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• **Supplementary File** •

Supplemental Material for "Probing quantum causality with geometric asymmetry in spatial-temporal correlations"

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Appendix .1 Derivation of Eq. (2)

We utilize the theoretical framework of the pseudo-density matrix (PDM) [1, 2] in which space and time are treated in an even-handed fashion. The two-point PDM induced by an initial state ρ_A and a quantum channel $\epsilon_{B|A}$ can be written as:

$$\begin{aligned} R_{AB} &= (\mathcal{I}_A \otimes \varepsilon_{B|A}) \left\{ \rho_A \otimes \frac{\mathrm{I}}{2}, SWAP \right\} \\ &= (\mathcal{I}_A \otimes \varepsilon_{B|A}) \sum_{i=0}^{3} \frac{1}{4} \left(\rho_A \cdot \sigma_i \otimes \sigma_i + \sigma_i \cdot \rho_A \otimes \sigma_i \right) \\ &= \sum_{i=0}^{3} \frac{1}{4} \left\{ \rho_A, \sigma_i \right\} \varepsilon_{B|A} \left(\sigma_i \right), \end{aligned} \tag{1}$$

where $\mathcal{I_A}$ is the identity super-operators acting on subsystem A and $SWAP = \frac{1}{2}\sum_{i=0}^{3} \sigma_i \otimes \sigma_i$. Here $\{p, q\} = pq + qp$ denotes the anti-commutator. Using the PDM formalism, the temporal quantum correlation can be written as:

$$\begin{aligned} \langle \sigma_k \sigma_k \rangle &= \mathrm{Tr}\left[\left(\sigma_k \otimes \sigma_k \right) R_{AB} \right] \\ &= \mathrm{Tr}\left[\left(\sigma_k \otimes \sigma_k \right) \sum_{i=0}^{3} \frac{1}{4} \left\{ \rho_A, \sigma_i \right\} \varepsilon_{B|A} \left(\sigma_i \right) \right] \\ &= \sum_{i=0}^{3} \frac{1}{4} \mathrm{Tr}\left[\sigma_k \left\{ \rho_A, \sigma_i \right\} \otimes \sigma_k \, \varepsilon_{B|A} \left(\sigma_i \right) \right]. \end{aligned} \tag{2}$$

When $i = 0$,

$$\begin{aligned} &\frac{1}{4} \mathrm{Tr}\left[\sigma_k \left\{ \rho_A, \sigma_0 \right\} \otimes \sigma_k \, \varepsilon_{B|A} \left(\sigma_0 \right) \right] \\ &= \frac{1}{4} \mathrm{Tr}\left[\left(\sigma_k \, \rho_A + \rho_A \, \sigma_k \right) \otimes \sigma_k \, \varepsilon_{B|A} \left(\sigma_0 \right) \right] \\ &= \mathrm{Tr}\left[\langle \sigma_k \rangle_{\rho_A} \varepsilon_{B|A} \left(\sigma_0 \right) \sigma_k \right]; \end{aligned} \tag{3}$$

when $i = k \neq 0$,

$$\frac{1}{4} \mathrm{Tr}\left[\sigma_k \left\{ \rho_A, \sigma_k \right\} \otimes \sigma_k \, \varepsilon_{B|A} \left(\sigma_k \right) \right]$$

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$$
\begin{aligned}
&= \frac{1}{4}\,\mathrm{Tr}\left[\sigma_k\left(\sigma_k\,\rho_A + \rho_A\,\sigma_k\right)\otimes\sigma_k\,\varepsilon_{B|A}\left(\sigma_k\right)\right]\\
&= \frac{1}{4}\,\mathrm{Tr}\left[2\,\sigma_k\sigma_k\,\rho_A\otimes\sigma_k\,\varepsilon_{B|A}\left(\sigma_k\right)\right]\\
&= \frac{1}{4}\,\mathrm{Tr}\left[2\,\sigma_k\,\rho_A\otimes\sigma_k\,\varepsilon_{B|A}\left(\sigma_0\right)\right]\\
&= \frac{1}{4}\,\mathrm{Tr}\left[2\,\sigma_k\,\rho_A\right]\mathrm{Tr}\left[\sigma_k\,\varepsilon_{B|A}\left(\sigma_k\right)\right]\\
&= \mathrm{Tr}\left[\varepsilon_{B|A}\left(\sigma_k\right)\sigma_k\right];
\end{aligned}
\tag{4}
$$

finally, when $k \neq i \neq 0$, the trace of these terms vanish. As such, the sum of the four terms equals to

$$
\langle\sigma_k\sigma_k\rangle = \mathrm{Tr}\left[\varepsilon_{B|A}\left(\sigma_k\right)\sigma_k + \langle\sigma_k\rangle_{\rho_0}\,\varepsilon_{B|A}\left(\sigma_0\right)\sigma_k\right] \tag{5}
$$

By explicitly writing out the superscripts A and B indicating the two points in the temporal correlation and substituting the index i by k, we recover eq.(2) in the main text as:

$$
\left\langle\sigma_i^A\sigma_i^B\right\rangle = \mathrm{Tr}\left[\varepsilon_{B|A}\left(\sigma_i\right)\sigma_i + \langle\sigma_i\rangle_{\rho_0}\,\varepsilon_{B|A}\left(\sigma_0\right)\sigma_i\right]. \tag{6}
$$

Appendix .2 Quantum non-unital channels

In many practical scenarios, quantum systems interact with their environments, leading to non-unitary transformations. The non-unital channels provide a more realistic model for these interactions, allowing researchers to capture the effects of decoherence, noise, and other environmental factors. Moreover, errors in quantum systems, such as bit flips or erasure, often appear in an asymmetric form and can be described by non-unital channels. Understanding and mitigating the impact of these errors are crucial for the development of robust quantum computation and communication protocols. Non-unital channels can also be used to study the dynamics of entanglement and correlations in quantum systems. They provide a framework for exploring how quantum states evolve and change due to interactions, leading to a better understanding of quantum correlations in realistic scenarios. The use of non-unital channels in quantum mechanics and quantum information science is motivated by the need to model realistic physical processes, address the challenges posed by noise and decoherence, and design robust quantum technologies for communication and computation. Their generality and flexibility make them valuable tools for the researche in these fields.

Appendix .3 Perspective plots of Fig. 3 in the main text

In this section, we use some perspective plots to describe more details about Fig.3a in the main text. We re-exhibit Fig. 3a of the main text as Fig. 1a in this supplemental material for convenient comparison and present the perspective plot viewing from the $(-1,-1,-1)$ to the $(1,1,1)$ direction of different regions of Fig. 1a as Fig. 1b-d.

Concretely, the red triangle in Fig. 1b is the projection of the red tetrahedron $\mathcal{T}_t$ in the original three-dimensional visualization. It specifies the bounding set of TQC induced by only unital channel or by arbitrary channel but with an initial state restricted to being maximally mixed. The blue triangle in Fig. 1b is the projections of the blue tetrahedron $\mathcal{T}_t$ in the three-dimensional visualization and specifies the set of all the two-point SQCs generated by any two-qubit quantum state. At the same time, the purple hexagon with gray border depicts the overlapping octahedron $\mathcal{O}_t$ between $\mathcal{T}_t$ and $\mathcal{T}_s$, which physically represents the set of two-point SQC generated by all two-qubit separable states.

A curved triangle enclosed by three red arcs in Fig. 1c is the projection of the inflated tetrahedron $\mathcal{Q}_t$. It surrounds $\mathcal{T}_t$ and represents the set of two-point TQCs resulted by arbitrary combinations of quantum channels and initial states. As presented in Fig. 1d, it is highlighted by the white circular segments that the projection of $\mathcal{Q}_t$ has some overlap with $\mathcal{T}_t$ under the condition that the region $\mathcal{O}_t$ counting out. Subsequently, the symmetry between $\mathcal{T}_t$ and $\mathcal{T}_s$ no longer exists and given this, any two-point TQC appearing in this $(\mathcal{Q}_t \cap \mathcal{T}_s) \setminus \mathcal{O}_s$ area has no spatial counterpart in $\mathcal{T}_s$.

References

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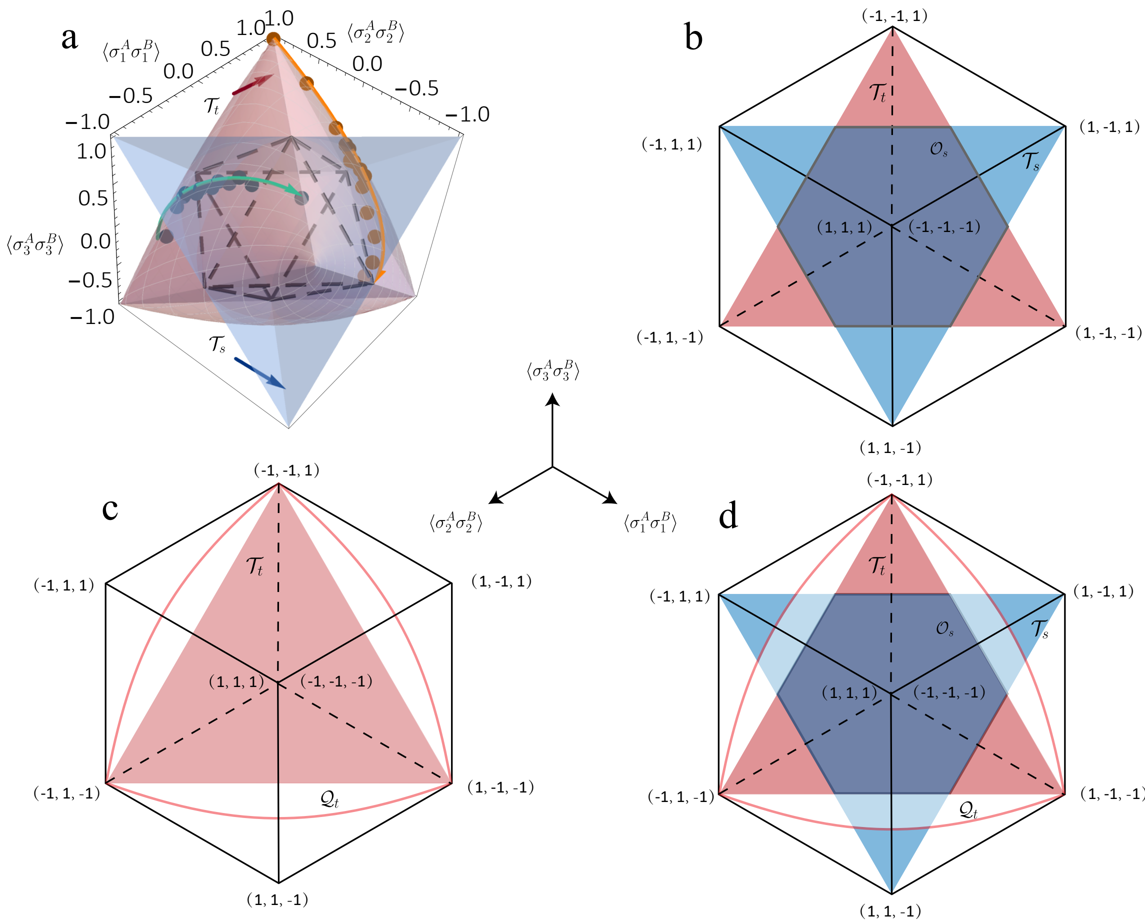

Figure 1 Perspective plot of Fig. 3a in the main text. See the main text for detailed description of the four subplots.