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Achieving Full Mutualism with Massive Passive Devices for Multiuser MIMO Symbiotic Radio

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Appendix A Proof of Lemma 1

By substituting $\mathbf{x}_k = \mathrm{vec}\left(\mathbf{X}_k\right) = \left[\mathrm{vec}^{\mathrm{T}}\left(\sqrt{P_1}\mathbf{g}_k\mathbf{h}_{1,k}^{\mathrm{H}}\mathbf{F}_1\right), \cdots, \mathrm{vec}^{\mathrm{T}}\left(\sqrt{P_M}\mathbf{g}_k\mathbf{h}_{M,k}^{\mathrm{H}}\mathbf{F}_M\right)\right]^{\mathrm{T}}$, $\sum_{k=1}^{K}\mathbf{x}_k\mathbf{x}_k^{\mathrm{H}}$ can be expressed as

$$\sum_{k=1}^{K}\mathbf{x}_k\mathbf{x}_k^{\mathrm{H}} = \begin{bmatrix} \sum_{k=1}^{K}\mathrm{vec}\left(\sqrt{P_1}\mathbf{g}_k\mathbf{h}_{1,k}^{\mathrm{H}}\mathbf{F}_1\right)\mathrm{vec}^{\mathrm{H}}\left(\sqrt{P_1}\mathbf{g}_k\mathbf{h}_{1,k}^{\mathrm{H}}\mathbf{F}_1\right) & \cdots & \sum_{k=1}^{K}\mathrm{vec}\left(\sqrt{P_1}\mathbf{g}_k\mathbf{h}_{1,k}^{\mathrm{H}}\mathbf{F}_1\right)\mathrm{vec}^{\mathrm{H}}\left(\sqrt{P_M}\mathbf{g}_k\mathbf{h}_{M,k}^{\mathrm{H}}\mathbf{F}_M\right) \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{K}\mathrm{vec}\left(\sqrt{P_M}\mathbf{g}_k\mathbf{h}_{M,k}^{\mathrm{H}}\mathbf{F}_M\right)\mathrm{vec}^{\mathrm{H}}\left(\sqrt{P_1}\mathbf{g}_k\mathbf{h}_{1,k}^{\mathrm{H}}\mathbf{F}_1\right) & \cdots & \sum_{k=1}^{K}\mathrm{vec}\left(\sqrt{P_M}\mathbf{g}_k\mathbf{h}_{M,k}^{\mathrm{H}}\mathbf{F}_M\right)\mathrm{vec}^{\mathrm{H}}\left(\sqrt{P_M}\mathbf{g}_k\mathbf{h}_{M,k}^{\mathrm{H}}\mathbf{F}_M\right) \end{bmatrix}. \tag{A1}$$

Furthermore, we have

$$\begin{aligned} \sum_{k=1}^{K}\mathrm{vec}\left(\sqrt{P_m}\mathbf{g}_k\mathbf{h}_{m,k}^{\mathrm{H}}\mathbf{F}_m\right)\mathrm{vec}^{\mathrm{H}}\left(\sqrt{P_i}\mathbf{g}_k\mathbf{h}_{i,k}^{\mathrm{H}}\mathbf{F}_i\right) &= \sum_{k=1}^{K}\sqrt{P_m}\sqrt{P_i}\left(\left(\mathbf{h}_{m,k}^{\mathrm{H}}\mathbf{F}_m\right)^{\mathrm{T}}\otimes\mathbf{g}_k\right)\left(\left(\mathbf{h}_{i,k}^{\mathrm{H}}\mathbf{F}_i\right)^{\mathrm{T}}\otimes\mathbf{g}_k\right)^{\mathrm{H}} \\ &= \sum_{k=1}^{K}\sqrt{P_m}\sqrt{P_i}\left(\mathbf{F}_m^{\mathrm{T}}\mathbf{h}_{m,k}^{*}\mathbf{h}_{i,k}^{\mathrm{T}}\mathbf{F}_i^{*}\right)\otimes\left(\mathbf{g}_k\mathbf{g}_k^{\mathrm{H}}\right), \end{aligned} \tag{A2}$$

in which the first equality holds from the identity $\mathrm{vec}(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) = \left(\mathbf{X}_3^{\mathrm{T}}\otimes\mathbf{X}_1\right)\mathrm{vec}(\mathbf{X}_2)$, and the last equality holds according to $\left(\mathbf{X}\otimes\mathbf{Y}\right)^{\mathrm{H}} = \mathbf{X}^{\mathrm{H}}\otimes\mathbf{Y}^{\mathrm{H}}$ and $\left(\mathbf{X}_1\otimes\mathbf{X}_2\right)\left(\mathbf{X}_3\otimes\mathbf{X}_4\right) = \left(\mathbf{X}_1\mathbf{X}_3\right)\otimes\left(\mathbf{X}_2\mathbf{X}_4\right)$. For $K \gg 1$, based on the law of large numbers, we have:

$$\begin{aligned} &\sum_{k=1}^{K}\sqrt{P_m}\sqrt{P_i}\left(\mathbf{F}_m^{\mathrm{T}}\mathbf{h}_{m,k}^{*}\mathbf{h}_{i,k}^{\mathrm{T}}\mathbf{F}_i^{*}\right)\otimes\mathbf{g}_k\mathbf{g}_k^{\mathrm{H}} \to K\sqrt{P_m}\sqrt{P_i}\mathbb{E}\left[\left(\mathbf{F}_m^{\mathrm{T}}\mathbf{h}_{m,k}^{*}\mathbf{h}_{i,k}^{\mathrm{T}}\mathbf{F}_i^{*}\right)\otimes\mathbf{g}_k\mathbf{g}_k^{\mathrm{H}}\right] \\ &= K\sqrt{P_m}\sqrt{P_i}\mathbb{E}\left[\mathbf{F}_m^{\mathrm{T}}\mathbf{h}_{m,k}^{*}\mathbf{h}_{i,k}^{\mathrm{T}}\mathbf{F}_i^{*}\right]\otimes\mathbb{E}\left[\mathbf{g}_k\mathbf{g}_k^{\mathrm{H}}\right] = KP_m\beta_{\mathrm{h},m}\beta_{\mathrm{g}}\delta\left(m-i\right)\left(\mathbf{F}_m^{\mathrm{T}}\mathbf{F}_i^{*}\right)\otimes\mathbf{I}_{N_r}, \end{aligned} \tag{A3}$$

where the second last equality holds due to the assumption that $\mathbf{h}_{m,k}$ and $\mathbf{g}_k$ are independent for all PTs and BDs, and the last equality holds since $\mathbf{h}_{m,k}$ are i.i.d. distributed with zero mean, i.e., $\mathbb{E}\left[\mathbf{h}_{m,k}^{*}\mathbf{h}_{i,k}^{\mathrm{T}}\right] \to \beta_{\mathrm{h},m}\mathbf{I}_{N_t}$ for $m = i$ and $\mathbb{E}\left[\mathbf{h}_{m,k}^{*}\mathbf{h}_{i,k}^{\mathrm{T}}\right] \to \mathbf{O}_{N_t}$ for $m \neq i$. As a result, $\sum_{k=1}^{K}\mathbf{x}_k\mathbf{x}_k^{\mathrm{H}}$ approaches to

$$\begin{aligned} \sum_{k=1}^{K}\mathbf{x}_k\mathbf{x}_k^{\mathrm{H}} &\to K\beta_{\mathrm{g}}\begin{bmatrix} P_1\beta_{\mathrm{h},1}\left(\mathbf{F}_1^{\mathrm{T}}\mathbf{F}_1^{*}\right)\otimes\mathbf{I}_{N_r} & \cdots & \mathbf{O}_{N_1N_r\times N_MN_r} \\ \vdots & \ddots & \vdots \\ \mathbf{O}_{N_MN_r\times N_1N_r} & \cdots & P_M\beta_{\mathrm{h},M}\left(\mathbf{F}_M^{\mathrm{T}}\mathbf{F}_M^{*}\right)\otimes\mathbf{I}_{N_r} \end{bmatrix} \\ &= K\beta_{\mathrm{g}}\begin{bmatrix} P_1\beta_{\mathrm{h},1}\left(\mathbf{F}_1^{\mathrm{T}}\mathbf{F}_1^{*}\right) & \cdots & \mathbf{O}_{N_1\times N_M} \\ \vdots & \ddots & \vdots \\ \mathbf{O}_{N_M\times N_1} & \cdots & P_M\beta_{\mathrm{h},M}\left(\mathbf{F}_M^{\mathrm{T}}\mathbf{F}_M^{*}\right) \end{bmatrix}\otimes\mathbf{I}_{N_r}. \end{aligned} \tag{A4}$$

For notational convenience, we let

$$\mathbf{\Phi} = \mathrm{blkdiag}\left\{P_1\beta_{\mathrm{h},1}\left(\mathbf{F}_1^{\mathrm{T}}\mathbf{F}_1^{*}\right), \cdots, P_M\beta_{\mathrm{h},M}\left(\mathbf{F}_M^{\mathrm{T}}\mathbf{F}_M^{*}\right)\right\}. \tag{A5}$$

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It then follows from (20) that

$$\begin{aligned}R_{\rm BD} &\to \frac{1}{L}\log_2\left|\left(\mathbf{I}_{N_{\rm sum}}+\frac{KL\alpha\beta_{\rm g}}{\sigma^2}\mathbf{\Phi}\right)\otimes\mathbf{I}_{N_r}\right| = \frac{1}{L}\log_2\left(\left|\mathbf{I}_{N_{\rm sum}}+\frac{KL\alpha\beta_{\rm g}}{\sigma^2}\mathbf{\Phi}\right|^{N_r}\left|\mathbf{I}_{N_r}\right|^{N_{\rm sum}}\right)\\ &= \frac{N_r}{L}\log_2\left|\mathbf{I}_{N_{\rm sum}}+\frac{KL\alpha\beta_{\rm g}}{\sigma^2}\mathbf{\Phi}\right| = \frac{N_r}{L}\sum_{m=1}^{M}\log_2\left|\mathbf{I}_{N_m}+KL\bar{P}_m\alpha\beta_{{\rm h},m}\beta_{\rm g}\mathbf{F}_m^{\rm T}\mathbf{F}_m^*\right|\\ &= \frac{N_r}{L}\sum_{m=1}^{M}\log_2\left|\mathbf{I}_{N_t}+KL\bar{P}_m\alpha\beta_{{\rm h},m}\beta_{\rm g}\mathbf{F}_m\mathbf{F}_m^{\rm H}\right|,\end{aligned} \tag{A6}$$

where the first equality holds due to the identity $|\mathbf{X}\otimes\mathbf{Y}| = |\mathbf{X}|^{\mathrm{rank}(\mathbf{Y})}|\mathbf{Y}|^{\mathrm{rank}(\mathbf{X})}$, the second last equality holds according to $|\mathrm{blkdiag}\{\mathbf{X},\mathbf{Y}\}| = |\mathbf{X}|\times|\mathbf{Y}|$, and the last equality holds according to the Weinstein-Aronszajn identity $|\mathbf{I}_m+\mathbf{XY}| = |\mathbf{I}_n+\mathbf{YX}|$ and $\left|\mathbf{X}^{\rm T}\right| = |\mathbf{X}|$. The proof is thus completed.

Appendix B Proof of Lemma 2

Let $\mathbf{B}(\mathbf{c}(n)) = \left[\sqrt{\bar{P}_1}\mathbf{H}_{{\rm eq},1}(\mathbf{c}(n))\mathbf{F}_1,\cdots,\sqrt{\bar{P}_M}\mathbf{H}_{{\rm eq},M}(\mathbf{c}(n))\mathbf{F}_M\right]\in\mathbb{C}^{N_r\times N_{\rm sum}}$. Then according to (7), we have

$$R_{\rm PT} = \mathbb{E}_{\mathbf{c}(n)}\left[\log_2|\mathbf{I}_{N_r}+\mathbf{B}(\mathbf{c}(n))\mathbf{B}^{\rm H}(\mathbf{c}(n))|\right] = \mathbb{E}_{\mathbf{c}(n)}\left[\log_2|\mathbf{I}_{N_{\rm sum}}+\mathbf{B}^{\rm H}(\mathbf{c}(n))\mathbf{B}(\mathbf{c}(n))|\right]. \tag{B1}$$

Let $g_{k,r}$ denote the rth element of $\mathbf{g}_k$, $\widetilde{\mathbf{h}}_{m,r}^{\rm T}$ and $\widetilde{\mathbf{h}}_{{\rm eq},m,r}^{\rm T}(\mathbf{c}(n))$ denote the rth row of $\mathbf{H}_m$ and $\mathbf{H}_{{\rm eq},m}(\mathbf{c}(n))$, respectively. Then $\mathbf{H}_{{\rm eq},m}(\mathbf{c}(n))$ can be expressed as $\mathbf{H}_{{\rm eq},m}(\mathbf{c}(n)) = [\widetilde{\mathbf{h}}_{{\rm eq},m,1}(\mathbf{c}(n)),\widetilde{\mathbf{h}}_{{\rm eq},m,2}(\mathbf{c}(n)),\cdots,\widetilde{\mathbf{h}}_{{\rm eq},m,N_r}(\mathbf{c}(n))]^{\rm T}$, where $\widetilde{\mathbf{h}}_{{\rm eq},m,N_r}^{\rm T}(\mathbf{c}(n)) = \widetilde{\mathbf{h}}_{m,r}^{\rm T}+\sum\limits_{k=1}^{K}\sqrt{\alpha}g_{k,r}\mathbf{h}_{m,k}^{\rm H}c_k(n)$. We have

$$\left(\sqrt{\bar{P}_m}\mathbf{H}_{{\rm eq},m}(\mathbf{c}(n))\mathbf{F}_m\right)^{\rm H}\sqrt{\bar{P}_i}\mathbf{H}_{{\rm eq},i}(\mathbf{c}(n))\mathbf{F}_i = \sqrt{\bar{P}_m\bar{P}_i}\mathbf{F}_m^{\rm H}\mathbf{H}_{{\rm eq},m}^{\rm H}(\mathbf{c}(n))\mathbf{H}_{{\rm eq},i}(\mathbf{c}(n))\mathbf{F}_i, \tag{B2}$$

where

$$\begin{aligned}&\mathbf{H}_{{\rm eq},m}^{\rm H}(\mathbf{c}(n))\mathbf{H}_{{\rm eq},i}(\mathbf{c}(n)) = \sum_{r=1}^{N_r}\widetilde{\mathbf{h}}_{{\rm eq},m,r}^{*}(\mathbf{c}(n))\widetilde{\mathbf{h}}_{{\rm eq},i,r}^{\rm T}(\mathbf{c}(n)) = \sum_{r=1}^{N_r}\left(\widetilde{\mathbf{h}}_{m,r}^{*}+\sum_{k=1}^{K}\sqrt{\alpha}g_{k,r}^{*}\mathbf{h}_{m,k}c_k^{*}(n)\right)\left(\widetilde{\mathbf{h}}_{i,r}^{\rm T}+\sum_{k=1}^{K}\sqrt{\alpha}g_{k,r}\mathbf{h}_{i,k}^{\rm H}c_k(n)\right)\\ &= \sum_{r=1}^{N_r}\widetilde{\mathbf{h}}_{m,r}^{*}\widetilde{\mathbf{h}}_{i,r}^{\rm T}+\sum_{r=1}^{N_r}\sum_{k=1}^{K}\sqrt{\alpha}g_{k,r}^{*}\mathbf{h}_{m,k}c_k^{*}(n)\widetilde{\mathbf{h}}_{i,r}^{\rm T}+\sum_{r=1}^{N_r}\widetilde{\mathbf{h}}_{m,r}^{*}\sum_{k=1}^{K}\sqrt{\alpha}g_{k,r}\mathbf{h}_{i,k}^{\rm H}c_k(n)+\sum_{r=1}^{N_r}\sum_{k=1}^{K}\sum_{k'=1}^{K}\alpha g_{k,r}^{*}g_{k',r}c_k^{*}(n)c_{k'}(n)\mathbf{h}_{m,k}\mathbf{h}_{i,k'}^{\rm H}.\end{aligned} \tag{B3}$$

Based on the law of large numbers, for $N_r\gg 1$ and $K\gg 1$, there is

$$\begin{aligned}&\sum_{r=1}^{N_r}\sum_{k=1}^{K}\sum_{k'=1}^{K}\alpha g_{k,r}^{*}g_{k',r}c_k^{*}(n)c_{k'}(n)\mathbf{h}_{m,k}\mathbf{h}_{i,k'}^{\rm H}\to N_r\mathbb{E}_g\left[\sum_{k=1}^{K}\sum_{k'=1}^{K}\alpha g_{k,r}^{*}g_{k',r}c_k^{*}(n)c_{k'}(n)\mathbf{h}_{m,k}\mathbf{h}_{i,k'}^{\rm H}\right]\\ &= N_r\sum_{k=1}^{K}\alpha|g_{k,r}|^2|c_k(n)|^2\mathbf{h}_{m,k}\mathbf{h}_{i,k}^{\rm H}\to KN_r\mathbb{E}\left[\alpha|g_{k,r}|^2|c_k(n)|^2\mathbf{h}_{m,k}\mathbf{h}_{i,k}^{\rm H}\right] = K\alpha N_r\beta_{\rm g}\beta_{{\rm h},m}\delta(m-i)\mathbf{I}_{N_t}.\end{aligned} \tag{B4}$$

Therefore,

$$\begin{aligned}&\mathbf{H}_{{\rm eq},m}^{\rm H}(\mathbf{c}(n))\mathbf{H}_{{\rm eq},i}(\mathbf{c}(n))\\ &\to\sum_{r=1}^{N_r}\widetilde{\mathbf{h}}_{m,r}^{*}\widetilde{\mathbf{h}}_{i,r}^{\rm T}+K\sum_{r=1}^{N_r}\mathbb{E}\left[\sqrt{\alpha}g_{k,r}^{*}\mathbf{h}_{m,k}c_k^{*}(n)\right]\widetilde{\mathbf{h}}_{i,r}^{\rm T}+K\sum_{r=1}^{N_r}\widetilde{\mathbf{h}}_{m,r}^{*}\mathbb{E}\left[\sqrt{\alpha}g_{k,r}\mathbf{h}_{i,k}^{\rm H}c_k(n)\right]+K\alpha N_r\beta_{\rm g}\beta_{{\rm h},m}\delta(m-i)\mathbf{I}_{N_t}\\ &= \mathbf{H}_m^{\rm H}\mathbf{H}_i+K\alpha N_r\beta_{\rm g}\beta_{{\rm h},m}\delta(m-i)\mathbf{I}_{N_t}.\end{aligned} \tag{B5}$$

Based on the law of large numbers, for $N_r\gg 1$, there is $\frac{1}{N_r}\mathbf{H}_m^{\rm H}\mathbf{H}_i\to\beta_{{\rm H},m}\mathbf{I}_{N_t}, m=i$ and $\frac{1}{N_r}\mathbf{H}_m^{\rm H}\mathbf{H}_i\to\mathbf{O}_{N_t}, m\neq i$. Then (B5) can be further expressed as

$$\mathbf{H}_{{\rm eq},m}^{\rm H}(\mathbf{c}(n))\mathbf{H}_{{\rm eq},i}(\mathbf{c}(n))\to\left(N_r\beta_{{\rm H},m}+K\alpha N_r\beta_{\rm g}\beta_{{\rm h},m}\right)\delta(m-i)\mathbf{I}_{N_t}. \tag{B6}$$

For notational convenience, denote $\mathbf{B}^{\rm H}(\mathbf{c}(n))\mathbf{B}(\mathbf{c}(n))\to\mathbf{\Gamma}$, where $\mathbf{\Gamma}$ is written as

$$\mathbf{\Gamma} = \begin{bmatrix}\bar{P}_1\left(N_r\beta_{{\rm H},1}+K\alpha N_r\beta_{\rm g}\beta_{{\rm h},1}\right)\mathbf{F}_1^{\rm H}\mathbf{F}_1 & \cdots & \mathbf{O}_{N_1}\\ \vdots & \ddots & \vdots\\ \mathbf{O}_{N_M} & \cdots & \bar{P}_M\left(N_r\beta_{{\rm H},M}+K\alpha N_r\beta_{\rm g}\beta_{{\rm h},M}\right)\mathbf{F}_M^{\rm H}\mathbf{F}_M\end{bmatrix}. \tag{B7}$$

Thus, $R_{\rm PT}$ in (7) approaches to

$$\begin{aligned}R_{\rm PT}&\to\log_2|\mathbf{I}_{N_{\rm sum}}+\mathbf{\Gamma}| = \sum_{m=1}^{M}\log_2\left|\mathbf{I}_{N_m}+\bar{P}_m\left(N_r\beta_{{\rm H},m}+K\alpha N_r\beta_{\rm g}\beta_{{\rm h},m}\right)\mathbf{F}_m^{\rm H}\mathbf{F}_m\right|\\ &= \sum_{m=1}^{M}\log_2\left|\mathbf{I}_{N_t}+\bar{P}_mN_r\left(\beta_{{\rm H},m}+K\alpha\beta_{\rm g}\beta_{{\rm h},m}\right)\mathbf{F}_m\mathbf{F}_m^{\rm H}\right|,\end{aligned} \tag{B8}$$

where the equality holds from the identity $|\mathrm{blkdiag}\{\mathbf{X},\mathbf{Y}\}| = |\mathbf{X}|\times|\mathbf{Y}|$, and the last equality holds according to the Weinstein-Aronszajn identity $|\mathbf{I}_m+\mathbf{XY}| = |\mathbf{I}_n+\mathbf{YX}|$. The proof is completed.

Appendix C Proof of Proposition 1

By differentiating (48), the resulting equation is expressed as

$$\begin{aligned} d\left(g_2\left(\mathbf{Q}_m\right)\right) &= d\Big(\frac{\log_2 e}{L}\ln\left|\mathbf{T}+a\mathbf{\Psi}_{k+1}^{\mathrm{H}}\left(\mathbf{Q}_m\otimes\mathbf{C}\right)\mathbf{\Psi}_{k+1}\right|\Big) \\ &= \frac{\log_2 e}{L}\mathrm{tr}\left(\left(\mathbf{T}+a\mathbf{\Psi}_{k+1}^{\mathrm{H}}\left(\mathbf{Q}_m\otimes\mathbf{C}\right)\mathbf{\Psi}_{k+1}\right)^{-1}d\left(\mathbf{T}+a\mathbf{\Psi}_{k+1}^{\mathrm{H}}\left(\mathbf{Q}_m\otimes\mathbf{C}\right)\mathbf{\Psi}_{k+1}\right)\right) \\ &= \frac{\log_2 e}{L}\mathrm{tr}\left(\left(\mathbf{T}+a\mathbf{\Psi}_{k+1}^{\mathrm{H}}\left(\mathbf{Q}_m\otimes\mathbf{C}\right)\mathbf{\Psi}_{k+1}\right)^{-1}a\mathbf{\Psi}_{k+1}^{\mathrm{H}}d\left(\mathbf{Q}_m\otimes\mathbf{C}\right)\mathbf{\Psi}_{k+1}\right) \\ &= \frac{\log_2 e}{L}\mathrm{tr}\left(\mathbf{\Psi}_{k+1}\left(\mathbf{T}+a\mathbf{\Psi}_{k+1}^{\mathrm{H}}\left(\mathbf{Q}_m\otimes\mathbf{C}\right)\mathbf{\Psi}_{k+1}\right)^{-1}a\mathbf{\Psi}_{k+1}^{\mathrm{H}}d\left(\mathbf{Q}_m\otimes\mathbf{C}\right)\right) \\ &= \frac{\log_2 e}{L}\mathrm{tr}\left(\mathbf{\Psi}_{k+1}\left(\mathbf{T}+a\mathbf{\Psi}_{k+1}^{\mathrm{H}}\left(\mathbf{Q}_m\otimes\mathbf{C}\right)\mathbf{\Psi}_{k+1}\right)^{-1}a\mathbf{\Psi}_{k+1}^{\mathrm{H}}d\left(\mathbf{Q}_m\right)\otimes\mathbf{C}\right), \end{aligned} \tag{C1}$$

where the second last equality holds due to the identity $\mathrm{tr}\left(\mathbf{XY}\right)=\mathrm{tr}\left(\mathbf{YX}\right)$. Furthermore, we have

$$\begin{aligned} d\left(\mathbf{Q}_m\right)\otimes\mathbf{C} &= \left(\mathbf{I}_{N_t}d\left(\mathbf{Q}_m\right)\right)\otimes\left(\mathbf{C}\mathbf{I}_{N_t(K-k)}\right)=\left(\mathbf{I}_{N_t}\otimes\mathbf{C}\right)\left(d\left(\mathbf{Q}_m\right)\otimes\mathbf{I}_{N_t(K-k)}\right) \\ &= \left(\mathbf{I}_{N_t}\otimes\mathbf{C}\right)\left(\mathbf{K}_{N_t^2(K-k)}\left(\mathbf{I}_{N_t(K-k)}\otimes d(\mathbf{Q}_m)\right)\mathbf{K}_{N_t^2(K-k)}\right), \end{aligned} \tag{C2}$$

where the second last equality holds according to the identity $\left(\mathbf{X}_1\otimes\mathbf{X}_2\right)\left(\mathbf{X}_3\otimes\mathbf{X}_4\right)=\left(\mathbf{X}_1\mathbf{X}_3\right)\otimes\left(\mathbf{X}_2\mathbf{X}_4\right)$, and the last equality follows from $\mathbf{K}_{pm}\left(\mathbf{X}\otimes\mathbf{Y}\right)\mathbf{K}_{np}=\mathbf{Y}\otimes\mathbf{X}$ with $\mathbf{X}\in\mathbb{C}^{m\times n}$, $\mathbf{Y}\in\mathbb{C}^{p\times q}$ and $\mathbf{K}_{mn}=\sum_{j=1}^{n}\left(\mathbf{e}_j^{\mathrm{T}}\otimes\mathbf{I}_m\otimes\mathbf{e}_j\right)\in\mathbb{C}^{mn\times mn}$. Thus, $d\left(g_2\left(\mathbf{Q}_m\right)\right)$ in (C1) can be expressed as

$$\begin{aligned} & d\left(g_2\left(\mathbf{Q}_m\right)\right) \\ &= \frac{\log_2 e}{L}\mathrm{tr}\left(\mathbf{\Psi}_{k+1}\left(\mathbf{T}+a\mathbf{\Psi}_{k+1}^{\mathrm{H}}\left(\mathbf{Q}_m\otimes\mathbf{C}\right)\mathbf{\Psi}_{k+1}\right)^{-1}a\mathbf{\Psi}_{k+1}^{\mathrm{H}}\left(\mathbf{I}_{N_t}\otimes\mathbf{C}\right)\mathbf{K}_{N_t^2(K-k)}\left(\mathbf{I}_{N_t(K-k)}\otimes d\left(\mathbf{Q}_m\right)\right)\mathbf{K}_{N_t^2(K-k)}\right) \\ &= \frac{\log_2 e}{L}\mathrm{tr}\left(\mathbf{K}_{N_t^2(K-k)}\mathbf{\Psi}_{k+1}\left(\mathbf{T}+a\mathbf{\Psi}_{k+1}^{\mathrm{H}}\left(\mathbf{Q}_m\otimes\mathbf{C}\right)\mathbf{\Psi}_{k+1}\right)^{-1}a\mathbf{\Psi}_{k+1}^{\mathrm{H}}\left(\mathbf{I}_{N_t}\otimes\mathbf{C}\right)\mathbf{K}_{N_t^2(K-k)}\left(\mathbf{I}_{N_t(K-k)}\otimes d\left(\mathbf{Q}_m\right)\right)\right), \end{aligned} \tag{C3}$$

where $\mathbf{I}_{N_t(K-k)}\otimes d\left(\mathbf{Q}_m\right)$ is a diagonal matrix that can be written as

$$\mathbf{I}_{N_t(K-k)}\otimes d\left(\mathbf{Q}_m\right)=\begin{bmatrix} d\left(\mathbf{Q}_m\right) & \mathbf{O}_{N_t} & \cdots & \mathbf{O}_{N_t} \\ \mathbf{O}_{N_t} & d\left(\mathbf{Q}_m\right) & \cdots & \mathbf{O}_{N_t} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O}_{N_t} & \mathbf{O}_{N_t} & \cdots & d\left(\mathbf{Q}_m\right) \end{bmatrix}. \tag{C4}$$

As defined above, $\mathbf{D}(\mathbf{Q}_m)=\mathbf{K}_{N_t^2(K-k)}\mathbf{\Psi}_{k+1}\left(\mathbf{T}+a\mathbf{\Psi}_{k+1}^{\mathrm{H}}\left(\mathbf{Q}_m\otimes\mathbf{C}\right)\mathbf{\Psi}_{k+1}\right)^{-1}a\mathbf{\Psi}_{k+1}^{\mathrm{H}}\left(\mathbf{I}_{N_t}\otimes\mathbf{C}\right)\mathbf{K}_{N_t^2(K-k)}\in\mathbb{C}^{N_t^2(K-k)\times N_t^2(K-k)}$, which can be written in the form of a block matrix with $\mathbf{E}_{p,g}(\mathbf{Q}_m)\in\mathbb{C}^{N_t\times N_t}$ denoting the sub-matrix at pth row and qth column. It follows from (C3) that

$$\begin{aligned} d\left(g_2\left(\mathbf{Q}_m\right)\right) &= \frac{\log_2 e}{L}\mathrm{tr}\left(\mathbf{D}(\mathbf{Q}_m)\left(\mathbf{I}_{N_t(K-k)}\otimes d\left(\mathbf{Q}_m\right)\right)\right) \\ &= \frac{\log_2 e}{L}\sum_{i=1}^{N_t(K-k)}\mathrm{tr}\left(\mathbf{E}_{i,i}(\mathbf{Q}_m)d\left(\mathbf{Q}_m\right)\right) \\ &= \mathrm{tr}\Big(\frac{\log_2 e}{L}\sum_{i=1}^{N_t(K-k)}\mathbf{E}_{i,i}(\mathbf{Q}_m)d\left(\mathbf{Q}_m\right)\Big). \end{aligned} \tag{C5}$$

According to the law of differentiation, we have

$$d\left(g_2\left(\mathbf{Q}_m\right)\right)=\mathrm{tr}\Big(\Big(\frac{\partial\left(g_2\left(\mathbf{Q}_m\right)\right)}{\partial\left(\mathbf{Q}_m\right)}\Big)^{\mathrm{T}}d\left(\mathbf{Q}_m\right)\Big). \tag{C6}$$

Comparing (C5) and (C6), we have

$$z_2\left(\mathbf{Q}_m\right)=\frac{\partial\left(g_2\left(\mathbf{Q}_m\right)\right)}{\partial\left(\mathbf{Q}_m\right)}=\left(\frac{\log_2 e}{L}\sum_{i=1}^{N_t(K-k)}\mathbf{E}_{i,i}(\mathbf{Q}_m)\right)^{\mathrm{T}}=\frac{\log_2 e}{L}\sum_{i=1}^{N_t(K-k)}\mathbf{E}_{i,i}^{\mathrm{T}}(\mathbf{Q}_m). \tag{C7}$$

The proof is thus completed.