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• **Supplementary File** •

Global output feedback regulation of time-varying nonlinear systems via the dual-gain method

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Appendix A Notations

$\mathbb{R}^+$ denotes the set of nonnegative real numbers. $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of all real numbers and the n-dimensional Euclidean space, separately. $|\cdot|$ denotes the absolute value of a real number. For a real vector $k = [k_1, \ldots, k_n]^{\mathrm{T}}$, the norm $\|k\|$ denotes its Euclidean norm. For a real matrix $K = (k_{ij})_{n\times m}$, K^{T} denotes its transpose; $\|K\|$ denotes its induced 2-norm and $\|K\|_\infty$ denotes its ∞-norm; $\lambda_{\max}(K)/\lambda_{\min}(K)$ are the maximum/minimum eigenvalues of K respectively. I denotes the n-dimensional identity matrix. A continuous function $\omega_1 : [0, +\infty) \to [0, +\infty)$ is said to be a class $\mathcal{K}_\infty$ if it is strictly increasing, $\omega_1(0) = 0$ and $\omega_1(\tau) \to \infty$ as $\tau \to \infty$. The arguments of functions might be simplified or omitted as long as there is no confusion in the context. For example, $\varsigma(t, x(t), u(t))$ could be denoted as $\varsigma(t, x, u)$, $\varsigma(\cdot)$ or ς.

Appendix B Proof of the inequality (10)

In fact, if $V_1(\varepsilon)=\beta\varepsilon^T P\varepsilon$ with the constant $\beta>0$ is chosen, then the time derivative of $V_1(\varepsilon)$ along (9) satisfies

$$\dot{V}_1 \leqslant -2\beta r\|\varepsilon\|^2 - \beta\frac{\dot{r}}{r}\varepsilon^{\mathrm{T}}(BP+PB)\varepsilon + 2\beta\frac{r}{r^v}\varepsilon^{\mathrm{T}}P\Lambda_2 x_1 + 2\beta\varepsilon^{\mathrm{T}}PF. \tag{B1}$$

Next, bounds for some terms on the right-hand side of (B1) are derived. Firstly, from Young's inequality, one gets

$$2\beta\frac{r}{r^v}\varepsilon^{\mathrm{T}}P\Lambda_2 x_1 \leqslant 2\beta r\|P\|\cdot\|\Lambda_2\|\cdot\|\varepsilon\|\cdot\Big|\frac{x_1}{r^v}\Big| \leqslant \beta r\|\varepsilon\|^2 + \beta r\|P\Lambda_2\|^2\frac{x_1^2}{r^{2v}}. \tag{B2}$$

Secondly, one has $\frac{\dot{r}}{r} \geqslant \frac{\dot{L}_1}{L_1}$ by (4) and (5). Thus, (4) and (7) imply

$$-\beta\frac{\dot{r}}{r}\varepsilon^{\mathrm{T}}(BP+PB)\varepsilon \leqslant -\beta\frac{\dot{L}_1}{L_1}\varepsilon^{\mathrm{T}}(BP+PB)\varepsilon \leqslant -\beta d_1\rho_2(1+|y|^p)^2\|\varepsilon\|^2 + \beta d_2\rho_1 L_1\|\varepsilon\|^2. \tag{B3}$$

Thirdly, it follows from (4) that $L_1(t) \geqslant 1$. If not, there is a time $t_1 \in (t_0, +\infty)$ which makes $L_1(t_1) < 1$ hold. Note that $L_1(t)$ is continuous and $L_1(t_0) = 1$. Thus there is a time $t_2 \in [t_0, t_1)$, which makes $L_1(t) < L_1(t_2) = 1$ hold for all $t \in (t_2, t_1]$. From this and (4), it is easily available that $\dot{L}_1(t) > 0$ for all $t \in [t_2, t_1]$, which means $L_1(t) > L_1(t_2) = 1$ for all $t \in (t_2, t_1]$. Obviously, a contradiction arises. This and (5) imply $r(t) \geqslant 1$ for all $t \geqslant t_0 \geqslant 0$. For $i = 1, \ldots, n$, with the help of Assumption 2, there holds

$$\Big|\frac{f_i}{r^{v+i-1}}\Big| \leqslant \frac{c\phi(1+|y|^p)}{r^{v+i-1}}\sum_{j=1}^{i}|x_j|. \tag{B4}$$

In addition, by (B4), one gets

$$2\beta\varepsilon^{\mathrm{T}}PF \leqslant 2\beta\|\varepsilon\|\cdot\|P\|\cdot\|F\| \leqslant 2nc\phi\beta\|\varepsilon\|\cdot\|P\|(1+|y|^p)\sum_{i=1}^{n}\frac{|x_i|}{r^{v+i-1}}. \tag{B5}$$

Substituting (B2), (B3) and (B5) into (B1), one gets

$$\dot{V}_1 \leqslant -\beta\Big(d_1\rho_2(1+|y|^p)^2 + r - d_2\rho_1 L_1\Big)\|\varepsilon\|^2 + \beta r\frac{x_1^2}{r^{2v}}\|P\Lambda_2\|^2 + 2nc\|P\|\beta\phi\|\varepsilon\|(1+|y|^p)\sum_{i=1}^{n}\frac{|x_i|}{r^{v+i-1}}, \tag{B6}$$

this implies the inequality (10) holds.

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Appendix C Proof of the inequality (15)

If $V_2(\zeta)=\zeta^{\mathrm{T}}Q\zeta$ is chosen, then its time derivative along (14) satisfies

$$\dot V_2 \leqslant -rM\|\zeta\|^2-\frac{\dot r}{r}\zeta^{\mathrm{T}}(QB+BQ)\zeta+2r\zeta^{\mathrm{T}}Q\Lambda_4(\varepsilon_1-\zeta_1)+2\zeta^{\mathrm{T}}Q(rJ\varepsilon_2+K)+2rMb_1(1-\theta)\zeta^{\mathrm{T}}QZ\zeta_1. \tag{C1}$$

Next, proper estimates for some terms of (C1) are carried out. Firstly, by (7) and $\frac{\dot r}{r}\geqslant\frac{\dot L_1}{L_1}$, one can get

$$-\frac{\dot r}{r}\zeta^{\mathrm{T}}(QB+BQ)\zeta\leqslant-\frac{\dot L_1}{L_1}\zeta^{\mathrm{T}}(QB+BQ)\zeta\leqslant-d_3\rho_2(1+|y|^p)^2\|\zeta\|^2+d_4\rho_1L_1\|\zeta\|^2. \tag{C2}$$

Secondly, in light of $\|J\|=\|Z\|=1$, $\|\Lambda_4\|\leqslant\|\Lambda_2\|$, and Young's inequality, one has

$$\begin{cases}2\zeta^TQJ\varepsilon_2\leqslant\|\varepsilon\|^2+\|Q\|^2\cdot\|\zeta\|^2,\\ 2\zeta^TQ\Lambda_4(\varepsilon_1-\zeta_1)\leqslant\|\zeta\|^2+\|Q\Lambda_2\|^2\cdot\|\varepsilon\|^2+2\|Q\Lambda_2\|\cdot\|\zeta\|^2,\\ 2Mb_1(1-\theta)\zeta^TQZ\zeta_1\leqslant 2Mb_1|1-\theta|\cdot\|Q\|\cdot\|\zeta\|^2.\end{cases} \tag{C3}$$

Thirdly, with the help of (B4), (11) and Young's inequality, one obtains

$$2\zeta^{\mathrm{T}}QK\leqslant2\|\zeta\|\cdot\|Q\|\cdot\|K\|\leqslant2(1+|y|^p)c\phi\|Q\|\cdot\|\zeta\|^2\leqslant(1+|y|^p)^2\|\zeta\|^2+m_{11}^2\phi^2\|\zeta\|^2, \tag{C4}$$

where the constant $m_{11}=c\|Q\|>0$. Substituting (C2)-(C4) into (C1), one gets

$$\begin{aligned}\dot V_2 \leqslant& -rM\Big(1-2b_1|1-\theta|\cdot\|Q\|\Big)\|\zeta\|^2+(1+|y|^p)^2\|\zeta\|^2+r\Big(1+\|Q\Lambda_2\|^2\Big)\|\varepsilon\|^2-d_3\rho_2(1+|y|^p)^2\|\zeta\|^2\\ &+d_4\rho_1L_1\|\zeta\|^2+\Big(\bar m_1(t)+2r\|Q\Lambda_2\|+r\|Q\|^2+r\Big)\|\zeta\|^2,\end{aligned} \tag{C5}$$

where $\bar m_1(t)=m_{11}^2\phi^2(t)$ is continuous and satisfies $\lim\limits_{t\to+\infty}\frac{\bar m_1(t)}{L_2(t)}=0$. Thus, the inequality (15) holds.

Appendix D Proof of the inequality (18)

By (8) and (11), for $i=2,\ldots,n$, there is

$$x_1=r^v\zeta_1,\ \ x_i=r^{v+i-1}\varepsilon_i+r^{v+i-1}M^{i-1}\zeta_i. \tag{D1}$$

Thus, for $i=1,\ldots,n$, there holds

$$\Big|\frac{x_i}{r^{v+i-1}}\Big|\leqslant|\varepsilon_i|+|M^{i-1}\zeta_i|.$$

From this, (B5) and Young's inequality, one has

$$\begin{aligned}2\beta\varepsilon^{\mathrm{T}}PF \leqslant& \ 2nc\phi\beta(1+|y|^p)\sum_{i=1}^{n}(|\varepsilon_i|+M^{i-1}|\zeta_i|)\|\varepsilon\|\cdot\|P\|\leqslant m_{21}\phi(1+|y|^p)(\|\varepsilon\|^2+\|\zeta\|^2)\\ \leqslant& \ (1+|y|^p)^2(\|\varepsilon\|^2+\|\zeta\|^2)+m_{21}^2\phi^2(\|\varepsilon\|^2+\|\zeta\|^2),\end{aligned} \tag{D2}$$

where $m_{21}=3\beta cn^{\frac32}M^{n-1}\|P\|$ is a positive constant. By (11) and (D2), (B6) can be further expressed as

$$\dot V_1\leqslant-\Big(\beta r-\bar m_2(t)-\beta d_2\rho_1L_1\Big)\|\varepsilon\|^2+\Big((1+|y|^p)^2+\bar m_2(t)+\beta r\|P\Lambda_2\|^2\Big)\|\zeta\|^2-(\beta d_1\rho_2-1)(1+|y|^p)^2\|\varepsilon\|^2, \tag{D3}$$

where $\bar m_2(t)=m_{21}^2\phi^2(t)$ is continuous and satisfies $\lim\limits_{t\to+\infty}\frac{\bar m_2(t)}{L_2(t)}=0$. Now, one chooses $\bar\theta<\tilde\theta=\min\big\{1,\frac{1}{2b_1\|Q\|}\big\}$, which together with Assumption 1 and $\sigma=1-2b_1\bar\theta\|Q\|$ indicates

$$1-2b_1|1-\theta|\cdot\|Q\|\geqslant\sigma.$$

Obviously, $\sigma\in(0,1)$, and (C5) is rewritten as

$$\dot V_2\leqslant rk_1\|\varepsilon\|^2+\bar m_1(t)\|\zeta\|^2+rk_2\|\zeta\|^2-rM\sigma\|\zeta\|^2+d_4\rho_1L_1\|\zeta\|^2-(d_3\rho_2-1)(1+|y|^p)^2\|\zeta\|^2, \tag{D4}$$

where $k_1=1+\|Q\Lambda_2\|^2$ and $k_2=1+2\|Q\Lambda_2\|+\|Q\|^2$. One constructs a continuously differentiable function $V_e(\varepsilon,\zeta)=V_1(\varepsilon)+V_2(\zeta)$. Then, (D3) and (D4) yield

$$\begin{aligned}\dot V_e \leqslant& -\Big(\beta r-rk_1-\bar m(t)-\beta d_2\rho_1L_1\Big)\|\varepsilon\|^2-\Big(rM\sigma-rk_3-\bar m(t)-d_4\rho_1L_1\Big)\|\zeta\|^2\\ &-(\beta d_1\rho_2-1)(1+|y|^p)^2\|\varepsilon\|^2-(d_3\rho_2-2)(1+|y|^p)^2\|\zeta\|^2,\end{aligned} \tag{D5}$$

where $\bar m(t)=\bar m_1(t)+\bar m_2(t)$ satisfies $\lim\limits_{t\to+\infty}\frac{\bar m(t)}{L_2(t)}=0$, and $k_3=k_2+\beta\|P\Lambda_2\|^2$. If the parameters ρ_1, ρ_2, b, M and β are chosen to satisfy (17), then (D5) is written as

$$\dot V_e\leqslant-(L_2^2L_1-\bar m)(\|\varepsilon\|^2+\|\zeta\|^2). \tag{D6}$$

Thus, inequality (18) holds.

Appendix E Proof of Claims 1-6

Claim 1. **$W(t)$ is bounded on $[t_0, T_f)$ with $T_f < +\infty$.** From (D6), for all $t \in [t_0, T_f)$, one gets

$$\dot{V}_e(t) \leqslant \bar{m}(t)(\|\varepsilon\|^2 + \|\zeta\|^2). \tag{E1}$$

Note that the function V_e satisfies the following inequality:

$$\bar{d}_1(\|\varepsilon\|^2 + \|\zeta\|^2) \leqslant V_e \leqslant \bar{d}_2(\|\varepsilon\|^2 + \|\zeta\|^2), \tag{E2}$$

where $\bar{d}_1 = \min\{\beta\lambda_{\min}(P), \lambda_{\min}(Q)\}$ and $\bar{d}_2 = \max\{\beta\lambda_{\max}(P), \lambda_{\max}(Q)\}$ are positive constants. Then, by (E2) and (E1), there holds for all $t \in [t_0, T_f)$ that

$$\bar{d}_1(\|\varepsilon(t)\|^2 + \|\zeta(t)\|^2) \leqslant V_e(t) \leqslant V_e(t_0)\exp\Big(\int_{t_0}^{t} \mu_2\bar{m}(s)ds\Big),\ \forall t \in [t_0, T_f). \tag{E3}$$

The continuity of $\bar{m}(t)$ on the domain $[0, +\infty)$ shows it is bounded on $[t_0, T_f)$. From this and (E3), it is not difficult to derive that $W(t)$ is bounded on $[t_0, T_f)$.

Claim 2. **$T_f = +\infty$.** Suppose $T_f < +\infty$, then T_f would be a finite escape time; that is, at least one component of $W(t)$ tends to $+\infty$ when $t \to T_f$. This contradicts the Claim 1. Thus, $T_f = +\infty$.

Claim 3. **$\lim_{t\to+\infty} W(t) = 0$.** Defining $\mu_1 = \frac{1}{\bar{d}_2}$, $\mu_2 = \frac{1}{\bar{d}_1}$ and using $L_1 \geqslant 1$, it follows from (D6) and (E2) that

$$\dot{V}_e \leqslant -(\mu_1 L_2^2 - \mu_2\bar{m})V_e. \tag{E4}$$

In addition, from the definition of $\bar{m}(t)$ and (5), one gets $\lim_{t\to+\infty} \frac{\bar{m}(t)}{L_2(t)} = 0$. Then, using (i) in (5), there is a sufficiently large time $T_0 (T_0 \geqslant t_0)$ which makes $\frac{1}{2}\mu_1 L_2^2(t) \geqslant \mu_2\bar{m}(t)$ hold for all $t \geqslant T_0$. In addition, (ii) in (5) guarantees that there is a sufficiently large time $T_1 > T_0$, which makes $L_2^2(t) \geqslant \dot{L}_2(t)$ hold for all $t \geqslant T_1$. Thus, (E4) is written as

$$\dot{V}_e(t) \leqslant -\tfrac{\mu_1}{2}\dot{L}_2(t)V_e(t),\ \forall t \geqslant T_1. \tag{E5}$$

A direct integration of (E5) from T_1 to t yields for all $t \geqslant T_1$ that

$$\bar{d}_1(\|\varepsilon(t)\|^2 + \|\zeta(t)\|^2) \leqslant V_e(t) \leqslant V_e(T_1)\exp\Big(\tfrac{\mu_1(L_2(T_1) - L_2(t))}{2}\Big). \tag{E6}$$

In other words, there holds for all $t \geqslant T_1$ and $i = 1, \ldots, n$ that

$$|\varepsilon_i(t)| \leqslant A_1\exp\Big(\frac{-\mu_1 L_2(t)}{4}\Big),\quad |\zeta_i(t)| \leqslant A_1\exp\Big(\frac{-\mu_1 L_2(t)}{4}\Big), \tag{E7}$$

where the constant $A_1 = \sqrt{\mu_2 V_e(T_1)}\exp\Big(\frac{\mu_1(L_2(T_1)}{4}\Big) > 0$. This implies that $\lim_{t\to+\infty}\varepsilon(t) = 0$ and $\lim_{t\to+\infty}\zeta(t) = 0$. Thus, $\lim_{t\to+\infty} W(t) = 0$.

Claim 4. **$\lim_{t\to+\infty} x(t) = 0$ and $\lim_{t\to+\infty}\hat{x}(t) = 0$.** By (D1), (E6) and Assumption 1, one gets

$$|y(t)| \leqslant 2\sqrt{\mu_2 V_e(T_1)}r^v(t),\ \forall t \geqslant T_1. \tag{E8}$$

Combining (4), (E8) with $L_1 \geqslant 1$, one deduces that

$$\begin{aligned} \dot{L}_1(t) &\leqslant -L_1(t)\Big(\rho_1 L_1(t) - \rho_2\Big(1 + 2^p(L_2^2(t) + b)^{pv}L_1^{pv}(t)(\mu_2 V_e(T_1))^{\frac{p}{2}}\Big)^2\Big) \\ &\leqslant -\rho_1 L_1^{1+2pv}(t)\Big(L_1^{1-2pv}(t) - \frac{\rho_2}{\rho_1}\Big(1 + 2^p(L_2^2(t) + b)^{pv}(\mu_2 V_e(T_1))^{\frac{p}{2}}\Big)^2\Big) \\ &= -\rho_1 L_1^{1+2pv}(t)\Big(L_1^{1-2pv}(t) - \varpi(t)\Big),\ \forall t \geqslant T_1, \end{aligned} \tag{E9}$$

where $\varpi(t) = \frac{\rho_2}{\rho_1}\Big(1 + 2^p(L_2^2(t) + b)^{pv}(\mu_2 V_e(T_1))^{\frac{p}{2}}\Big)^2$ is monotonically increasing on $[T_1, +\infty)$ and satisfies $\lim_{t\to+\infty}\varpi(t) = +\infty$. Next, we prove that there is a time $T_2 (T_2 \geqslant T_1)$ ensuring the following inequality

$$L_1^{1-2pv}(t) \leqslant \varpi(t),\ \forall t \geqslant T_2 \tag{E10}$$

holds on the hypothesis of $1 - 2pv > 0$. In fact, the conclusion can be drawn by considering two contrary cases. (i) If there is a time $\bar{T}_1 (\bar{T}_1 \geqslant T_1)$, which makes $L_1^{1-2pv}(t) > \varpi(t)$ hold for all $t \geqslant \bar{T}_1$, then $\lim_{t\to+\infty}\varpi(t) = +\infty$ implies $\lim_{t\to+\infty} L_1^{1-2pv}(t) = +\infty$. However, using such an inequality in (E9) leads to $\dot{L}_1(t) < 0, \forall t \geqslant \bar{T}_1$. This contradicts $\lim_{t\to+\infty} L_1^{1-2pv}(t) = +\infty$. (ii) If there exist two time moments $\bar{T}_1', \bar{T}_2' (\bar{T}_2' > \bar{T}_1' \geqslant T_1)$ which make the following relationship hold

$$\begin{cases} L_1^{1-2pv}(\bar{T}_1') = \varpi(\bar{T}_1'),\quad L_1^{1-2pv}(\bar{T}_2') = \varpi(\bar{T}_2'), \\ L_1^{1-2pv}(t) > \varpi(t),\ \forall t \in (\bar{T}_1', \bar{T}_2'), \end{cases} \tag{E11}$$

then, according to (E9) and the inequality in (E11), it can be concluded that $\dot{L}_1(t) < 0$ for all $t \in (\bar{T}_1', \bar{T}_2')$, which demonstrates $L_1^{1-2pv}(t) < L_1^{1-2pv}(\bar{T}_1')$ for all $t \in (\bar{T}_1', \bar{T}_2')$. Substituting the first equation in (E11) into the obtained inequality, one immediately has $L_1^{1-2pv}(t) < \varpi(\bar{T}_1'), \forall t \in (\bar{T}_1', \bar{T}_2')$. Again, by the monotonically increasing of $\varpi(t)$ on $(\bar{T}_1', \bar{T}_2') \subseteq [T_1, +\infty)$, one finally achieves $L_1^{1-2pv}(t) < \varpi(t)$ for all $t \in (\bar{T}_1', \bar{T}_2')$. This inequality contradicts the one in (E11). Of course, if there are more than two time instants; namely, $\bar{T}_1' < \bar{T}_2' < \cdots < \bar{T}_i' < \cdots$ with $L_1^{1-2pv}(\bar{T}_i') = \varpi(\bar{T}_i')$ and $L_1^{1-2pv}(t) > \varpi(t), \forall t \in (\bar{T}_i', \bar{T}_{i+1}')$, the method in (ii) is still applicable. Thus (E10) holds. Then, using simple calculations, one deduces from (8), (11), (E7) and (E8) that for all $t \geqslant T_2$:

$$\begin{aligned} |\hat{x}_1(t)| &\leqslant r^v(t)(|\zeta_1(t)| + |\varepsilon_1(t)|) \leqslant (L_2^2(t)+b)^v \varpi^{\frac{v}{1-2pv}}(t)(|\zeta_1(t)| + |\varepsilon_1(t)|) \\ &\leqslant 2A_1 \exp\Big(\frac{-\mu_1 L_2(t)}{4}\Big)(L_2^2(t)+b)^v \varpi^{\frac{v}{1-2pv}}(t), \end{aligned} \tag{E12}$$

and for $i = 2, 3, \ldots, n$:

$$\begin{aligned} |\hat{x}_i(t)| &\leqslant r^{v+i-1}(t) M^{i-1} |\zeta_i(t)| \leqslant (L_2^2(t)+b)^{v+i-1} M^{i-1} \varpi^{\frac{v+i-1}{1-2pv}}(t) |\zeta_i(t)| \\ &\leqslant A_1 M^{i-1} \exp\Big(\frac{-\mu_1 L_2(t)}{4}\Big)(L_2^2(t)+b)^{v+i-1} \varpi^{\frac{v+i-1}{1-2pv}}(t). \end{aligned} \tag{E13}$$

It follows from (i) in (5) and the definition of $\varpi(t)$ that $\lim\limits_{t\to+\infty} \frac{\varpi(t)}{(L_2^2(t)+b)^{2pv}} < +\infty$; that is, there is a finite time $T_3 (T_3 \geqslant T_2)$ which makes the following inequality hold

$$\varpi(t) \leqslant A_2 (L_2^2(t)+b)^{2pv}, \ \forall t \geqslant T_3, \tag{E14}$$

where A_2 is a finite positive constant. Then, using (E14) in (E12) gives rise to the following inequality:

$$|\hat{x}_1(t)| \leqslant \bar{A}_1 \exp\Big(\frac{-\mu_1 L_2(t)}{4}\Big)(L_2^2(t)+b)^{\frac{v}{1-2pv}}, \ \forall t \geqslant T_3, \tag{E15}$$

where $\bar{A}_1 = 2A_1 A_2^{\frac{v}{1-2pv}}$ is a positive constant. Notice that $1 - 2pv > 0$ and

$$\lim_{t\to+\infty} \exp\Big(\frac{-\mu_1 L_2(t)}{4}\Big)(L_2^2(t)+b)^{\frac{v}{1-2pv}} = \lim_{s\to+\infty} \exp\Big(\frac{-\mu_1 s}{4}\Big)(s^2+b)^{\frac{v}{1-2pv}} = 0.$$

Thus, there holds $\lim\limits_{t\to+\infty} \hat{x}_1(t) = 0$. Taking similar calculations to (E13), there also holds $\lim\limits_{t\to+\infty} \hat{x}_i(t) = 0$ for $i = 2, 3, \ldots, n$. Moreover, it can be obtained from (8), (E7), (E10) and (E14) that for $i = 1, \ldots, n$,

$$\begin{aligned} |x_i(t)| &\leqslant (L_2^2(t)+b)^{v+i-1} A_2^{\frac{v+i-1}{1-2pv}} (L_2^2(t)+b)^{\frac{2pv(v+i-1)}{1-2pv}} A_1 \exp\Big(\frac{-\mu_1 L_2(t)}{4}\Big) + |\hat{x}_i(t)| \\ &= \bar{A}_i (L_2^2(t)+b)^{\frac{v+i-1}{1-2pv}} \exp\Big(\frac{-\mu_1 L_2(t)}{4}\Big) + |\hat{x}_i(t)|, \ \forall t \geqslant T_3, \end{aligned} \tag{E16}$$

where $\bar{A}_i = A_1 A_2^{\frac{v+i-1}{1-2pv}}$ is a positive constant. Letting $t \to +\infty$ on both sides of (E16), one immediately has $\lim\limits_{t\to+\infty} x_i(t) = 0$. So far, there hold $\lim\limits_{t\to+\infty} x(t) = 0$ and $\lim\limits_{t\to+\infty} \hat{x}(t) = 0$.

Claim 5. $\lim\limits_{t\to+\infty} u(t) = 0$**.** By (13), (E10) and (E14)-(E16), one has

$$|u(t)| \leqslant r^n(t) M^n b_1 |\theta(t)| \cdot |x_1(t)| + \sum_{i=2}^{n} r^{n-i+1}(t) b_i |\hat{x}_i(t)| \leqslant \sum_{i=1}^{n} \tilde{A}_i (L_2^2(t)+b)^{\frac{n+v+2i-2}{1-2pv}} \exp\Big(\frac{-\mu_1 L_2(t)}{4}\Big), \forall t \geqslant T_3, \tag{E17}$$

where $\tilde{A}_1 = 2\bar{A}_1 A_2^{\frac{n}{1-2pv}} M^n b_1 (1+\bar{\theta})$ and $\tilde{A}_i = A_1 M^n b_i A_2^{\frac{n+v+2i-2}{1-2pv}}, i = 2\ldots, n$ are positive constants. Letting $t \to +\infty$ on both sides of (E17) again, one achieves $\lim\limits_{t\to+\infty} u(t) = 0$.

Claim 6. $L_1(t)$ **is bounded on** $[t_0, +\infty)$**.** By the definition of $L_1(t)$ and the boundedness of $y(t)$, one gets

$$\dot{L}_1(t) \leqslant -\rho_1 L_1^2(t) + \rho_3 L_1(t), \tag{E18}$$

where $\rho_3 = \sup\limits_{t\in[t_0,+\infty)} \rho_2 (1 + |y(t)|^p)^2$ is a positive constant. Notice that the solution of the equation $\dot{L}_1(t) = \rho_1 L_1^2(t) + \rho_3 L_1(t)$ with initial condition $L_1(t_0) = 1$ is $L_1(t) = \frac{\rho_3}{\rho_1 + (\rho_3-\rho_1)\exp(-\rho_3(t-t_0))}$. As a result, the solution of the inequality (E18) satisfies

$$L_1(t) \leqslant \frac{\rho_3}{\rho_1 + (\rho_3-\rho_1)\exp(-\rho_3(t-t_0))} \leqslant \lim_{t\to+\infty} \frac{\rho_3}{\rho_1 + (\rho_3-\rho_1)\exp(-\rho_3(t-t_0))} = \frac{\rho_3}{\rho_1},$$

which shows that $L_1(t)$ is bounded on $[t_0, +\infty)$. This completes the proof. □

Appendix F Simulation examples

Example 1: an application example.

To demonstrate the potential application of the presented control scheme, we take into account the motion of an object with a mass of 1000 g which is connected with the wall by a nonlinear spring in a lubricant horizon. Such a motion can be modeled as

$$\ddot{\varrho}(t) + F_{\sigma_1}(\dot{\varrho}(t)) + F_{\sigma_2}(\varrho(t)) = F(t), \tag{F1}$$

where $\varrho(t)$ is the displacement of the object, $F(t)$ is the driving force, $F_{\sigma_1}(\dot{\varrho}(t)) = \check{a}\dot{\varrho}(t)$ is a resistive force due to the friction with $\check{a}$ being an viscosity coefficient, $F_{\sigma_2}(\varrho(t))$ is a restoring force of the spring. It is known that the spring constant does not change within the range of the restoring force. However, if a higher pull breaks through the range of the restoring force, then a small displacement increment produces a larger force increment. In other words, the spring constant keeps invariable within a certain displacement, and changes continuously as time increases beyond the certain displacement. Of course, the physical feature of the spring shows that the spring constant is bounded. In this example, the restoring force is characterized by $F_{\sigma_2}(\varrho(t)) = \frac{\pi}{4}\varrho(t)(1+\varrho^2(t))$ for all $t \in [0,1]$ and $F_{\sigma_2}(\varrho(t)) = \varrho(t)(1+\varrho^2(t))\arctan t$ for all $t \geqslant 1$. In the design process, the displacement $\varrho(t)$ is measured by the displacement sensor which is likely subject to $\pm 10\%$ error because of the limited fabrication technology [1]. If the designer chooses $x_1(t) = \varrho(t), x_2(t) = \dot{\varrho}(t), u(t) = F(t)$ and lets $\phi(t) = \frac{\pi}{4}, \forall t \in [0,1]$ and $\phi(t) = \arctan t, \forall t \geqslant 1$, then (F1) can be considered a particular case of system (1). Specifically, $f_1 = 0, f_2 = -\phi(t)(1+x_1^2(t))x_1(t) - \breve{a}x_2(t)$ and $\theta(t)$ varies continuously on the interval $[0.9, 1.1]$. Before determining the regulator, the choice of $a_1 = 1, a_2 = 1, b_1 = 2, b_2 = 3, \breve{a} = 1$ ensures that Assumption 2 holds for $c = 2, p = 2$ as well as Assumption 1 is satisfied with $\bar{\theta} = 0.1 < 0.1890$ respectively. According to (8), one selects the constant parameters as $M = 9, b = 3.8$, and determines $L_1(t)$ via $\dot{L}_1(t) = -0.1L_1^2(t) + 20(1+|y(t)|)^2 L_1(t), L_1(0) = 1$, and $L_2(t) = t^{\frac{1}{2}}$. Then the actual regulator is constructed as

$$\begin{cases} u(t) = -162(t+3.8)^2 L_1^2(t) y(t) - 27(t+3.8)L_1(t)\hat{x}_2(t), \\ \dot{\hat{x}}_1(t) = \hat{x}_2(t) - (t+3.8)L_1(t)\hat{x}_1(t), \\ \dot{\hat{x}}_2(t) = u(t) - (t+3.8)^2 L_1^2(t)\hat{x}_1(t). \end{cases}$$

To run the simulation properly, we choose $[x_1(0), x_2(0), \hat{x}_1(0), \hat{x}_2(0)]^{\mathrm{T}} = [-0.2, -1, 0.6, 0.5]^{\mathrm{T}}$ as the initial values. In Figures.1-4, the simulation results are given which indicate the validity of the presented control scheme.

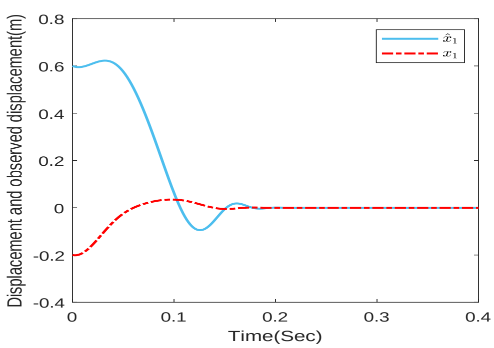

Fig. 1 The curves of displacement and observed displacement.

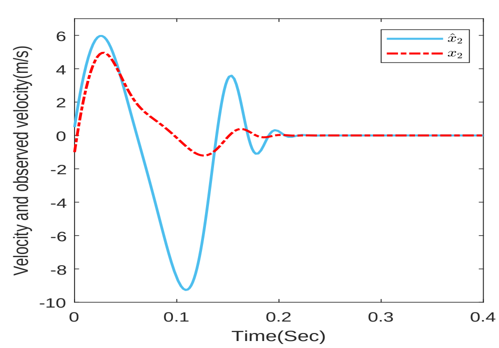

Fig. 2 The curves of velocity and observed velocity.

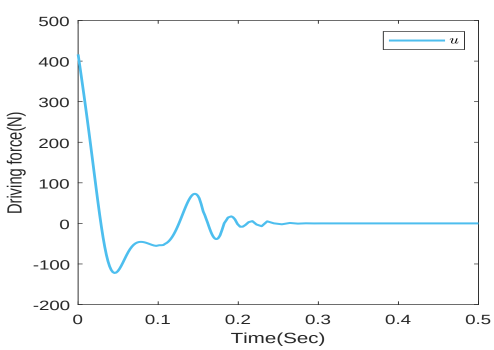

Fig. 3 The curve of driving force.

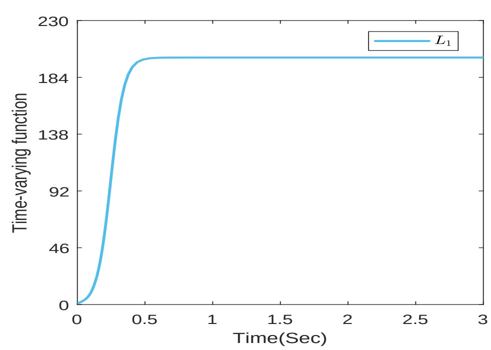

Fig. 4 The curve of time-varying function.

Example 2: a numerical example. To detailedly illustrate the influence on system performance in the existence of unbounded time-varying function $\phi(t)$, we consider the nonlinear system descried by

$$\begin{cases} \dot{x}_1(t) = x_2(t) + 2(t+1)(1+y(t))x_1(t), \\ \dot{x}_2(t) = u(t) + x_2(t)(1+y(t))\sin(6x_1(t)), \end{cases}$$

where $y(t) = \theta(t)x_1(t)$, and $\theta(t) = 1 + 0.1|\sin(10t)|$. Assumption 2 holds with $c = 2, p = 1, \phi(t) = t+1$, and the choice of $a_1 = \frac{1}{2}, a_2 = 1, b_1 = 1, b_2 = \frac{1}{2}$ ensures that Assumption 1 holds with $\bar{\theta} = 0.1 < 0.1120$. Similarly, one selects the constant parameters as $M = 9, b = 2.3$, and further designs $L_1(t)$ based on $\dot{L}_1(t) = -0.02L_1^2(t) + 5(1+|y(t)|)^2 L_1(t), L_1(0) = 1$ and $L_2(t) = t^3$, which evokes the actual regulator as

$$\begin{cases} u(t) = -81(t^4+2.3)^2 L_1^2(t)y(t) - \frac{9}{2}(t^4+2.3)L_1(t)\hat{x}_2(t), \\ \dot{\hat{x}}_1(t) = \hat{x}_2(t) - \frac{1}{2}(t^4+2.3)L_1(t)\hat{x}_1(t), \\ \dot{\hat{x}}_2(t) = u(t) - (t^4+2.3)^2 L_1^2(t)\hat{x}_1(t). \end{cases}$$

To illustrate the superiority of the presented control scheme in this note, we also consider the following controller based on the dual-domination scheme in [2]:

$$\begin{cases} u(t) = -72y(t) - 12\hat{x}_2(t), \\ \dot{\hat{x}}_1(t) = \hat{x}_2(t) - 2\hat{x}_1(t), \\ \dot{\hat{x}}_2(t) = u(t) - 16\hat{x}_1(t). \end{cases}$$

To run the simulation, we use the same initial values as $[x_1(0), x_2(0), \hat{x}_1(0), \hat{x}_2(0)]^{\mathrm{T}} = [0.5, -1, -2, 3]^{\mathrm{T}}$. Figs.1-3 show some comparisons between [2] and this paper. To be specific, Figs. 5-6 show that the presented control scheme in this note ensures that $x(t)$ and $\hat{x}(t)$ are bounded and eventually converge to zero, while the dual-domination scheme in [2] fails to do that. Fig. 7 implies that $u(t)$ using the dual-domination scheme in [2] tends to infinity in a finite time, whereas the scheme in this note renders $u(t)$ to converge to zero. Fig. 8 indicates that $L_1(t)$ is bounded.

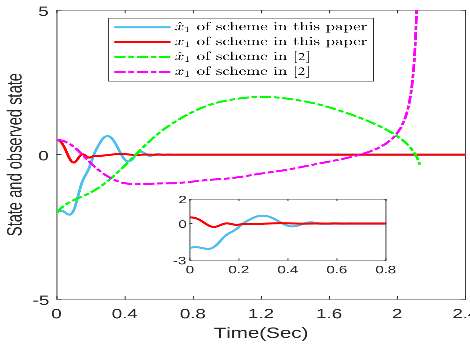

Fig. 5 The curves of x_1 and $\hat{x}_1$.

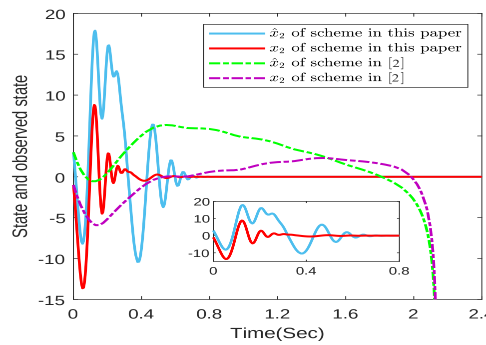

Fig. 6 The curves of x_2 and $\hat{x}_2$.

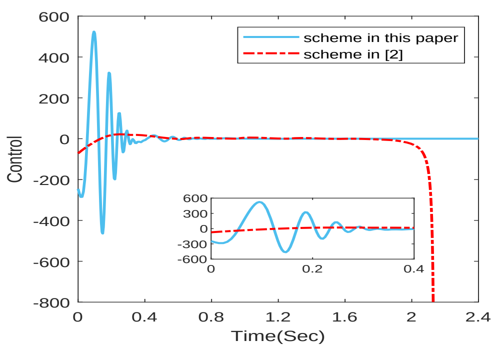

Fig. 7 The curve of u.

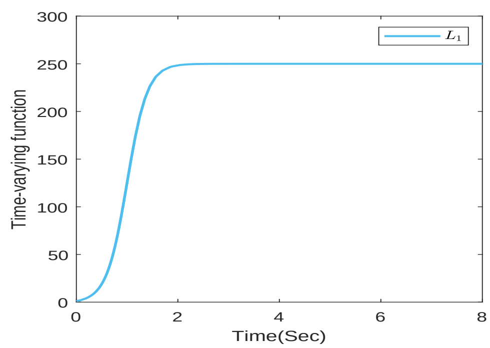

Fig. 8 The curve of L_1.

References

1 Lantto E. Robust control of magnetic bearings in subcritical machines. Dept, Elect. Eng., Helsinki Univ. Technol., Espoo, Finland, Feb. 1999.

2 Chen C C, Qian C J, Sun Z Y, et al. Global output feedback stabilization of a class of nonlinear systems with unknown measurement sensitivity. IEEE Trans. Autom. Control, 2017, 63: 2212-2217.