZD strategies for NRGs are investigated. By proposing and using the fictitious opponent player (FOP), an NRG can be transferred to a two-player game, where a player, say, player i, plays with the FOP, who represents the whole network except player i. The ZD strategies for player i are designed for i vs. FOP.

The rest of this paper is organized as follows: A brief survey on STP is given in Section 2. Then it is used to develop a PEE of finite repeated games. Finally, some properties of the transition matrix of PPE are investigated, which are important for designing ZD strategies. Section 3 deduces a general formula for designing ZD strategies. A necessary and sufficient condition for the designed ZD strategies to be available is presented. Thereafter, some numerical examples are discussed to illustrate the design procedure. The FOP is proposed in Section 4 for NRGs. Using FOP, the technique of ZD strategies becomes applicable to NRGs. Section 5 is a brief conclusion.

Before ending this section, the notations used in this paper are presented. $\mathcal{M}_{m\times n}$ is the set of $m \times n$ dimensional real matrices. M^* is the adjoint matrix of M. $\sigma(M)$ is the set of eigenvalues of M. $\rho(M)$ is the spectral radius of M. $M > 0$ ($M \geqslant 0$) represents that all entries of M are positive (non-negative). $\ltimes$ is the STP of matrices. $\mathrm{Col}(A)$ ($\mathrm{Row}(A)$) is the set of columns (rows) of A; $\mathrm{Col}_i(A)$ ($\mathrm{Row}_i(A)$): the i-th column (row) of A. $\mathcal{D}_k = \{1, 2, \ldots, k\}$. δ_k^i is the i-th column of identity matrix I_k. δ_k^0 is for a zero vector of dimension k. $\mathcal{B} = \{0, 1\}$; and $\mathcal{B}^k = \{(b_1, \ldots, b_k)^{\mathrm{T}} \mid b_i \in \mathcal{B}, \forall i\}$. $\Delta_k = \mathrm{Col}(I_k) = \{\delta_k^i \mid i = 1, \ldots, k\}$ $L \in \mathcal{M}_{m\times n}$ is called a logical matrix, if $\mathrm{Col}(L) \subset \Delta_m$. Let $L = [\delta_m^{i_1}, \delta_m^{i_2}, \ldots, \delta_m^{i_n}]$, it is briefly denoted by $L = \delta_m[i_1, i_2, \ldots, i_n]$. $\mathcal{L}_{m\times n}$ is the set of $m \times n$ logical matrices. Υ^m is the set of m dimensional (column) random vectors. That is, $x = (x_1, x_2, \ldots, x_m)^{\mathrm{T}} \in \Upsilon^m$ means $x_i \geqslant 0$, $\forall i$, and $\sum_{i=1}^m x_i = 1$. $\Upsilon_{m\times n}$ is the set of $m \times n$ (column) random matrices. That is, $A \in \Upsilon_{m\times n}$, if and only if, columns of A, i.e., $\mathrm{Col}_j(A)$, $j = 1, 2, \ldots, n$, are random vectors. $\mathcal{G}_{[n;k_1,k_2,\ldots,k_n]}$ is the set of finite non-cooperative games with n players, and player i has k_i strategies, $i = 1, 2, \ldots, n$.

2 Modeling of finite repeated games

2.1 A brief survey on STP

Definition 1 ([25, 26]). Let $M \in \mathcal{M}_{m\times n}$, $N \in \mathcal{M}_{p\times q}$, and $t := \mathrm{lcm}(n, p)$ be the least common multiple of n and p. Then the STP of M and N is defined as

$$M \ltimes N := \left(M \otimes I_{t/n}\right) \left(N \otimes I_{t/p}\right) \in \mathcal{M}_{(mt/n)\times(qt/p)}, \tag{1}$$

where $\otimes$ is the Kronecker product.

Remark 1. (i) STP is a generalization of conventional matrix products. That is, if $n = p$, then $M \ltimes N = MN$. It is not necessary (and almost impossible) to distinguish STP from conventional matrix products, because, in a computing process, the product might shift from one to the other because of the changes in dimensions. Hence in most cases, the symbol $\ltimes$ is omitted.

(ii) As a generalization, STP keeps all major properties of conventional matrix products available, including associativity and distributivity. All the properties of the matrix product used in this paper are the same for both conventional matrix products and STPs.

(iii) Since conventional matrix products can be considered as a special case of STP, all the matrix products used in this paper without product symbols are assumed to be STPs.

Next, we consider how to express a finite-valued mapping (or logical mapping) into a matrix form using STP.

Let $f : \mathcal{D}_m \to \mathcal{D}_n$ be a mapping from a finite set to another finite set. Then we can identify $j \in \mathcal{D}_m$ with its vector form $\vec{j} := \delta_m^j \in \Delta_m$. In this way, f can be regarded as a mapping $f : \Delta_m \to \Delta_n$. In the sequel $\vec{j}$ is simply denoted by j again if there is no possible confusion.

Proposition 1. Let $f : \mathcal{D}_m \to \mathcal{D}_n$. Then there exists a unique matrix $M_f \in \mathcal{L}_{m\times n}$, called the structure matrix of f, such that as the arguments are expressed into their vector forms, we have

$$f(x) = M_f x. \tag{2}$$

As a corollary, Proposition 1 can be extended into a more general form.

Corollary 1. Let $x_i \in \mathcal{D}_{k_i}$, $i = 1, 2, \dots, n$, $y_j \in \mathcal{D}_{p_j}$, $j = 1, 2, \dots, m$, and $x = \ltimes_{i=1}^n x_i$, $y = \ltimes_{j=1}^m y_j$. Assume

$$y_j = f_j(x_1, x_2, \dots, x_n), \quad j = 1, 2, \dots, m,$$

which have their vector forms as

$$y_j = M_j \ltimes_{i=1}^n x_i, \quad j = 1, 2, \dots, m. \tag{3}$$

Then there exists a unique matrix M_F, called the structure matrix of the mapping $F = (f_1, \dots, f_m)$, such that

$$y = M_F x, \tag{4}$$

where

$$M_F = M_1 * M_2 * \cdots * M_n \in \mathcal{L}_{\rho\times\kappa},$$

and $\rho = \prod_{j=1}^m p_j$, $\kappa = \prod_{i=1}^n k_i$, and $*$ is Kratri-Rao product of matrices[1)].

Similarly, we have the following result.

Corollary 2. Let $x_i \in \Upsilon_{k_i}$, $i = 1, 2, \dots, n$ and $y_j \in \Upsilon_{p_j}$, $j = 1, 2, \dots, m$, and

$$y_j = M_j x, \quad j = 1, 2, \dots, m, \tag{5}$$

where $M_j \in \Upsilon_{p_j\times\kappa}$. If the random variables $y_j, j = 1, 2, \dots, m$ are conditional independent on $x_1, x_2, \dots, x_n$. Then there exists a unique matrix M_F such that

$$y = M_F x, \tag{6}$$

where

$$M_F = M_1 * M_2 * \cdots * M_n \in \Upsilon_{\rho\times\kappa},$$

which is also called the structure matrix of the mapping $F = (f_1, f_2, \dots, f_m)$.

1) Let $A \in \mathcal{M}_{s\times n}$, $B \in \mathcal{M}_{t\times n}$. Then the Khatri-Rao product of A and B, denoted by $A * B \in \mathcal{M}_{st\times n}$, is defined by $\mathrm{Col}_i(A * B) = \mathrm{Col}_i(A)\,\mathrm{Col}_i(B)$, $i = 1, 2, \dots, n$ [26].

2.2 PEE of finite games

Definition 2. Consider a finite game $G=(N,S,C)$, where
(i) $N=\{1,2,\dots,n\}$ is the set of players;
(ii) $S=\prod_{i=1}^n S_i$ is the profile, where $S_i=\{1,2,\dots,k_i\},\ i=1,2,\dots,n$, is the strategies (or actions) of player i;
(iii) $C=(c_1,c_2,\dots,c_n)$, where $c_i:S\to\mathbb{R}$ is the payoff (or utility, cost) function of player i, $i=1,2,\dots,n$.

The set of such finite games is denoted by $\mathcal{G}_{[n;k_1,k_2,\dots,k_n]}$. A matrix formulation of the repeated game $G\in\mathcal{G}_{[n;k_1,k_2,\dots,k_n]}$ is described as follows [27]:
(i) Identifying $j\in S_i$ with $\delta_{k_i}^j\in\Delta_{k_i}$, then $S_i\sim\Delta_{k_i}$.
(ii) Setting $\kappa=\prod_{i=1}^n k_i$, then $S\sim\Delta_\kappa=\prod_{i=1}^n\Delta_{k_i}$.
(iii) Let $x_i\in\Delta_{k_i}$ be the vector form of a strategy for player i. Then $x=\ltimes_{i=1}^n x_i\in\Delta_\kappa$ is a profile.
(iv) For each player's payoff function c_i, there exists a unique row vector $V_i^c\in\mathbb{R}^\kappa$ such that

$$c_i(x)=V_i^c x,\quad i=1,2,\dots,n. \tag{7}$$

Now consider a repeated game G^r of G, which stands for (infinitely) repeated G. Then each player can determine its action at $t+1$ using historical knowledge. It was proved in [1] (see also [2]) that "the shortest memory player sets the rule of the game, which means the long-memory strategies have no advantages over the memory-one strategies". Based on this observation, the strategy updating rule is assumed Markov-like. That is, the strategy of player i at time $t+1$ depends on the profile at t only. Then, we have [27]

$$x_i(t+1)=L_i x(t),\quad i=1,2,\dots,n. \tag{8}$$

Two types of strategies are commonly used.
- Pure strategy:
$$L_i\in\mathcal{L}_{k_i\times\kappa},\quad i=1,2,\dots,n;$$
- Mixed strategy:
$$L_i\in\Upsilon_{k_i\times\kappa},\quad i=1,2,\dots,n.$$

Multiplying (by STP) all equations in (8) together yields

$$x(t+1)=Lx(t), \tag{9}$$

where

$$L=L_1*L_2*\cdots*L_n.$$

In pure strategy case $L\in\mathcal{L}_{\kappa\times\kappa}$ and in mixed strategy case $L\in\Upsilon_{\kappa\times\kappa}$. In the mixed strategy case, $x(t)$ can be considered as a distribution of profiles at time t. If we take into consideration that δ_κ^i is used to express the i-th profile, then $x(t)$ can also be considered as the expected value of the profile at time t.

In this paper, we are concerned only with mixed strategy cases. Now what a player can manipulate is its own strategy updating rule. That is, player i can only choose its L_i.

We arrange profiles in alphabetic order as

$$\begin{aligned} S\ &=\ \{(s_1,s_2,\dots,s_n)\mid s_i\in S_i,\ i=1,2,\dots,n\}\\ &=\ \{(1,1,\dots,1),(1,1,\dots,2),\dots,(k_1,k_2,\dots,k_n)\}\\ &:=\{s^1,s^2,\dots,s^\kappa\}.\end{aligned}$$

Denote the probability of player i choosing strategy j at time $t+1$ under the situation that the profile at time t is s^r as

$$p_{i,j}^r=\mathrm{Prob}(x_i(t+1)=j\mid x(t)=s^r). \tag{10}$$

Then we have the strategy evolutionary equation (SEE) of player i as

$$x_i(t+1)=L_i x(t), \tag{11}$$

where

$$L_i = \begin{bmatrix} p_{i,1}^1 & p_{i,1}^2 & \cdots & p_{i,1}^\kappa \\ p_{i,2}^1 & p_{i,2}^2 & \cdots & p_{i,2}^\kappa \\ \vdots & & & \\ p_{i,k_i}^1 & p_{i,k_i}^2 & \cdots & p_{i,k_i}^\kappa \end{bmatrix} \in \Upsilon_{k_i \times \kappa}, \ i = 1, 2, \dots, n. \tag{12}$$

According to Corollary 2, we have a PEE as

$$x(t+1) = Lx(t), \tag{13}$$

where the transition matrix

$$L = L_1 * L_2 * \cdots * L_n. \tag{14}$$

We give a simple example to calculate L.

Example 1. Consider the repeated prisoners' dilemma. Let $p_{i,j}^r$ be the probability of player i taking strategy $j \in \{C, D\} \sim \{1, 2\}$ under the condition s^r. Then a straightforward computation shows that

$$\begin{cases} x_1(t+1) = L_1 x(t), \\ x_2(t+1) = L_2 x(t), \end{cases}$$

where

$$L_1 = \begin{bmatrix} p_{1,1}^1 & p_{1,1}^2 & p_{1,1}^3 & p_{1,1}^4 \\ p_{1,2}^1 & p_{1,2}^2 & p_{1,2}^3 & p_{1,2}^4 \end{bmatrix}, \ L_2 = \begin{bmatrix} p_{2,1}^1 & p_{2,1}^2 & p_{2,1}^3 & p_{2,1}^4 \\ p_{2,2}^1 & p_{2,2}^2 & p_{2,2}^3 & p_{2,2}^4 \end{bmatrix}.$$

Let

$$p_i = p_{1,1}^i, \ q_i = p_{2,1}^i, \quad i = 1, 2, 3, 4.$$

It follows that

$$p_{1,2}^i = 1 - p_{1,1}^i = 1 - p_i, \ p_{2,2}^i = 1 - p_{2,1}^i = 1 - q_i, \quad i = 1, 2, 3, 4.$$

Then we have

$$\begin{aligned} L &= L_1 * L_2 \\ &= \begin{bmatrix} p_{1,1}^1 p_{2,1}^1 & p_{1,1}^2 p_{2,1}^2 & p_{1,1}^3 p_{2,1}^3 & p_{1,1}^4 p_{2,1}^4 \\ p_{1,1}^1 p_{2,2}^1 & p_{1,1}^2 p_{2,2}^2 & p_{1,1}^3 p_{2,2}^3 & p_{1,1}^4 p_{2,2}^4 \\ p_{1,2}^1 p_{2,1}^1 & p_{1,2}^2 p_{2,1}^2 & p_{1,2}^3 p_{2,1}^3 & p_{1,2}^4 p_{2,1}^4 \\ p_{1,2}^1 p_{2,2}^1 & p_{1,2}^2 p_{2,2}^2 & p_{1,2}^3 p_{2,2}^3 & p_{1,2}^4 p_{2,2}^1 \end{bmatrix} \\ &= \begin{bmatrix} p_1 q_1 & p_1 q_2 & p_2 q_1 & p_2 q_2 \\ p_1(1-q_1) & p_1(1-q_2) & p_2(1-q_1) & p_2(1-q_2) \\ (1-p_1)q_1 & (1-p_1)q_2 & (1-p_2)q_1 & (1-p_2)q_2 \\ (1-p_1)(1-q_1) & (1-p_1)(1-q_2) & (1-p_2)(1-q_1) & (1-p_2)(1-q_2) \end{bmatrix}. \end{aligned} \tag{15}$$

Remark 2. It is easy to verify that the transition matrix in PEE (refer to (15)) is essentially the transpose of the Markov matrix for the memory one game in [1]. Corresponding to the "column order" of [1] there is a "row order" change in (15). This is because our profiles are ordered in alphabetic as CC, CD, DC, DD, while Ref. [1] uses the order CC, DC, CD, DD.

2.3 Properties of PEE

In this subsection, we investigate some properties of the transition matrix L of PEE, which are required for designing ZD strategies. As aforementioned in Remark 2, L is the same as the Markov transition matrix for the memory one game in [1] (only with a transpose). So L is a column random matrix. This difference does not affect the following discussion. Hence the following argument is a mimic of the corresponding argument in [1]. What we are going to do is extend it to a general case and put it on a solid mathematical foundation.

In the sequel, we need an assumption on L. To present it, some preparation is necessary.

A random square matrix M is called a primitive matrix if there exists a finite integer $s > 0$ such that $M^s > 0$ [37]. Some nice properties of primitive matrix are cited as follows.

Proposition 2 (Perron-Frobenius theorem [37])**.** Let L be a primitive stochastic matrix. Then

(i) $\rho(L) = 1$ and there exists a unique $\lambda \in \sigma(L)$ such that $|\lambda| = 1$;

(ii)

$$\lim_{t\to\infty} L^t = P > 0. \tag{16}$$

Moreover, $P = uv^{\mathrm{T}}$, where $Lu = u$, $u > 0$, $L^{\mathrm{T}}v = v$, $v > 0$.

We are ready to present our fundamental assumption.

Assumption A-1. L is primitive.

Remark 3. (i) A-1 is not always true. For instance, consider (15) and let $p^2_{1,1} = 0$, $p^3_{1,1} = 0$, $p^4_{1,1} = 0$, and $p^4_{2,1} = 0$. Then L is not primitive.

(ii) If $0 < p^r_{i,j} < 1$, $\forall r, i, j$, then a straightforward verification shows that L is primitive. So A-1 is always true except for a zero-measure set.

(iii) According to Proposition 2, we have the following immediate conclusions.

(a) If L is primitive, then

$$\mathrm{rank}(L - I_\kappa) = \kappa - 1. \tag{17}$$

(b) There exists $P = uv^{\mathrm{T}}$, where $Lu = u$, $u > 0$, $L^{\mathrm{T}}v = v$, $v > 0$, such that Eq. (16) holds. That is,

$$\lim_{t\to\infty} L^t = uv^{\mathrm{T}}. \tag{18}$$

Proposition 3. Let L be a $\kappa \times \kappa$ column primitive stochastic matrix. Define $M := L - I_\kappa$ and let M^* be its adjoint matrix. Then

(i) $\mathrm{rank}(M^*) = 1$;

(ii)

$$\mathrm{Col}_j(M^*) \neq 0, \quad j = 1, 2, \ldots, \kappa. \tag{19}$$

The proof of this proposition and all other proofs can be found in Appendix A.

Proposition 4. Consider the PEE (13). If L is primitive, then

$$x^* := \lim_{t\to\infty} x(t) = u/\|u\|, \tag{20}$$

where u comes from (18).

Hereafter, we assume u has been normalized. Then $x^* = u$ is the only normalized eigenvector of L corresponding to eigenvalue 1.

Proposition 5. Assume L is primitive, and then

$$\mathrm{Col}_j(M^*) \propto u, \quad \forall j. \tag{21}$$

Combining (19) and (21) yields

$$\mathrm{Col}_j(M^*) = \mu_j u, \quad \mu_j \neq 0, \ \ j = 1, 2, \ldots, \kappa. \tag{22}$$

3 Design of ZD strategies for repeated games

3.1 A universal formula for ZD-strategies

Consider the transition matrix L of PEE (13). Recall the finite game G. For player i with action j, define an indicative vector $\xi_{i,j} \in \mathbb{R}^\kappa$ as follows:

$$\xi_{i,j} = \ltimes_{\tau=1}^{n} \gamma_\tau, \tag{23}$$

where

$$\gamma_\tau = \begin{cases} \mathbf{1}_{k_\tau}, \tau \neq i, \\ \delta_{k_i}^j, \ \tau = i. \end{cases}$$

$\xi_{i,j} \in \mathbb{R}^\kappa$ is called a strategy extraction vector, which has the following property.

Lemma 1. Consider the FRG G^r. Strategy extraction vector $\xi_{i,j} \in \mathbb{R}^\kappa$ has the following property:

$$\begin{aligned} \xi_{i,j}^{\mathrm{T}} L &= \textstyle\sum_{a \in \Phi_{i,j}} \mathrm{Row}_a(L) \\ &= [p_{i,j}^1, p_{i,j}^2, \ldots, p_{i,j}^\kappa], \ \forall i \in N, \forall j \in A_i, \end{aligned} \tag{24}$$

where $\Phi_{i,j} = \{a = (a_1, \ldots, a_n) \in A \mid a_i = j\} \subseteq A$.

Proof. According to the definition of $\xi_{i,j}$ and L, we have

$$\begin{aligned} \xi_{i,j}^{\mathrm{T}} L &= \xi_{i,j}^{\mathrm{T}} [\mathrm{Col}_1(L), \mathrm{Col}_2(L), \ldots, \mathrm{Col}_k(L)] \\ &= [\xi_{i,j}^{\mathrm{T}} \mathrm{Col}_1(L), \xi_{i,j}^{\mathrm{T}} \mathrm{Col}_2(L), \ldots, \xi_{i,j}^{\mathrm{T}} \mathrm{Col}_k(L)], \end{aligned}$$

where for each column

$$\begin{aligned} \xi_{i,j}^{\mathrm{T}} \mathrm{Col}_r(L) &= (\ltimes_{\tau=1}^n \gamma_\tau^{\mathrm{T}})(\ltimes_{s=1}^n \mathrm{Col}_r(L_s)) \\ &= (\otimes_{\tau=1}^n \gamma_\tau^{\mathrm{T}})(\otimes_{s=1}^n \mathrm{Col}_r(L_s)) \\ &= \otimes_{s=1}^n (\gamma_s^{\mathrm{T}} \mathrm{Col}_r(L_s)) \\ &= p_{i,j}^r. \end{aligned}$$

The second equality comes from STP's property. The third equality comes from the property of the Kronecker product.

Remark 4. (i) The strategy extraction vector $\xi_{i,j} \in \mathbb{R}^\kappa$ is called "Repeat" strategy in [5, 10]. (ii) The purpose of (24) is to pick out the set of rows from matrix L, which involve $p_{i,j}^r$. The row labels of such a set are denoted by $\Phi_{i,j}$. Then $\xi_{i,j}^{\mathrm{T}} L$ is the summation of the rows in L, which are labeled by $\Phi_{i,j}$. For each pair (i,j), $\xi_{i,j}^{\mathrm{T}} L$ realizes an elementary (equivalent) transformation for L, which results in a row of L which contains $p_{i,j}$ only, i.e., this new row does not involve $p_{s,r}^d$, $(s,r) \neq (i,j)$.

If L is primitive, then it has a stationary distribution $\mu \in \Upsilon^\kappa$ satisfying

$$L\mu = \mu \Leftrightarrow (L - I)\mu = 0. \tag{25}$$

Multiplying $\xi_{i,j}$ to both sides of (25) yields that

$$[\xi_{i,j}^{\mathrm{T}} L - \xi_{i,j}^{\mathrm{T}}]\mu = 0. \tag{26}$$

Let $T_i = [\xi_i^{\mathrm{T}} L - \xi_i^{\mathrm{T}}]$, where $\xi_i = [\xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,k_i}]$. According to [10], the ZD strategy of player i belongs to the interaction of two subspaces.

Definition 3 ([10]). The ZD strategy L_i of player i exists if and only if

$$\mathrm{Span}(V) \cap \mathrm{Span}(T_i^{\mathrm{T}}) \neq \{\mathbf{0}_\kappa\}, \tag{27}$$

where $V = [\mathbf{1}_\kappa, (V_1^c)^{\mathrm{T}}, (V_2^c)^{\mathrm{T}}, \ldots, (V_n^c)^{\mathrm{T}}]$.

Remark 5. Definition 3 can be used to detect whether a given strategy L_i is a ZD strategy or not. However, it is difficult to design a ZD strategy for a given game.

In the following, we only consider how to derive player i's ZD strategy $p_{i,j}$ associated with action j using $\xi_{i,j}$, where $p_{i,j} = [p_{i,j}^1, p_{i,j}^2, \ldots, p_{i,j}^\kappa]$. A general design formula is presented in the following.

Proposition 6. Consider a repeated game G^r, where $G \in \mathcal{G}_{[n;k_1,k_2,\ldots,k_n]}$. Assume player i aims at a set of linear relations on the expected payoffs as

$$\ell_{i,j}(Ec_1, Ec_2, \ldots, Ec_n, 1) = 0, \quad 1 \leqslant i \leqslant n, \ j = 1, 2, \ldots, k_i - 1, \tag{28}$$

where $\ell_{i,j}$ is a linear function and Ec_i is the expected payoff of player i. Then its ZD strategies can be designed as

$$\begin{aligned} p_{i,j} &= (p^1_{i,j}, p^2_{i,j}, \ldots, p^\kappa_{i,j}) \\ &= \mu_{i,j}\ell_{i,j}\left(V_1^c, V_2^c, \ldots, V_n^c, \mathbf{1}_\kappa^{\mathrm{T}}\right) + \xi_{i,j}, \quad j = 1, 2, \ldots, k_i - 1, \end{aligned} \tag{29}$$

where $\mu_{i,j} \neq 0$ are adjustable parameters.

Proof. If $p_{i,j}$ satisfies (29), then we have

$$\begin{aligned} p_{i,j} - \xi_{i,j}^{\mathrm{T}} &= \mu_{i,j}\ell_{i,j}\left(V_1^c, V_2^c, \ldots, V_n^c, \mathbf{1}_\kappa^{\mathrm{T}}\right) \\ &= \mu_{i,j}\ell_{i,j}\left(Ec_1, Ec_2, \ldots, Ec_n\right) + c \\ &= 0, \end{aligned} \tag{30}$$

where c is a constant. Eq. (30) implies that

$$\ell_{i,j}\left(Ec_1, Ec_2, \ldots, Ec_n\right) = 0,\ 1 \leqslant i \leqslant n,\ j = 1, 2, \ldots, k_i - 1.$$

Remark 6. Eq. (29) is a fundamental formula, which provides a convenient way to design ZD strategies for our preassigned purposes. One may be concerned about the time complexity of the proposed formula. We point out that the complexity is related to the number of players n and the number of strategies k_i for each player. The method of reducing the complexity of designing ZD strategies and FOP method is proposed in Section 4.

Definition 4. A set of ZD strategies is permissible, if the following two conditions are satisfied:

$$\text{(i)} \quad 0 \leqslant p_{i,j} \leqslant 1, \quad j = 1, 2, \ldots, k_i - 1; \tag{31}$$

$$\text{(ii)} \quad 0 \leqslant \sum_{j=1}^{k_i-1} p_{i,j} \leqslant 1. \tag{32}$$

Remark 7. (i) It is obvious that permissibility is a fundamental requirement. Non-permissible strategies are meaningless.

(ii) It is clear that player i can unilaterally design at most $|S_i| - 1$ linear relations. Because when $p_{i,j}$, $j < |S_i|$ are all determined, $p_{i,|S_i|}$ is uniquely determined by

$$p_{i,|S_i|} = \mathbf{1}_\kappa^{\mathrm{T}} - \sum_{j=1}^{|S_i|-1} p_{i,j}.$$

(iii) Indeed, player i can design $|S_i| - 1$ linear relations as it wishes. This is an advantage of (29), because it clearly tells how many linear relations a player may design. It was pointed out by [10] that "when the number M_n of possible actions for player n is more than two, player n may be able to employ a ZD strategy with dim $V_n \geqslant 2$ to simultaneously enforce more than one linear relations. Such a possibility has never been reported in the context of ZD strategies".

(iv) Of course, player i needs not to design $|S_i| - 1$ relations. If it intends to design $r < |S_i| - 1$ relations, Eq. (32) has to be modified by reducing the summation to r items.

(v) The ZD design formula (29) can be used simultaneously by multiple players, or even all n players.

(vi) Eq. (22) is extremely important for (29) to be available, because it ensures that each row in the $M = L - I_\kappa$ is replaceable by a designed linear relation to get zero determinant.

Even though a set of ZD strategies is permissible, it may not be available, which means the goal (28) may not be reached. We need the following results.

Theorem 1. Consider a repeated game, G^r, where $G \in \mathcal{G}_{[n;k_1,k_2,\ldots,k_n]}$. The stationary distribution exists, if and only if,

(i) there exists a $\mu \in \Upsilon^\kappa$ such that

$$\lim_{t\to\infty} L^t = \mu\mathbf{1}_\kappa^{\mathrm{T}}; \tag{33}$$

(ii)

$$\operatorname{rank}(L - I_\kappa) = \kappa - 1. \tag{34}$$

Remark 8. The existence of stationary distribution μ is only a sufficient condition for a set of ZD strategies designed by (29) to be available. As pointed by [3], it can be replaced by $\lim_{t\to\infty}\sum_{k=1}^{t} x(k)$, which is the same as μ provided μ exists.

Remark 9. To see that permissibility is not enough to ensure (33) and (34), we recall Example 1. Assume

$$L_1 = \begin{bmatrix} p^1_{1,1} & 0 & p^3_{1,1} & 0 \\ p^1_{1,2} & 1 & p^3_{1,2} & 1 \end{bmatrix}, \ L_2 = \begin{bmatrix} p^1_{2,1} & 0 & p^3_{2,1} & 0 \\ p^1_{2,2} & 1 & p^3_{2,2} & 1 \end{bmatrix}.$$

Then it is easy to verify that

$$L = L_1 * L_2 = \begin{bmatrix} p^1_{1,1}p^1_{2,1} & 0 & p^3_{1,1}p^3_{2,1} & 0 \\ p^1_{1,1}p^1_{2,2} & 0 & p^3_{1,1}p^3_{2,2} & 0 \\ p^1_{1,2}p^1_{2,1} & 0 & p^3_{1,2}p^3_{2,1} & 0 \\ p^1_{1,2}p^1_{2,2} & 1 & p^3_{1,2}p^3_{2,2} & 1 \end{bmatrix} \sim \begin{bmatrix} p^1_{1,1}p^1_{2,1} & p^3_{1,1}p^3_{2,1} & 0 & 0 \\ p^1_{1,2}p^1_{2,1} & p^3_{1,2}p^3_{2,1} & 0 & 0 \\ p^1_{1,1}p^1_{2,2} & p^3_{1,1}p^3_{2,2} & 0 & 0 \\ p^1_{1,2}p^1_{2,2} & p^3_{1,2}p^3_{2,2} & 1 & 1 \end{bmatrix},$$

where "$\sim$" stands for similar, which is caused by swapping the second row with the third row and the second column with the third column. Then it is clear that L^t is always similar to a block lower triangular matrix. Hence, Eq. (33) can never be satisfied. While using (29), by choosing suitable V^c_i, $i = 1, 2$ and parameter $\mu_{1,1}$, a permissible set of ZD strategies can easily be constructed, which provides a counter-example to show permissibility is not enough to ensure availability.

Note that verifying the two conditions in Theorem 1 is not an easy job. Hence we may replace them with the following one.

Corollary 3. Consider a repeated game, G^r, where $G \in \mathcal{G}_{[n;k_1,k_2,\dots,k_n]}$. Assume the PEE of G^r is (9), where L is primitive, and then the set of ZD strategies designed by (29) is available.

Remark 10. (i) Even though primitivity of L is only a sufficient condition, it is almost necessary because only a zero-measure set of L may not be primitive. That is, if L is not primitive then there must be some (r, i, j) with $p^r_{i,j} \in \mathcal{B}$. So the designer, who intends to use ZD strategies, is better to avoid using such values.

(ii) Any player cannot unilaterally make the conditions in Theorem 1 satisfied. It depends on other players' strategies. What the player i can do is to do its best, that is, to ensure its designed rows, $\xi_{i,j}$, $j = 1, 2, \dots, k_i$ are linearly independent. (A Chinese idiom says that "Mou Shi Zai Ren, Cheng Shi Zai Tian" (Man proposes, God disposes).) That is the situation for a ZD strategy designer.

3.2 Numerical examples

In the following, we discuss some numerical examples.

Example 2. Consider a $G \in \mathcal{G}_{[3;2,3,2]}$. Since $k_1 = 2$, $k_2 = 3$, and $k_3 = 2$, using (24), it is easy to calculate that

$$\begin{aligned}
&\Phi_{1,1} = \{1, 2, 3, 4, 5, 6\}, \\
&\Phi_{1,2} = \{7, 8, 9, 10, 11, 12\}, \\
&\Phi_{2,1} = \{1, 2, 7, 8\}, \\
&\Phi_{2,2} = \{3, 4, 9, 10\}, \\
&\Phi_{2,3} = \{5, 6, 11, 12\}, \\
&\Phi_{3,1} = \{1, 3, 5, 7, 9, 11\}, \\
&\Phi_{3,2} = \{2, 4, 6, 8, 10, 12\}. \\
&\xi_{1,1} = [1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0], \\
&\xi_{1,2} = [0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1], \\
&\xi_{2,1} = [1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0], \\
&\xi_{2,2} = [0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0], \\
&\xi_{2,3} = [0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1], \\
&\xi_{3,1} = [1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0], \\
&\xi_{3,2} = [0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1].
\end{aligned} \tag{35}$$

These parameters depend on the type of games; precisely speaking, they depend on the parameters $\{n; k_1, \dots, k_n\}$ only. They are independent of particular games.

(i) Pinning strategy. Assume the payoff vectors are

$$\begin{aligned}
V_1^c &= [-3, -0.5, 6, 9, 8, 7, -4, -4.5, 5, 6.5, 5, 7],\\
V_2^c &= [4, -1, -5, 7.5, 2, 3.5, 8, -4, 5, 8, 9, -2],\\
V_3^c &= [9, 5, -6, -5.5, 5.5, 8, 8.5, 5.5, -0, -3.5, 4.5, 7].
\end{aligned}$$

Assume player 2 wants to design pinning strategies that enforce the average payoffs of players 1 and 3 to be

$$\begin{aligned}
Ec_1 = r_1 = 4,\\
Ec_2 = r_2 = -3.
\end{aligned}$$

It may choose $\mu_{2,1} = \mu_{2,2} = 0.1$ and then set

$$\begin{aligned}
p_{2,1} &:= (0.1) * V_1^c - (0.4) * \mathbf{1}_{12}^{\mathrm{T}} + \xi_{2,1}\\
&= [0.3, 0.55, 0.2, 0.5, 0.4, 0.3, 0.2, 0.15, 0.1, 0.25, 0.1, 0.3],\\
p_{2,2} &= (0.1) * V_3^c + (0.3) * \mathbf{1}_{12}^{\mathrm{T}} + \xi_{2,2}\\
&= [0.6, 0.2, 0.1, 0.15, 0.25, 0.5, 0.55, 0.25, 0.7, 0.35, 0.15, 0.4].
\end{aligned} \tag{36}$$

It is ready to verify that the ZD strategies designed in (36) are permissible.

(ii) Extortion strategy. Consider a $G \in \mathcal{G}_{[3;2,3,2]}$ again. Assume the payoff structure vectors are as follows:

$$\begin{aligned}
V_1^c &= [16, 11, -4, -8, -2, -10.3, 11.4, 18.5, 1.2, -3, -2.5, 1.5],\\
V_2^c &= [3, 2, -1, 0, 5, -6, 4, 3, 3, 1, -1, 7],\\
V_3^c &= [-2.9, 0, 6.8, 7.1, 2, -9.4, -8.2, 0.4, 4.6, 6.1, -2, 2.3].
\end{aligned}$$

Player 2 plans to design an extortion strategy against both players 1 and 3. It may design

$$\begin{aligned}
Ec_2 - r = k_1(Ec_1 - r),\\
Ec_2 - r = k_2(Ec_3 - r).
\end{aligned}$$

To this end, it needs to design

$$\begin{aligned}
p_{2,1} - \xi_{2,1} = \mu_1 \left[(V_2^c - r\mathbf{1}_{12}^{\mathrm{T}}) - k_1(V_1^c - r\mathbf{1}_{12}^{\mathrm{T}})\right],\\
p_{2,3} - \xi_{2,3} = \mu_2 \left[(V_2^c - r\mathbf{1}_{12}^{\mathrm{T}}) - k_2(V_3^c - r\mathbf{1}_{12}^{\mathrm{T}})\right].
\end{aligned}$$

Choosing $\mu_1 = 0.05$, $\mu_2 = 0.1$, $r = 1$, $k_1 = 1.1$, $k_2 = 1.2$, it follows that

$$\begin{aligned}
p_{2,1} &= [0.275, 0.5, 0.175, 0.445, 0.365, 0.2715, 0.178, 0.1375, 0.0890, 0.22, 0.0925, 0.2725],\\
p_{2,2} &= [0.668, 0.22, 0.104, 0.168, 0.28, 0.548, 0.604, 0.272, 0.768, 0.388, 0.16, 0.444].
\end{aligned} \tag{37}$$

The ZD strategies designed in (37) are also permissible.

Remark 11. (i) In general, designing a set of permissible ZD strategies is not an easy job. To determine related parameters we need to solve a set of linear inequalities.

(ii) To verify Lemma 1, we calculate the matrix $M = L - I_\kappa$ for Example 2 as follows:

$$M = \begin{bmatrix} p^1_{1,1}p^1_{2,1}p^1_{3,1} - 1 & p^2_{1,1}p^2_{2,1}p^2_{3,1} & \cdots & p^{12}_{1,1}p^{12}_{2,1}p^{12}_{3,1} \\ p^1_{1,1}p^1_{2,1}p^1_{3,2} & p^2_{1,1}p^2_{2,1}p^2_{3,2} - 1 & \cdots & p^{12}_{1,1}p^{12}_{2,1}p^{12}_{3,2} \\ p^1_{1,1}p^1_{2,2}p^1_{3,1} & p^2_{1,1}p^2_{2,2}p^2_{3,1} & \cdots & p^{12}_{1,1}p^{12}_{2,2}p^{12}_{3,1} \\ p^1_{1,1}p^1_{2,2}p^1_{3,2} & p^2_{1,1}p^2_{2,2}p^2_{3,2} & \cdots & p^{12}_{1,1}p^{12}_{2,2}p^{12}_{3,2} \\ p^1_{1,1}p^1_{2,3}p^1_{3,1} & p^2_{1,1}p^2_{2,3}p^2_{3,1} & \cdots & p^{12}_{1,1}p^{12}_{2,3}p^{12}_{3,1} \\ p^1_{1,1}p^1_{2,3}p^1_{3,2} & p^2_{1,1}p^2_{2,1}p^2_{3,2} & \cdots & p^{12}_{1,1}p^{12}_{2,1}p^{12}_{3,2} \\ p^1_{1,2}p^1_{2,1}p^1_{3,1} & p^2_{1,2}p^2_{2,1}p^2_{3,1} & \cdots & p^{12}_{1,2}p^{12}_{2,1}p^{12}_{3,1} \\ p^1_{1,2}p^1_{2,1}p^1_{3,2} & p^2_{1,2}p^2_{2,1}p^2_{3,2} & \cdots & p^{12}_{1,2}p^{12}_{2,1}p^{12}_{3,2} \\ p^1_{1,2}p^1_{2,2}p^1_{3,1} & p^2_{1,2}p^2_{2,2}p^2_{3,1} & \cdots & p^{12}_{1,2}p^{12}_{2,2}p^{12}_{3,1} \\ p^1_{1,2}p^1_{2,2}p^1_{3,2} & p^2_{1,2}p^2_{2,2}p^2_{3,2} & \cdots & p^{12}_{1,2}p^{12}_{2,2}p^{12}_{3,2} \\ p^1_{1,2}p^1_{2,3}p^1_{3,1} & p^2_{1,2}p^2_{2,3}p^2_{3,1} & \cdots & p^{12}_{1,2}p^{12}_{2,3}p^{12}_{3,1} \\ p^1_{1,2}p^1_{2,3}p^1_{3,2} & p^2_{1,2}p^2_{2,1}p^2_{3,2} & \cdots & p^{12}_{1,2}p^{12}_{2,1}p^{12}_{3,2} - 1 \end{bmatrix}. \tag{38}$$

Then it is easy to verify that $\Phi_{i,j}$ is the row with each component containing $p^t_{i,j}$ as a factor. Moreover, a simple calculation shows that Lemma 1 is correct.

(iii) To verify the availability of ZD-strategies in (36), we assume the strategy for player 1 is

$$p_{1,1} = [0.2, 0.3, 0.8, 0.7, 0.5, 0.4, 0.7, 0.9, 0.2, 0.2, 0.1, 0.9];$$

the strategy for player 3 is

$$p_{3,1} = [0.15, 0.2, 0.8, 0.85, 0.2, 0.35, 0.7, 0.9, 0.2, 0.15, 0.55, 0.35].$$

Then the strategy profile dynamics is

$$x(t+1) = Lx(t), \quad t \geqslant 0,$$

where

$$L = \begin{bmatrix} 0.009 & 0.033 & 0.128 & 0.2975 & 0.04 & 0.042 & 0.098 & 0.1215 & 0.004 & 0.0075 & 0.0055 & 0.0945 \\ 0.051 & 0.132 & 0.032 & 0.0525 & 0.16 & 0.078 & 0.042 & 0.0135 & 0.016 & 0.0425 & 0.0045 & 0.1755 \\ 0.018 & 0.012 & 0.064 & 0.0892 & 0.025 & 0.07 & 0.2695 & 0.2025 & 0.028 & 0.0105 & 0.0083 & 0.126 \\ 0.102 & 0.048 & 0.016 & 0.0158 & 0.1 & 0.13 & 0.1155 & 0.0225 & 0.112 & 0.0595 & 0.0067 & 0.234 \\ 0.003 & 0.015 & 0.448 & 0.2082 & 0.035 & 0.028 & 0.1225 & 0.486 & 0.008 & 0.012 & 0.0413 & 0.0945 \\ 0.017 & 0.06 & 0.112 & 0.0367 & 0.14 & 0.052 & 0.0525 & 0.054 & 0.032 & 0.068 & 0.0338 & 0.1755 \\ 0.036 & 0.077 & 0.032 & 0.1275 & 0.04 & 0.063 & 0.042 & 0.0135 & 0.016 & 0.03 & 0.0495 & 0.0105 \\ 0.204 & 0.308 & 0.008 & 0.0225 & 0.16 & 0.117 & 0.018 & 0.0015 & 0.064 & 0.17 & 0.0405 & 0.0195 \\ 0.072 & 0.028 & 0.016 & 0.0383 & 0.025 & 0.105 & 0.1155 & 0.0225 & 0.112 & 0.0420 & 0.0743 & 0.014 \\ 0.408 & 0.112 & 0.004 & 0.0068 & 0.1 & 0.195 & 0.0495 & 0.0025 & 0.448 & 0.238 & 0.0607 & 0.0260 \\ 0.012 & 0.035 & 0.112 & 0.0893 & 0.035 & 0.042 & 0.0525 & 0.054 & 0.032 & 0.048 & 0.3713 & 0.0105 \\ 0.068 & 0.14 & 0.028 & 0.0158 & 0.14 & 0.078 & 0.0225 & 0.006 & 0.128 & 0.272 & 0.3037 & 0.0195 \end{bmatrix}.$$

It is easy to verify that

$$\mathrm{rank}(L - I_{12}) = 11.$$

Moreover, we also have that

$$\lim_{t\to\infty} L^t = u\mathbf{1}^{\mathrm{T}}_{12},$$

where

$$u = [0.0731, 0.075, 0.0715, 0.082, 0.126, 0.0775, 0.0434, 0.1002, 0.0475, 0.1278, 0.0683, 0.1077]^{\mathrm{T}},$$

which is the normalized eigenvector of L with respect to its (unique) eigenvalue 1.

4 Application to NRGs

This section considers how to design ZD strategies for a player, i, in an NRG. We propose a method, called an FOP.

4.1 FOP

Definition 5 ([27]). An NRG is a triple $((N,E),G,\Pi)$, where (N,E) is a network graph; N is the set of players; $G \in \mathcal{G}_{[2;k,k]}$ is a symmetric game with two players, called the fundamental network game; Π is the strategy updating rule, which describes how each player to update its strategies using its neighborhood information.

Remark 12. (i) $G \in \mathcal{G}_{[2;k,k]}$ is symmetric, if $S_1 = S_2 := S_0$ and for any $x, y \in S_0$,

$$c_1(x,y) = c_2(y,x).$$

(ii) If $(i,j) \in E$, then players i and j will play game G repeatedly. In this paper, only the fixed graph is considered. Since G is symmetric, then the order of two players does not affect the result.

(iii) Such an NRG is denoted by $G^{nr} = ((N,E),G,\Pi)$.

Let player $i \in N$, and $\deg(i) = d$. Then it may consider $N\backslash\{i\}$ as one player, called the FOP of i, denoted by p_{-i}. Assuming $|S_0| = k$, the neighbors' strategies can be considered as the strategies of p_{-i}. That is, p_{-i} has totally k^d strategies.

In fact, we do not need to distinct different neighbors; hence if $S_0 = \{s_1, s_2, \dots, s_k\}$, then the set of strategies of p_{-i}, denoted by S_{-i}, is

$$S_{-i} = \{\underbrace{s_1 s_1 \cdots s_1}_{d}, \underbrace{s_1 s_1 \cdots s_2}_{d}, \dots, \underbrace{s_k s_k \cdots s_k}_{d}\}. \tag{39}$$

Each $s_* \in S_{-i}$ can be expressed as

$$s_* = (\underbrace{s_1 s_1 \cdots s_1}_{d_1}, \underbrace{s_2 s_2 \cdots s_2}_{d_2}, \dots, \underbrace{s_k s_k \cdots s_k}_{d_k}),$$

where $d_i \geqslant 0$ and $d_1 + d_2 + \cdots + d_k = d$. Hence, we can also express s_* using $(d_1, d_2, \dots, d_k)$, which means s_j has been used by d_j neighbors, $1 \leqslant j \leqslant k$. Using this notation, we have

$$S_{-i} = \left\{ (d_1, d_2, \dots, d_k) \mid d_j \geqslant 0,\ \forall j;\ \sum_{j=1}^{k} d_j = d \right\}. \tag{40}$$

It is easy to verify that defining the strategies of p_{-i} in this way, by ignoring the order of neighbors, the total number of strategies is reduced from k^d to

$$|S_{-i}| = \frac{(k+d-1)!}{(k-1)!d!}.$$

Hence this treatment reduces the computational complexity.

From the point of view of player i, the NRG is equivalent to a game between it and p_{-i}, who has the set of strategies S_{-i} defined by (40). Let $s_* = (d_1, d_2, \dots, d_k) \in S_{-i}$. Then the payoff functions for c_i and player p_{-i}, denoted by c_{-i}, are

$$\begin{aligned} c_i(x_i, s_*) &= \sum_{j=1}^{k} d_j c_i(x_i, s_j), \\ c_{-i}(x_i, s_*) &= \sum_{j=1}^{k} d_j c_j(x_i, s_j). \end{aligned} \tag{41}$$

Note that the FOP formulation is particularly suitable for using ZD strategies because it is not affected by the structure and size of the network graph, even though the size might be ∞. As long as the stationary distribution of the overall network exists, ZD strategies are still applicable. Moreover, it is easily designable.

Table 1 Payoff bi-matrix of prisoner's dilemma

$P_1\backslash P_2$	C	D
C	$R,\ R$	$S,\ T$
D	$T,\ S$	P, P

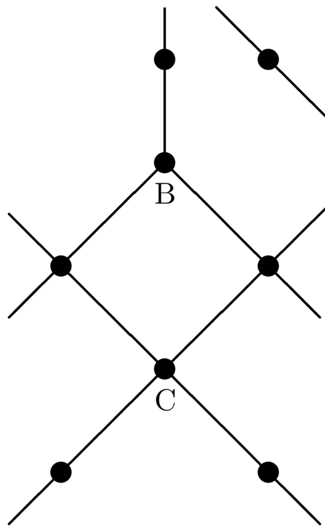

Figure 1 Networked prisoners' dilemma.

4.2 ZD strategies for NRGs

This subsection considers how to design ZD strategies for NRGs. We describe the process through two examples.

Example 3. Consider prisoner's dilemma G. The two strategies for both players are cooperation (C) and defect (D). Their payoffs are described in Table 1, where, as a convention, $T > R > P > S$.

Consider a networked repeated prisoners' dilemma, denoted by G^{nr}. The network graph, depicted by Figure 1, is non-homogeneous.

(1) Consider player A. Since $\deg(A) = 2$, the set of strategies of p_{-A} is

$$S_{-A} = \{(CC), (CD), (DD)\}.$$

Using (41), the payoff vectors for c_A and c_{-A} are, respectively,

$$V^c_A = (2R, R + S, 2S, 2T, T + P, 2P),$$
$$V^c_{-A} = (2R, R + T, 2T, 2S, S + P, 2P).$$

It is easy to calculate that $\kappa = 6$, and

$$\Phi_{1,1} = \{1, 2, 3\},\ \xi_{1,1} = (1, 1, 1, 0, 0, 0).$$

• Pinning strategy. To get $Ec_{-A} = r$, the ZD strategy of player A can be designed as

$$(p^1_{1,1}, p^2_{1,1}, \ldots, p^6_{1,1}) = \mu(V^c_{-A} - r\mathbf{1}^{\mathrm{T}}_6) - \xi_{1,1}.$$

• Extortion strategy. To get $Ec_A - r = \ell(Ec_{-A} - r)$ with $\ell > 1$, the ZD strategy of player A can be designed as

$$(p^1_{1,1}, p^2_{1,1}, \ldots, p^6_{1,1}) = \mu\left((V^c_A - r\mathbf{1}^{\mathrm{T}}_6) - \ell(V^c_{-A} - r\mathbf{1}^{\mathrm{T}}_6)\right) - \xi_{1,1}.$$

(2) Consider player B. Since $\deg(B) = 3$, the set of strategies of p_{-B} is

$$S_{-B} = \{(CCC), (CCD), (CDD), (DDD)\}.$$

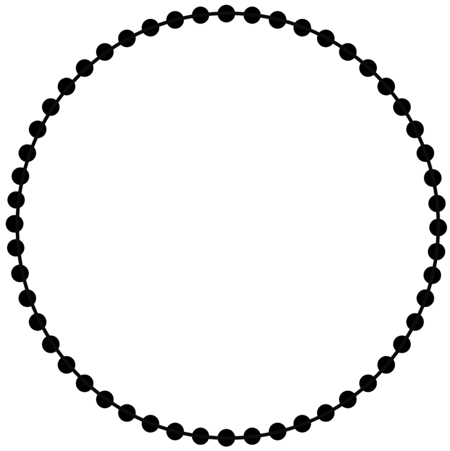

Figure 2 Cycle ring graph.

Using (41), the payoff vectors for c_B and c_{-B} are, respectively,

$$\begin{aligned} V_B^c &= (3R, 2R+S, R+2S, 3S, 3T, 2T+P, T+2P, 3P), \\ V_{-B}^c &= (3R, 2R+T, R+2T, 3T, 3S, 2S+P, S+2P, 3P). \end{aligned}$$

We have $\kappa = 8$ and

$$\Phi_{1,1} = \{1,2,3,4\},\ \xi_{1,1} = (1,1,1,1,0,0,0,0).$$

The design of ZD strategies is similar to the one for A.

(3) Consider player C. Since $\deg(C) = 4$, the set of strategies of p_{-C} is

$$S_{-C} = \{(CCCC), (CCCD), (CCDD), (CDDD), (DDDD)\}.$$

Using (41), the payoff vectors for c_C and c_{-C} are, respectively,

$$\begin{aligned} V_C^c &= (4R, 3R+S, 2R+2S, R+3S, 4S, 4T, 3T+P, 2T+2P, T+3P, 4P), \\ V_{-C}^c &= (4R, 3R+T, 2R+2T, R+3T, 4T, 4S, 3S+P, 2S+2P, S+3P, 4P). \end{aligned}$$

It is easy to calculate that $\kappa = 10$ and

$$\Phi_{1,1} = \{1,2,3,4,5\},\ \xi_{1,1} = (1,1,1,1,1,0,0,0,0,0).$$

The design of ZD strategies is similar to the one for A, or B.

To illustrate the effectiveness of the proposed method for large games, we provide Example 4.

Example 4. Consider a networked repeated prisoners' dilemma, denoted by G^{nr}. The network graph, depicted by Figure 2, is a circular ring with a large number of nodes.

For any given player i, $\deg(i) = 2$. The set of strategies of p_{-i} is

$$S_{-i} = \{(CC), (CD), (DD)\}.$$

Using (41), the payoff vectors for c_i and c_{-i} are, respectively,

$$V_i^c = (2R, R+S, 2S, 2T, T+P, 2P),\ V_{-i}^c = (2R, R+T, 2T, 2S, S+P, 2P).$$

It is easy to calculate that $\kappa = 6$, and $\Phi_{1,1} = \{1,2,3\},\ \xi_{1,1} = (1,1,1,0,0,0)$.

To realize $Ec_i - r = 0.5(Ec_{-i} - r)$, the ZD strategy of player i can be designed as

$$(p_{1,1}^1, p_{1,1}^2, \ldots, p_{1,1}^6) = \mu\left[(V_i^c - r\mathbf{1}_6^{\mathrm{T}}) - 0.5(V_{-i}^c - r\mathbf{1}_6^{\mathrm{T}})\right] - \xi_{1,1}.$$

Remark 13. (i) In the above discussion we only provide the formula for designing ZD strategies. A problem is: is the solution $\{p^i_{j,k}, k \in [1, k_i]\}$ obtained from the formula permissible? From the designer's point of view, it can be seen immediately from the numerical result. As for the theoretical discussion, it is a challenging problem and is out of the scope of this paper. We refer to [10] for the existence of the proper solution.

(ii) When an individual player i in an NRG using ZD strategies, it can "manipulate" its immediate neighbors' payoffs from it. That is the payoff the rest of the network got from it. Though it is by no means the player i can manipulate the whole network's payoffs, from its individual perspective, it might be enough.

(iii) Under our FOP formulation, the ZD strategies in the NRG are exactly the same as the ones for non-NRGs.

(iv) The existence of the ZD strategies is not trivial. We refer to [38] for some related discussion. Further investigation on the existence of ZD strategies for NRGs seems to be necessary and interesting.

5 Conclusion

This paper considers the design of ZD strategies proposed by Press and Dyson for general finite games. Using STP, a fundamental formula is presented to numerically realize ZD strategies for finite games with multiply players and asymmetric strategies. In addition to the generality, it simplifies the design procedure. Then, a necessary and sufficient condition for the availability of the designed ZD strategies is also obtained, which puts the ZD technique on a solid foundation. Some numerical examples are presented to demonstrate the efficiency of the method proposed in this paper.

Finally, as an application of the general formula, the NRGs are considered. A new concept, called FOP, is proposed as the opponent player for a preassigned player i. Using it, the ZD strategies for player i are designed for the game between itself and its FOP. It is interesting that one single player may be able to "control" the payoff of the rest part of the network from it by using ZD strategies, no matter how large the network is.

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Appendix A

(1) Proof of Proposition 3.

(i) Since L has unique eigenvalue 1, $\mathrm{rank}(M) = \kappa - 1$. Hence there exists at least one $(\kappa-1)\times(\kappa-1)$ miner of M, which is nonsingular. Hence, $M^* \neq 0$. Observing that

$$MM^* = \det(M) = 0, \tag{A1}$$

and $\mathrm{rank}(M) = \kappa - 1$, it follows that $\mathrm{rank}(M^*) = 1$.

(ii) Assume there exists $1 \leqslant j \leqslant n$ such that $\mathrm{Col}_j(M^*) = 0$. Consider $M\backslash\{\mathrm{Row}_j(M)\}$, which is obtained from M by deleting its j-th row. Then all its $(\kappa-1)\times(\kappa-1)$ minors have zero determinants. That is, $M\backslash\{\mathrm{Row}_j(M)\}$ is row-dependent.

To get a contradiction, we show that any $\kappa - 1$ rows of M are linearly independent. Since $\sum_{i=1}^{\kappa}\mathrm{Row}_i(M) = \mathbf{0}_\kappa^{\mathrm{T}}$, $\mathrm{Row}_j(M) = -\sum_{i\neq j}\mathrm{Row}_i(M)$. If $\mathrm{rank}(M\backslash\{\mathrm{Row}_j(M)\}) < \kappa - 1$, then $\mathrm{rank}(M) < \kappa - 1$, which is a contradiction.

(2) Proof of Proposition 4.

First, we show that the limit exists. Since $\{x(t) \mid t = 1, 2, \ldots\} \subset \Upsilon^\kappa$ and Υ^κ is a compact set, there exists a subsequence $\{x_{t_i} \mid i = 1, 2, \ldots\}$ such that

$$\lim_{i\to\infty} x_{t_i} = x^* \in \Upsilon^\kappa.$$

Note that $\lim_{t\to\infty} L^t = P$, denoted by $x^0 = Px^*$. We claim that

$$\lim_{t\to\infty} x_t = x^0. \tag{A2}$$

Given any $\epsilon > 0$, there exists N_1 such that when $t_i > N_1$,

$$\|x_{t_i} - x^*\| < \sqrt{\epsilon};$$

and there exists $N_2 > 0$ such that when $t > N_2$,

$$\|M^t - P\| < \sqrt{\epsilon}.$$

Choose an element $t_{i_0} > N_1$ from the subsequence and set

$$N_3 = t_{i_0} > N_1.$$

Assume $t > N_2 + N_3$, and then

$$x(t) = M^{t-N_3}x(t_{i_0}).$$

Since $t - N_3 > N_2$ and $N_3 = t_{i_0} > N_1$, it follows that

$$\|x(t) - x^0\| < (\sqrt{\epsilon})^2 = \epsilon.$$

Eq. (A2) follows. It is also clear that $x^0 = x^*$. Moreover, $Lx^* = x^*$ and $Px^* = x^*$. Now since $P = uv^{\mathrm{T}} \in \Upsilon_{\kappa\times\kappa}$, without loss of generality, we can normalize u by replacing u by $u/\|u\|$. Then $v = \mathbf{1}_\kappa$. Moreover,

$$x* = Px^* = uv^{\mathrm{T}}x^* = u.$$

(3) Proof of Proposition 5.

Note that $MM^* = \det(M) = 0$, and $Mu = 0$. Since $\mathrm{rank}(M) = \kappa - 1$, the solution of equation $Mx = 0$ is a one-dimensional subspace. Now each column of M^* is a solution, the conclusion follows.

(4) Proof of Theorem 1.

(Necessity) It is obvious that μ is a stationary distribution if it satisfies the following equation:

$$\lim_{t\to\infty} L^t x_0 = u, \quad \forall x_0 \in \Upsilon^\kappa. \tag{A3}$$

We first prove $\lim_{t\to\infty} L^t$ exists. Since $\Upsilon_{\kappa\times\kappa}$ is a compact set, if $\lim_{t\to\infty} L^t$ does not exist, there must be at least two subsequences $\{L^{n_i}\}$, and $\{L^{m_i}\}$, such that

$$\lim_{i\to\infty} L^{n_i} = P_1, \lim_{i\to\infty} L^{m_i} = P_2,$$

and $P_1 \neq P_2$. Say, $\mathrm{Col}_s(P_1) \neq \mathrm{Col}_s(P_2)$, choosing $x_0 = \delta_\kappa^s$, then it violates (A3).

Hence we have the decomposition

$$\lim_{t\to\infty} L^t = P.$$

Again, because of (A3), P should have the form that $P = [u, u, \ldots, u]$. The conclusion is obvious.

As for the condition (ii), if $\mathrm{rank}(L - I_\kappa) < \kappa - 1$, then $M^* = 0$ is a zero matrix. Then Eq. (28) fails. Hence Eq. (28) can never be obtained from (29), and ZD strategies do not work.

(Sufficiency) Replacing any row $s \in \Phi_{i,j}$ of matrix $M = L - I_\kappa$ by $\xi_{i,j}$, then condition (ii) ensures (22). Using (29) and expanding the determinant via the replaced row, Eq. (28) follows.