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On equivalence of state-based potential games

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Abstract In this paper, we explore state-based potential games using the semi-tensor product of matrices. First, applying the potential equation, we derive both a necessary and sufficient condition as well as a sufficient condition to verify whether a state-based game qualifies as a potential game. Next, we present two static equivalence conditions of state-based potential games. We then delve into dynamic equivalence. We propose a criterion that allows us to identify state-based games that are dynamically equivalent to state-based potential games and share similar dynamic properties. Ultimately, we introduce the concept of state-based networked evolutionary games. We provide a necessary and sufficient condition to ensure that a state-based networked evolutionary game can be classified as a state-based potential game.

Keywords semi-tensor product of matrices, state-based potential game, verification, equivalence, state-based networked evolutionary game

1 Introduction

John von Neumann was the pioneering force behind game theory, a discipline he developed to study the behavior within economic societies [1]. Since its inception, game theory has expanded rapidly and is now widely applied across various communities, including biology [2, 3], economics [4], and engineering [5,6]. The concept of potential games was initially introduced by Rosenthal [7] as "congestion game". Then, Monderer and Shapley [8] later coined the term "potential games" in 1996. One of the defining characteristics of potential games is that each one possesses at least one pure Nash equilibrium (NE). Moreover, these games will converge to NE under certain strategy updating rules (SURs). These unique properties have made potential games useful in addressing various practical problems, including resource allocation [9], wireless sensor networks [10] and traffic congestion [11].

Verifying potential games has historically posed a significant challenge. Using the semi-tensor product (STP) of matrices, Ref. [12] proposed the potential equation to verify potential games. Using the theory of vector space, Qi et al. [13] provided the static equivalence of finite games. Two static equivalences are used to divide finite games into equivalence classes. The concept of networked evolutionary games was explored using the STP method [14]. This research led to the discovery that if the fundamental network game is potential, then the network evolutionary game is also potential [12]. Building on these findings, the dynamic equivalence of finite games was introduced, and the properties of near-potential games were studied [15]. Ref. [16] proposed the concept of quasi-potential games and provided an algorithm to ensure the convergence of the quasi-potential game to the NE.

A shortcoming of this field is that potential games only form a subspace of finite games [15]. This restriction means that many practical problems cannot be modeled as potential games. To address this issue, Marden proposed the concept of state-based games. A special type of state-based game, known as the state-based potential game (SPG), has at least one recurrent state equilibrium (RSE) and will converge to RSE under certain SUR [17]. Li et al. [18] proposed two memory better reply learning rule and demonstrated that the learning rule guarantees the convergence of state-based games with RSE.

SPGs serve as an extension of potential games and have found extensive application in solving numerous practical problems. In one instance, Ref. [19] and his team modeled the multiple-association problem

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as an SPG, enabling ultradense networks to achieve global optimization through a specifically designed association policy. Using game theory, the optimal control problem under time-varying topology is modeled as an SPG [20]. In addition, Liang et al. [21] employed ordinal SPGs to address the demand-side energy management problem within smart grids. In the biological field, models related to state-based games have also emerged. An evolutionary dynamic model featuring game transitions was considered by [22]. This model viewed the switch between games as a transition between states and the evolution of cooperation in structured populations. An environmental feedback model, which incorporates state-dependent strategies, was developed to explore the interplay between the environment and individual behavior. This model was subsequently analyzed using both stochastic game theory and evolutionary dynamics theory [23].

The STP of matrices serves as a crucial tool in our analysis. It is a generalization of the traditional matrix product [24] and has been applied various contexts, including Boolean networks [25–28], finite automaton [29], and game theory [12, 18, 30]. In this article, we employ STP to obtain the algebraic representation of state-based games and provide verifiable algebraic conditions that allow a state-based game to be classified as an SPG.

Researching SPGs presents a unique set of challenges owing to the introduction of state space. To the best of our knowledge, no effective method currently exists to verify whether a state-based game qualifies as an SPG. This gap in understanding underscores the importance of investigating the algebraic verification of SPGs. While it is understood that all SPGs only constitute a subspace of state-based games, their practical significance in engineering cannot be overlooked. This fact drives our interest in studying the static and dynamic equivalence of SPGs. By doing so, we can identify state-based games equivalent to SPGs, which share the same properties, thereby expanding the applicability of SPGs. Given the widespread use of networked evolutionary games in practice, we also aim to delve into the nature of SPGs within these networks. With this in mind, this paper offers three key contributions. (i) Using a potential equation, we provide an algebraic inequality to verify whether a state-based game is an SPG. (ii) Using STP, we offer conditions for the static and dynamic equivalence conditions of SPGs. (iii) We introduce the concept of state-based networked evolutionary games (SNEGs) and provide a sufficient condition to guarantee an SNEG is potential. Specifically, if the fundamental state-based network game (FSNG) is potential, then the SNEG is also potential, and vice versa under a mild condition. By applying STP, we present verifiable algebraic conditions for the verification of SPGs, static/dynamic equivalence of SPGs, and determination of SNEGs. The verification of these conditions requires only the identification of the corresponding structure vectors and matrices, simplifying the process significantly.

This paper is structured as follows: Section 2 lays out the necessary preliminaries and problem formulation. In Section 3, we present our major results. Section 4 provides an illustrative example of the theoretical results, while Section 5 concludes the paper.

2 Preliminaries

2.1 Notations

We first give some notations.

(1) $\mathbf{1}_n = (\underbrace{1, 1, \ldots, 1}_{n})^{\mathrm{T}}$.

(2) $\mathrm{Col}(M)$ is the set of columns of M, and $\mathrm{Col}_i(M)$ is the i-th column of M.

(3) $\mathcal{D}_k := \{1, 2, \ldots, k\},\ k \geqslant 2$.

(4) $\Delta_n := \{\delta_n^i | i = 1, \ldots, n\}$, where δ_n^i is the i-th column of I_n.

(5) If $\mathrm{Col}(L) \subseteq \Delta_n$, then the matrix L is called a logical matrix. $\mathcal{L}_{m\times n}$ is the set of $m \times n$ logical matrices.

(6) $\Upsilon_k := \{(r_1, r_2, \ldots, r_k) \mid r_i \geqslant 0,\ \sum_{i=1}^{k} r_i = 1\}$ is the set of k-dimensional probabilistic vectors.

(7) $\Upsilon_{m\times n} := \{M \in \mathcal{M}_{m\times n} \mid \mathrm{Col} \subset \Upsilon_m\}$ is the set of $m \times n$ probabilistic matrices.

(8) $|\Omega|$ is the number of elements in the set Ω.

(9) $[k, h]$ is the set of $\{k, k + 1, \ldots, h\}$.

2.2 STP of matrices

Definition 1 ([24]). Let $M \in \mathcal{M}_{m\times n}$, $N \in \mathcal{M}_{p\times q}$. The STP of M and N is defined as

$$M \ltimes N := (M \otimes I_{s/n})(N \otimes I_{s/p}) \in \mathcal{M}_{ms/n\times qs/p},$$

where s is the least common multiple of n and p.

Remark 1. STP is used as the default matrix product in this paper. More details on STP are included in [24].

Lemma 1 ([24]). Let $D_f^{[p,q]} = \mathbf{1}_p^{\mathrm{T}} \otimes I_q$, $D_r^{[p,q]} = I_p \otimes \mathbf{1}_q^{\mathrm{T}}$, $M \in \Delta_p, N \in \Delta_q$, then

$$D_f^{[p,q]}MN = N,\ D_r^{[p,q]}MN = M,$$

where D_f and D_r are called dummy matrices.

Lemma 2 ([24]). Let $f : \mathcal{D}_k^n \to \mathcal{D}_k$ (or $f : \mathcal{D}_k^n \to \mathbb{R}$) be a k-valued logical (or pseudo-logical) function. Then there exists a unique structure matrix (or structure vector) $M_f \in \mathcal{L}_{k\times k^n}$ (or $M_f \in \mathcal{M}_{1\times k^n}$) satisfying

$$f(a_1, a_2, \ldots, a_n) = M_f \ltimes_{i=1}^n a_i.$$

2.3 Problem formulation

The definition of state-based games can be obtained by introducing state space and state transition functions in the definition of normal finite games.

Definition 2 ([17]). A finite state-based game is a quintuple $G = (N, \mathcal{A}, U, X, P)$, where

(1) $N = \{1, 2, \ldots, n\}$ is the set of players;

(2) $\mathcal{A} := \Pi_{i=1}^n \mathcal{A}_i$ is the action profile set and $\mathcal{A}_i = \{1, 2, \ldots, k_i\}$ is player i's action set;

(3) $U = \{u_1, u_2, \ldots, u_n\}$ and $u_i : \mathcal{A} \times X \to \mathbb{R}$ is player i's payoff function;

(4) $X = \{x_1, x_2, \ldots, x_r\}$ is the finite state set;

(5) $P : \mathcal{A} \times X \to \Delta(X)$ is the Markovian state transition function, where $\Delta(X)$ denotes the set of probability distributions over the finite state space X.

Remark 2. The state-based game can be seen as a simplified form of stochastic games proposed by Shapley [31]. The key difference between stochastic games and state-based games is the calculation of payoff value. For stochastic games, the total reward of each player is calculated by the discounted value of the reward at each stage. Hence, stochastic games focus on the set of stationary strategies, while in state-based games, action state pairs and recurrent state equilibria are more concerned.

For each player $i \in N$ and the strategy $a_i = j \in \mathcal{A}_i$, by identifying $j \sim \delta_{k_i}^j$, a_i can be denoted as the vector form $\delta_{k_i}^j \in \Delta_{k_i},\ j \in [1, k_i]$. Similarly, the state can also be denoted as $x_i = \delta_r^i \in \Delta_r$.

By Lemma 2, the payoff function of player i of normal finite games is

$$u_i(a_1, \ldots, a_n) = V_i^u \ltimes_{l=1}^n a_l,\ i \in N,$$

where V_i^u is the structure vector of u_i.

The vector form of the finite game G is

$$V_G^u = (V_1^u, V_2^u, \ldots, V_n^u) \in \mathbb{R}^{nk}, k = \Pi_{i=1}^n k_i.$$

In [13], the set of games with n players and $|\mathcal{A}_i| = k_i,\ i \in N$, is denoted by

$$\mathcal{G}_{[n;k_1,k_2,\ldots,k_n]} \sim \mathbb{R}^{nk}.$$

In the same way, the player i's payoff function under the state x_j can be expressed as

$$u_i(a_1, \ldots, a_n, x_j) = [V_{1i}^u, \ldots, V_{ji}^u, \ldots, V_{ri}^u] \ltimes x_j \ltimes_{l=1}^n a_l,\ i \in N,\ x_j \in X,$$

where V_{ji}^u is the structure vector of u_i under the state x_j. The structure vector of the state-based game G is

$$V_G^u = (V_{11}^u, \ldots, V_{1n}^u, \ldots, \ldots, V_{r1}^u, \ldots, V_{rn}^c) \in \mathbb{R}^{rnk}. \tag{1}$$

Therefore, the space structure of state-based games with n players, r states and $|\mathcal{A}_i| = k_i,\ i \in N$, is

$$\mathcal{G}_{[n;r;k_1,k_2,\ldots,k_n]} \sim \mathbb{R}^{rnk}.$$

Definition 3 ([17])**.** The action state pair $[a^*, x^*]$ is an RSE regarding the state transition process $P(\cdot)$ if the following two conditions are satisfied:

(1) The state x^* satisfies $x^* \in X(a^* \mid x)$ for every state $x \in X(a^* \mid x^*)$.

(2) For every agent $i \in N$ and every state $x \in X(a^* \mid x^*)$,

$$u_i(a_i^*, a_{-i}^*, x) = \max_{a_i \in \mathcal{A}_i} u_i(a_i, a_{-i}^*, x),$$

where $a^* = (a_1^*, a_2^*, \ldots, a_n^*)$, and $a_{-i}^* = (a_1^*, \ldots, a_{i-1}^*, a_{i+1}^*, \ldots, a_n^*)$.

Definition 4 ([17])**.** A state-based game $G = (N, \mathcal{A}, U, X, P)$ is an SPG if for each $[a, x] \in \mathcal{A} \times X$, there is a potential function $\varphi:\ \mathcal{A} \times X \to \mathbb{R}$ satisfying the following conditions.

(1) For any player $i \in N$ and action $b_i \in \mathcal{A}_i$,

$$u_i(b_i, a_{-i}, x) - u_i(a, x) = \varphi(b_i, a_{-i}, x) - \varphi(a, x), \tag{2}$$

where $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in \mathcal{A}_{-i} := \Pi_{j \neq i} \mathcal{A}_j$.

(2) For any state y in the support of $P(a, x)$,

$$\varphi(a, y) \geqslant \varphi(a, x). \tag{3}$$

Relaxing the requirement in (2) to

$$u_i(b_i, a_{-i}, x) > u_i(a, x) \Rightarrow \varphi(b_i, a_{-i}, x) > \varphi(a, x),$$

the class of ordinal SPG can be similarly defined.

Remark 3. From the definition of SPGs, Eq. (2) reveals that the finite game under any fixed state $x \in X$ is potential. Therefore, each potential game, such as snowdrift game and prisoner's dilemma game, can be viewed as a special SPG with only one state.

3 Main results

3.1 Verification of SPGs

Since SPGs have many good properties, it is very meaningful to study the verification of SPGs. In this subsection, a necessary and sufficient condition and a sufficient condition are constructed, respectively, to verify the SPG.

Similar to Lemma 15 in [12], we draw the following result.

Theorem 1. For a state-based game $G \in \mathcal{G}_{[n;r;k_1,k_2,\ldots,k_n]}$, Eq. (2) is true if and only if for any state x_j and action profile $a = (a_1, a_2, \ldots, a_n)$, there exist $d_{ji} = d_{ji}(a_1, \ldots, \hat{a}_i, \ldots, a_n, x_j)$, where d_{ji} is independent of a_i, such that

$$u_i(a_1, a_2, \ldots, a_n, x_j) = \varphi(a_1, a_2, \ldots, a_n, x_j) + d_{ji}(a_1, \ldots, \hat{a}_i, \ldots, a_n, x_j),\ i \in N. \tag{4}$$

Using STP, Eq. (4) can be expressed in its vector form

$$[V_{1i}^u, \ldots, V_{ri}^u] \ltimes x_j \ltimes_{l=1}^n a_l = [V^{\varphi(\cdot, x_1)}, \ldots, V^{\varphi(\cdot, x_r)}] \ltimes x_j \ltimes_{l=1}^n a_l + [V_{1i}^d, \ldots, V_{ri}^d] \ltimes x_j \ltimes_{l \neq i} a_l, \tag{5}$$

where $V_{ji}^u \in \mathbb{R}^k,\ V^{\varphi(\cdot, x_j)} \in \mathbb{R}^k,\ V_{ji}^d \in \mathbb{R}^{\frac{k}{k_i}}$, $V^{\varphi(\cdot, x_j)}$ is the structure vector of φ under the state x_j.

Using Lemma 1, from (5), we can get

$$V_{ji}^u = V^{\varphi(\cdot, x_j)} + V_{ji}^d M_i,\ i \in N,\ j \in [1, r],$$

where

$$M_i = \begin{cases} D_f^{[k_1, k_2]}, & i = 1, \\ D_r^{[\Pi_{j=1}^{i-1} k_j, k_i]}, & i \neq 1. \end{cases} \tag{6}$$

Theorem 2 ([12]). A finite game is potential if and only if there exists a solution of the potential equation $\psi\xi = b$. Furthermore, the structure vector of the potential function P is

$$V_P = V_1^u - V_1^d M_1 = V_1^u - \xi_1^{\mathrm{T}} D_f^{[k,k]}.$$

Definition 5. Let $M \in \Upsilon_{r\times r}$. Define a matrix $M' = (M'_{ij})_{r\times r}$ as

$$M'_{ij} = \begin{cases} 1, \ M_{ij} > 0, i \neq j, \\ 1, \ M_{ij} = 1, i = j, \\ 0, \ \text{otherwise}. \end{cases}$$

The matrix M' is decomposed into several logical submatrices $M'_1, M'_2, \ldots, M'_\kappa$, where $M'_j, \ j \in [1,\kappa]$ are obtained by keeping only one entry in each column of M' as 1 and the others as 0, that is, $\mathrm{Col}_i(M'_j) \in \Delta_r, \ i \in [1,r], \ j \in [1,\kappa], \ \kappa = \prod_{i=1}^r \kappa_i, \ \kappa_i$ is the number of 1 in $\mathrm{Col}_i(M')$. $\{M'_1, M'_2, \ldots, M'_\kappa\}$ is called a deterministic decomposition with respect to the probabilistic matrix M.

Similarly, for the game $G = (N, \mathcal{A}, U, X, P)$, the deterministic decomposition regarding the state transition matrix $P(a,\cdot,\cdot)$ under the action profile a is defined as $\{P_\kappa(a,\cdot,\cdot), \ \kappa = 1, 2, \ldots\}$.

For a state-based game, we get that for each given state $x_j \in X = \{x_1, \ldots, x_r\}$, if the potential equation of the corresponding normal finite game has a solution, then the state-based game satisfies Theorem 1. Because the potential function is unique up to a constant, $V^{\varphi(\cdot,x_j)} + m_j \mathbf{1}_k$ is also a structure vector of $\varphi(\cdot, x_j)$. Denote

$$V^\varphi = \left[V^{\varphi(\cdot,x_1)} + m_1\mathbf{1}_k^{\mathrm{T}}, V^{\varphi(\cdot,x_2)} + m_2\mathbf{1}_k^{\mathrm{T}}, \ldots, V^{\varphi(\cdot,x_r)} + m_r\mathbf{1}_k^{\mathrm{T}}\right], \tag{7}$$

where $m_j \in \mathbb{R}, \ j \in [1,r], \ k = \Pi_{i=1}^n k_i$.

Therefore, using the potential equation, we establish Theorem 3.

Theorem 3. Consider a state-based game $G = (N, \mathcal{A}, U, X, P) \in \mathcal{G}_{[n;r;k_1,k_2,\ldots,k_n]}$. Then G is an SPG if and only if the following two conditions hold.

(1) For any given state $x_j \in X$, the potential equation of the corresponding normal finite game has a solution.

(2) For any action profile $a = (a_1, a_2, \ldots, a_n) \in \mathcal{A}$, for each matrix $P_\kappa(a,\cdot,\cdot)$ in the deterministic decomposition of the state transition matrix $P(a,\cdot,\cdot)$,

$$V^\varphi(P_\kappa(a,\cdot,\cdot) - I_r) \ltimes_{l=1}^n a_l \geqslant 0_{1\times r} \tag{8}$$

has at least one solution, that is, there exist $m_1, m_2, \ldots, m_r \in \mathbb{R}$ such that V^φ defined in (7) satisfies (8).

Proof. From the analysis above Theorem 3, we just need to show that Condition (2) of Theorem 3 is equivalent to (3).

For any action profile $a \in \mathcal{A}$, Eq. (3) can be rewritten as its vector form

$$V^\varphi y \ltimes_{l=1}^n a_l \geqslant V^\varphi x \ltimes_{l=1}^n a_l, \tag{9}$$

where y is the supported state of $P(a,x)$, $V^\varphi \in \mathbb{R}^{rk}, \ x, y \in \mathbb{R}^r, \ a \in \mathbb{R}^k$. That is,

$$V^\varphi(y - x) \ltimes_{l=1}^n a_l \geqslant 0.$$

Then, put the state y_j supported by $P(a, x_j)$, $j \in [1,r]$ together, we have

$$V^\varphi(y_1 - x_1, y_2 - x_2, \ldots, y_r - x_r) \ltimes_{l=1}^n a_l \geqslant 0_{1\times r}.$$

In other words, we can get that

$$V^\varphi([y_1, y_2, \ldots, y_r] - [x_1, x_2, \ldots, x_r]) \ltimes_{l=1}^n a_l \geqslant 0_{1\times r}, \tag{10}$$

where the matrix $[y_1, y_2, \ldots, y_r]$ represents a possible state transition under action profile $a \in \mathcal{A}$.

Note that $[x_1, x_2, \ldots, x_r] = I_r$ and the definition of deterministic decomposition with respect to the state transition matrix, Eq. (10) is true if and only if for each deterministic decomposition $P_\kappa(a,\cdot,\cdot)$ of the state transition matrix $P(a,\cdot,\cdot)$, Eq. (8) has at least one solution. Furthermore, by plugging the solution of (8) into (7), we get the structure vector V^φ of the potential function of the SPG G.

Next, a sufficient condition for verification of SPGs is given.

Theorem 4. Assume the state-based game $G = (N, \mathcal{A}, U, X, P) \in \mathcal{G}_{[n;r;k_1,k_2,\ldots,k_n]}$ satisfies the following conditions.

(1) For any action profile $a \in \mathcal{A}$, the state always transfers towards the direction of making the players more profitable, that means for each state y dependent on $P(a, x)$, one has $u_i(a, y) \geqslant u_i(a, x),\ i \in N$.

(2) There exist functions $\varphi : \mathcal{A} \times X \to \mathbb{R}$ and $d_i = d_i(a_{-i}),\ i \in N$, where d_i is independent of $x \in X$ and $a_i \in \mathcal{A}_i$, such that

$$u_i(a_i, a_{-i}, x) - \varphi(a_i, a_{-i}, x) = d_i(a_{-i}),\ i \in N. \tag{11}$$

Then G is an SPG.

Proof. According to (11), for any $b_i \in \mathcal{A}_i$, $a = (a_i, a_{-i}) \in \mathcal{A}$ and $x \in X$, we obtain

$$u_i(b_i, a_{-i}, x) - u_i(a, x) = \varphi(b_i, a_{-i}, x) - \varphi(a, x),\ i \in N. \tag{12}$$

Replacing x with $y \in X$ in (11), it holds

$$u_i(a, y) - \varphi(a, y) = d_i(a_{-i}). \tag{13}$$

By (11) and (13), for any states $x, y \in X$, we can obtain that

$$u_i(a, x) - u_i(a, y) = \varphi(a, x) - \varphi(a, y). \tag{14}$$

According to Condition (1) and (14), for each state y dependent on $P(a, x)$, we get

$$\varphi(a, y) \geqslant \varphi(a, x). \tag{15}$$

From (12) and (15), it is easily obtained that G is an SPG with φ as a potential function.

In fact, the potential function φ satisfying (11) is not easy to find. A method is given in Remark 4 to reduce the complexity.

Remark 4. From Theorem 4, we can see that for any $x, y \in X$ and the fixed $a \in \mathcal{A}$, the state-based game G satisfying Theorem 4 has the same $u_i(a, y) - u_i(a, x)$ for all $i \in N$, that is, $u_i(a, y) - u_i(a, x) = \varphi(a, y) - \varphi(a, x),\ i \in N$. Therefore, if a game G satisfying Condition (1) of Theorem 4 has the same $u_i(a, y) - u_i(a, x)$, for all $i \in N$, we only need to prove whether the game G under a fixed state is a potential game. If yes, then associating with (14) and Condition (1) of Theorem 4, it is also an SPG and the corresponding potential function can be calculated by (14).

An example is given below to illustrate the process described in Remark 4.

Example 1. Consider a state-based game $G = (N, \mathcal{A}, U, X, P)$, where $N = \{1, 2, 3, 4\}$, $\mathcal{A}_i = \{1, 2\},\ i \in N,\ X = \{x_1, x_2, x_3\}$. The payoff matrices are given in Tables 1–3.

The state transition matrices are given by (16)–(18).

$$V^{P(\cdot,x_1)} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \tag{16}$$

$$V^{P(\cdot,x_2)} = \begin{bmatrix} 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 1 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \tag{17}$$

$$V^{P(\cdot,x_3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{18}$$

Table 1 Payoff matrix under x_1

Profile	1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
u_1	3	2	3	2	3	2	3	2	0	1	0	1	0	1	0	1
u_2	3	3	2	2	0	0	1	1	3	3	2	2	0	0	1	1
u_3	4	3	0	1	3	2	1	2	4	3	0	1	3	2	1	2
u_4	4	0	3	1	4	0	3	1	3	1	2	2	3	1	2	2

Table 2 Payoff matrix under x_2

Profile	1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
u_1	4	3	3	2	4	3	3	2	0	1	1	2	0	1	1	2
u_2	4	4	2	2	1	1	1	1	3	3	3	3	0	0	2	2
u_3	5	4	0	1	4	3	1	2	4	3	1	2	3	2	2	3
u_4	5	1	3	1	5	1	3	1	3	1	3	3	3	1	3	3

Table 3 Payoff matrix under x_3

Profile	1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
u_1	4	3	4	3	3	2	3	2	0	1	0	1	1	2	1	2
u_2	4	4	3	3	0	0	1	1	3	3	2	2	1	1	2	2
u_3	5	4	1	2	3	2	1	2	4	3	0	1	4	3	2	3
u_4	5	1	4	2	4	0	3	1	3	1	2	2	4	2	3	3

From (17), we obtain that under the action profile $a = (1,1,1,1) \in \mathcal{A}$, states $x_2, x_3 \in X$ are the supported states of x_2 and $u_i(a, x_j) \geqslant u_i(a, x_2)$, where $i \in N,\ j = 2, 3$. By verifying successively, we obtain that the state-based game satisfies Condition (1) of Theorem 4. The structure vectors of φ and $d_i(a_{-i}),\ i \in N$ satisfying (11) can be chosen as

$$V^{\varphi(\cdot,x_1)} = [12, 8, 8, 6, 9, 5, 7, 5, 9, 7, 5, 5, 6, 4, 4, 4],$$

$$V^{\varphi(\cdot,x_2)} = [13, 9, 8, 6, 10, 6, 7, 5, 9, 7, 6, 6, 6, 4, 5, 5],$$

$$V^{\varphi(\cdot,x_3)} = [13, 9, 9, 7, 9, 5, 7, 5, 9, 7, 5, 5, 7, 5, 5, 5],$$

$$V^{d_1} = [-9, -6, -5, -4, -6, -3, -4, -3],$$

$$V^{d_2} = [-9, -5, -6, -4, -6, -4, -3, -3],$$

$$V^{d_3} = [-8, -5, -6, -3, -5, -4, -3, -2],$$

$$V^{d_4} = [-8, -5, -5, -4, -6, -3, -3, -2].$$

Then the game G is an SPG.

In fact, Example 1 satisfies Condition (1) of Theorem 4, and for a given $a \in \mathcal{A}$ and any fixed $x, y \in X$, the difference $u_i(a, y) - u_i(a, x)$ is the same for all players $i \in N$. Hence, we can first prove that the game under fixed state x_1 is potential and get the structure vector $V^{\varphi(\cdot,x_1)}$ of the potential function. Using (14), $V^{\varphi(\cdot,x_2)}$ and $V^{\varphi(\cdot,x_3)}$ can be obtained, respectively. Therefore, the game G is an SPG.

By calculation, we obtain that for each fixed $x_i \in X$, $a^*(x_i) = (1,1,1,1),\ i \in [1,3]$ is a maximum point of the potential function $\varphi(\cdot, x_i)$, which is an NE of the game under state x_i. The state transition under action profile $a^* = (1,1,1,1)$ is shown in Figure 1, from which we get that x_2 and x_3 are recurrent states under $a^* = (1,1,1,1)$. Hence, the action state pairs $[a^*, x_2]$ and $[a^*, x_3]$ are RSEs.

Remark 5. Example 1 is given to illustrate Theorem 4. We can verify that this example satisfies the conditions of Theorem 4, then the state-based game is an SPG. Actually, from the proof of Theorem 4, we can find a more simple method to verify this state-based game to be an SPG. As mentioned in Remark 4, for a given action profile $a \in \mathcal{A}$, when a state-based game satisfies Condition (1) of Theorem 4 and has the same $u_i(a, y) - u_i(a, x)$ for all $i \in N, x, y \in X$, this game is an SPG if it is potential under a certain fixed state.

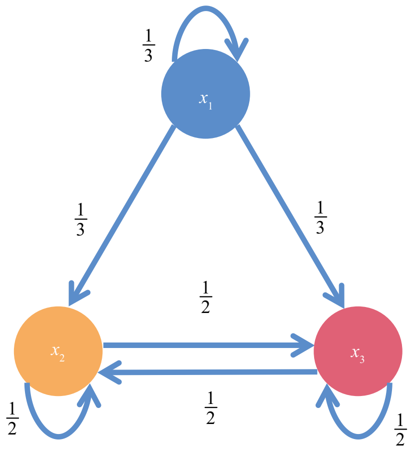

Figure 1 (Color online) State transition under action profile $a^* = (1, 1, 1, 1)$.

3.2 Static equivalence of SPGs

The static equivalence conditions of SPGs are given in this part, which enhances the freedom for the design of SPGs.

Theorem 5. Consider two state-based games $G = (N, \mathcal{A}, U, X, P)$ and $G' = (N, \mathcal{A}, U', X, P')$, $G,\ G' \in \mathcal{G}_{[n;r;k_1,k_2,\ldots,k_n]}$. Assume the two games satisfy the following conditions.

(1) For each action profile $a \in \mathcal{A}$ and any states $x_i, x_j \in X$, x_j is the supported state of $P(a, x_i)$ if and only if x_j is the supported state of $P'(a, x_i)$.

(2) There exists a constant vector $W \in \mathbb{R}^{rn}$ such that

$$V_G^u - V_{G'}^{u'} = W \otimes \mathbf{1}_k^{\mathrm{T}}, \tag{19}$$

where $k = \prod_{i=1}^n k_i$, V_G^u ($V_G^{u'}$) is the structure vectors of G (G').

Then, the state-based game G is an SPG if and only if the state-based game G' is an SPG. Moreover, they have the same potential function φ.

Proof. Let $W = [w_{11}, w_{12}, \ldots, w_{1n}, w_{21}, w_{22}, \ldots, w_{2n}, \ldots, \ldots, w_{r1}, w_{r2}, \ldots, w_{rn}] \in \mathbb{R}^{rn}$. From (1) and (19), for any $i \in N$ and $x_j \in X$, we can obtain that

$$V_{ji}^u - V_{ji}^{u'} = w_{ji}\mathbf{1}_k^{\mathrm{T}},\ j \in [1, r]. \tag{20}$$

Suppose G is an SPG with φ as a potential function. Then, for each $i \in N$ and $b_i \in \mathcal{A}_i$, it holds

$$u_i(b_i, a_{-i}, x) - u_i(a, x) = \varphi(b_i, a_{-i}, x) - \varphi(a, x).$$

From (20), for any $a \in \mathcal{A}$ under the state $x_j \in X$, we get

$$u_i(a, x_j) - u_i'(a, x_j) = w_{ji},\ j \in [1, r],\ i \in N, \tag{21}$$

that is, the difference $u_i(a, x_j) - u_i'(a, x_j)$ is independent of the action profile $a \in \mathcal{A}$.

By (21), for any $b_i \in \mathcal{A}_i$ under the state $x \in X$, we can obtain

$$u_i'(b_i, a_{-i}, x) - u_i'(a, x) = u_i(b_i, a_{-i}, x) - u_i(a, x) = \varphi(b_i, a_{-i}, x) - \varphi(a, x),\ i \in N. \tag{22}$$

From Condition (1), if y is a supported state of $P'(a, x)$, then y is also a supported state of $P(a, x)$. Since G is an SPG, for any state y in the support of $P(a, x)$, we have

$$\varphi(a, y) \geqslant \varphi(a, x), \tag{23}$$

where y is also a supported state of $P'(a, x)$.

From (22) and (23), the game G' is an SPG with φ as its potential function.

Conversely, if the state-based game G' is an SPG, the state-based game G is also an SPG. And the two games have the same potential function φ.

Theorem 6. Consider two state-based games $G = (N, \mathcal{A}, U, X, P)$ and $G' = (N, \mathcal{A}, U', X, P')$, $G,\ G' \in \mathcal{G}_{[n;r;k_1,k_2,\ldots,k_n]}$. If the Condition (2) in Theorem 5 is replaced with

$$V_G^u - V_{G'}^{u'} = \mathbf{1}_n^{\mathrm{T}} \otimes v \otimes \mathbf{1}_r^{\mathrm{T}}, \quad v \in \mathbb{R}^k, \tag{24}$$

then the state-based game G is an SPG if and only if the state-based game G' is an SPG.

Proof. (Necessity) From (24), we can obtain that for each $i \in N$ and $x_j \in X$,

$$V_{ji}^u - V_{ji}^{u'} = v,\ i \in N,\ j \in [1, r]. \tag{25}$$

Assume G is an SPG with φ as a potential function. From Theorem 1, for any $x_j \in X$ and $a = (a_1, a_2, \ldots, a_n) \in \mathcal{A}$, we have

$$V_{ji}^u = V^{\varphi(\cdot,x_j)} + V_{ji}^d M_i,\ i \in N,\ j \in [1, r], \tag{26}$$

where M_i is shown in (6).

From (25) and (26), we get

$$V_{ji}^{u'} - (V^{\varphi(\cdot,x_j)} - v) = (V_{ji}^u - v) - (V^{\varphi(\cdot,x_j)} - v) = V_{ji}^d M_i. \tag{27}$$

Similar to Theorem 5, for each state x_j in the support of $P'(a, x_i)$, we have

$$\varphi(a, x_j) \geqslant \varphi(a, x_i). \tag{28}$$

Eq. (28) can be expressed in its vector form, $V^{\varphi(\cdot,x_j)} \ltimes_{l=1}^n a_l \geqslant V^{\varphi(\cdot,x_i)} \ltimes_{l=1}^n a_l$. Thus

$$(V^{\varphi(\cdot,x_j)} - v) \ltimes_{l=1}^n a_l \geqslant (V^{\varphi(\cdot,x_i)} - v) \ltimes_{l=1}^n a_l. \tag{29}$$

From (27) and (29), the game G' is an SPG with $V^\varphi - 1_n^{\mathrm{T}} \otimes v \otimes 1_r^{\mathrm{T}}$ as the structure vector of the potential function, where V^φ is the structure vector of φ.

(Sufficiency) Similar to the proof of the necessity, we can get that if the game G' is an SPG with $V^{\varphi'}$ as the structure vector of its potential function, then G is also an SPG, with $V^{\varphi'} + 1_n^{\mathrm{T}} \otimes v \otimes 1_r^{\mathrm{T}}$ as the structure vector of its potential function.

Remark 6. Theorems 5 and 6 have provided two verifiable static equivalent conditions. The two static equivalence conditions of Theorems 5 and 6 divide state-based games into equivalence classes such that the state-based games in the same equivalence class have the same or similar properties. Hence, for the state-based games in the same equivalence class, we only need to study one of the games to know the properties of the others. In practical applications, if we want to model a problem as an SPG, the construction of the SPG is not unique. We can find a suitable SPG that meets the characteristics of the problem through the equivalence class.

3.3 Dynamic equivalence of SPGs

To expand the application of SPGs, the dynamic equivalence of SPGs is studied in this part, such that some state-based games that are dynamically equivalent to an SPG also have similar dynamic properties.

An important action evolutionary process was shown in [17]. Define a player's strict better reply set for each $[a, x] \in \mathcal{A} \times X$ as

$$\mathcal{O}_i(a, x) := \{b_i \in \mathcal{A}_i | u_i(b_i, a_{-i}, x) > u_i(a, x)\},\ i \in N.$$

Then the action evolutionary process is as follows:

$$\begin{cases} p_i^{a_i(t-1)} = 1, & \text{if } \mathcal{O}_i(a(t-1), x(t)) = \emptyset, \\ p_i^{a_i(t-1)} = \epsilon, & \text{if } \mathcal{O}_i(a(t-1), x(t)) \neq \emptyset, \\ p_i^{a_i'} = \dfrac{1-\epsilon}{|\mathcal{O}_i(a(t-1), x(t))|}, & \text{if } a_i' \in \mathcal{O}_i(a(t-1), x(t)) \neq \emptyset, \\ p_i^{a_i''} = 0, & \text{otherwise}, \end{cases} \tag{30}$$

where $p_i^{a_i(t-1)}$ is the probability that $a_i = a_i(t-1)$, and $\epsilon \in (0, 1)$ is the inertia of the player.

In [17], Marden has pointed out the convergence of ordinal SPGs under the action evolutionary process (30).

Theorem 7 ([17]). The ordinal SPG with the action evolutionary process (30) will almost surely converge to an action invariant set of recurrent state equilibria (RSEs).

For a game $G = (N, \mathcal{A}, U, X, P) \in \mathcal{G}_{[n;r;k_1,k_2,\ldots,k_n]}$, based on the action-depending state transition matrix $P(a, \cdot, \cdot)$ and the action evolutionary process (30), by STP and Lemma 1, the dynamics of the game is

$$\begin{cases} \mathbb{E}x(t+1) = M_P \mathbb{E}(x(t)a(t)), \\ \mathbb{E}a(t+1) = M_F \mathbb{E}(x(t+1)a(t)), \end{cases} \tag{31}$$

where $a(t)$ and $x(t)$ are the action profile and state of the game at time t, respectively, and $M_P \in \Upsilon_{r\times rk}$, $M_F \in \Upsilon_{k\times rk}$ are the structure matrices.

Definition 6. Two state-based games $G = (N, \mathcal{A}, U, X, P)$ and $G' = (N, \mathcal{A}, U', X, P')$ are said to be dynamically equivalent if they satisfy the following conditions.

(1) For any action profile $a \in \mathcal{A}$ and any states $x_i, x_j \in X$, x_j is the supported state of $P(a, x_i)$ if and only if x_j is the supported state of $P'(a, x_i)$.

(2) They have the same action evolution dynamic.

Theorem 8. Assume a state-based game $G' = (N, \mathcal{A}, U', X, P')$ and an SPG $G = (N, \mathcal{A}, U, X, P)$ are dynamically equivalent and satisfy the action updating rule (30), then G' can almost surely converge to its action invariant set of RSEs.

Proof. Using STP, under the action updating rule (30) the evolution dynamic of the SPG G can be expressed as (31).

Similarly, the dynamic process of the game G' is

$$\begin{cases} \mathbb{E}x(t+1) = M'_{P'} \mathbb{E}(x(t)a(t)), \\ \mathbb{E}a(t+1) = M'_F \mathbb{E}(x(t+1)a(t)), \end{cases}$$

where $M'_{P'}$, M'_F are the structure matrices of the state and action dynamic of the game G', respectively.

Since G and G' are dynamically equivalent and update according to the same rule (30), then $M_F = M'_F$, which means that the action transition of the two games is consistent. Combined with Condition (1) of Definition 6, we have

$$\{b_i \in \mathcal{A}_i | u_i(b_i, a_{-i}, x) > u_i(a, x)\} = \{b_i \in \mathcal{A}_i | u'_i(b_i, a_{-i}, x) > u'_i(a, x)\}.$$

Hence,

$$u_i(b_i, a_{-i}, x) > u_i(a, x) \Leftrightarrow u'_i(b_i, a_{-i}, x) > u'_i(a, x).$$

By Definition 4, for the SPG G, one has

$$u_i(b_i, a_{-i}, x) - u_i(a, x) = \varphi(b_i, a_{-i}, x) - \varphi(a, x),$$

where φ is the potential function.

And for each state y dependent on $P(a, x)$,

$$\varphi(a, y) \geqslant \varphi(a, x). \tag{32}$$

Therefore, for the state-based game G', one has

$$u'_i(b_i, a_{-i}, x) > u'_i(a, x) \Leftrightarrow \varphi(b_i, a_{-i}, x) > \varphi(a, x). \tag{33}$$

Since y is a supported state of $P'(a, x)$ if and only if y is also a supported state of $P(a, x)$, then

$$\varphi(a, y) \geqslant \varphi(a, x), \tag{34}$$

where the state y is dependent on $P'(a, x)$.

Hence, from (33) and (34), G' is an ordinal SPG. By Theorem 7, the conclusion is obviously established.

Remark 7. From the above proof process, it is not difficult to see that under the action updating rule (30), if a state-based game and an SPG are dynamically equivalent, the state-based game is an ordinal SPG.

Remark 8. According to the dynamic equivalence condition in Theorem 8, it can be seen that the state-based games dynamic equivalent to SPGs can also converge to an action invariant set of RSEs under a certain action evolutionary process. Hence, when we study the evolutionary dynamics of a practical problem, by dynamic equivalence condition, we can model it as a state-based game instead of an SPG, which expands the freedom of the game design.

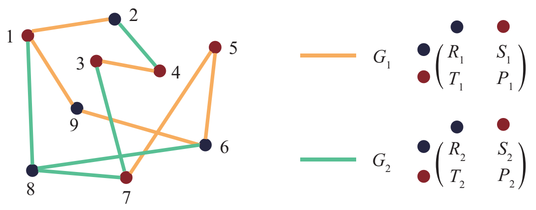

Figure 2 (Color online) Networked evolutionary dynamics with game transitions.

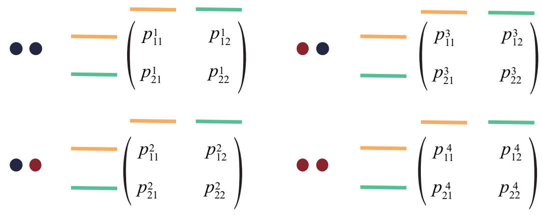

Figure 3 (Color online) State transition matrix of the networked evolutionary game.

3.4 SNEGs

In nature, the type of the game between populations is not invariable, but alters with the change in the ecological environment. Furthermore, the change in the ecological environment is related to the strategy chosen by the population in the game process [22]. Therefore, we consider an evolutionary dynamics with game transition as shown in Figure 2, where the set of players $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, the blue and red points represent the actions of players, and the orange and green lines are two states of each symmetric game with two players. The state transition matrix of each symmetric game with two players is shown in Figure 3. This kind of game can be modeled as an SNEG. Based on this, we discuss SNEGs in this part.

Definition 7. An SNEG $\mathcal{G} = ((N, E), G, \Pi)$ consists of

(1) A network graph (N, E), which is a simple graph;

(2) An FSNG G, such that if $(i, j) \in E$, then i and j play FSNG with actions $a_i(t)$ and $a_j(t)$, respectively, under state $x_{ij}(t)$, where $x_{ij}(t)$ represents the state at time t of the FSNG played between players i and j;

(3) An action updating rule Π.

The action updating rule used in this part is the better reply with inertia (30).

Definition 8. For an edge $e_{ij} = (i, j) \in E$ of an SNEG $\mathcal{G}$, define the index matrix $\Gamma_{e_{ij}} := \otimes_{l=1}^{n} \gamma_l$, where

$$\gamma_l = \begin{cases} I_{k_l}, & l = i, j, \\ \mathbf{1}_{k_l}^{\mathrm{T}}, & l \neq i, j, \end{cases}$$

$k_l = |\mathcal{A}_l|$, and $\mathcal{A}_l$ is all of player l's strategies.

Corollary 1. For an edge $e_{ij} = (i, j) \in E$ of an SNEG $\mathcal{G}$, we have $a_i \ltimes a_j = \Gamma_{e_{ij}} \ltimes_{l=1}^{n} a_l$, where $a_l \in \mathcal{A}_l$, $l \in N$.

Theorem 9. Consider an SNEG $\mathcal{G} = ((N, E), G, \Pi)$. If the FSNG is potential, then the SNEG $\mathcal{G}$ is also potential, called the state-based potential networked evolutionary game (SPNEG), and the potential function is $\varphi_{\mathcal{G}} = \sum_{e_\ell \in E} \varphi_{e_\ell}$, where φ_{e_ℓ} is the potential function of the FSNG on the edge e_ℓ.

Proof. Assume the network graph has q edges. Each edge is given a label, denoted as $e_1, e_2, \ldots, e_q$, then the set of edges is $E = \{e_1, e_2, \ldots, e_q\}$. The state space corresponding to edge e_ℓ is X_ℓ, $\ell \in [1, q]$.

The state space of the SNEG is defined as $X = \prod_{\ell=1}^{q} X_\ell$ (where "Π" is the Cartesian product), that is, for $x \in X$, $x = \ltimes_{\ell=1}^{q} x_\ell$, $x_\ell \in X_\ell$. Then the probability of state transition from x^i to x^j under action a of the game $\mathcal{G}$ can be defined as

$$P(a, x^i, x^j) = P_1(\Gamma_{e_1} \ltimes_{l=1}^{n} a_l, x_1^i, x_1^j) P_2(\Gamma_{e_2} \ltimes_{l=1}^{n} a_l, x_2^i, x_2^j) \cdots P_q(\Gamma_{e_q} \ltimes_{l=1}^{n} a_l, x_q^i, x_q^j),$$

where $x^i, x^j \in X$, $x_\ell^i, x_\ell^j \in X_\ell$, P_ℓ is the state transition function of the FSNG on the edge e_ℓ, $\ell \in [1, q]$.

Assume the edge $(i, j) = e_{ij} \in E$. Then the player i's payoff under the action profile $a \in \mathcal{A}$ and state $x \in X$ is

$$u_i(a, x) = \sum_{j \in \mathcal{N}_i} u_{ij}(a_i, a_j, x_{ij}),$$

where x_{ij} is the state corresponding to edge e_{ij}, $\mathcal{N}_i = \{j \mid (i, j) \in E\}$ is all of player i's neighborhood.

Since the FSNG is an SPG, then for the edge e_{ij}, we have

$$u_{ij}(b_i, a_j, x_{ij}) - u_{ij}(a_i, a_j, x_{ij}) = \varphi_{e_{ij}}(b_i, a_j, x_{ij}) - \varphi_{e_{ij}}(a_i, a_j, x_{ij}). \tag{35}$$

According to (35), we have

$$\begin{aligned}&\varphi_{\mathcal{G}}(b_i,a_{-i},x)-\varphi_{\mathcal{G}}(a_i,a_{-i},x)\\&=\sum_{(m,n)=e_{mn}\in E}(\varphi_{e_{mn}}(a_m,a_n,x_{mn})-\varphi_{e_{mn}}(a_m,a_n,x_{mn}))\\&=\sum_{j\in\mathcal{N}_i}(\varphi_{e_{ij}}(b_i,a_j,x_{ij})-\varphi_{e_{ij}}(a_i,a_j,x_{ij}))\\&=\sum_{j\in\mathcal{N}_i}(u_{ij}(b_i,a_j,x_{ij})-u_{ij}(a_i,a_j,x_{ij}))\\&=u_i(b_i,a_{-i},x)-u_i(a_i,a_{-i},x).\end{aligned}\tag{36}$$

Then we will prove that for each state y dependent on $P(a,x)$ of the SNEG, we have $\varphi_{\mathcal{G}}(a,y)\geqslant\varphi_{\mathcal{G}}(a,x)$. Since y is the supported state of $P(a,x)$, then $P(a,x,y)>0$. That is,

$$P(a,x,y)=P_1(\Gamma_{e_1}\ltimes_{l=1}^n a_l,x_1,y_1)P_2(\Gamma_{e_2}\ltimes_{l=1}^n a_l,x_2,y_2)\cdots P_q(\Gamma_{e_q}\ltimes_{l=1}^n a_l,x_q,y_q)>0,\tag{37}$$

where $P(a,x,y)$ represents the probability of transferring from state x to state y under action $a\in\mathcal{A}$.

From (37), we can obtain that $P_1(\Gamma_{e_1}\ltimes_{l=1}^n a_l,x_1,y_1)>0,P_2(\Gamma_{e_2}\ltimes_{l=1}^n a_l,x_2,y_2)>0,\ldots,P_q(\Gamma_{e_q}\ltimes_{l=1}^n a_l,x_q,y_q)>0$.

Since the FSNG is potential, we have

$$\begin{aligned}\varphi_{e_1}(\Gamma_{e_1}\ltimes_{l=1}^n a_l,y_1)&\geqslant\varphi_{e_1}(\Gamma_{e_1}\ltimes_{l=1}^n a_l,x_1),\\\varphi_{e_2}(\Gamma_{e_2}\ltimes_{l=1}^n a_l,y_2)&\geqslant\varphi_{e_2}(\Gamma_{e_2}\ltimes_{l=1}^n a_l,x_2),\\&\vdots\\\varphi_{e_q}(\Gamma_{e_q}\ltimes_{l=1}^n a_l,y_q)&\geqslant\varphi_{e_q}(\Gamma_{e_q}a\ltimes_{l=1}^n a_l,x_q).\end{aligned}$$

Then, we can get that $\sum_{e_\ell\in E}\varphi_{e_\ell}(\Gamma_{e_\ell}\ltimes_{l=1}^n a_l,y_\ell)\geqslant\sum_{e_\ell\in E}\varphi_{e_\ell}(\Gamma_{e_\ell}\ltimes_{l=1}^n a_l,x_\ell)$.

Therefore,

$$\varphi_{\mathcal{G}}(a,y)\geqslant\varphi_{\mathcal{G}}(a,x).\tag{38}$$

According to (36) and (38), it can be proved that the SNEG $\mathcal{G}=((N,E),G,\Pi)$ is potential with $\varphi_{\mathcal{G}}=\sum_{e_\ell\in E}\varphi_{e_\ell}$ as a potential function.

Corollary 2. For an SNEG, assume that for any consistent action profile $a\in\mathcal{A}$, that is, $a_i=a_j,\ \forall i,j\in N,\ P(a,x,x)>0,\ \forall x\in X$. Then if the SNEG is potential, the FSNG is also potential and the relationship between their potential functions is the same as Theorem 9.

Proof. The definitions of the state space (state transition) of FNSG and SNEG are consistent with those defined in the proof of Theorem 9.

According to [12], we can obtain that under a fixed state $x\in X$, if the SNEG is a potential game, then the FNSG is also a potential game, and their potential function satisfies

$$\varphi_{\mathcal{G}}=\sum_{(m,n)=e_{mn}\in E}\varphi_{e_{mn}},$$

where $\varphi_{\mathcal{G}}$ and $\varphi_{e_{mn}}$ are defined in Theorem 9.

In the following, we will prove that for any $i\in N,\ j\in\mathcal{N}_i,\ a_i\in\mathcal{A}_i,\ a_j\in\mathcal{A}_j$,

$$\varphi_{e_{ij}}(a_i,a_j,y_{ij})\geqslant\varphi_{e_{ij}}(a_i,a_j,x_{ij}),$$

where $x_{ij},y_{ij}\in X_{ij}$, and y_{ij} is the supported state of $P_{ij}(a_i,a_j,x_{ij})$.

For any $(m,n)\in E$, given an action profile $a\in\mathcal{A}$ of SNEG, satisfying $\Gamma_{e_{mn}}\ltimes_{l=1}^n a_l=a_i\ltimes a_j$, $\Gamma_{e_{mn}}\ltimes_{l=1}^n a_l=a_i\ltimes a_i$ or $\Gamma_{e_{mn}}\ltimes_{l=1}^n a_l=a_j\ltimes a_j$. Then, we choose two states $x,y\in X$ of SNEG, such that

$$\begin{cases}x_{mn}=x_{ij},y_{mn}=y_{ij}, & \text{when }\ \Gamma_{e_{mn}}\ltimes_{l=1}^n a_l=a_i\ltimes a_j,\\x_{mn}=y_{mn}, & \text{when }\ \Gamma_{e_{mn}}\ltimes_{l=1}^n a_l=a_i\ltimes a_i\text{ or }\Gamma_{e_{mn}}\ltimes_{l=1}^n a_l=a_j\ltimes a_j.\end{cases}\tag{39}$$

Table 4 Payoff matrix under x_1 of game G

Profile	1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
u_1	$3+l_1$	$2+l_1$	$3+l_1$	$2+l_1$	$3+l_1$	$2+l_1$	$3+l_1$	$2+l_1$	l_1	$1+l_1$	l_1	$1+l_1$	l_1	$1+l_1$	l_1	$1+l_1$
u_2	$3+l_2$	$3+l_2$	$2+l_2$	$2+l_2$	l_2	l_2	$1+l_2$	$1+l_2$	$3+l_2$	$3+l_2$	$2+l_2$	$2+l_2$	l_2	l_2	$1+l_2$	$1+l_2$
u_3	$4+l_3$	$3+l_3$	l_3	$1+l_3$	$3+l_3$	$2+l_3$	$1+l_3$	$2+l_3$	$4+l_3$	$3+l_3$	l_3	$1+l_3$	$3+l_3$	$2+l_3$	$1+l_3$	$2+l_3$
u_4	$4+l_4$	l_4	$3+l_4$	$1+l_4$	$4+l_4$	l_4	$3+l_4$	$1+l_4$	$3+l_4$	$1+l_4$	$2+l_4$	$2+l_4$	$3+l_4$	$1+l_4$	$2+l_4$	$2+l_4$

Table 5 Payoff matrix under x_2 of game G

Profile	1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
u_1	$4+l_5$	$3+l_5$	$3+l_5$	$2+l_5$	$4+l_5$	$3+l_5$	$3+l_5$	$2+l_5$	l_5	$1+l_5$	$1+l_5$	$2+l_5$	l_5	$1+l_5$	$1+l_5$	$2+l_5$
u_2	$4+l_6$	$4+l_6$	$2+l_6$	$2+l_6$	$1+l_6$	$1+l_6$	$1+l_6$	$1+l_6$	$3+l_6$	$3+l_6$	$3+l_6$	$3+l_6$	l_6	l_6	$2+l_6$	$2+l_6$
u_3	$5+l_7$	$4+l_7$	l_7	$1+l_7$	$4+l_7$	$3+l_7$	$1+l_7$	$2+l_7$	$4+l_7$	$3+l_7$	$1+l_7$	$2+l_7$	$3+l_7$	$2+l_7$	$2+l_7$	$3+l_7$
u_4	$5+l_8$	$1+l_8$	$3+l_8$	$1+l_8$	$5+l_8$	$1+l_8$	$3+l_8$	$1+l_8$	$3+l_8$	$1+l_8$	$3+l_8$	$3+l_8$	$3+l_8$	$1+l_8$	$3+l_8$	$3+l_8$

Table 6 Payoff matrix under x_3 of game G

Profile	1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
u_1	$4+l_9$	$3+l_9$	$4+l_9$	$3+l_9$	$3+l_9$	$2+l_9$	$3+l_9$	$2+l_9$	l_9	$1+l_9$	l_9	$1+l_9$	$1+l_9$	$2+l_9$	$1+l_9$	$2+l_9$
u_2	$4+l_{10}$	$4+l_{10}$	$3+l_{10}$	$3+l_{10}$	l_{10}	l_{10}	$1+l_{10}$	$1+l_{10}$	$3+l_{10}$	$3+l_{10}$	$2+l_{10}$	$2+l_{10}$	$1+l_{10}$	$1+l_{10}$	$2+l_{10}$	$2+l_{10}$
u_3	$5+l_{11}$	$4+l_{11}$	$1+l_{11}$	$2+l_{11}$	$3+l_{11}$	$2+l_{11}$	$1+l_{11}$	$2+l_{11}$	$4+l_{11}$	$3+l_{11}+l_{11}$	l_{11}	$1+l_{11}$	$4+l_{11}$	$3+l_{11}$	$2+l_{11}$	$3+l_{11}$
u_4	$5+l_{12}$	$1+l_{12}$	$4+l_{12}$	$2+l_{12}$	$4+l_{12}$	l_{12}	$3+l_{12}$	$1+l_{12}$	$3+l_{12}$	$1+l_{12}$	$2+l_{12}$	$2+l_{12}$	$4+l_{12}$	$2+l_{12}$	$3+l_{12}$	$3+l_{12}$

Assume there are s edges satisfying $\Gamma_{e_{mn}} \ltimes_{l=1}^{n} a_l = a_i \ltimes a_j$. From the assumption, we have

$$\varphi_{\mathcal{G}}(a, y) \geqslant \varphi_{\mathcal{G}}(a, x),$$

where y is the supported state of $P(a, x)$, that is,

$$\sum_{(m,n)=e_{mn} \in E} \varphi_{e_{mn}}(a_m, a_n, y_{mn}) \geqslant \sum_{(m,n)=e_{mn} \in E} \varphi_{e_{mn}}(a_m, a_n, x_{mn}).$$

From (39), we have

$$s\varphi_{e_{ij}}(a_i, a_j, y_{ij}) \geqslant s\varphi_{e_{ij}}(a_i, a_j, x_{ij}).$$

Hence,

$$\varphi_{e_{ij}}(a_i, a_j, y_{ij}) \geqslant \varphi_{e_{ij}}(a_i, a_j, x_{ij}). \tag{40}$$

According to (40) and [12], it can be proved that the FSNG is an SPG and the potential function satisfies $\varphi_{\mathcal{G}} = \sum_{(m,n)=e_{mn} \in E} \varphi_{e_{mn}}$.

Remark 9. Theorem 9 provides a convenient method to verify an SNEG to be an SPNEG. According to Theorem 9, for some practical problems, such as the above mentioned game among populations with game transition, to achieve the convergence of the SNEG, instead of designing the whole SNEG, we only need to design the FSNG to be the SPG.

4 Simulation

In this section, we give an example to illustrate the obtained theoretical results.

Example 2. Consider a game among companies, where the external economic environment will have an effect on the competitive relationship among companies and the competitive relationship among companies will also alter the external economic environment. Then we model this game as a state-based game $G = (N, \mathcal{A}, U, X, P)$, where $N = \{1, 2, 3, 4\}$ is the set of companies (called players), $\mathcal{A} := \Pi_{i=1}^{n} \mathcal{A}_i$, $\mathcal{A}_i = \{1, 2\}$, $i \in N$ is the strategy set of the company i, denoting cooperation or competition, and $X = \{x_1, x_2, x_3\}$ is the state set representing three kinds of economic environments.

The state transition matrices are given by (16)–(18) in Example 1, which represent the switching of the economic environments.

We aim to stabilize this state-based game to $[a, x] = [(1, 1, 1, 1), x_j]$, $j \in \{2, 3\}$, that is, we want a balance among the four companies. We can achieve this balance by designing the payoff matrices. Actually, the payoff matrices given in Example 1 are one of the design approaches.

Table 7 Payoff matrix under x_1 of game G'

Profile	1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
u_1	3	2	4	3	2	1	3	2	2	0	3	1	1	0	0	1
u_2	3	2	3	3	2	1	2	1	3	3	2	2	2	2	1	1
u_3	4	3	3	1	4	3	2	3	4	3	0	1	2	2	0	2
u_4	4	3	2	1	4	2	3	1	4	2	1	1	3	1	2	2

Table 8 Payoff matrix under x_2 of game G'

Profile	1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
u_1	4	4	2	2	3	3	3	2	3	1	1	2	2	2	1	2
u_2	4	3	3	2	3	2	1	1	1	2	3	3	0	0	1	2
u_3	4	2	1	1	3	3	2	2	4	3	1	2	2	1	1	2
u_4	3	1	2	1	4	2	3	2	2	0	2	2	3	1	3	3

Table 9 Payoff matrix under x_3 of game G'

Profile	1111	1112	1121	1122	1211	1212	1221	1222	2111	2112	2121	2122	2211	2212	2221	2222
u_1	3	3	3	3	4	2	3	1	2	2	2	1	2	2	1	1
u_2	4	3	2	3	0	1	0	1	4	4	1	1	2	2	1	1
u_3	4	4	1	3	3	2	1	2	4	2	2	1	4	2	3	2
u_4	3	1	3	1	3	1	4	0	2	1	3	3	4	3	2	2

On one hand, using Theorem 5, without changing the state transition matrix, we can also achieve this balance by designing the payoff matrices as Tables 4–6, where $l_1, l_2, \ldots, l_{12} \in \mathbb{R}$. We find that the state-based game $G = (N, \mathcal{A}, U, X, P)$ is an SPG for any $l_1, l_2, \ldots, l_{12} \in \mathbb{R}$. These SPGs have the same potential function, then they can almost surely converge to the RSEs $[a, x] = [(1, 1, 1, 1), x_j],\ j \in \{2, 3\}$ under the action updating rule (30). That is, these SPGs with payoff matrices in Tables 4–6 satisfy the conditions of Theorem 5, then they are in the same equivalence class and have the same potential function. All of them can be used to model this problem.

On the other hand, using Definition 6 and Theorem 8, we can further design a state-based game that is dynamically equivalent to the above designed SPG to achieve this balance. Consider the state-based game $G' = (N, \mathcal{A}, U', X, P)$, without changing the state transition matrix, the payoff matrices of the state-based game G' can be designed as Tables 7–9. Hence, we can obtain that under the action updating rule (30), they have the same action evolution dynamic M_F as shown in (41). That is, the state-based game G' and the above designed SPG satisfy the conditions of Theorem 8, therefore the state-based game can almost surely converge to $[a, x] = [(1, 1, 1, 1), x_j],\ j \in \{2, 3\}$ under the action updating rule (30) and can also be used to model the problem, which enhance the freedom of game design.

$$M_F = \begin{bmatrix} 1.000 & 0.90 & 0.90 & 0.81 & \cdots & 0 & 0 \\ 0 & 0.10 & 0 & 0.09 & \cdots & 0 & 0 \\ 0 & 0 & 0.10 & 0.09 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0.01 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{41}$$

5 Conclusion

This paper delves into the study of SPGs using the STP method. Our research began with a verifiable algebraic condition derived from the potential equation of normal finite games, ensuring that the state-based game is potential. Following this, we introduced two static equivalence conditions specific to SPGs. Then, we explored the dynamic equivalence of SPGs, revealing that state-based games dynamically equivalent to an SPG possess dynamic properties. Ultimately, we examined the network structure and established that an SNEG is potential if its FSNG is an SPG. By constructing the state space and state transition matrix of the SPNEG, we constructed the potential function.

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