SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

July 2024, Vol. 67, Iss. 7, 172204:1–172204:15
https://doi.org/10.1007/s11432-023-4052-4

Prescribed time control based on the periodic delayed sliding mode surface without singularities

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Received 22 June 2023/Revised 27 November 2023/Accepted 16 January 2024/Published online 27 June 2024

Abstract In this paper, nonsingular prescribed-time control is studied based on periodic delayed sliding mode surfaces. Different from the existing sliding mode control, where either singularity problems may appear or the convergence time depends on the initial state, the proposed sliding mode control approaches can achieve prescribed-time convergence without singularity. The proposed nonsingular sliding mode control approaches can be applied to both second-order and high-order nonlinear systems with prescribed-time convergence. As applications of the proposed sliding mode control approaches, the control of hypersonic vehicle systems is revisited. Numerical simulations on the nonlinear model of the hypersonic vehicle system show the effectiveness of the proposed methods.

Keywords prescribed-time control, smooth periodic delayed feedback, sliding mode control, high-order nonlinear systems, hypersonic vehicle systems

1 Introduction

As a powerful design tool for nonlinear systems, sliding mode control has received extensive attention due to its simple structure and strong robustness to disturbances [1–5]. Through the efforts of a large number of researchers, various forms of sliding modes have been developed. Inspired by terminal attractors introduced in [6], a new class of sliding modes termed as terminal sliding modes was designed in [7]. Compared with linear sliding mode, the terminal sliding mode has the advantage of fast convergence speed, finite time convergence, and much more stronger robustness to disturbances. However, the traditional terminal sliding mode has a singularity problem [8, 9]. To overcome this problem, a new non-singular terminal sliding mode surface is proposed in [8]. In order to improve the convergence rate in the sliding mode control, many other types of sliding mode surfaces, such as fast finite-time [9], fixed-time [10–15], and prescribed-time sliding mode surfaces have been proposed. For more related work on sliding mode control, see [4,13,16–19] and the references therein. However, the convergence time (rather than its upper bound) in both the finite-time and fixed-time sliding mode surfaces depends on the initial condition, which may be difficult to compute exactly in practice. In addition, for some time-varying high-gain based prescribed-time sliding mode control methods, there may exist some singularity problems. Moreover, due to the existence of power functions, most traditional nonsingular finite-time sliding mode control methods (see [8–10]) can only be applied to second-order systems, which might limit their applications.

Although the analysis and design of time-varying systems are more complex than those of time-invariant systems, time-varying feedback can solve some problems that time-invariant feedback cannot, for example, prescribed-time control. Compared with the traditional finite-time/fixed-time control, the convergence time of prescribed-time control can be prescribed in advance and does not depend on the initial state. One of the prescribed-time control methods is time-varying high-gain feedback, which can be traced back at least to the traditional optimal control with a terminal constraint (see [20,21]), and has received extensive attention in recent years (see [22–26] and the references therein). On the other hand, periodic feedback can solve problems that aperiodic feedback cannot. Recently, a periodic delayed feedback approach and a

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predictor-based periodic delayed feedback approach for prescribed-time stabilization of controllable linear systems with input delay were established in [27].

In this paper, different from the existing sliding mode surfaces, where either singularity problems may appear or the convergence time depends on the initial state, novel prescribed-time time-varying sliding mode surfaces are designed based on smooth periodic delayed (SPD) feedback initially proposed in [27]. Compared with the traditional power functions-based nonsingular finite-time/fixed-time sliding mode surfaces, which may only be applied to second-order nonlinear systems, the nonsingular sliding mode surfaces proposed in this paper can not only achieve prescribed-time convergence, but also can be applied to second-order nonlinear systems as well as high-order nonlinear systems, for which both integral sliding mode surface and hierarchical sliding mode surface are designed. It is shown that the proposed SPD sliding mode control can achieve the prescribed-time convergence. As an applications of the proposed SPD sliding mode control approaches, the controller design of hypersonic vehicle systems is revisited. Numerical simulations with the nonlinear model of the hypersonic vehicle system show the effectiveness of the proposed methods.

The remainder of this paper is organized as follows. We state the problem to be studied in Section 2, where we also present the main idea for solving our problem. The SPD sliding mode control methods are then introduced in detail in Section 3. The proposed SPD sliding mode control methods are used to solve the tracking control problem for the hypersonic vehicle system in Section 4. The paper is finally concluded in Section 5.

Notation. For $x \in \mathbb{R}$, and $\alpha \in (0, \infty)$, let

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$$

and $\mathrm{sig}^{\alpha}(x) = |x|^{\alpha}\mathrm{sgn}(x)$.

2 Motivation and problem description

2.1 Motivation

Consider the following nonlinear system:

$$\dot{x} = f(t, x, u), \quad x(0) = x_0, \quad t \geqslant 0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $f(t, x, u) : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous nonlinear function such that $f(t, 0, 0) = 0, \forall t \geqslant 0$. Assume that the nonlinear system (1) has a unique solution. For sliding mode control of (1), there are plenty of sliding mode surfaces (manifolds) in the form of

$$s = g(t, \dot{x}, x),$$

where $g(t, \dot{x}, x) : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a function, and the state x can converge to zero along the sliding surface (manifold) $s = 0$. The sliding mode surfaces commonly used in the literature are collected in Table 1 [7–10, 12, 13, 28–30], where $\alpha, \beta > 0, \gamma \in (0, 1), \nu > \gamma, \mu > 0, \lambda > 0, \frac{\lambda}{1+\mu} > 1, \eta > 0, T > 0$ are some constants, and (q, p) is a pair of positive odd numbers satisfying $q < p$.

Due to space limitations, there are still many other sliding surfaces that are not listed in Table 1. Readers can further refer to [10] for finite-time sliding mode surfaces, to [11] for fixed-time sliding mode surfaces, and to [31] for prescribed-time sliding mode surfaces. It can be observed from Table 1 that only the sliding modes with fixed-time convergence in Nos. 7, 8, and 9 are non-singular, yet the convergence time still depends on the initial state, which may be difficult to obtain in practical engineering. For the above reasons, one of the purposes of this paper is to explore a new prescribed-time sliding mode surfaces without singularities, and the other is to use the proposed sliding mode surfaces to study the prescribed-time control problem of high-order nonlinear systems.

Table 1 Different choices of the function $g(t, \dot{x}, x)$

No.	$g(t, \dot{x}, x)$	Nonsingularity	Stability type	Ref.
1	$g_1 = \dot{x} + \alpha x$	✓	Exponential	[28]
2	$g_2 = \dot{x} + \beta x^{\frac{q}{p}}$	×	Finite time	[7]
3	$g_3 = \dot{x} + \alpha x + \beta x^{\frac{q}{p}}$	×	Fast Finite time	[9]
4	$g_4 = x + \frac{1}{\beta}\dot{x}^{\frac{p}{q}}$	✓	Finite time	[8]
5	$g_5 = \dot{x} + \alpha x + \beta \lvert x\rvert^{\gamma} \operatorname{sgn}(x)$	×	Fast Finite time	[10]
6	$g_6 = x + \beta \lvert\dot{x}\rvert^{\gamma+1} \operatorname{sgn}(\dot{x})$	✓	Finite time	[10]
7	$g_7 = x + \alpha \lvert x\rvert^{\nu+1} \operatorname{sgn}(x) + \beta \lvert\dot{x}\rvert^{\gamma+1} \operatorname{sgn}(x)$	✓	Fixed time	[29]
8	$g_8 = \dot{x} + 2\beta\sqrt{\lvert\arctan(x)\rvert}\left(1+x^2\right)\operatorname{sgn}(x)$	✓	Fixed time	[12]
9	$g_9 = \dot{x} + \beta \lvert x\rvert^{\frac{\lambda x^2}{1+\mu x^2}} \operatorname{sgn}(x)$	✓	Fixed time	[13]
10	$g_{10} = (T-t)\dot{x} + \eta x$	×	Prescribed time	[30]

2.2 Problem description and the key idea

Let us consider the following single input nonlinear system in the normal form:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \vdots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = f(t,x) + g(t,x)u + \delta(t,x), \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, $f(t,x) : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ and $g(t,x)(\neq 0) : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ are known functions, and $\delta(t,x) : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is an unknown disturbance. For convenience, the above nonlinear system in the normal form is further written in the following compact form:

$$\dot{x} = Ax + b\left(f(t,x) + g(t,x)u + \delta(t,x)\right), \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^{n\times 1}$ are given by

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{4}$$

The following assumption is imposed on system (2).

Assumption 1 ([12])**.** There exists a known function $\rho(t,x) : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ such that $|\delta(t,x)| \leqslant \rho(t,x), \forall t \geqslant 0, \forall x \in \mathbb{R}^n$.

In this paper, the control objective is to design a sliding mode controller such that the state of the closed-loop system converges to zero at a prescribed time $T > 0$, and maintains there forever. We will solve the problem in two steps:

(1) Design a non-singular sliding mode control law

$$u(t) = u(t, T, s(t), x_{[t-T,t]}),$$

where $x_{[a,b]}$ denotes the function $x(\theta), \theta \in [a,b]$, such that the state of the closed-loop system converges to sliding mode surface $s(t) = 0$ (to be designed) in a fixed time $\tau < T$, where $\tau \in (0,T)$ is some constant, and maintains at the surface forever, namely, $s(t) = 0, \forall t \geqslant \tau$.

(2) Design an SPD sliding mode surface

$$s(t) = s(t, T, x_{[t-\tau,t]}),$$

such that, if $s(t) = 0, \forall t \geqslant \tau$, the state of the closed-loop system converges to zero in the next $T - \tau$ s, and maintains there forever, namely, $x(t) = 0, \forall t \geqslant T$.

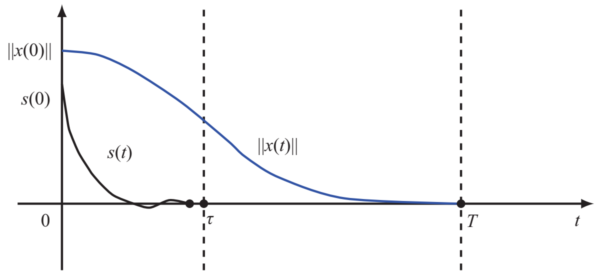

Figure 1 (Color online) Convergence of $s(t)$ and $x(t)$ by the SPD sliding mode control.

It follows that the state converges to the sliding mode surface $s(t) = 0$ in the fixed time τ, and then converges to zero in the next $T - \tau$ s.

The histories of the sliding mode surface $s(t)$ and the state are illustrated in Figure 1.

2.3 Preliminaries

To present our method, we need some preliminary results.

Lemma 1 ([32])**.** Consider the nonlinear system (1). Suppose that there is a positive definite function $V(x)$ such that

$$\dot{V}(x) \leqslant -cV^{\alpha}(x) - dV^{\beta}(x), \quad \forall t \geqslant 0,$$

where $c > 0$, $d > 0$, $0 < \alpha < 1$, and $\beta > 1$ are some constants. Then $x(t) = 0, \forall t \geqslant T_\tau$, where

$$T_\tau = T_\tau(x_0) \leqslant \tau \triangleq \frac{1}{c\,(1-\alpha)} + \frac{1}{d\,(\beta-1)}. \tag{5}$$

Definition 1 ([27])**.** Let $h > 0$ be a given constant and $r \geqslant 0$ be a given integer (can be infinity). A symmetric function $R_h(\cdot) : \mathbb{R} \to \mathbb{R}$ is a said to be an $\boldsymbol{S}^{(r)}(h)$ function if

(1) $R_h(\cdot)$ is $2h$-periodic and $R_h(t) = 0, \forall t \in [0, h]$;

(2) $R_h(t) \geqslant 0, \forall t \in [h, 2h]$ and there is an $h^* \in (h, 2h)$, such that $R_h(h^*) > 0$;

(3) $R_h(\cdot) \in \boldsymbol{C}^r$, which implies that $R_h^{(i)}(h) = R_h^{(i)}(2h) = 0, i = 0, 1, \ldots, r$.

Consider the following linear system:

$$\dot{z}(t) = Az(t) + bv(t), \quad t \geqslant 0, \tag{6}$$

where $z \in \mathbb{R}^n$ is the state vector, $v(t) \in \mathbb{R}$ is the control input, and (A, b) is a controllable pair in the form of (4). Regarding this system, a smooth periodic delayed feedback that can achieve prescribed-time convergence is given as follows.

Lemma 2 ([27])**.** Let $T > 0$ be a prescribed time, $k \in \mathbb{R}^{1\times n}$ be any vector, and $h = \frac{1}{2}T$. Consider the following periodic delayed feedback control:

$$v(t) = kz(t) - K_{(A_\mathrm{c},h)}(t)z(t-h), \tag{7}$$

where $A_\mathrm{c} = A + bk$, $z(\theta)$ is an arbitrary bounded function for $\theta \in [-h, 0)$, $K_{(A_\mathrm{c},h)}(t)$ is a $2h$-periodic function defined as (we only consider $t \in [0, 2h]$)

$$K_{(A_\mathrm{c},h)}(t) = R_h(t)b^\mathrm{T}\mathrm{e}^{-A_\mathrm{c}^\mathrm{T}t}W\mathrm{e}^{A_\mathrm{c}(h-t)}, \tag{8}$$

in which $W = W_\mathrm{c}^{-1}(A_\mathrm{c}, h)$ is defined as

$$W_\mathrm{c}(A_\mathrm{c}, h) = \int_h^{2h} \mathrm{e}^{-A_\mathrm{c}s}bR_h(s)b^\mathrm{T}\mathrm{e}^{-A_\mathrm{c}^\mathrm{T}s}\mathrm{d}s > 0, \tag{9}$$

with $R_h(t) \in \boldsymbol{S}^{(r)}(h)$. Then the state of the closed-loop system

$$\dot{z}(t) = A_\mathrm{c}z(t) - bK_{(A_\mathrm{c},h)}(t)z(t-h), \quad t \geqslant 0, \tag{10}$$

consisting of (6) and (7) satisfies $z(t) = 0, \forall t \geqslant 2h = T$. Moreover,

$$z(0) \neq 0 \Rightarrow z(t) \neq 0, \quad \forall t \in [0, 2h). \tag{11}$$

3 SPD sliding mode control

3.1 Integral SPD sliding mode control

Based on Lemma 2, an integral SPD sliding mode surface can be designed as

$$s(t) = x_n(t) - \int_0^t (kx(\sigma) - K(\sigma)x(\sigma - h))\,\mathrm{d}\sigma, \quad t \geqslant 0, \tag{12}$$

where $h > 0$ is a constant, and $x(\theta) = \phi_1(\theta), \theta \in [-h, 0]$ is an arbitrary vector-valued bounded function. To avoid possible negative effects of $x(t-h) = \phi_1(t-h), t \in [0, h]$ on the convergence of the sliding mode, we let

$$K(t) = \begin{cases} 0, & t \leqslant h, \\ K_{(A_c,h)}(t-h), & t > h, \end{cases} \tag{13}$$

where $K_{(A_c,h)}(t-h)$ is defined in (8), $A_c = A + bk$, with $k \in \mathbb{R}^{1\times n}$ being any vector. The function $K(t)$ with $n = 1$ is illustrated in Figure 2. Since $K(t) = 0, \forall t \in [0, h]$, the term $K(t)x(t-h) = 0, \forall t \in [0, h]$, which implies the effect of the initial function $\phi_1(\theta), \theta \in [-h, 0]$ is completely eliminated by the time-varying function $K(t)$.

The result can then be stated as follows.

Theorem 1. Let Assumption 1 be satisfied, $T > 0$ be a prescribed time. Choose $\alpha \in (1/2, 1)$, $c > 0$, $d > 0$, and $\beta > 1$ such that τ defined in (5) satisfies $\tau = \frac{1}{3}T$. Let

$$h = \tau = \frac{1}{3}T.$$

Consider the sliding mode control law

$$\begin{aligned} u(t) =& g^{-1}(t,x)\left(-\frac{c}{2^\alpha}\mathrm{sig}^{2\alpha-1}(s(t)) - \frac{d}{2^\beta}\mathrm{sig}^{2\beta-1}(s(t)) - \rho(t,x)\,\mathrm{sgn}(s(t))\right. \\ &\left. - f(t,x) + kx(t) - K(t)x(t-h)\right), \quad t \geqslant 0, \end{aligned} \tag{14}$$

where $K(t)$ is defined in (13). Then the state of the closed-loop system consisting of (3) and (14) satisfies $x(t) = 0$, $\forall t \geqslant T$.

Proof. For clarity, we divide the proof into two steps.

Step 1. The convergence of $s(t)$. Consider the Lyapunov function

$$V(s) = \frac{1}{2}s^2 = \frac{1}{2}s^2(t),$$

whose time-derivative can be written as

$$\dot{V}(s) = s\left(f(t,x) + g(t,x)u(t) + \delta(t,x) - kx(t) + K(t)x(t-h)\right), \tag{15}$$

where $x(\theta) = \phi_1(\theta), \theta \in [-h, 0]$. Substituting (14) into (15) and noting that $K(t) = 0, t \in [0, h]$ yield

$$\dot{V}(s) = -\frac{c|s(t)|^{2\alpha}}{2^\alpha} - \frac{d|s(t)|^{2\beta}}{2^\beta} + s(t)\delta(t,x) - \rho(t,x)|s(t)| \leqslant -cV^\alpha(s) - dV^\beta(s), \quad \forall t \geqslant 0.$$

Using Lemma 1, we know that $s(t) = 0, \forall t \geqslant T_\tau$, which implies $s(t) = 0, \forall t \in [\tau, \infty) = [h, \infty)$.

Step 2. The convergence of $x(t)$. When $t \geqslant h$, there holds $s(t) = 0$, namely,

$$x_n(t) = \int_0^t \left(kx(\sigma) - K_{(A_c,h)}(\sigma - h)\,x(\sigma - h)\right)\mathrm{d}\sigma.$$

Taking the time-derivative of both sides on the above equation yields

$$\dot{x}_n(t) = kx(t) - K_{(A_c,h)}(t-h)x(t-h), \quad \forall t \in [h, \infty). \tag{16}$$

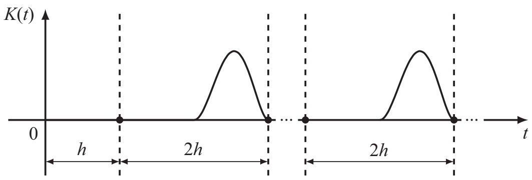

Figure 2 Function $K(t)$ defined in (13) with $n = 1$.

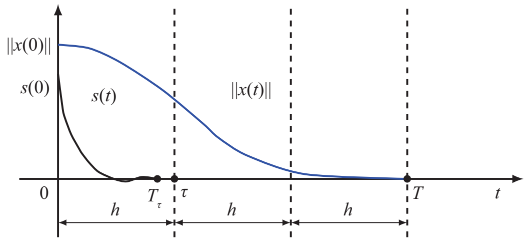

Figure 3 (Color online) Convergence of the SPD integral sliding mode (12) and the state.

By combining (3) with (16), we can get

$$\dot{x}(t) = A_{\rm c}x(t) - bK_{(A_{\rm c},h)}(t-h)x(t-h), \quad \forall t \in [h, \infty),$$

which can be further written as

$$\dot{\chi}(\kappa) = A_{\rm c}\chi(\kappa) - bK_{(A_{\rm c},h)}(\kappa)\chi(\kappa - h), \quad \forall \kappa \in [0, \infty), \tag{17}$$

letting $\kappa = t - h$ and $\chi(\kappa) = \chi(t-h) = x(t)$. It is noticed that Eq. (17) is exactly in the form of (10). Thus, according to Lemma 2, we have $\chi(\kappa) = 0, \forall \kappa \geqslant 2h$, or equivalently, $x(t) = 0$, $\forall t \geqslant T$.

The histories of $s(t)$ and $x(t)$ are shown in Figure 3.

Remark 1. We give some explanations on the exact settling time of the closed-loop system. For simplicity, we suppose that $\delta(t, x) = 0$. If $x(T_\tau) = 0$, then it follows from (14) that $\dot{x}(T_\tau) = 0$, which in turn implies $x(t) = 0, \forall t \geqslant T_\tau$. If $x(T_\tau) \neq 0$, then it follows from (14) that $\dot{x}(t) = (A+bk)x(t), \forall t \in [T_\tau, h]$, which implies that $x(h) \neq 0$. Thus, by (11) we know that $x(t) \neq 0, \forall t \in [h, 3h)$ and $x(t) = 0, \forall t \geqslant 3h = T$, namely, the settling time is exactly $3h = T$. Therefore, expect for the very particular case that $x(T_\tau) = 0$, the settling time of the closed-loop system is exactly T.

Remark 2. In view of (12), similarly to [33], we can also take the integral sliding mode as

$$s(t) = x_n(t) - x_n(0) - \int_0^t (kx(\sigma) - K(t)x(\sigma - h))\,{\rm d}\sigma,$$

which implies $s(0) = 0$. Clearly, in this case, we have $T_\tau = 0$.

Remark 3. Since the expression in (9) contains integrals, it is sometimes difficult to calculate. According to [27], if the feedback gain k is designed as $k = -b^{\rm T}P$, where $P > 0$ is the unique positive solution to the following parametric Lyapunov equation (PLE) [34]:

$$A^{\rm T}P + PA - Pbb^{\rm T}P = -\gamma P, \quad \gamma > 0,$$

we can use the following expression instead:

$$K_{(A_{\rm c},h)}(t) = {\rm e}^{\gamma(2t-5h)}R_h(t)b^{\rm T}P{\rm e}^{A(2h-t)}S_\gamma^{-1}(h)\,{\rm e}^{A(3h-t)}P,$$

where $S_\gamma(\sigma), \sigma \in [0, h]$ is the solution to the following linear Lyapunov differential equation:

$$\dot{S}_\gamma(\sigma) = -(A + \gamma I_n)^{\rm T} S_\gamma(\sigma) - S_\gamma(\sigma)(A + \gamma I_n) + PbR_h(h+\sigma)b^{\rm T}P, \quad S_\gamma(0) = 0.$$

3.2 SPD sliding mode control for second-order systems

We now consider system (2) with $n = 2$, namely,

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f(t,x) + g(t,x)u + \delta(t,x). \end{cases} \tag{18}$$

Due to the special structure of the second-order system, the integral operation in the original sliding mode surface (12) can be avoided, and thus a simpler sliding surface can be proposed as follows:

$$s(t) = x_2(t) - ax_1(t) + K(t)x_1(t-h), \quad t \geqslant 0, \tag{19}$$

where $x_1(\theta) = \phi_2(\theta), \theta \in [-h, 0]$ is an arbitrary bounded function, $a \leqslant 0$ and $h > 0$ are constants, and

$$K(t) = \begin{cases} 0, & t \leqslant h, \\ K_{(a,h)}(t-h), & t > h, \end{cases} \tag{20}$$

in which $K_{(a,h)}(t)$ is determined by (8) with $A_{\mathrm{c}} = a$, $b = 1$, and $R_h(t) \in \boldsymbol{S}^{(r)}$, $r \geqslant 1$.

Theorem 2. Let Assumption 1 be satisfied, $T > 0$ be a prescribed time. Choose $\alpha \in (1/2, 1)$, $c > 0$, $d > 0$, and $\beta > 1$ such that τ defined in (5) satisfies $\tau = \frac{1}{3}T$. Let

$$h = \tau = \frac{1}{3}T.$$

Consider the sliding mode control law

$$\begin{aligned} u(t) =& g^{-1}(t,x)\left(-\frac{c}{2^{\alpha}}\mathrm{sig}^{2\alpha-1}(s(t)) - \frac{d}{2^{\beta}}\mathrm{sig}^{2\beta-1}(s(t)) - \rho(t,x)\,\mathrm{sgn}(s(t)) - f(t,x)\right. \\ &\left. + ax_2 - K(t)\,x_2(t-h) - \dot{K}(t)\,x_1(t-h)\right), \quad t \geqslant 0, \end{aligned} \tag{21}$$

where $K(t)$ is defined in (20). Then the state of the closed-loop system consisting of (18) and (21) satisfies $x(t) = 0, \forall t \geqslant T$.

Proof. Similarly to the proof of Theorem 1, the proof is also divided into two steps.

Step 1. The convergence of $s(t)$. Consider the Lyapunov function $V(s) = \frac{1}{2}s^2$ whose time-derivative along (18) and (21) can be written as

$$\begin{aligned} \dot{V}(s) =& s(f(t,x) + g(t,x)u(t) + \delta(t,x) - ax_2(t) + K(t)\,x_2(t-h) + \dot{K}(t)\,x_1(t-h)) \\ =& -\frac{c|s(t)|^{2\alpha}}{2^{\alpha}} - \frac{d|s(t)|^{2\beta}}{2^{\beta}} + s(t)\delta(t,x) - \rho(t,x)\,|s(t)| \\ \leqslant& -cV^{\alpha}(s) - dV^{\beta}(s), \quad \forall t \geqslant 0, \end{aligned}$$

where we have used the facts that $K(t) = \dot{K}(t) = 0, t \in [0, h]$ and $\dot{x}_1(t-h) = x_2(t-h), t \geqslant h$. Using Lemma 1, we know that $s(t) = 0, \forall t \in [\tau, \infty) = [h, \infty)$.

Step 2. The convergence of $x(t)$. It follows from $s(t) = 0$ that

$$\dot{x}_1(t) = ax_1(t) - K_{(a,h)}(t-h)x_1(t-h), \quad \forall t \geqslant h. \tag{22}$$

As done in the proof of Theorem 1, Eq. (22) can be rewritten as

$$\dot{\chi}_1(\kappa) = a\chi_1(\kappa) - K_{(a,h)}(\kappa)\,\chi_1(\kappa-h), \quad \forall \kappa \in [0, \infty), \tag{23}$$

with $\kappa = t - h$ and $\chi_1(\kappa) = \chi_1(t-h) = x_1(t)$. It is noticed that Eq. (23) is also in the form of (10) with $A_{\mathrm{c}} = a$ and $b = 1$. Thus, according to Lemma 2, we have $x_1(\kappa) = 0, \forall \kappa \geqslant h$, or equivalently, $x_1(t) = 0, \forall t \geqslant 3h = T$. Since $s(t) = 0, \forall t \geqslant h$, we have

$$x_2(t) = ax_1(t) - K_{(a,h)}(t-h)x_1(t-h), \quad \forall t \geqslant T. \tag{24}$$

It is observed from (8) that $K_{(a,h)}(t-h) = 0, \forall t \in [T, T+h] = [h+2h, h+3h]$. Then we have $x_2(t) = 0, t \in [T, T+h]$. For $t \geqslant T+h$, we have $x_1(t) = 0$ and $x_1(t-h) = 0$, which, in view of (24), implies that $x_2(t) = 0$. Thus the state of the closed-loop system consisting of (18) and (21) satisfies $x(t) = 0, \forall t \geqslant T$.

Remark 4. In the traditional sliding mode methods for second-order systems, either there are singularity problems [30] or the convergence times are related to the initial value of the state [8, 10, 12, 13, 29]. Unlike the existing methods, the SPD sliding mode surface (19) proposed in this subsection not only avoids the singularity of the sliding mode surface, but also guarantees the prescribed time convergence of the state of the closed-loop system.

Table 2 Different choices of the function $R(t)$

No.	$k(t)$ $(a=0)$	$k(t)$ $(a \neq 0)$	$R(t)$
1	$k_1 = \frac{2}{h}\sin^2\left(\frac{\pi t}{h}\right)$	$k_1 = \frac{4a(a^2h^2+\pi^2)}{\pi^2(\mathrm{e}^{2ah}-1)}\sin^2\left(\frac{\pi t}{h}\right)\mathrm{e}^{a(5h-2t)}$	$R_1 = \sin^2\left(\frac{\pi t}{h}\right)$
2	$k_2 = \frac{6(\mathrm{e}^{t-h}-1)^2(\mathrm{e}^{t-2h}-1)^2}{6h+\sinh(2h)-8\sinh(h)}$	$k_2 = \frac{2a\mathrm{e}^{a(3h-2t)}\mathrm{e}^{2h}(\mathrm{e}^{t-h}-1)^2(\mathrm{e}^{t-2h}-1)^2\sigma_1}{6\mathrm{e}^{2h}(1-\mathrm{e}^{-2ah})+a^2\sigma_2-a\sigma_3}$	$R_2 = (\mathrm{e}^{t-h}-1)^2(1-\mathrm{e}^{t-2h})^2$
3	$k_3 = \frac{30}{h^5}(t-h)^2(2h-t)^2$	$k_3 = \frac{-4a^5\mathrm{e}^{a(5h-2t)}(h-t)^2(2h-t)^2}{ah(3+3\mathrm{e}^{2ah}+ah(1-\mathrm{e}^{2ah}))+3-3\mathrm{e}^{2ah}}$	$R_3 = (t-h)^2(2h-t)^2$

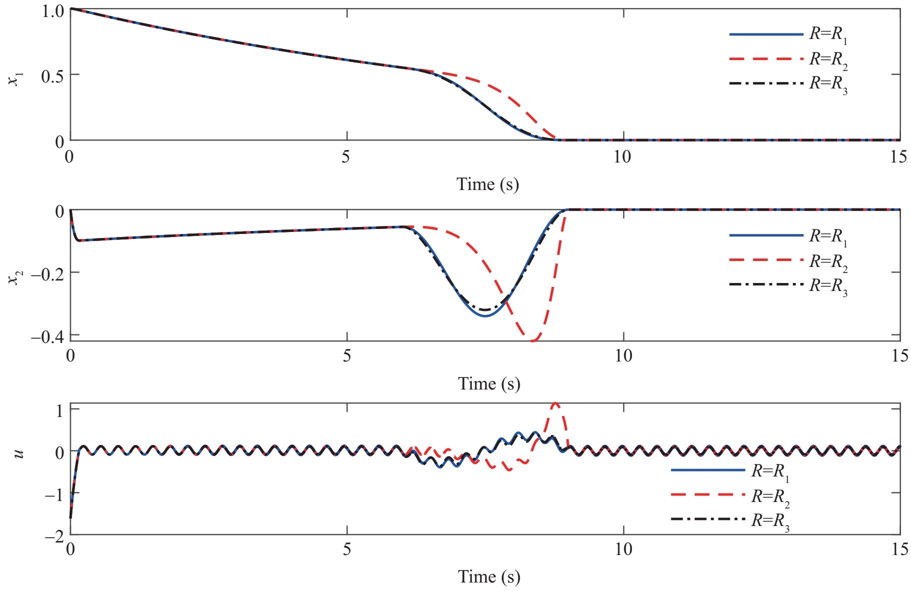

Figure 4 (Color online) States and control of the second-order system (25).

Remark 5. The periodic functions $K_{(a,h)}(t)$ and $R_h(t)$ can be chosen as (we only consider $t \in [0, 2h]$)

$$K_{(a,h)}(t) = \begin{cases} 0, & t \in [0,h), \\ k(t), & t \in [h,2h], \end{cases} \quad R_h(t) = \begin{cases} 0, & t \in [0,h), \\ R(t), & t \in [h,2h]. \end{cases}$$

Parameters in the function $R(t) \in \boldsymbol{S}^{(1)}$ are collected in Table 2, where $\sigma_1 = 4a^4 - 20a^3 + 35a^2 - 25a + 6$, $\sigma_2 = 2\mathrm{e}^{2h} - 4\mathrm{e}^{h} - 2\mathrm{e}^{2h}\mathrm{e}^{-2ah} + 4\mathrm{e}^{3h}\mathrm{e}^{-2ah} - 2\mathrm{e}^{4h}\mathrm{e}^{-2ah} + 2$, and $\sigma_3 = 7\mathrm{e}^{2h} - 8\mathrm{e}^{h} - 7\mathrm{e}^{2h}\mathrm{e}^{-2ah} + 8\mathrm{e}^{3h}\mathrm{e}^{-2ah} - \mathrm{e}^{4h}\mathrm{e}^{-2ah} + 1$. In addition to those listed in Table 2, there are many other types of $\boldsymbol{S}^{(r)}$ functions, for example, $R_4 = (\sqrt{(t-h)(2h-t)+1} - 1)^{r+1}$, $R_5 = (t-h)^{r+1}(2h-t)^{r+1}$, $R_6 = (\ln((t-h)(2h-t)+1))^{r+1}$, and $R_7 = (\mathrm{e}^{t-h} - 1)^{r+1}(1-\mathrm{e}^{t-2h})^{r+1}$. Furthermore, we can increase the power of $R(t)$ to increase its smoothness, such as $R_3 = (t-h)^2(2h-t)^2 \in \boldsymbol{S}^{(1)}$, and $R_5 = (t-h)^{r+1}(2h-t)^{r+1} \in \boldsymbol{S}^{(r)}$.

Example 1. Consider the following second-order system [8]:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = 0.1\sin(20t) + u. \end{cases} \tag{25}$$

According to Theorem 2, we can apply the control law (21) to system (25) with $a = -0.1$, $T = 9$ s, $h = 3$ s, and $R(t)$ being selected as $R_1(t)$, $R_2(t)$, and $R_3(t)$ listed in Table 2, respectively. Let $x_0 = [1, 0]^{\mathrm{T}}$ and $\phi_3(s) = \phi_4(s) = 0, s \in [-h, 0]$. The simulation results are shown in Figure 4, from which we can see that the state of the closed-loop system converges to zero at the prescribed time $T = 9$ s. We notice that the control performance of the closed-loop system with $R = R_1$ and $R = R_3$ is better than that with $R = R_2$.

3.3 Hierarchical SPD sliding mode control

Different from Subsection 3.1, this subsection proposes a hierarchical SPD sliding mode control method, which generalizes the method in Subsection 3.2 to n-order nonlinear systems in the normal form with $n \geqslant 2$.

Consider the following hierarchical SPD sliding mode surfaces:

$$\begin{aligned}
&s_0(t) = x_1(t),\\
&s_1(t) = \dot{s}_0(t) - a_0 s_0(t) + K_0(t) s_0\left(t-h\right),\\
&s_2(t) = \dot{s}_1(t) - a_1 s_1(t) + K_1(t) s_1\left(t-h\right),\\
&\vdots\\
&s_{n-2}(t) = \dot{s}_{n-3}(t) - a_{n-3} s_{n-3}(t) + K_{n-3}(t) s_{n-3}(t-h),\\
&s_{n-1}(t) = \dot{s}_{n-2}(t) - a_{n-2} s_{n-2}(t) + K_{n-2}(t) s_{n-2}(t-h),
\end{aligned} \tag{26}$$

where $s_k(\theta) = \psi_k(\theta), \theta \in [-h, 0], k = 0, 1, \ldots, n-2$ are some arbitrary bounded functions, $a_i \leqslant 0$ are some constants, and

$$K_i(t) = \begin{cases} 0, & t \leqslant (n-1)\,h,\\ K_{(a_i,h)}(t-(n-1)\,h), & t > (n-1)\,h, \end{cases} \tag{27}$$

in which $K_{(a_i,h)}(t)$ is determined by (8) with $b = 1$,

$$h = \frac{T}{3\,(n-1)}, \tag{28}$$

and $R_i \in \boldsymbol{S}^{(r_i)}, r_i \geqslant n-(i+1), i = 0, 1, \ldots, n-2$. Using the fact that $K_i^{(n-k-1)}(t) = 0, t \in [0, (n-1)h], k = n-1, n-2, \ldots, i, i = 0, 1, \ldots, n-2$, we have, for all $t \geqslant 0$,

$$\begin{aligned}
&s_0^{(k)}(t) = \begin{cases} x_1^{(k)}(t) = x_{k+1}(t), & k = 1, 2, \ldots, n-1,\\ f(t,x) + g(t,x)u + \delta(t,x), & k = n, \end{cases}\\
&s_1^{(k)}(t) = s_0^{(k+1)}(t) + \frac{\mathrm{d}^k}{\mathrm{d}t^k}(K_0(t)s_0\left(t-h\right) - a_0 s_0(t)), \quad k = 1, 2, \ldots, n-1,\\
&\vdots\\
&s_{n-1}^{(k)}(t) = s_{n-2}^{(k+1)}(t) + \frac{\mathrm{d}^k}{\mathrm{d}t^k}(K_{n-2}(t)s_{n-2}(t-h) - a_{n-2}s_{n-2}(t)), \quad k = 1,
\end{aligned} \tag{29}$$

which implies that $s_i(t)$ is $(n-i)$-th differentiable, $i = 0, 1, \ldots, n-1$. We notice that the hierarchical SPD sliding mode surfaces (26) reduce to (19) exactly if $n = 2$.

Our main result in this subsection can then be stated as follows.

Theorem 3. Let Assumption 1 be satisfied, $T > 0$ be a prescribed time and h be given by (28). Choose $\alpha \in (1/2, 1)$, $c > 0$, $d > 0$, and $\beta > 1$ such that τ defined in (5) satisfies

$$\tau = (n-1)\,h = \frac{T}{3}.$$

Consider the sliding mode control law

$$\begin{aligned}
u(t) = g^{-1}(t,x)\Bigg(&-\frac{c}{2^\alpha}\mathrm{sig}^{2\alpha-1}(s_{n-1}(t)) - \frac{d}{2^\beta}\mathrm{sig}^{2\beta-1}(s_{n-1}(t)) - f(t,x)\\
&-\sum_{i=1}^{n-1}\frac{\mathrm{d}^{n-i}}{\mathrm{d}t^{n-i}}(K_{i-1}(t)s_{i-1}\left(t-h\right) - a_{i-1}s_{i-1}(t)) - \rho\left(t,x\right)\mathrm{sgn}(s_{n-1}(t))\Bigg),
\end{aligned} \tag{30}$$

where $s_k(\theta) = \varphi_k(\theta), \theta \in [-h, 0], k = 0, 1, \ldots, n-2$ are some arbitrary bounded functions, and $K_i(t), i = 0, 1, \ldots, n-2$ are defined in (27). Then the state of the closed-loop system consisting of (3) and (30) satisfies $x(t) = 0, \forall t \geqslant T$.

Proof. For clarity, we prove this theorem in $n+1$ steps.

Step 1. The convergence of s_{n-1}. It follows from (29) that, for all $t \geqslant 0$,

$$\dot{s}_{n-1}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(K_{n-2}(t)s_{n-2}(t-h) - a_{n-2}s_{n-2}(t)) + s_{n-2}^{(2)}(t)$$

$$
\begin{aligned}
&=\frac{\mathrm{d}^2}{\mathrm{d}t^2}(K_{n-3}(t)s_{n-3}(t-h)-a_{n-3}s_{n-3}(t))+\frac{\mathrm{d}}{\mathrm{d}t}(K_{n-2}(t)s_{n-2}(t-h)-a_{n-2}s_{n-2}(t))+s_{n-3}^{(3)}(t)\\
&=\cdots\\
&=\sum_{i=1}^{n-1}\frac{\mathrm{d}^{n-i}}{\mathrm{d}t^{n-i}}(K_{i-1}(t)s_{i-1}\left(t-h\right)-a_{i-1}s_{i-1}(t))+s_0^{(n)}(t).
\end{aligned}
$$

The time-derivative of the Lyapunov function $V_{n-1}\left(s_{n-1}\right)=\frac{1}{2}s_{n-1}^2$ along (3) and (30) can be written as

$$
\begin{aligned}
\dot{V}_{n-1}\left(s_{n-1}\right)=&s_{n-1}\dot{s}_{n-1}\\
=&s_{n-1}\left(f(t,x)+g(t,x)u+\delta(t,x)+\sum_{i=1}^{n-1}\frac{\mathrm{d}^{n-i}}{\mathrm{d}t^{n-i}}(K_{i-1}(t)s_{i-1}\left(t-h\right)-a_{i-1}s_{i-1}(t))\right)\\
=&-\frac{c}{2^\alpha}|s_{n-1}(t)|^{2\alpha}-\frac{d}{2^\beta}|s_{n-1}(t)|^{2\beta}+s_{n-1}(t)\delta(t,x)-\rho\left(t,x\right)|s_{n-1}(t)|\\
\leqslant&-cV^\alpha\left(s_{n-1}\right)-dV^\beta\left(s_{n-1}\right),\quad\forall t\in[0,\infty).
\end{aligned}
$$

Using Lemma 1, we know that $s_{n-1}=0,\forall t\in[\tau,\infty)=[(n-1)h,\infty)$.

Step 2. The convergence of s_{n-2}. It follows from $s_{n-1}(t)=0$ and (27) that, for all $t\in[(n-1)h,\infty)$,

$$
\dot{s}_{n-2}(t)=a_{n-2}s_{n-2}(t)-K_{(a_{n-2},h)}(t-(n-1)h)s_{n-2}(t-h). \tag{31}
$$

Similarly to (22), the above system is also in the form of (10). Thus we have $s_{n-2}(t)=0,\forall t\in[\tau{+}2h,\infty)=[(n{+}1)h,\infty)$. Besides, it is observed from (8) that $K_{(a_{n-2},h)}(t-(n-1)\,h)=0,\forall t\in[(n+1)\,h,(n+2)\,h]$. For $t\geqslant(n+2)\,h$, we have $s_{n-2}(t-h)=0$, which, in view of (31), implies that $\dot{s}_{n-2}(t)=0,\forall t\in[(n+1)\,h,\infty)$.

Step 3. The convergence of s_{n-3}. It follows from $s_{n-2}(t)=\dot{s}_{n-2}(t)=0$ that, for all $t\in[(n+1)\,h,\infty)$,

$$
\begin{aligned}
\dot{s}_{n-3}(t)&=a_{n-3}s_{n-3}(t)-K_{(a_{n-3},h)}(t-(n-1)\,h)s_{n-3}(t-h)\\
&=a_{n-3}s_{n-3}(t)-K_{(a_{n-3},h)}(t-(n+1)\,h)s_{n-3}\left(t-h\right),
\end{aligned}\tag{32}
$$

which, by taking time-derivatives on both sides, implies

$$
\ddot{s}_{n-3}(t)=a_{n-3}\dot{s}_{n-3}(t)-K_{(a_{n-3},h)}(t-(n+1)\,h)\dot{s}_{n-3}(t-h)-\dot{K}_{(a_{n-3},h)}(t-(n+1)\,h)s_{n-3}(t-h). \tag{33}
$$

Similarly to (31), system (32) is also in the form of (10). Thus we have $s_{n-3}(t)=0,\forall t\in[(n+3)\,h,\infty)$. Besides, it is observed from (8) and $R_{n-3}\in\boldsymbol{S}^{(r_{n-3})},r_{n-3}\geqslant2$ that, for all $t\in[(n+3)\,h,(n+4)\,h]$,

$$
K_{(a_{n-3},h)}(t-(n+1)\,h)=0,\quad\dot{K}_{(a_{n-3},h)}(t-(n+1)\,h)=0. \tag{34}
$$

For $t\geqslant(n+4)\,h$, we have $s_{n-3}(t-h)=0$, which, in view of (32) and (34), implies that $\dot{s}_{n-3}(t)=0,\forall t\in[(n+3)\,h,\infty)$. Similarly, for $t\geqslant(n+4)\,h$, we have $s_{n-3}(t-h)=\dot{s}_{n-3}(t-h)=0$, which, in view of (33) and (34), implies that $\ddot{s}_{n-3}(t)=0,\forall t\in[(n+3)\,h,\infty)$.

Step $i,i=4,5,\ldots,n$. The convergence of s_{n-i}. It follows from $s_{n-i+1}(t)=\dot{s}_{n-i+1}(t)=\cdots=s_{n-i+1}^{(i-2)}(t)=0$ that, for all $t\in[(2i+n-5)\,h,\infty)$,

$$
\begin{aligned}
\dot{s}_{n-i}(t)&=a_{n-i}s_{n-i}(t)-K_{(a_{n-i},h)}(t-(n-1)\,h)s_{n-i}(t-h),\\
&=a_{n-i}s_{n-i}(t)-K_{(a_{n-i},h)}(t-(2i+n-5)\,h)s_{n-i}(t-h),\\
\ddot{s}_{n-i}(t)&=a_{n-i}\dot{s}_{n-i}(t)-\frac{\mathrm{d}}{\mathrm{d}t}(K_{(a_{n-i},h)}(t-(2i+n-5)\,h)s_{n-i}(t-h)),\\
&\vdots\\
s_{n-i}^{(i-1)}(t)&=a_{n-i}s_{n-i}^{(i-2)}(t)-\frac{\mathrm{d}^{(i-2)}}{\mathrm{d}t^{(i-2)}}(K_{(a_{n-i},h)}(t-(2i+n-5)\,h)s_{n-i}(t-h)).
\end{aligned}
$$

Similarly to Steps 2 and 3, we can show that $s_{n-i}(t)=\dot{s}_{n-i}(t)=\cdots=s_{n-i}^{(i-1)}(t)=0,\forall t\in[(2i+n-3)\,h,\infty)$.

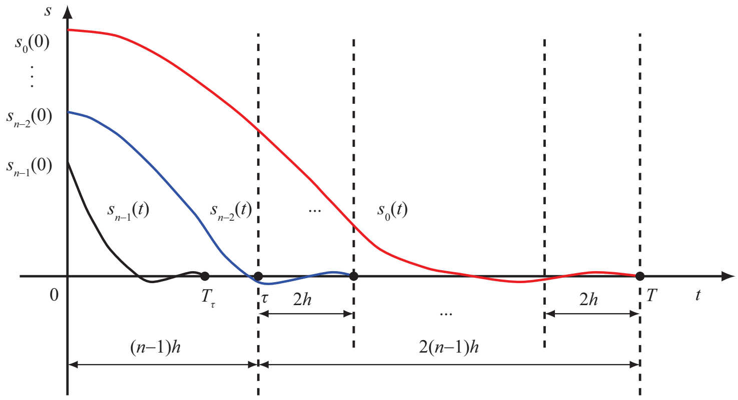

Figure 5 (Color online) Convergence of the hierarchical sliding mode.

Step $n+1$**.** The convergence of $x(t)$. Notice that $x_1(t) = s_0(t)$ and $x_i(t) = x_1^{(i-1)}(t) = s_0^{(i-1)}(t), i = 2, 3, \ldots, n$. Thus it follows from $s_0(t) = \dot{s}_0(t) = \cdots = s_0^{(n-1)}(t) = 0, \forall t \in [(3n-3)h, \infty) = [T, \infty)$ that $x_i(t) = 0, i = 1, 2, \ldots, n, \forall t \in [T, \infty)$, that is, the state of the closed-loop system consisting of (3) and (30) satisfies $x(t) = 0, \forall t \geqslant T$.

Remark 6. Different from traditional hierarchical sliding mode surfaces such as $s_i = \dot{s}_{i-1} + cs_{i-1}^{\alpha}$, $s_0 = x_1$, $i = 1, 2, \ldots, n-1$, where singularity phenomenon may appear, the SPD sliding mode surfaces proposed in this paper are essentially linear time-varying sliding mode surfaces with respect to $x(t)$, which can avoid the singularity phenomenon by choosing suitable parameters.

Remark 7. We point out that we can also use other types of scalar periodic delay systems to design hierarchical sliding mode surfaces, for example (see Section V in [27] for detail),

$$\dot{x}(t) = ax(t) + b_0u(t) + b_1u(t-\tau).$$

4 Applications to the hypersonic vehicle system

Consider the longitudinal model of the hypersonic vehicle [35]

$$\begin{cases} \dot{V} &= \frac{F\cos\alpha - D}{m} - \frac{\mu\sin\gamma}{r^2}, \\ \dot{\gamma} &= \frac{L + F\sin\alpha}{mV} - \frac{(\mu - V^2r)\cos\gamma}{Vr^2}, \\ \dot{H} &= V\sin\gamma, \\ \dot{\alpha} &= q - \dot{\gamma}, \\ \dot{q} &= \frac{M_{yy}}{I_{yy}}, \end{cases} \tag{35}$$

where V, γ, H, α, q are respectively the velocity, flight path angle, altitude, angle of attack, and pitch rate, m and I_{yy} are the mass and moment of inertia of the hypersonic vehicle, μ is the gravitational constant of the earth, and L, D, F, M_{yy} are, respectively, the lift, drag, thrust, and pitching moment received by the aircraft, shown in Appendix A. The control object is to make the altitude H and velocity V track the reference command signals in prescribed time $T = 60$ s. The reference velocity is $V_{\rm d} = 15160$ ft/s and the reference altitude is $H_{\rm d} = 112000$ ft.

Letting the states $z = [z_1, z_2, z_3]^{\rm T}$ with $z_i = V^{(i-1)} - V_{\rm d}^{(i-1)}, i = 1, 2, 3$, and $y = [y_1, y_2, y_3, y_4]^{\rm T}$ with $y_j = H^{(j-1)} - H_{\rm d}^{(j-1)}, j = 1, 2, 3, 4$, system (35) can be equivalently written as

$$\begin{cases} \dot{z} = A_3z + b_3(f_V - V_{\rm d}^{(3)} + v_1 + d_1), \\ \dot{y} = A_4y + b_4(f_H - H_{\rm d}^{(4)} + v_2 + d_2), \end{cases} \tag{36}$$

where f_V and f_H are some known functions (see f_V and f_H defined in [35]), $b_3 = [0, 0, 1]^{\mathrm{T}}$, $b_4 = [0, 0, 0, 1]^{\mathrm{T}}$,

$$A_3 = \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix},$$

and $v = [v_1, v_2]^{\mathrm{T}} = B\left[\beta_{\mathrm{c}}, \delta_{\mathrm{E}}\right]^{\mathrm{T}}$ is the control input, $[d_1, d_2]^{\mathrm{T}} = B[d_{\mathrm{s}1}, d_{\mathrm{s}2}]^{\mathrm{T}}$ in which $B = [b_{ij}]_{2\times 2}$ is defined in Appendix A, β_{c} is defined in (A1) in Appendix A, and δ_{E} is the elevator deflection angle. We notice that these two subsystems in (36) are exactly in the form of (3).

According to Theorem 1, the controllers can be designed as

$$v_1 = -\frac{c}{2^{\alpha}}\mathrm{sig}^{2\alpha-1}(s_V(t)) - \frac{d}{2^{\beta}}\mathrm{sig}^{2\beta-1}(s_V(t)) - f_V + V_{\mathrm{d}}^{(3)} - b_3^{\mathrm{T}} P_V z(t) - \rho_V \mathrm{sgn}\left(s_V(t)\right) - K_V(t) z(t-h), \tag{37}$$

$$v_2 = -\frac{c}{2^{\alpha}}\mathrm{sig}^{2\alpha-1}(s_H(t)) - \frac{d}{2^{\beta}}\mathrm{sig}^{2\beta-1}(s_H(t)) - f_H + H_{\mathrm{d}}^{(4)} - b_4^{\mathrm{T}} P_H y(t) - \rho_H \mathrm{sgn}\left(s_H(t)\right) - K_H(t) y(t-h), \tag{38}$$

where $\rho_V = |b_{11}|\rho_1 + |b_{12}|\rho_2$, $\rho_H = |b_{21}|\rho_1 + |b_{22}|\rho_2$ in which $\rho_i = \sup_{t\geqslant 0}\{|d_{si}(t)|\}, i = 1, 2$, (s_V, s_H) are defined according to (12), P_V and P_H are the solutions to the following PLEs:

$$\begin{aligned} A_3^{\mathrm{T}} P_V + P_V A_3 - P_V b_3 b_3^{\mathrm{T}} P_V &= -\gamma_V P_V, \\ A_4^{\mathrm{T}} P_H + P_H A_4 - P_H b_4 b_4^{\mathrm{T}} P_H &= -\gamma_H P_H, \end{aligned}$$

$A_{3\mathrm{c}} = A_3 - b_3 b_3^{\mathrm{T}} P_V$, $A_{4\mathrm{c}} = A_4 - b_4 b_4^{\mathrm{T}} P_H$, and $K_V(t)$ and $K_H(t)$ are constructed according to (13) by choosing appropriate $R_h(t) \in \boldsymbol{S}^{(1)}(h)$. To reduce control chattering, we replace $\mathrm{sgn}(s_V)$ by $\mathrm{sat}(s_V/\phi_V)$ and $\mathrm{sgn}(s_H)$ by $\mathrm{sat}(s_H/\phi_H)$ in control law (37) and (38), where $\mathrm{sat}(x) = x$ if $|x| \leqslant 1$ and $\mathrm{sat}(x) = \mathrm{sgn}(x)$ otherwise.

According to Theorem 3, the controllers can also be designed as

$$\begin{aligned} v_1 = &-\frac{c_1}{2^{\alpha_1}}\mathrm{sig}^{2\alpha_1-1}(s_{V2}(t)) - \frac{d_1}{2^{\beta_1}}\mathrm{sig}^{2\beta_1-1}(s_{V2}(t)) - \sum_{i=1}^{2}\frac{\mathrm{d}^{3-i}}{\mathrm{d}t^{3-i}}(K_{V(i-1)}(t) s_{V(i-1)}(t-h_V) \\ &- f_V + V_{\mathrm{d}}^{(3)} - a_{V(i-1)} s_{V(i-1)}(t)) - \rho_V \mathrm{sgn}\left(s_{V2}(t)\right), \end{aligned} \tag{39}$$

$$\begin{aligned} v_2 = &-\frac{c_2}{2^{\alpha_2}}\mathrm{sig}^{2\alpha_2-1}(s_{H3}(t)) - \frac{d_2}{2^{\beta_2}}\mathrm{sig}^{2\beta_2-1}(s_{H3}(t)) - \sum_{i=1}^{3}\frac{\mathrm{d}^{4-i}}{\mathrm{d}t^{4-i}}(K_{H(i-1)}(t) s_{H(i-1)}(t-h_H) \\ &- f_H + H_{\mathrm{d}}^{(4)} - a_{H(i-1)} s_{H(i-1)}(t)) - \rho_H \mathrm{sgn}\left(s_{H3}(t)\right), \end{aligned} \tag{40}$$

where $s_{V(i-1)}$, $i = 1, 2, 3$ and $s_{H(i-1)}$, $i = 1, 2, 3, 4$ are defined according to (26), $K_{V(i-1)}(t), i = 1, 2, 3$ and $K_{H(i-1)}(t), i = 1, 2, 3, 4$ are constructed according to (8) by choosing appropriate $R_{Vh}(t) \in \boldsymbol{S}^{(2)}(h)$ and $R_{Hh}(t) \in \boldsymbol{S}^{(3)}(h)$. Similarly, we replace $\mathrm{sgn}(s_{V2})$ by $\mathrm{sat}(s_{V2}/\phi_{V2})$ and $\mathrm{sgn}(s_{H3})$ by $\mathrm{sat}(s_{H3}/\phi_{H3})$ in the control laws (39) and (40).

The external disturbances suffered by the hypersonic vehicle are assumed to be [36]

$$d_{\mathrm{s}1}(t) = 0.0024\sin\left(0.2t\right), \quad d_{\mathrm{s}2}(t) = 0.012\sin\left(0.2t\right).$$

The initial conditions of the hypersonic vehicle are chosen as [36,37] $V(0) = 15060$ ft/s, $H(0) = 110000$ ft, $\gamma(0) = 0$ rad, $\alpha(0) = 0.0334$ rad, $\beta(0) = 0.1802$, and $q(0) = 0$ rad/s. The parameters in control laws (37) and (38) are selected as $\gamma_V = \gamma_H = 0.2$, $\alpha = 0.9$, $\beta = 1.2$, $c = 1$, $d = 0.5$, $h = 20$, $\rho_1 = 0.005$, $\rho_2 = 0.02$, $\phi_V = 0.01$, $\phi_H = 0.1$, and (we only consider $t \in [0, 2h]$)

$$R_h(t) = \begin{cases} 0, & t \in [0, h], \\ \sin^2\left(\frac{\pi t}{h}\right), & t \in [h, 2h]. \end{cases}$$

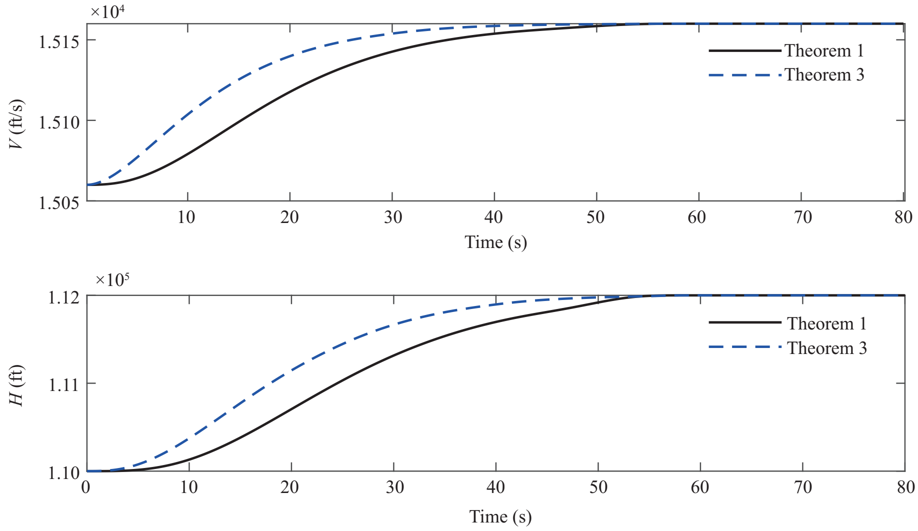

Figure 6 (Color online) Outputs of the hypersonic vehicle.

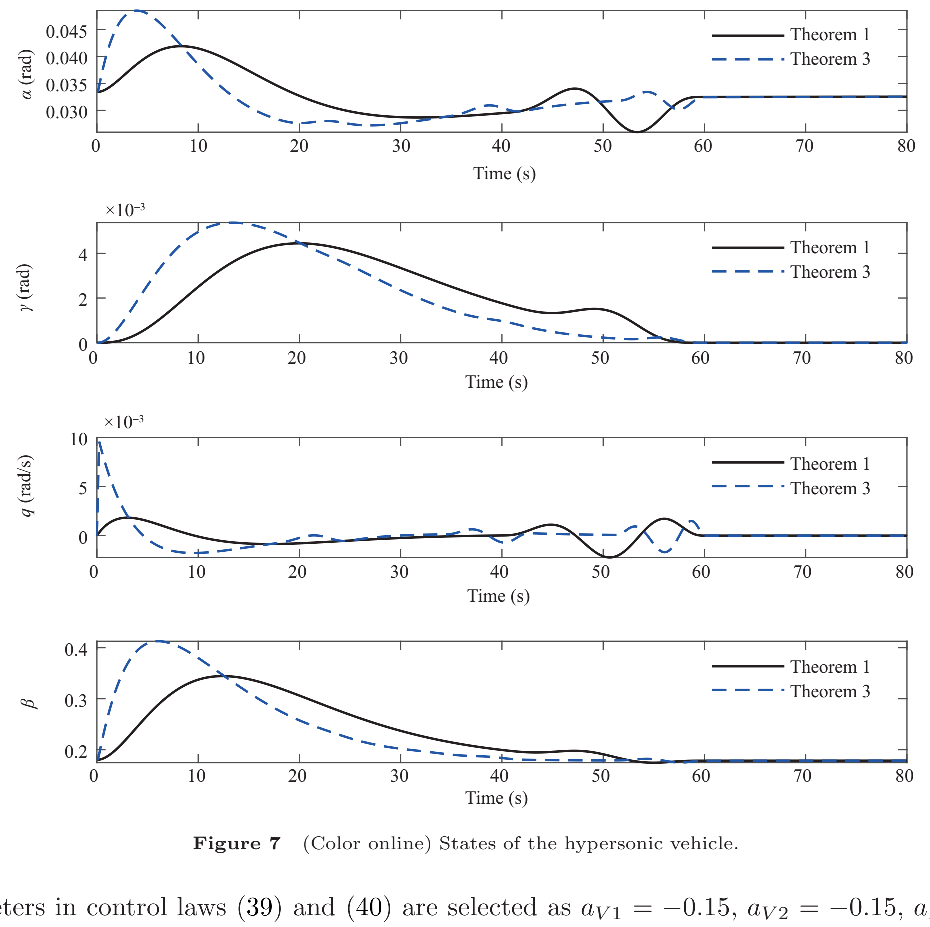

Figure 7 (Color online) States of the hypersonic vehicle.

The parameters in control laws (39) and (40) are selected as $a_{V1} = -0.15$, $a_{V2} = -0.15$, $a_{H1} = a_{H2} = a_{H3} = -0.2$, $\alpha_1 = \alpha_2 = 0.9$, $\beta_1 = \beta_2 = 1.2$, $c_1 = c_2 = 1$, $d_1 = d_2 = 0.5$, $h_H = 20/3$, $h_V = 10$, $\rho_1 = 0.005$, $\rho_2 = 0.02$, $\phi_{V2} = 0.01$, $\phi_{H3} = 0.1$, and (we also only consider $t \in [0, 2h]$)

$$R_{Vh}(t) = \begin{cases} 0, & t \in [0, h], \\ (t-h)^3 (2h-t)^3, & t \in [h, 2h], \end{cases}$$

$$R_{Hh}(t) = \begin{cases} 0, & t \in [0, h], \\ (t-h)^4 (2h-t)^4, & t \in [h, 2h]. \end{cases}$$

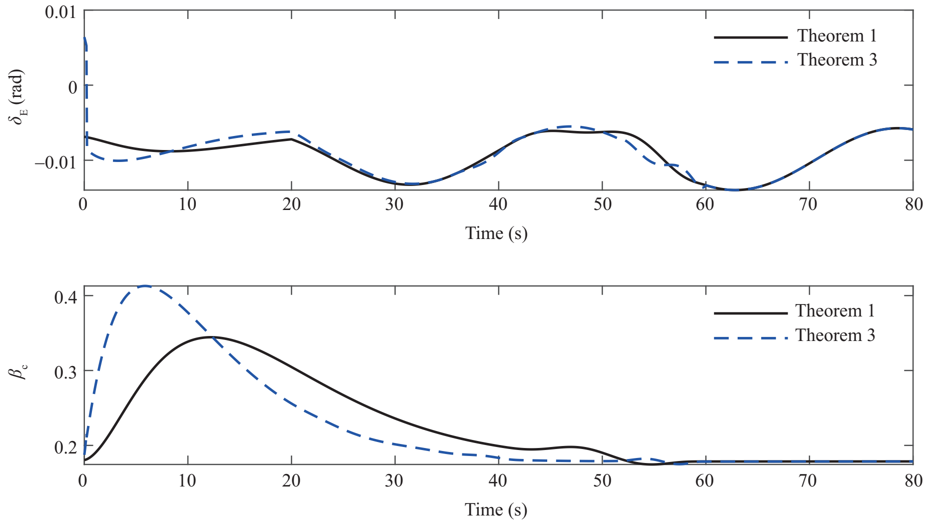

Figure 8 (Color online) Inputs for the hypersonic vehicle.

The simulation results are shown in Figures 6–8. It can be observed that the actual velocity and altitude track the reference signals at the prescribed time $T = 60$ s, while $\alpha, \gamma, q, \beta, \beta_c$, and δ_E are all within the allowable range. In addition, roughly speaking, we find that the control performance associated with Theorem 1 is better than that of Theorem 3.

5 Conclusion

In this paper novel prescribed-time time-varying sliding mode surfaces were designed based on smooth periodic delayed feedback. Compared with the traditional power functions-based nonsingular finite-time/fixed-time sliding mode surfaces, which might only be applicable to second-order nonlinear systems, the nonsingular sliding mode surfaces proposed in this paper can not only achieve prescribed-time convergence, but also can be applied to second-order nonlinear systems as well as high-order nonlinear systems. It was shown that the proposed sliding mode control can achieve the prescribed-time convergence. As an application of the proposed sliding mode control approaches, the controller design of hypersonic vehicle systems was revisited. Numerical simulations verified the effectiveness of the proposed methods.

Acknowledgements This work was supported in part by National Natural Science Foundation of China for Distinguished Young Scholars (Grant No. 62125303), Science Center Program of National Natural Science Foundation of China (Grant No. 62188101), and Fundamental Research Funds for the Central Universities (Grant No. HIT.BRET.2021008).

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Appendix A Longitudinal model of hypersonic vehicle

The detailed expressions of L, D, F, M_{yy} are given as follows:

$$\begin{aligned}
&L = \frac{1}{2}\rho V^2 S C_L, \quad D = \frac{1}{2}\rho V^2 S C_D, \quad F = \frac{1}{2}\rho V^2 S C_T,\\
&M_{yy} = \frac{1}{2}\rho V^2 S\bar{c}\left(C_M(\alpha) + C_M(\delta_{\rm E}) + C_M(q)\right),\\
&r = H + R_E, \quad C_L = 0.6203\alpha,\\
&C_D = 0.6450\alpha^2 + 0.0043378\alpha + 0.003772,\\
&C_T = \begin{cases} 0.02576\beta, & \beta < 1,\\ 0.0224 + 0.00336\beta, & \beta > 1,\end{cases}\\
&C_M(\alpha) = -0.035\alpha^2 + 0.036617\alpha + 5.3261\times 10^{-6},\\
&C_M(q) = \frac{\bar{c}}{2V} q\left(-6.796\alpha^2 + 0.3015\alpha - 0.2289\right),\\
&C_M(\delta_{\rm E}) = c_{\rm e}\left(\delta_{\rm E} + d_{\rm s2}(t) - \alpha\right),
\end{aligned}$$

where R_E is the radius of the earth, S is the reference area, β and $\delta_{\rm E}$ are respectively the throttle setting and the elevator deflection angle, ρ is the density of air, and the engine dynamics can be described as a second-order system [35]

$$\ddot{\beta} = -2\xi\omega_n\dot{\beta} - \omega_n^2\beta + \omega_n^2\left(\beta_{\rm c} + d_{\rm s1}(t)\right), \tag{A1}$$

where $d_{\rm s1}(t)$ and $d_{\rm s2}(t)$ are external disturbance, and $\bar{c}$ and $c_{\rm e}$ are some constants. Besides, according to [35], the detailed expressions of $B = [b_{ij}]_{2\times 2}$ are

$$\begin{aligned}
&b_{11} = \frac{\rho V^2 S c_\beta \omega_n^2 \cos\alpha}{2m}, \quad b_{12} = -\frac{c_e\bar{c}\rho V^2 S}{2mI_{yy}}\left(F\sin\alpha + \frac{\partial D}{\partial\alpha}\right),\\
&b_{21} = \frac{\rho V^2 S c_\beta \omega_n^2 \sin(\alpha+\gamma)}{2m}, \quad b_{22} = \frac{c_e\bar{c}\rho V^2 S}{2mI_{yy}}\left(F\cos(\alpha+\beta) + \frac{\partial L}{\partial\alpha}\cos\gamma - \frac{\partial D}{\partial\alpha}\sin\gamma\right),
\end{aligned}$$

in which

$$c_\beta = \begin{cases} 0.02576, & \beta < 1,\\ 0.00336, & \beta > 1.\end{cases}$$