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Optimal rejection of bounded perturbations in linear leader-following consensus protocol: invariant ellipsoid method

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Abstract The objective of the invariant ellipsoid method is to minimize the smallest invariant and attractive set of a linear control system operating under the influence of bounded external disturbances. This study extends the application of this method to address the leader-following consensus problem. Initially, a linear control protocol is designed for the multi-agent system in the absence of disturbances. Subsequently, in the presence of bounded disturbances, by employing a similar linear control protocol, a necessary and sufficient condition is introduced to derive the optimal control parameters for the multi-agent system such that the state of followers converges to and remains in a minimal invariant ellipsoid around the state of the leader.

Keywords multi-agent system, leader-following consensus, invariant ellipsoid, disturbance

1 Introduction

A multi-agent system (MAS) consists of multiple agents that can achieve global behavior through local communication. During task execution, MAS performs better working efficiency, reduced sensitivity and increased flexibility. Consequently, MAS has received considerable academic attention, particularly regarding control issues, such as consensus [1], formation [2, 3], containment [4], and estimation problems such as distributed estimation [5]. The theory obtained has been applied to some application scenarios such as mobile robot swarms [6], network security [7], and smart grids [8].

The consensus problem is one of the most fundamental research issues within MASs and can be subdivided into leaderless consensus and leader-following consensus. In the case of leaderless consensus, the ultimate shared position cannot be preselected. However, in certain scenarios, as demonstrated in [9], the leader-following problem might be more useful, wherein a virtual or real leader can be introduced to guide the MAS in following a predefined or unknown trajectory.

From a practical application perspective, while formulating MAS consensus problems, discrepancies inevitably arise between the actual system and the mathematical model used for control design. These discrepancies stem from unmodeled dynamics, uncertainties in MAS parameters, or simplified complex plant dynamics. Therefore, how to guarantee the required performance of MAS under certain (matched and mismatched) disturbances is one of the critical challenges in real-world applications. This challenge has spurred the development of robust control methods to address this issue.

The robust control design of MAS is an approach focused on mitigating disturbances within a nominal system. Its main objective is to achieve a specified level of performance or stability in the system, even in the presence of bounded disturbances. Numerous established methodologies for robust control design have been developed, including techniques such as sliding mode control (SMC) [10, 11], H_∞ approach [12, 13], and attractive/invariant ellipsoid method [14, 15].

SMC has been recognized as a robust control design methodology since the 1960s in Russia. Utkin [10] published the first survey in 1977. The sliding mode method could provide a good performance, especially

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for the system with bounded matched disturbance, since this kind of disturbance can be compensated by the discontinuity of the SMC [16–18]. However, SMC may not provide a satisfied performance for the system with unmatched disturbance [19]. H_∞ techniques offer distinct advantages over classical control methods, particularly for multivariable systems with cross-coupling between the channels [12, 20–22]. However, employing H_∞ techniques presents certain challenges including a high level of mathematical proficiency and a reasonably accurate model of the system being controlled.

The basic concept of the attractive/invariant ellipsoid method is introduced in [23–25]. This method has found widespread application in addressing a range of control and estimation challenges, across linear systems [26] and nonlinear plant models [27]. An important characteristic of this method is its capacity to ensure the system state converges to a minimal ellipsoid regardless of perturbations or uncertainty, provided that they satisfy certain bounds. The utilization of invariant ellipsoids has simplified the process of reformulating the original problems into matrix inequalities, thereby reducing optimal control design problems to solving semi-definite programs. Many researchers have applied the invariant set method to solve problems in MAS. A notable advancement in MASs, utilizing the invariant set methodology is presented by Deng et al. [15], who provids a sufficient condition for being an invariant set of MAS. In [28], a finite-time control protocol of reaching the invariant set is designed. In [29], the invariant set method is proposed to analyze the limit behaviors of all the synchronous trajectories. However, these studies above primarily provide or use sufficient condition for establishing invariant set of MAS, which means that the proposed control protocol may not be optimal.

The primary contribution of this article lies in expanding the attractive/invariant ellipsoid method to MAS. We propose a necessary and sufficient condition that ensures the control protocol is optimal, and specifically designed to minimize the impact of external disturbances. Furthermore, we derived a sufficient condition for characterizing an attractive ellipsoid.

The rest of the paper is structured as follows: Section 2 delves into the concepts of graph theory and invariant set theories. The problem statement is presented in detail in Section 3. Section 4 discusses the main results regarding the necessary and sufficient conditions for an attractive/invariant ellipsoid of the MAS under bounded disturbance. Finally, Section 5 presents simulation results that support the proposed theories.

Notation. $\mathbb{R}$ is the set of real numbers; $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$; $\|x\| = \sqrt{x^{\mathrm{T}}x}$ is the Euclidean norm in $\mathbb{R}^n$; $\|x\|_P = \sqrt{x^{\mathrm{T}}Px}$ is the weight norm in $\mathbb{R}^n$; $P \succ 0$ ($\prec 0, \succeq 0, \preceq 0$) for a symmetric matrix $P \in \mathbb{R}^{n\times n}$ means that the matrix P is positive (negative) definite (semi-definite); $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ represent the minimal and maximal eigenvalue of matrix P, respectively; for $P \succeq 0$ the square root of P is a matrix $M = P^{\frac{1}{2}}$ such that $M^2 = P$; $\mathrm{tr}(P)$ denotes the trace of matrix P; L^∞ is the space of Lebesgue measurable essentially bounded function $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}^n$ with norm defined as $\|\sigma\|_{L^\infty} := \operatorname{ess\,sup}_{t\in\mathbb{R}_+} \|\sigma(t)\|_\infty < +\infty$; $1_N \in \mathbb{R}^N$ is a vector with all elements equal to 1.

2 Preliminary

2.1 Graph theory

A fixed graph is usually characterized by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{C}\}$, where $\mathcal{V} = \{0, \dots, N\}$ is the node set; $\mathcal{E} = \{(i,j)|i,j \in \mathcal{V}\}$ is the edge set, $(i,j) \in \mathcal{E}$ if the local information of node j could convey to node i, and node j makes a neighbor of node i, n_i denotes the quantity of neighbors of node i; $\mathcal{C}$ is the weighted adjacency matrix whose elements are denoted by c_{ij}, $i,j \in \mathcal{V}$, $c_{ij} > 0$ if $(i,j) \in \mathcal{E}$, and $c_{ij} = 0$ otherwise. The self-loop phenomenon is excluded in this work, i.e., $c_{ii} = 0$. The graph $\mathcal{G}$ is directed if $(i,j) \in \mathcal{E}$ does not imply $(j,i) \in \mathcal{E}$, while $\mathcal{G}$ is undirected if $(i,j) \in \mathcal{E} \Leftrightarrow (j,i) \in \mathcal{E}$. A directed path from node i to node j in the graph $\mathcal{G}$ is a sequence of nodes $\overline{i_0, i_s}$, where $i_0 = i$, $i_s = j$, $(i_{\kappa+1}, i_\kappa) \in \mathcal{E}$, and $\kappa = \overline{0, s-1}$. A directed tree is a directed graph with a node called root, which possesses a unique directed path to each of the rest nodes. $\mathcal{T}_{\mathcal{G}} = \{\mathcal{V}_{\mathcal{T}}, \mathcal{E}_{\mathcal{T}}, \mathcal{C}_{\mathcal{T}}\}$ is a spanning tree of graph $\mathcal{G}$ if $\mathcal{T}_{\mathcal{G}}$ is a directed tree and $\mathcal{V}_{\mathcal{T}} = \mathcal{V}$. The Laplacian matrix $\mathcal{L}$, induced by $\mathcal{C}$, with elements $l_{ij}, i,j \in \mathcal{V}$ defined as

$$l_{ij} = \begin{cases} -c_{ij}, & \text{if} \quad i \neq j, \\ \sum_{k=0}^{N} c_{ik}, & \text{if} \quad i = j. \end{cases}$$

Thereby it is obvious one Laplacian matrix corresponds to one specified $\mathcal{G}$. The Laplacian matrix has some particular properties, e.g., all eigenvalues of $\mathcal{L}$ have non-negative real parts; if $\mathcal{G}$ is an undirected graph, then the associated Laplacian matrix $\mathcal{L} \succeq 0$. A directed graph $\mathcal{G}$ has a spanning tree if and only if the associated Laplacian matrix $\mathcal{L}$ has exactly one zero eigenvalue [30].

Lemma 1 ([31])**.** Let

$$\hat{\mathcal{L}} = \begin{bmatrix} l_{11} - l_{01} & \cdots & l_{1N} - l_{0N} \\ \vdots & \ddots & \vdots \\ l_N - l_{01} & \cdots & l_{NN} - l_{0N} \end{bmatrix},$$

ζ_i, $i = 1, 2, \ldots, N$ be the eigenvalues of $\hat{\mathcal{L}}$, and μ_j, $j = 0, 1, \ldots, N$ be the eigenvalues of $\mathcal{L}$, respectively, where $|\zeta_1| \leqslant |\zeta_2| \leqslant \cdots \leqslant |\zeta_N|$ and $0 = |\mu_0| \leqslant |\mu_1| \leqslant \cdots \leqslant |\mu_N|$. Then, $\zeta_1 = \mu_1, \ldots, \zeta_N = \mu_N$.

2.2 Invariant set theory

Recall the definition of attractive/invariant ellipsoid.

Definition 1 ([26])**.** Let symmetric matrix $P \succ 0$. The ellipsoid

$$\varepsilon(P) = \{e \in \mathbb{R}^{Nn} : \|e\|_P \leqslant 1\}, \quad P \succ 0 \tag{1}$$

centered at the origin is called

- invariant for system (3) and (4), if $e(0) \in \varepsilon(P)$ implies $e(t) \in \varepsilon(P)$ for $\forall t \geqslant 0$.
- attractive for system (3) and (4) if $e(0) \notin \varepsilon(P)$ then $e(t) \to \varepsilon(P)$ as $t \to \infty$.

In other words, for an invariant ellipsoid $\varepsilon(P)$, the system state originating from within it will remain inside for the entire duration. Another property is that the invariant ellipsoid is attracting if the initial position $e(0)$ is outside the invariant set $\varepsilon(P)$ and $e(t) \to \varepsilon(P)$, as $t \to \infty$.

Since the matrix P is referred to as the matrix used to characterize the ellipsoid $\varepsilon(P)$, it can be regarded as the characteristic of the impact of external disturbances on the trajectory of system. Therefore, minimizing the impact of disturbance implies minimizing the invariant ellipsoid $\varepsilon(P)$. In this paper, we use the trace criterion

$$\operatorname{tr}(P) \tag{2}$$

to characterize the size of ellipsoid, which represents the sum of the squares of the semiaxes of the ellipsoid invariant.

3 Problem statement

In this paper, we deal with MAS where the communication topology is modeled by a fixed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{C}\}$. In $\mathcal{G}$, there is a root node which conveys its local information to the nearest nodes, and the rest nodes interact under an undirected graph. It means that $\mathcal{G}$ is composed of a root node, a set of nodes which are undirectedly connected, and the directed connection from the root node to its nearest ones. In the sense of MAS, agent 0 corresponds to the root node, which is designated as the leader, while the others, labeled from 1 to N, perform as followers. The dynamic of the leader is given in the following form:

$$\dot{\sigma}_0(t) = A\sigma_0 + Bu_0, \quad \sigma_0 + Bu_0 \in \mathbb{R}^n, \tag{3}$$

where the leader input u_0 is assumed to be known, and the followers are modeled as

$$\dot{\sigma}_i(t) = A\sigma_i(t) + Bu_i(t), \quad i = \overline{1, N}, \tag{4}$$

where $\sigma_i(t) \in \mathbb{R}^n$ is the ith agent state, $u_i(t) \in \mathbb{R}^m$ is the control input of agent i, and the system matrix $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. In addition, the control protocol applied in this research is presented in the following form:

$$u_i = K\sum_{j=0}^{N} c_{ij}(\sigma_j - \sigma_i) + u_0,\ K \in \mathbb{R}^{m\times n},\ i = \overline{1, N}. \tag{5}$$

Therefore, based on (3), (4), and (5), we have the error system between the state of followers and the leader as follows:

$$\dot{e} = (I_N \otimes A - \tilde{\mathcal{L}} \otimes BK)e, \tag{6}$$

where $e = [e_1, e_2, \ldots, e_N]^{\mathrm{T}}$, $e_i = \sigma_i - \sigma_0$,

$$\tilde{\mathcal{L}} = \begin{bmatrix} l_{11} & \cdots & l_{1N} \\ \vdots & \ddots & \vdots \\ l_{N1} & \cdots & l_{NN} \end{bmatrix}$$

corresponds to the undirected graph, and we let $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$ be its eigenvalues. According to Lemma 1, $\tilde{\mathcal{L}}$ is a special case of $\hat{\mathcal{L}}$ by letting $l_{01} = \cdots = l_{0N} = 0$. Thus we have $\lambda_i \geqslant 0$.

Definition 2 (Inspired by [1]). The MAS admits the linear consensus protocol (5) if there exists a feedback gain K such that the error equation (6) is globally asymptotically stable.

In this paper we study the MAS operating in the presence of bounded external disturbances. For example, the dynamics of MAS of flying robots may be perturbed by a wind. The impact of disturbance on each agent can be considered uniformly bounded. In this paper we assume that the leader of MAS is virtual, so its dynamic is not perturbed. The model of such an MAS can be presented as follows:

$$\dot{\sigma}_0(t) = A\sigma_0 + Bu_0, \quad \sigma_0 + Bu_0 \in \mathbb{R}^n, \tag{7}$$

$$\dot{\sigma}_i(t) = A\sigma_i(t) + Bu_i(t) + E\omega(t), \quad i = 1, 2, \ldots, N, \tag{8}$$

where $E \in \mathbb{R}^{n\times p}$ is constant matrix and $\omega(t) \in \mathbb{R}^p$ is the bounded external disturbance such that

$$\omega^{\mathrm{T}} Q\omega \leqslant 1, \quad \omega \in L^{\infty}(\mathbb{R}_+, \mathbb{R}^p), \quad 0 \prec Q = Q^{\top} \in \mathbb{R}^{p\times p}. \tag{9}$$

Similar to the MAS without disturbances, the error system can be written as follows:

$$\dot{e} = (I_N \otimes A - \tilde{\mathcal{L}} \otimes BK)e + 1_N \otimes E\omega. \tag{10}$$

The main objective is to design a linear consensus protocol of MAS (3) and (4) without disturbance such that in the case of disturbance this protocol is able to minimize the invariant ellipsoid of error system (10) under constraint (9).

4 Main results

4.1 Linear consensus protocol in the absence of disturbance

A result on linear leader-following consensus is introduced in this subsection, which is refined from [9].

Lemma 2. The following three claims are equivalent:

(a) MAS (3) and (4) admits linear leader-following consensus,

(b) $\mathcal{G}$ has a spanning tree, and the pair (A, B) is stabilizable,

(c) Matrix inequality

$$P(I_N \otimes A) + (I_N \otimes A^{\mathrm{T}})P - P(\tilde{\mathcal{L}} \otimes BK) - (\tilde{\mathcal{L}} \otimes BK)^{\mathrm{T}} P \prec 0 \tag{11}$$

with $0 \prec P \in \mathbb{R}^{nN\times nN}$ is feasible.

Proof. • (a) ⇔ (c): The closed-loop error equation (6) is globally asymptotically stable if and only if the matrix inequality (11) is feasible. Combining Definition 2 about linear leader-following consensus, we can obtain that (a) and (c) are equivalent.

• (a) ⇔ (b): According to Definition 2, linear leader-following consensus is achieved when Eq. (6) is globally asymptotically stable. According to [32, 33], the stability of (6) is equivalent to the stability of N systems $A - \lambda_i BK$, $i = 1, 2, \ldots, N$, where λ_i is the eigenvalue of $\tilde{\mathcal{L}}$, $\lambda_i \geqslant 0$. Inspired by Finsler's lemma [34–36], if (A, B) is stabilizable, then there exist solutions $(X, \gamma) \in (\mathbb{R}^{n\times n}, \mathbb{R})$ such that the following inequality holds:

$$AX + XA^{\mathrm{T}} - \gamma BB^{\mathrm{T}} \prec 0. \tag{12}$$

Taking $K = \frac{\gamma}{2\lambda_1} B^{\mathrm{T}} X^{-1}$, $\gamma > 0$, we obtain the following inequality:

$$(A - \lambda_i BK)X + X(A - \lambda_i BK)^{\mathrm{T}} \prec 0, \ \lambda_i \neq 0, \ \forall i = 1, 2, \ldots, N.$$

Therefore, we can deduce that $\lambda_i > 0$, $\forall i = 1, 2, \ldots, N$, which is met if and only if the graph $\mathcal{G}$ has a spanning tree.

4.2 Invariant ellipsoid of MAS with disturbance

Consider the leader-following system (7) and (8) with control protocol (5) under constrain (9), the following theorem inspired by [14] is proposed to provide a necessary and sufficient condition for being an invariant ellipsoid of system (10).

Theorem 1. Let $P^{\mathrm{T}}=P\in\mathbb{R}^{nN\times nN}$ be positive definite. The ellipsoid $\varepsilon(P)$ of system (10) is an invariant set under bounded external disturbance ω if and only if there exists a real number $\beta>0$ such that the following matrix inequalities hold:

$$\begin{bmatrix} P(I_N\otimes A)+(I_N\otimes A)^{\mathrm{T}}P-P(\tilde{\mathcal{L}}\otimes BK)-(\tilde{\mathcal{L}}\otimes BK)^{\mathrm{T}}P+\beta P & P(1_N\otimes E)\\ (1_N\otimes E)^{\mathrm{T}}P & -\beta Q\end{bmatrix}\preceq 0. \tag{13}$$

Proof. We first prove that Eq. (13) implies

$$\left.\begin{matrix} e^{\mathrm{T}}Pe\geqslant 1,\\ \omega^{\mathrm{T}}Q\omega\leqslant 1,\end{matrix}\right\}\rightarrow\ \frac{\mathrm{d}}{\mathrm{d}t}e^{\mathrm{T}}Pe\leqslant 0. \tag{14}$$

From (13), one derives for any vector $[e^{\mathrm{T}},\omega^{\mathrm{T}}]^{\mathrm{T}}$. The following inequality holds

$$\begin{bmatrix} e\\ \omega\end{bmatrix}^{\mathrm{T}} M \begin{bmatrix} e\\ \omega\end{bmatrix}\leqslant 0,$$

with

$$M=\begin{bmatrix} P(I_N\otimes A)+(I_N\otimes A)^{\mathrm{T}}P-P(\tilde{\mathcal{L}}\otimes BK)-(\tilde{\mathcal{L}}\otimes BK)^{\mathrm{T}}P+\beta P & P(1_N\otimes E)\\ (1_N\otimes E)^{\mathrm{T}}P & -\beta Q\end{bmatrix},$$

which implies

$$\begin{aligned}&e^{\mathrm{T}}(P(I_N\otimes A)+(I_N\otimes A)^{\mathrm{T}}P-P(\tilde{\mathcal{L}}\otimes BK)-(\tilde{\mathcal{L}}\otimes BK)^{\mathrm{T}}P)e+e^{\mathrm{T}}P(1_N\otimes E)\omega+\omega^{\mathrm{T}}(1_N\otimes E)^{\mathrm{T}}Pe\\ &\quad\leqslant\beta(\omega^{\mathrm{T}}Q\omega-e^{\mathrm{T}}Pe).\end{aligned}$$

Thus for $e^{\mathrm{T}}Pe\geqslant 1$, $\omega^{\mathrm{T}}Q\omega\leqslant 1$, one derives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}e^{\mathrm{T}}Pe&=e^{\mathrm{T}}(P(I_N\otimes A)+(I_N\otimes A)^{\mathrm{T}}P-P(\tilde{\mathcal{L}}\otimes BK)-(\tilde{\mathcal{L}}\otimes BK)^{\mathrm{T}}P)e+e^{\mathrm{T}}P(1_N\otimes E)\omega+\omega^{\mathrm{T}}(1_N\otimes E)^{\mathrm{T}}Pe\\ &\leqslant\beta(\omega^{\mathrm{T}}Q\omega-e^{\mathrm{T}}Pe)\leqslant 0.\end{aligned}$$

Therefore the first claim is proven.

Next, suppose that Eq. (14) holds, implying the ellipsoid $\varepsilon(P)$ is not invariant. This means that there exists a trajectory starting from the initial state $e(0)$ with $g(0)=e^{\mathrm{T}}(0)Pe(0)=1$ and ending at an $e(T)\in\varepsilon(P)$ such that $g(T)=e^{\mathrm{T}}(T)Pe(T)>1$. Thus there exists an instant $\bar{t}\in(0,T)$ such that $g(\bar{t})\geqslant 1$ and $\frac{\mathrm{d}}{\mathrm{d}t}g(\bar{t})>0$. This contradicts the condition (14). So we can conclude that $\varepsilon(P)$ is an invariant set.

Necessity: We first prove that if $\varepsilon(P)$ is an invariant ellipsoid of system (13); then we have

$$e^{\mathrm{T}}(P(I_N\otimes A)+(I_N\otimes A)^{\mathrm{T}}P-P(\tilde{\mathcal{L}}\otimes BK)-(\tilde{\mathcal{L}}\otimes BK)^{\mathrm{T}}P)e+e^{\mathrm{T}}P(1_N\otimes E)\omega+\omega^{\mathrm{T}}(1_N\otimes E)^{\mathrm{T}}Pe\leqslant 0,$$

whenever $\omega^{\mathrm{T}}Q\omega\leqslant e^{\mathrm{T}}Pe$. To do this, suppose this condition does not hold, which means there exist $\hat{e}$ and $\hat{\omega}$ such that $\hat{\omega}^{\mathrm{T}}Q\hat{\omega}\leqslant\hat{e}^{\mathrm{T}}P\hat{e}$ but

$$\hat{e}^{\mathrm{T}}(P(I_N\otimes A)+(I_N\otimes A)^{\mathrm{T}}P-P(\tilde{\mathcal{L}}\otimes BK)-(\tilde{\mathcal{L}}\otimes BK)^{\mathrm{T}}P)\hat{e}+\hat{e}^{\mathrm{T}}P(1_N\otimes E)\hat{\omega}+\hat{\omega}^{\mathrm{T}}(1_N\otimes E)^{\mathrm{T}}P\hat{e}>0.$$

This implies $P\hat{e}\neq\mathbf{0}$; thus we have $\hat{e}^{\mathrm{T}}P\hat{e}\neq 0$. Then let $\bar{e}:=\frac{\hat{e}}{\sqrt{\hat{e}^{\mathrm{T}}P\hat{e}}}$ and $\bar{\omega}:=\frac{\hat{\omega}}{\sqrt{\hat{e}^{\mathrm{T}}P\hat{e}}}$. Obviously, we have

$$\bar{e}^{\mathrm{T}}(P(I_N\otimes A)+(I_N\otimes A)^{\mathrm{T}}P-P(\tilde{\mathcal{L}}\otimes BK)-(\tilde{\mathcal{L}}\otimes BK)^{\mathrm{T}}P)\bar{e}+\bar{e}^{\mathrm{T}}P(1_N\otimes E)\bar{\omega}+\bar{\omega}^{\mathrm{T}}(1_N\otimes E)^{\mathrm{T}}P\bar{e}>0,$$

and $\bar{e}$ is on the boundary of $\varepsilon(P)$. For the system (10) with initial state $e(0)=\bar{e}$, we derive that $g(0):=e(0)^{\mathrm{T}}Pe(0)=1$ and $\frac{\mathrm{d}}{\mathrm{d}t}g(0)>0$. Therefore there must exists a small $\xi>0$, such that $g(\xi)>1$;

this implies that $\varepsilon(P)$ is not an invariant set which contradicts the assumption. The first claim is proven. Thus we have

$$s^{\mathrm{T}}\begin{bmatrix} P(I_N\otimes A)+(I_N\otimes A)^{\mathrm{T}}P-P(\tilde{\mathcal{L}}\otimes BK)-(\tilde{\mathcal{L}}\otimes BK)^{\mathrm{T}}P & P(1_N\otimes E) \\ (1_N\otimes E)^{\mathrm{T}}P & \mathbf{0} \end{bmatrix}s\leqslant 0,$$

for $\forall s: s^{\mathrm{T}}\left(\begin{smallmatrix} 0 & 0 \\ 0 & Q \end{smallmatrix}\right)s\leqslant 1$ and $s^{\mathrm{T}}\left(\begin{smallmatrix} -P & 0 \\ 0 & 0 \end{smallmatrix}\right)s\leqslant -1$ with $s=[e^{\mathrm{T}},\omega^{\mathrm{T}}]^{\mathrm{T}}$. By S-procedure [37], one derives Eq. (13) holds for $\beta\geqslant 0$. For the case $\beta=0$, the following inequality must hold:

$$\begin{bmatrix} P(I_N\otimes A)+(I_N\otimes A)^{\mathrm{T}}P-P(\tilde{\mathcal{L}}\otimes BK)-(\tilde{\mathcal{L}}\otimes BK)^{\mathrm{T}}P & P(1_N\otimes E) \\ (1_N\otimes E)^{\mathrm{T}}P & \mathbf{0} \end{bmatrix}\preceq 0. \tag{15}$$

The latter inequality holds only if $E=\mathbf{0}$, which contradicts the assumption of this theorem. Therefore we obtain that β should be positive.

Corollary 1. Let the gain K, as defined in (5), be selected such that the MAS (3) and (4) achieves leader-following consensus without disturbances. Then any invariant ellipsoid of perturbed leader-following system (10) is attractive.

Proof. Since the leader-following consensus of system is achieved, then one derives for the system (6) without disturbance

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{\mathrm{T}}Pe=e^{\mathrm{T}}(P(I_N\otimes A)+(I_N\otimes A)^{\mathrm{T}}P-P(\tilde{\mathcal{L}}\otimes BK)-(\tilde{\mathcal{L}}\otimes BK)^{\mathrm{T}}P)e\leqslant 0.$$

From Theorem 1, any invariant ellipsoid of perturbed leader-following system (10) implies that there exists a $\beta>0$ such that

$$\begin{aligned} & e^{\mathrm{T}}(P(I_N\otimes A)+(I_N\otimes A)^{\mathrm{T}}P-P(\tilde{\mathcal{L}}\otimes BK)-(\tilde{\mathcal{L}}\otimes BK)^{\mathrm{T}}P)e+e^{\mathrm{T}}P(1_N\otimes E)\omega+\omega^{\mathrm{T}}(1_N\otimes E)^{\mathrm{T}}Pe \\ & \quad \leqslant\beta(\omega^{\mathrm{T}}Q\omega-e^{\mathrm{T}}Pe). \end{aligned}$$

Therefore, we obtain that for the system (10) with disturbance,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}e^{\mathrm{T}}Pe &= e^{\mathrm{T}}(P(I_N\otimes A)+(I_N\otimes A)^{\mathrm{T}}P-P(\tilde{\mathcal{L}}\otimes BK)-(\tilde{\mathcal{L}}\otimes BK)^{\mathrm{T}}P)e+e^{\mathrm{T}}P(1_N\otimes E)\omega+\omega^{\mathrm{T}}(1_N\otimes E)^{\mathrm{T}}Pe \\ & \leqslant\beta(\omega^{\mathrm{T}}Q\omega-e^{\mathrm{T}}Pe)<0 \end{aligned}$$

holds whenever $e^{\mathrm{T}}Pe>1$ and $\omega^{\mathrm{T}}Q\omega\leqslant 1$, which is sufficient condition of being an attractive ellipsoid of system (10).

From the practical application point of view, it is natural to introduce the constraint of input within the framework of the proposed approach.

Corollary 2. The linear control protocol satisfies the following constraint:

$$(u(e)-u_0)^{\mathrm{T}}(u(e)-u_0)\leqslant\eta^2,\quad \forall e\in\mathbb{R}^{nN}:\quad e^{\mathrm{T}}Pe\leqslant 1, \tag{16}$$

for some $\eta>0$ if and only if the following LMI holds:

$$\begin{bmatrix} P & \tilde{\mathcal{L}}^{\mathrm{T}}\otimes K^{\mathrm{T}} \\ \tilde{\mathcal{L}}\otimes K & \eta^2 I_{nN} \end{bmatrix}\succeq 0. \tag{17}$$

Proof. From (5), one derives that $u(e)=(\tilde{\mathcal{L}}\otimes K)e$ [9]. To satisfy the control constraints, it requires

$$e^{\mathrm{T}}(\tilde{\mathcal{L}}\otimes K)^{\mathrm{T}}(\tilde{\mathcal{L}}\otimes K)e\leqslant\eta^2,\quad \forall e:\quad e^{\mathrm{T}}Pe\leqslant 1, \tag{18}$$

which is a classical S-procedure for two quadratic forms. It is equivalent to that there exists $\tau\geqslant 0$ such that

$$(\tilde{\mathcal{L}}\otimes K)^{\mathrm{T}}(\tilde{\mathcal{L}}\otimes K)\preceq\tau P,\quad \tau\leqslant\eta^2. \tag{19}$$

Since the minimal invariant ellipsoid is interesting to us, we take

$$\tau=\tau_{\max}=\eta^2; \tag{20}$$

then one derives

$$(\tilde{\mathcal{L}} \otimes K)^{\mathrm{T}}(\tilde{\mathcal{L}} \otimes K) \preceq \eta^2 P. \tag{21}$$

Using Schur complement, we obtain (17).

Corollary 3. The worst case of external disturbance ω^* for system (10) can be given in the following form:

$$\omega^* = \frac{Q^{-1}(1_N \otimes E)^{\mathrm{T}} Pe}{\|Q^{-\frac{1}{2}}(1_N \otimes E)^{\mathrm{T}} Pe\|}. \tag{22}$$

Proof. The worst disturbance will push the system trajectories to the boundary of the invariant ellipsoid, which requires

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\mathrm{T}} Pe \to \mathrm{Max}. \tag{23}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\mathrm{T}} Pe = e^{\mathrm{T}}(P(I_N \otimes A) + (I_N \otimes A)^{\mathrm{T}} P - P(\tilde{\mathcal{L}} \otimes BK) - (\tilde{\mathcal{L}} \otimes BK)^{\mathrm{T}} P)e + e^{\mathrm{T}} P(1_N \otimes E)\omega + \omega^{\mathrm{T}}(1_N \otimes E)^{\mathrm{T}} Pe.$$

Thus it becomes the problem

$$\max_{\omega^{\mathrm{T}} Q \omega = 1} \langle \omega, (1_N \otimes E)^{\mathrm{T}} Pe \rangle. \tag{24}$$

Let $\bar{\omega} = Q^{\frac{1}{2}} \omega$. Since $0 \prec Q = Q^{\mathrm{T}}$, Eq. (24) can be rewritten as

$$\max_{\bar{\omega}^{\mathrm{T}} \bar{\omega} = 1} \langle \bar{\omega}, Q^{-\frac{1}{2}}(1_N \otimes E)^{\mathrm{T}} Pe \rangle. \tag{25}$$

The obvious solution of $\bar{\omega}$ is

$$\bar{\omega}^* = \frac{Q^{-\frac{1}{2}}(1_N \otimes E)^{\mathrm{T}} Pe}{\|Q^{-\frac{1}{2}}(1_N \otimes E)^{\mathrm{T}} Pe\|}. \tag{26}$$

Finally one derives the solution ω^* as follows:

$$\omega^* = \frac{Q^{-1}(1_N \otimes E)^{\mathrm{T}} Pe}{\|Q^{-\frac{1}{2}}(1_N \otimes E)^{\mathrm{T}} Pe\|}. \tag{27}$$

4.3 Minimization of invariant ellipsoid

Since Eq. (13) guarantees $\varepsilon(P)$ to be attractive/invariant ellipsoid, and the trace of matrix P characterizes the size of invariant ellipsoid (representing the impact of external disturbance). Therefore, the external disturbance of system (10) can be optimally rejected by solving the following optimal problem:

$$\begin{aligned} &\mathrm{tr}(X) \to \mathrm{Min}, \quad \text{where } X = P^{-1}, \\ &\text{under constraints (13) and (17)}, \end{aligned} \tag{28}$$

where the constraint (17) is added to avoid the infinite input. The following corollary allows one to limit the search for the minimum invariant ellipsoid to the one-parameter family (13), simplifying the problem to a one-dimensional minimization over a finite interval.

Corollary 4. For any given matrix gain K that makes $(I_N \otimes A - \tilde{\mathcal{L}} \otimes BK)$ Hurwitz, the minimal (in the sense of (28)) invariant ellipsoid of system (13) belongs to the one-parameter family of ellipsoid generated by matrices P and K satisfying

$$\begin{bmatrix} P(I_N \otimes A) + (I_N \otimes A)^{\mathrm{T}} P - P(\tilde{\mathcal{L}} \otimes BK) - (\tilde{\mathcal{L}} \otimes BK)^{\mathrm{T}} P + \beta P & P(1_N \otimes E) \\ (1_N \otimes E)^{\mathrm{T}} P & -\beta Q \end{bmatrix} = 0, \tag{29}$$

over the interval $0 < \beta < -2\mathrm{Max}(\mathrm{Re}\lambda_i(I_N \otimes A - \tilde{\mathcal{L}} \otimes BK))$.

Proof. Following the Lemma A.16 of [38], let A be Hurwitz matrix and (A, E) be controllable; then the solution of Lyapunov equation

$$AX + XA^{\mathrm{T}} + EE^{\mathrm{T}} = 0 \tag{30}$$

is the solution of optimal problem

$$\mathrm{tr}(X) \to \mathrm{Min}, \tag{31}$$

under constraint

$$AX + XA^{\mathrm{T}} + EE^{\mathrm{T}} \preceq 0. \tag{32}$$

Then Eq. (29) can be rewritten as

$$P\left(I_N \otimes A - \tilde{\mathcal{L}} \otimes BK + \frac{\beta}{2} I_{nN}\right) + \left(I_N \otimes A - \tilde{\mathcal{L}} \otimes BK + \frac{\beta}{2} I_{nN}\right)^{\mathrm{T}} P + \frac{1}{\beta} P(1_N \otimes E)Q^{-1}(1_N \otimes E)^{\mathrm{T}} P = 0. \tag{33}$$

Similarly, we require that $(I_N \otimes A - \tilde{\mathcal{L}} \otimes BK + \frac{\beta}{2} I_{nN})$ is Hurwitz matrix with

$$\mathrm{Re}\lambda_i \left(I_N \otimes A - \tilde{\mathcal{L}} \otimes BK + \frac{\beta}{2} I_{nN}\right) < 0; \tag{34}$$

then we obtain that Eq. (29) has unique solution when $0 < \beta < -2\mathrm{Max}(\mathrm{Re}\lambda_i(I_N \otimes A - \tilde{\mathcal{L}} \otimes BK))$.

5 Simulation examples

In the following simulation example, we first define an MAS model with bounded external disturbance and an input constraint. Secondly, the optimal problem (28) is solved to obtain the optimal solution K. Finally, the MAS is simulated based on the optimal linear control protocol (5).

Consider the MAS (3) and (4) with 4 agents regulated by the linear consensus protocol (5). The fixed communication topology is depicted in Figure 1. The agents are labeled by $i = 0, 1, 2, 3$, where the agent 0 is the leader and the agent $1, 2, 3$ are followers. In this example, the dynamic of leader and follower is modeled by (3) and (4), respectively. The dynamic system matrices A, B, and E are randomly chosen as follows:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{35}$$

Let the weight $\omega_{ij} = 1$ if the agent i receives the information from agent j, otherwise $\omega_{ij} = 0$. Thus the Laplacian matrix can be written as follows:

$$\mathcal{L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}. \tag{36}$$

Then one derives

$$\tilde{\mathcal{L}} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \tag{37}$$

The external disturbance is bounded with $\omega^{\mathrm{T}} Q \omega \leqslant 1$ where

$$Q = \begin{bmatrix} 800 & 0 \\ 0 & 4000 \end{bmatrix}, \quad \omega = \begin{bmatrix} \frac{1}{50} \\ \frac{1}{80} \end{bmatrix} \sin \frac{t}{2}. \tag{38}$$

The bounded input is defined by $\eta = 50000$. Solving the optimal problem (28) with bounded input (17) by the solver BMIBNB of Yalmip[1)], we obtain the optimal control protocol:

$$K = [46.6001 \ 25.6217], \tag{39}$$

1) Yalmip bmi solver: Bmibnb. https://yalmip.github.io/solver/bmibnb/.

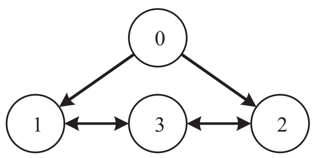

Figure 1 Topology of MAS.

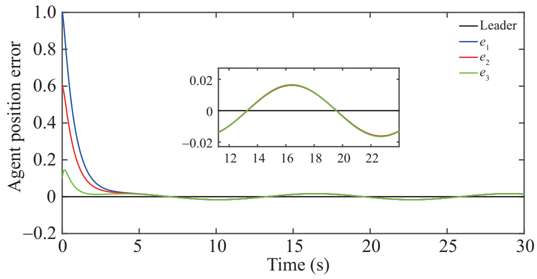

Figure 2 (Color online) Consensus trajectories of three agents under bounded disturbance.

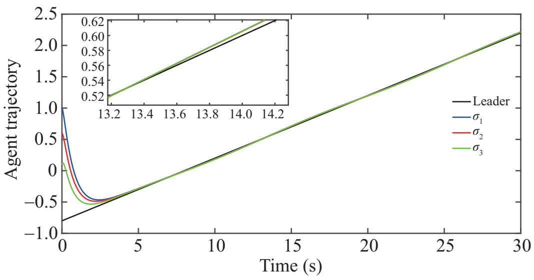

Figure 3 (Color online) Trajectories of three agents tracking leader under bounded disturbance.

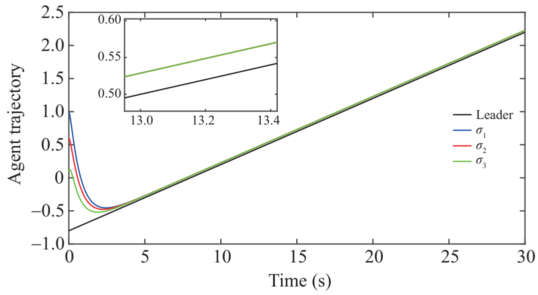

Figure 4 (Color online) Trajectories of three agents tracking leader under worst disturbance.

$$P = 10^3 \times \begin{bmatrix} 1.9963 & 0.0008 & -1.2544 & -0.0003 & -0.6919 & -0.0018 \\ 0.0008 & 0.0188 & -0.0005 & -0.0135 & 0.0014 & -0.0037 \\ -1.2544 & -0.0005 & 1.9291 & -0.0000 & -0.6245 & -0.0027 \\ -0.0003 & -0.0135 & -0.0000 & 0.0186 & 0.0030 & -0.0030 \\ -0.6919 & 0.0014 & -0.6245 & 0.0030 & 1.2549 & 0.0049 \\ -0.0018 & -0.0037 & -0.0027 & -0.0030 & 0.0049 & 0.0080 \end{bmatrix}. \tag{40}$$

In the following simulation examples, the optimal K above is applied. In the first example, the leader is assumed to be at the origin. The initial state of MAS is given by

$$\sigma = [\sigma_0, \sigma_1, \sigma_2, \sigma_3]^{\mathrm{T}} = [0, 0, 1, 0, 0.6, 0, 0.1, 0.5]^{\mathrm{T}},$$

with $u_0 = 0$. Figure 2 shows the consensus trajectory of three different agents under bounded disturbance. It is obvious to see that the trajectory converges to a certain invariant set centered by the leader. The maximum position error between leader and followers is from the agent 3 and the maximum absolute error is about 0.01659. This means the position errors of followers converge to the invariant set $[-0.01659, 0.01659]$.

In the second example, it is assumed that the leader is moving over time. The initial state of MAS is given by

$$\sigma = [\sigma_0, \sigma_1, \sigma_2, \sigma_3]^{\mathrm{T}} = [-0.8, 0.1, 1, 0, 0.6, 0, 0.1, 0.5]^{\mathrm{T}},$$

with $u_0 = 0.01$. Figure 3 depicts the trajectories of three agents as they follow the leader. Similarly to the previous example, the maximum absolute position error comes from the agent 3, which is about 0.0243.

In the third example, the MAS works in the case of worst disturbance as defined in (22). To make the trajectory more clear, the following initial state is applied:

$$\sigma = [\sigma_0, \sigma_1, \sigma_2, \sigma_3]^{\mathrm{T}} = [0, 0, 1, 0, 0.6, 0, 0.1, 0.5]^{\mathrm{T}},$$

with $u_0 = 0$.

It is evident from Figure 4 that the worst disturbance pushes the system's trajectory farthest away from the reference position. This maximum absolute position error is constant about 0.0397.

6 Conclusion

This study extends the invariant/attractive ellipsoid method [14, 26, 27] to MAS. The optimal control protocol is derived by solving semi-definite matrix inequalities that characterize the minimal invariant/attractive ellipsoid. From a practical standpoint, a bounded input is introduced. The simulation results demonstrate that the MAS state converges to a minimal invariant ellipsoid in the presence of bounded disturbances. Remarkably, in the worst case of disturbance, the system trajectories converge and subsequently remain inside the invariant ellipsoid. This outcome signifies the effectiveness of the optimal control protocol in minimizing the impact of disturbances on the MAS.

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