SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

April 2024, Vol. 67, Iss. 4, 142202:1–142202:21
https://doi.org/10.1007/s11432-023-3874-6

Higher-order properties and extensions for indirect MRAC and APPC of linear systems

Yanjun ZHANG[1,2], Zhipeng ZHANG[1,2], Jian SUN[1,2,3*],
Lei WANG[4] & Kangkang ZHANG[5,6]

[1]*National Key Lab of Autonomous Intelligent Unmanned Systems, Beijing Institute of Technology, Beijing* 100081*, China;*
[2]*School of Automation, Beijing Institute of Technology, Beijing* 100081*, China;*
[3]*Beijing Institute of Technology Chongqing Innovation Center, Chongqing* 401120*, China;*
[4]*College of Control Science and Engineering, Zhejiang University, Hangzhou* 310058*, China;*
[5]*Department of Electrical and Electronic Engineering, Imperial College London, London* SW72AZ*, UK;*
[6]*KIOS Research and Innovation Center of Excellence, University of Cyprus, P.O. Box* 20537*, Nicosia* 1678*, Cyprus*

Received 10 April 2023/Revised 19 June 2023/Accepted 7 August 2023/Published online 29 February 2024

Abstract Recently, a reference derived some new higher-order output tracking properties for direct model reference adaptive control (MRAC) of linear time-invariant (LTI) systems: $\lim_{t\to\infty} e^{(i)}(t) = 0, i = 1, \ldots, n^*-1$, where n^* and $e^{(i)}(t)$ denote the relative degree of the system and the i-th derivative of the output tracking error, respectively. However, a naturally arising question involves whether indirect adaptive control (including indirect MRAC and indirect adaptive pole placement control) of LTI systems still has higher-order tracking properties. Such properties have not been reported in the literature. Therefore, this paper provides an affirmative answer to this question. Such higher-order tracking properties are new discoveries since they hold without any additional design conditions and, in particular, without the persistent excitation condition. Given the higher-order properties, a new adaptive control system is developed with stronger tracking features. (1) It can track a reference signal with any order derivatives being unknown. (2) It has higher-order exponential or practical output tracking properties. (3) Finally, it is different from the usual MRAC system, whose reference signal's derivatives up to the n^* order are assumed to be known. Finally, two simulation examples are provided to verify the theoretical results obtained in this paper.

Keywords higher-order tracking, indirect adaptive control, model reference adaptive control, adaptive pole placement control

1 Introduction

Let a general class of continuous-time linear time-invariant (LTI) systems be $y(s) = G(s)u(s)$ for $G(s) = \frac{Z(s)}{P(s)}$, where $Z(s)$ and $P(s)$ are zero and pole polynomials with unknown coefficients, respectively, and s is the Laplace transform variable. Two basic approaches, namely, direct and indirect adaptive control approaches, are developed in designing adaptive output tracking controllers. The former directly estimates the controller's unknown parameters. However, the latter first estimates the unknown system parameters and then maps the estimated system parameters to the controller parameters using an algebraic equation. The direct model reference adaptive control (MRAC), which is the main branch of direct adaptive control, has been systematically developed, thus making it mature. The indirect adaptive control mainly contains indirect MRAC and indirect adaptive pole placement control (APPC). Furthermore, the MRAC covering direct and indirect cases forces the closed-loop system to match a reference system and impose zero-pole cancellation. Such cancellation must be stable to ensure closed-loop stability. Thus, one key design condition of MRAC is that the zeros of the control system are stable. For LTI systems with unstable zeros and poles, the APPC is the main strategy to achieve stable output tracking under an internal model design condition on the reference system.

MRAC and APPC have been extensively studied in the literature. For instance, some basic adaptive control theory references [1–6] presented complete proofs of MRAC and APPC system properties, while

* Corresponding author (email: sunjian@bit.edu.cn)

Ref. [7] surveyed some representative references on adaptive control. Recently, Ref. [8] derived some new higher-order properties for a direct MRAC system: $\lim_{t\to\infty} e^{(i)}(t) = 0$, $i = 1, \ldots, n^* - 1$, where n^* and $e^{(i)}(t)$ denote the relative degree of the system and the i-th derivative of the output tracking error, respectively. The new results presented in [8] bring the direct MRAC system performance closer to that of a nominal control system (assuming all system parameters are known): $\lim_{t\to\infty} e^{(i)}(t) = 0$, $i = 1, \ldots, n^* - 1$, exponentially. Additionally, in [9], the authors extended the results in [8] to nonlinear systems and derived new higher-order tracking properties of nonlinear adaptive control systems. The results in [10] indicate that the well-known backstepping technique in [11] is also effective in handling the issue discussed in [8]. So far, adaptive control is still prevalent in the control community, and new progress is constantly made in this direction (see [12–27]) and some studies of the authors of this paper ([28–31]).

However, there has been no report on whether an indirect adaptive control system under an MRAC or an APPC framework has higher-order tracking properties. As mentioned, MRAC only deals with minimum-phase systems, while APPC deals with minimum-phase systems and effectively controls non-minimum-phase systems. Moreover, compared with direct adaptive control systems, indirect adaptive control is better for integration with other advanced control techniques to achieve the desired system performance. This is because the indirect adaptive control strategy first designs the parameter update laws to estimate the unknown system parameters and then uses these estimated parameters to develop an estimated system model with known signals and parameters. The structure of the estimated system model is similar to the original system and is suitable to be integrated with other advanced control techniques to design the controllers. Hence, it is important to further address the higher-order output tracking properties of the indirect adaptive control systems to promote the theory. Additionally, the higher-order convergence of the tracking error is meaningful in some applications. For example, the pitch angle tracks a reference angle to avoid extreme oscillations that are harmful to the aircraft. For the pitch angle to smoothly track a reference angle, its first or second derivative should track some prescribed trajectories. The higher-order output tracking properties favorably fulfill the real control requirement. In other words, it is theoretically and practically significant to address the higher-order tracking properties of indirect adaptive control. Based on the above consideration, this paper systematically addresses the higher-order tracking properties of indirect adaptive control systems, covering indirect MRAC systems and indirect APPC systems, by demonstrating that the higher-order derivatives of the tracking error also converge to zero asymptotically under the usual indirect adaptive control design conditions.

Furthermore, based on the higher-order tracking properties discovered in this paper, a new adaptive control system is developed, which is required to track any given reference signal under the condition that the derivatives of the reference signal are all unknown. It is shown that some higher-order exponential or practical output tracking properties are still ensured. One may think that the standard high-gain differential observer in [32] may be used to estimate the derivatives of the reference signal and that the usual MRAC law can still be effective by replacing the derivatives of the reference signal with their estimates generated from the high-gain differential observer. However, the exponential convergence of the tracking error cannot be realized even under the persistent excitation (PE) condition (PE definition is given in Appendix A). This is because the mismatch between the derivatives of the reference signal and their estimates cannot be eliminated, as commonly seen in the literature ([32]). By contrast, the new adaptive control system developed in this paper ensures that the tracking error and some of its certain order derivatives converge to zero exponentially fast under the PE condition. The new adaptive control system also has some application prospects in engineering. For example, considering an operational scenario in which an unmanned aerial vehicle is required to track an enemy aircraft, the new adaptive control strategy can be used for constructing an adaptive controller that ensures globally exponential or practical tracking (including position tracking, velocity tracking, and accelerated velocity tracking) under the condition that the velocity and accelerated velocity of the enemy aircraft are unmeasurable. In summary, the novelties and contributions of this paper are as follows.

(1) For the indirect adaptive control of continuous-time LTI systems described by a general input-output form, this paper shows that the tracking error and some of its certain order derivatives converge to zero asymptotically. These higher-order tracking properties have not been reported in the literature.

(2) The higher-order tracking properties are new discoveries for the indirect adaptive control systems that cover indirect MRAC and APPC systems. These higher-order properties only depend on the usual conditions similar to the usual indirect MRAC and APPC designs, particularly without the PE condition.

(3) This paper develops a new adaptive control system based on the higher-order tracking properties of the indirect MRAC system. This system exhibits some new stronger tracking capabilities. (i) It can track

any given reference signal with any order derivatives being unknown. (ii) It has higher-order exponential or practical output tracking properties under different design conditions. (iii) Finally, it is different from the usual MRAC system whose reference output's derivatives up to the n^* order are known.

The remainder of this paper is organized as follows. Section 1 provides a notation description. Section 2 introduces the background of indirect MRAC and indirect APPC. Section 3 addresses some new higher-order properties of indirect MRAC and APPC systems and gives the extensions of the higher-order properties to the development of an adaptive control system with stronger tracking properties. Section 4 presents two simulation examples to verify the theoretical results. Finally, Section 5 concludes the main work of this paper.

Notation. In this paper, s denotes the Laplace transform variable or the time differentiation operator, i.e., $s[x](t) = \dot{x}(t)$, where $x(t)$ refers to any signal of any finite dimension. The signal spaces L^2 and L^∞ are defined as $L^2 = \{x(t) : \|x(\cdot)\|_2 < \infty\}$ and $L^\infty = \{x(t) : \|x(\cdot)\|_\infty < \infty\}$, where $\|x(\cdot)\|_2 = \sqrt{\int_0^\infty \|x(t)\|_2^2 \mathrm{d}t}$ and $\|x(\cdot)\|_\infty = \sup_{t \geqslant 0} \|x(t)\|_\infty$. Moreover, $y(t) = G(s)[u](t) \triangleq \mathcal{L}^{-1}[G(s)u(s)]$ is the output $y(t)$ of a continuous-time LTI system represented by a transfer function $G(s)$ with an input $u(t)$, where $\mathcal{L}^{-1}[\cdot]$ is the inverse Laplace transform operator. This notation is simple, combining time and s-domain signal operations and avoiding complex convolution expressions for control system presentation. A similar notation is exploited in [5, 6].

2 Review of indirect MRAC and APPC systems

Consider an LTI system described by the input-output form:

$$y(s) = G(s)u(s),\ G(s) = \frac{Z(s)}{P(s)}, \tag{1}$$

where $Z(s)$ and $P(s)$ are polynomials: $P(s) = s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0$ and $Z(s) = z_{n-1}s^{n-1} + z_{n-2}s^{n-2} + \cdots + z_1 s + z_0$. The coefficients of $P(s)$ and $Z(s)$ are all unknown. To represent the system using differential equations, the system (1) can also be defined as

$$P(s)[y](t) = Z(s)[u](t),\ t \geqslant 0, \tag{2}$$

where $P(s)$ and $Z(s)$ are differential operators. This notation is particularly useful for signal operations in adaptive control systems [6, 7, 9].

All derivations presented in this paper are based on the input-output model (2), which does not require any information about the system matrices. Especially for the output feedback adaptive control, there is no need to design a full state observer for the control design when using (2). It should be noted that for some black-box systems, it may be challenging to build a state-space system model when no information about the internal state variables is available. Thus, it is of significance to address adaptive control problems using the input-output model (2).

Reference model. The reference model for indirect MRAC is

$$y_m(t) = W_m(s)[r](t), \tag{3}$$

where $y_m(t)$ is the reference output, $r(t)$ is the reference input, and $W_m(s)$ is a known and stable transfer function with its relative degree equal to that of the system (2). As shown in [5, 6], the model (3) is a standard choice for the reference signal, and $W_m(s)$ is commonly chosen as $1/P_m(s)$ with $P_m(s)$ being a monic stable polynomial.

The reference model for the indirect APPC is

$$Q_m(s)[y_m](t) = 0, \tag{4}$$

where $Q_m(s) = s^{n_q} + q_{n_q-1}s^{n_q-1} + \cdots + q_1 s + q_0$, as the internal model of y_m, is a known monic polynomial of degree n_q with all roots in $\Re[s] \leqslant 0$ and with no repeated roots on the $j\omega$-axis. It should be noted that there are no restrictions on the degree of $Q_m(s)$. As clarified in [5, 6], the condition (4) is necessary for the APPC design.

The control objective of the general MRAC and APPC is the same: designing an output feedback adaptive control law $u(t)$ to ensure closed-loop stability and asymptotic output tracking $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$.

The following assumptions are for the indirect MRAC design.

Assumption A1. $Z(s)$ is a stable polynomial.

Assumption A2. The degree n of $P(s)$ is known.

Assumption A3. The system relative degree, defined as n^*, is known.

As clarified in Section 1, the closed-loop MRAC system involves zero-pole cancellation. Assumption A1 ensures that the cancellation is stable. Assumption A2 is used to construct a parametrized model for the system (2) to estimate the unknown system parameters. Assumption A3 is used to determine the degree of $P_m(s)$, and indicates that $z_{m+1} = \cdots = z_{n-1} = 0$, i.e., $Z(s) = z_m s^m + \cdots + z_1 s + z_0,\ z_m \neq 0,\ m = n - n^*$.

The following assumptions are for the APPC design.

Assumption B1. $Q_m(s)P(s)$ and $Z(s)$ are coprime.

Assumption B2. The order n of $P(s)$ is known.

Assumption B1 is used to calculate the control law parameters and necessary for pole placement-based tracking control design and analysis. Similar to Assumption A2, Assumption B2 is used to construct a parametrized model for parameter adaptation. Both assumptions are standard for APPC (the readers are referred to [5, 6]). Note that for a standard APPC system, there is no requirement to know the relative degree information n^*.

Next, we provide a brief introduction to the indirect adaptive control design procedure containing MRAC and APPC:

(i) construction of a parametrized model of the system (1);

(ii) development of a parameter update law;

(iii) derivation of an estimated system model; and

(iv) construction of an adaptive control law ensuring desired system performance.

Parametrized model. Define $\Lambda_e(s) = s^n + \lambda^e_{n-1}s^{n-1} + \cdots + \lambda^e_1 s + \lambda^e_0$ and $\Lambda_{n-1}(s) = \lambda^e_{n-1}s^{n-1} + \cdots + \lambda^e_1 s + \lambda^e_0$. A parametrized model for the system (2) is constructed as

$$y(t) - \frac{\Lambda_{n-1}(s)}{\Lambda_e(s)}[y](t) = \theta_p^{*\mathrm{T}}\phi(t), \tag{5}$$

where

$$\theta_p^* = [z_0, z_1, \ldots, z_{n-1}, -p_0, -p_1, \ldots, -p_{n-1}]^{\mathrm{T}}, \tag{6}$$

$$\phi(t) = \left[\frac{1}{\Lambda_e(s)}[u](t), \frac{s}{\Lambda_e(s)}[u](t), \ldots, \frac{s^{n-1}}{\Lambda_e(s)}[u](t), \frac{1}{\Lambda_e(s)}[y](t), \frac{s}{\Lambda_e(s)}[y](t), \ldots, \frac{s^{n-1}}{\Lambda_e(s)}[y](t)\right]^{\mathrm{T}}. \tag{7}$$

The elements in $\phi(t)$ are available signals obtained through filtering u or y by stable filters $\frac{s^i}{\Lambda_e(s)}$, $i = 0, \ldots, n-1$. For the indirect MRAC, the $n-m-1$ elements z_i, $i = m+1, \ldots, n-1$, in θ_p^* are zero. Thus, θ_p^* can be further specified as $\theta_p^* = [z_0, z_1, \ldots, z_m, 0, \ldots, 0, -p_0, -p_1, \ldots, -p_{n-1}]^{\mathrm{T}}$. Given that it does not cause any confusion, we use (6) and (7) to design and analyze the MRAC and APPC simultaneously.

Parameter update law. Let $\theta_p(t)$ denote the estimate of θ_p^*. First, we define an estimation error as

$$\epsilon(t) = \theta_p^{\mathrm{T}}(t)\phi(t) - y(t) + \frac{\Lambda_{n-1}(s)}{\Lambda_e(s)}[y](t),\ t \geqslant 0. \tag{8}$$

A typical parameter update law in the literature is

$$\dot{\theta}_p(t) = -\frac{\Gamma\phi(t)\epsilon(t)}{m^2(t)} + f(t),\ \theta_p(0) = \theta_0,\ t \geqslant 0, \tag{9}$$

where $\Gamma = \mathrm{diag}\{\Gamma_1, \Gamma_2\}$ with $\Gamma_i \in \mathbb{R}^{n\times n}$ and $\Gamma_i = \Gamma_i^{\mathrm{T}} > 0$, θ_0 is an initial estimate of θ_p^*, $f(t)$ is a modification term used to avoid the singularity problem of the adaptive control laws for MRAC and APPC, respectively. Moreover, $m(t) = \sqrt{1 + \kappa\phi^{\mathrm{T}}(t)\phi(t)}$ with $\kappa > 0$.

Estimated system model. Let

$$\theta_p(t) = [\hat{z}_0(t), \ldots, \hat{z}_{n-1}(t), -\hat{p}_0(t), \ldots, -\hat{p}_{n-1}(t)]^{\mathrm{T}}. \tag{10}$$

Using $\theta_p(t)$, we construct the estimates of $P(s)$ and $Z(s)$, respectively:

$$\hat{P}(s,\hat{p}) = s^n + \hat{p}_{n-1}s^{n-1} + \cdots + \hat{p}_1 s + \hat{p}_0,\ \hat{Z}(s,\hat{z}) = \hat{z}_{n-1}s^{n-1} + \hat{z}_{n-2}s^{n-2} + \cdots + \hat{z}_1 s + \hat{z}_0, \tag{11}$$

where $\hat{p} = [\hat{p}_0, \ldots, \hat{p}_{n-1}]^{\mathrm{T}}$ and $\hat{z} = [\hat{z}_0, \ldots, \hat{z}_{n-1}]^{\mathrm{T}}$ are the estimates of $p^* = [p_0, \ldots, p_{n-1}]^{\mathrm{T}}$ and $z^* = [z_0, \ldots, z_{n-1}]^{\mathrm{T}}$.

As presented above, there is almost no difference between the forms of the parametrized model, estimation error, parameter update law, and estimated system model of indirect MRAC and APPC. Nevertheless, a key point must be clarified for the control law singularity problem.

Clarifications for the singularity problem. Designing $f(t)$ for MRAC ensures that the estimate of z_m is away from zero and thus the MRAC law is nonsingular in the process of parameter adaptation. To this end, let $f(t) = [0_{1\times m}, f_{m+1}(t), 0_{1\times(2n-m-2)}]$. The following information about z_m is needed.

Assumption A4. A non-zero lower bound on $|z_m|$ is known, and so is the sign of z_m.

The purpose of designing $f(t)$ for APPC is to ensure that $Q_m(s)\hat{P}(s,\hat{p})$ and $\hat{Z}(s,\hat{p})$ are coprime so that the Diophantine equation

$$C(s,\psi_c)Q_m(s)\hat{P}(s,\hat{p}) + D(s,\psi_d)\hat{Z}(s,\hat{z}) = A^*(s) \tag{12}$$

has a unique solution $\{C(s,\psi_c), D(s,\psi_d)\}$, where $\psi_c = [c_0, c_1, \ldots, c_{n-2}]^{\mathrm{T}}, \psi_d = [d_0, d_1, \ldots, d_{n+n_q-1}]^{\mathrm{T}}$, and $A^*(s)$ is chosen as a stable monic polynomial of degree $2n + n_q - 1$. However, to design $f(t)$ for APPC, some additional information about θ_p^* would be needed. Such information can be given in a form suitable for performing parameter projection on the parameter update law constraining the parameters in $\theta_p(t)$ within some given intervals. Then, $Q_m(s)\hat{P}(s,\hat{p})$ and $\hat{Z}(s,\hat{z})$ are always coprime during parameter adaptation. In this paper, we employ the following singularity-free condition.

Assumption B3. $Q_m(s)\hat{P}(s,\hat{p})$ and $\hat{Z}(s,\hat{z})$ are coprime during parameter adaptation.

For the details about how to ensure that Assumption (B3) holds by designing $f(t)$, the readers are referred to [3–6].

Considering parameter adaptation, we have the following result.

Lemma 1 ([6])**.** The parameter update law (9) ensures that

(1) $\theta_p(t)$, $\dot{\theta}_p(t)$, and $\frac{\epsilon(t)}{m(t)}$ belong to L^∞; and

(2) $\frac{\epsilon(t)}{m(t)}$ and $\dot{\theta}_p(t)$ belong to L^2.

The proof of Lemma 1 is performed based on a Lyapunov-based analysis, where the Lyapunov function is chosen as $V = \tilde{\theta}_p^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}_p$ with $\tilde{\theta}_p = \theta_p - \theta_p^*$ and Γ in (9). The readers are referred to [3–6] for details.

Indirect MRAC law. The indirect MRAC law is

$$u(t) = \theta_1^{\mathrm{T}}(t)\omega_1(t) + \theta_2^{\mathrm{T}}(t)\omega_2(t) + \theta_{20}(t)y(t) + \theta_3(t)r(t), \tag{13}$$

where $\omega_1(t) = \frac{a(s)}{\Lambda_c(s)}[u](t) \in \mathbb{R}^{n-1}$ and $\omega_2(t) = \frac{a(s)}{\Lambda_c(s)}[y](t) \in \mathbb{R}^{n-1}$ with $a(s) = [1, s, \ldots, s^{n-2}]$ and $\Lambda_c(s)$ being a monic stable polynomial of degree $n-1$. With $\theta_3(t) = \frac{1}{\hat{z}_m(t)}$, θ_1, θ_2, and θ_{20} are obtained from

$$\theta_1^{\mathrm{T}}(t)a(\lambda)\hat{P}(\lambda,\hat{p}) + (\theta_2^{\mathrm{T}}(t)a(\lambda) + \theta_{20}(t)\Lambda_c(\lambda))\hat{Z}(\lambda,\hat{z}) = \Lambda_c(\lambda)(\hat{P}(\lambda,\hat{p}) - \theta_3(t)\hat{Z}(\lambda,\hat{z})P_m(\lambda)). \tag{14}$$

Regardless if $\hat{P}(\lambda,\hat{p})$ and $\hat{Z}(\lambda,\hat{z})$ are coprime or not, the above equation always has a solution $\{\theta_1, \theta_2, \theta_{20}\}$ with $\theta_3 = \frac{1}{\hat{z}_m}$. As demonstrated in [3–6], under Assumptions (A1)–(A4), the MRAC law (13) with the update law (9) ensures closed-loop stability and

$$\theta(t) \in L^\infty,\ \dot{\theta}(t) \in L^2,\ \lim_{t\to\infty} e(t) = 0,\ e(t) \in L^2,\ \dot{e}(t) \in L^\infty \tag{15}$$

with $e(t) = y(t) - y_m(t),\ \theta(t) = [\theta_1^{\mathrm{T}}(t), \theta_2^{\mathrm{T}}(t), \theta_{20}(t), \theta_3(t)]^{\mathrm{T}}$.

Indirect APPC law. The indirect APPC law is

$$u(t) = \psi_1^{\mathrm{T}}(t)\eta_1(t) + \psi_2^{\mathrm{T}}(t)\eta_2(t) + \psi_3(t)(y(t) - y_m(t)), \tag{16}$$

where $\eta_1(t) = \frac{b(s)}{\Lambda_d(s)}[u](t) \in \mathbb{R}^{n_q+n-1}$ and $\eta_2(t) = \frac{b(s)}{\Lambda_d(s)}[y-y_m](t) \in \mathbb{R}^{n_q+n-1}$ with $b(s) = [1, s, \ldots, s^{n_q+n-2}]$

$\in \mathbb{R}^{n_q+n-1}$ and $\Lambda_d(s) = s^{n_q+n-1} + \lambda^c_{n_q+n-2} s^{n_q+n-2} + \cdots + \lambda^c_1 s + \lambda^c_0$ being a chosen stable polynomial,

$$\psi_1(t) = -\begin{bmatrix} c_0(t) & 0 & \cdots & 0 \\ c_1(t) & c_0(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-2}(t) & c_{n-3}(t) & \cdots & 0 \\ 1 & c_{n-2}(t) & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & c_{n-3}(t) \\ 0 & 0 & \cdots & c_{n-2}(t) \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{n_q-1} \\ 1 \end{bmatrix} + \begin{bmatrix} \lambda^c_0 \\ \lambda^c_1 \\ \vdots \\ \lambda^c_{n_q+n-2} \end{bmatrix}, \ \psi_3(t) = -d_{n_q+n-1}(t), \tag{17}$$

$$\psi_2(t) = -\begin{bmatrix} d_0(t) \\ d_1(t) \\ \vdots \\ d_{n_q+n-2}(t) \end{bmatrix} + d_{n_q+n-1}(t) \begin{bmatrix} \lambda^c_0 \\ \lambda^c_1 \\ \vdots \\ \lambda^c_{n_q+n-2} \end{bmatrix}. \tag{18}$$

Note that $c_i(t)$ and $d_j(t)$, $i = 0, \ldots, n-2$, $j = 0, \ldots, n_q + n - 2$, are time-varying signals derived from the Diophantine equation (12) and q_i, $i = 0, \ldots, n_q - 1$, in $\psi_1(t)$ are the coefficients of $Q_m(s)$ below (4). As proven in [3–6], under Assumptions (B1)–(B3), the APPC law (16) ensures closed-loop stability and

$$\psi(t) \in L^\infty, \ \dot{\psi}(t) \in L^2, \ \lim_{t\to\infty} e(t) = 0, \ e(t) \in L^2, \ \dot{e}(t) \in L^\infty, \tag{19}$$

with $\psi(t) = [\psi_1^{\mathrm{T}}(t), \psi_2^{\mathrm{T}}(t), \psi_3(t)]^{\mathrm{T}}$.

Higher-order tracking error convergence question. Recently, for direct MRAC systems, the higher-order properties of the output tracking error $\lim_{t\to\infty} \frac{\mathrm{d}^i e(t)}{\mathrm{d}t^i} = 0$, $i = 1, \ldots, n^* - 1$, have been studied in [8, 33]. However, does an indirect MRAC or an indirect APPC system have higher-order tracking properties? This question has not been explicitly answered in the literature, with Section 3 providing an affirmative "yes" to this open question.

Extensions to adaptive stronger tracking control. The higher-order tracking properties motivate us to consider a new adaptive control problem: the construction of an adaptive control system that can track a reference output with any order derivatives being unavailable. Particularly, the constructed adaptive control system is required to ensure higher-order exponential or practical output tracking properties. Section 3 demonstrates that the new adaptive control problem can be solved based on the higher-order tracking properties.

3 New properties and extensions of indirect MRAC and APPC systems

This section contains two parts. First, we demonstrate the tracking errors of indirect MRAC and APPC systems afford stronger convergence properties than typically presented in the literature. Second, we construct a new adaptive control system that can track a reference output exponentially or practically under the condition that any order derivatives of the reference output are unknown.

3.1 New higher-order tracking properties

In a direct MRAC system, Ref. [8] showed that the system has the following higher-order tracking properties:

$$\lim_{t\to\infty} e^{(i)}(t) = 0, \ i = 1, 2, \ldots, n^* - 1. \tag{20}$$

This motivates us to derive the following theorem, which clarifies that the indirect MRAC and APPC systems have similar higher-order tracking properties.

Theorem 1. Suppose that Assumptions (A1)–(A4) hold for the indirect MRAC system and Assumptions (B1)–(B3) hold for the indirect APPC system. Then, the higher-order tracking properties (20) hold for the two systems, respectively.

Proof. This proof contains two steps addressing the MRAC and APPC cases, respectively.

Step 1: MRAC case. Let $\omega(t) = [\omega_1^{\mathrm{T}}(t), \omega_2^{\mathrm{T}}(t), y(t), r(t)]^{\mathrm{T}} \in \mathbb{R}^{2n}$ and $\theta^* = [\theta_1^{*\mathrm{T}}, \theta_2^{*\mathrm{T}}, \theta_{20}^*, \theta_3^*]^{\mathrm{T}} \in \mathbb{R}^{2n}$, where θ_i^*, $i = 1, 2, 20, 3$, are the so-called matching parameters such that

$$\theta_1^{*\mathrm{T}} a(s)P(s) + (\theta_2^{*\mathrm{T}} a(s) + \theta_{20}^* \Lambda_c(s))Z(s) = \Lambda_c(s)(P(s) - \theta_3^* Z(s)P_m(s)). \tag{21}$$

Lemma A1 in Appendix A clarifies that Eq. (21) always has a non-trivial solution with respect to $\{\theta_1^*, \theta_2^*, \theta_{20}^*, \theta_3^*\}$. Operating both sides of (21) on $y(t)$ yields

$$\theta_1^{*\mathrm{T}} a(s)P(s)[y](t) + (\theta_2^{*\mathrm{T}} a(s) + \theta_{20}^* \Lambda_c(s))Z(s)[y](t) = \Lambda_c(s)(P(s) - \theta_3^* Z(s)P_m(s))[y](t).$$

It follows from $P(s)[y](t) = Z(s)[u](t)$ that

$$\theta_1^{*\mathrm{T}} a(s)Z(s)[u](t) + (\theta_2^{*\mathrm{T}} a(s) + \theta_{20}^* \Lambda_c(s))Z(s)[y](t) = \Lambda_c(s)Z(s)[u](t) - \theta_3^* Z(s)P_m(s)[y](t).$$

Since $Z(s)$ and $\Lambda_c(s)$ are both stable, we obtain

$$u(t) = \theta_1^{*\mathrm{T}} \frac{a(s)}{\Lambda_c(s)}[u](t) + \theta_2^{*\mathrm{T}} \frac{a(s)}{\Lambda_c(s)}[y](t) + \theta_{20}^*[y](t) + \theta_3^* P_m(s)[y](t) - \epsilon_0(t), \tag{22}$$

where $\epsilon_0(t)$ is an exponentially decaying signal associated with initial conditions. Substituting (13) to the above identical equation (22) indicates that

$$\theta_3^* P_m(s)[e](t) = \tilde{\theta}^{\mathrm{T}}(t)\omega(t) + \epsilon_0(t),\ \tilde{\theta}(t) = \theta(t) - \theta^*. \tag{23}$$

From (23), we demonstrate that $\lim_{t\to\infty} e^{(i)}(t) = 0,\ i = 1, 2, \ldots, n^* - 1$. To this end, we decompose $e^{(i)}(t)$ into two parts: one converging to zero asymptotically and one being arbitrarily small (we show that it also converges to zero based on the limit definition). Firstly, we introduce two virtual filters $K(s)$ and $H(s)$:

$$K(s) = \frac{\gamma^{n^*}}{(s+\gamma)^{n^*}},\ sH(s) = 1 - K(s), \tag{24}$$

where $\gamma > 0$ is a designed parameter to be specified. It follows from (24) that $H(s) = \frac{\sum_{i=1}^{n^*} C_{n^*}^i s^{i-1}\gamma^{n^*-i}}{(s+\gamma)^{n^*}}$, where $C_{n^*}^i = \frac{n^*(n^*-1)\cdots(n^*-i+1)}{i!}$. Then, its impulse response function is

$$h(t) = \mathcal{L}^{-1}[H(s)] = \sum_{i=1}^{n^*} \frac{\gamma^{n^*-i}}{(n^*-i)!} t^{n^*-i} \exp(-\gamma t). \tag{25}$$

Using $\int_0^\infty t^{n^*-i}\exp(-\gamma t)\mathrm{d}t = \frac{(n^*-i)!}{\gamma^{n^*-i}\gamma}$, one can derive that

$$\|h(\cdot)\|_1 = \int_0^\infty |h(t)|\mathrm{d}t = \frac{n^*}{\gamma}. \tag{26}$$

From (23) with $\theta_3^* = \frac{1}{z_m}$, we have

$$e(t) = \frac{z_m}{P_m(s)}[\tilde{\theta}^{\mathrm{T}}\omega](t) + \frac{z_m}{P_m(s)}[\epsilon_0](t) + \epsilon_1(t), \tag{27}$$

where $\epsilon_1(t)$ is an exponentially decaying signal. Note that ϵ_0 is an exponentially decaying signal. Thus, the signal $\frac{z_m}{P_m(s)}[\epsilon_0](t)$ and its i-th order derivatives for $i = 1, 2, \ldots, n^*$ converge to zero exponentially. Furthermore, from (23) and (27), it is evident that the Laplace transform of $\epsilon_1(t)$ is in the form of $\frac{c_0}{P_m(s)}[1](t)$, where c_0 depends on the initial value of $e(t)$. This indicates that the i-th order derivatives of $\epsilon_1(t)$, $i = 1, 2, \ldots, n^* - 1$, converge to zero exponentially. Thus, we ignore the effect of $\epsilon_0(t)$ and $\epsilon_1(t)$ in the following analysis.

By ignoring the effect of the two decaying terms $\epsilon_0(t)$ and $\epsilon_1(t)$ and using (24), we decompose $\dot{e}(t)$ into

$$\dot{e}(t) = (sH(s) + K(s))\frac{z_m s}{P_m(s)}[\tilde{\theta}^{\mathrm{T}}\omega](t)H(s)\frac{z_m s^2}{P_m(s)}[\tilde{\theta}^{\mathrm{T}}\omega](t) + sK(s)[e](t). \tag{28}$$

Firstly, we consider the case of $n^* = 2$. In this case, $sK(s)$ is stable and strictly proper. Recalling that $\lim_{t\to\infty} e(t) = 0$, one can readily verify that $\lim_{t\to\infty} sK(s)[e](t) = 0$.

Note that the closed-loop stability has been confirmed above, and $\frac{z_m s^2}{P_m(s)}$ is stable and proper. Thus, $\frac{z_m s^2}{P_m(s)}[\tilde{\theta}^{\mathrm{T}}\omega](t)$ is also bounded. Along with (26), it indicates that

$$\left| H(s)\frac{z_m s^2}{P_m(s)}[\tilde{\theta}^{\mathrm{T}}\omega](t) \right| \leqslant \frac{c_1}{\gamma} \tag{29}$$

for some constant $c_1 > 0$ independent of γ. Now, we use the limit definition to show $\lim_{t\to\infty} \dot{e}(t) = 0$.

For every $\varepsilon > 0$, we set $\gamma = \frac{2c_1}{\varepsilon}$. Then, from (29), we have

$$\left| H(s)\frac{z_m s^2}{P_m(s)}[\tilde{\theta}^{\mathrm{T}}\omega](t) \right| \leqslant \frac{\varepsilon}{2}. \tag{30}$$

Note that $\lim_{t\to\infty} sK(s)[e](t) = 0$ for all $\gamma > 0$. Thus, for the above chosen parameters ε and γ, there always exists a time $T > 0$ such that

$$|sK(s)[e](t)| < \frac{\varepsilon}{2},\ \forall t \geqslant T. \tag{31}$$

Therefore, for every $\varepsilon > 0$ and the above $T > 0$, combining (28), (30), and (31) indicates that $|\dot{e}(t)| < \varepsilon,\ \forall t \geqslant T$, which implies that $\lim_{t\to\infty} \dot{e}(t) = 0$.

For $n^* = 3$, similar to (28), we decompose $\ddot{e}(t)$ into $\ddot{e}(t) = H(s)\frac{z_m s^3}{P_m(s)}[\tilde{\theta}^{\mathrm{T}}\omega](t) + s^2K(s)[e](t)$. Note that $\frac{z_m s^3}{P_m(s)}$ is stable and proper, and $s^2K(s)$ is stable and strictly proper. Thus, following a procedure similar to the case of $n^* = 2$, we conclude that $\lim_{t\to\infty} \ddot{e}(t) = 0$.

For the MRAC system with an arbitrary relative degree n^*, we decompose $e^{(i)}(t) = H(s)\frac{z_m s^{i+1}}{P_m(s)}[\tilde{\theta}^{\mathrm{T}}\omega](t)+ s^iK(s)[e](t)$, $i = 1, 2, \ldots, n^* - 1$. Note that $\frac{z_m s^{i+1}}{P_m(s)}$ is stable and proper, and $s^iK(s)$ is stable and strictly proper. Thus, we can also follow from the case of $n^* = 2$ that $\lim_{t\to\infty} e^{(i)}(t) = 0$, $i = 1, 2, \ldots, n^* - 1$.

Step 2: APPC case. We derive the ideal version of the Diophantine equation (12) as

$$C(s)Q_m(s)P(s) + D(s)Z(s) = A^*(s). \tag{32}$$

Lemma A2 in Appendix A indicates that Eq. (32) has a unique solution $\{C(s), D(s)\}$, the degree of $C(s)$ is $n-1$, and the degree of $D(s)$ is not larger than $n_q + n - 1$. Operating both sides of (32) on $e(t)$ results in

$$A^*(s)[e](t) = C(s)Q_m(s)P(s)[e](t) + D(s)Z(s)[e](t).$$

Then, from (4), $P(s)[y](t) = Z(s)[u](t)$ and $e(t) = y(t) - y_m(t)$, it follows that

$$A^*(s)[e](t) = C(s)Q_m(s)Z(s)[u](t) + D(s)Z(s)[e](t). \tag{33}$$

Since $A^*(s)$ is stable with degree $2n+n_q-1$, we express $A^*(s)$ into a product of two stable filters, that is, $A^*(s) = \Lambda_1(s)\Lambda_2(s)$, where $\Lambda_1(s)$ is stable with degree n^* and $\Lambda_2(s)$ is stable with degree $2n+n_q-1-n^*$. Then, from (33), it leads to

$$\Lambda_1(s)[e](t) = \frac{C(s)Q_m(s)Z(s)}{\Lambda_2(s)}[u](t) + \frac{D(s)Z(s)}{\Lambda_2(s)}[e](t) + \epsilon_2(t), \tag{34}$$

where $\epsilon_2(t)$ is an exponentially decaying signal associated with initial conditions. Based on the relative degree information (although the relative degree information is unknown for the APPC case), the degree of $Z(s)$ is $n - n^*$. Recalling that the degrees of $Q_m(s)$ and $C(s)$ are n_q and $n-1$, respectively, and the degree of $D(s)$ is not larger than $n_q + n - 1$, one can verify that $\frac{C(s)Q_m(s)Z(s)}{\Lambda_2(s)}$ and $\frac{D(s)Z(s)}{\Lambda_2(s)}$ are both stable and proper. Thus, with (34) to hand, it follows from the stability analysis of the MRAC case (the contents below (23)) that Eq. (20) also holds for the APPC case.

Remark 1. The higher-order properties (20) for the indirect MRAC and APPC systems are new discoveries, as these have never been reported before in the literature and hold without additional design conditions. Especially for the APPC system, Eq. (20) holds regardless of whether the relative degree information is known. This is because the APPC design is independent of the system's relative degree. Opposing, the MRAC design needs relative degree information. The only difference between the higher-order tracking properties of the MRAC and APPC systems is that the order n^* of the former case is known, while for the latter case, it is not.

So far, we have given an affirmative "yes" answer to the open question: whether an indirect MRAC or an indirect APPC system has the higher-order tracking properties: $\lim_{t\to\infty} e^{(i)}(t) = 0$, $i = 1, \ldots, n^* - 1$.

3.2 Extensions to adaptive stronger tracking control

The higher-order tracking properties (20) motivate us to consider whether the indirect MRAC/APPC systems can be extended to solve a new control problem: how to design an adaptive control system that can track a reference output with unknown derivatives. In this subsection, we demonstrate that, with the higher-order tracking properties (20) and two high-gain differential observers, the new tracking control problem can be solved, and the higher-order tracking properties are still ensured. Here, we present the details for the indirect MRAC case, and leave the APPC case as a future study.

Control objective. To proceed, we first clarify the characteristics of the reference output considered in this part. Let $y^*(t)$ denote the reference output which satisfies that

(1) $y^{*(i)}(t) \in L^\infty$, $i = 0, 1, 2, \ldots, n^*$, and

(2) $y^*(t)$ is known, but its derivatives are all unknown.

Let $\bar{e}(t) = y(t) - y^*(t)$ denote the tracking error. Then, the control objective is to design an adaptive control law $u(t)$ for the system (2) such that all closed-loop signals are bounded, and $\bar{e}(t)$ and its some certain order derivatives converge to zero exponentially or practically.

Remark 2. As clarified in Section 1, the high-gain differential observer in [32] can be employed to estimate the derivatives of $y^*(t)$, and the MRAC law (13) can still be effective by replacing the derivatives of $y^*(t)$ with their estimates generated from the high-gain differential observer. However, the exponential convergence of the tracking error cannot be realized even assuming that all system parameters are known. By contrast, the new adaptive control law developed in this part can ensure exponential convergence of the tracking error and its some certain order derivatives under the PE condition. Moreover, the new adaptive control law ensures higher-order practical output tracking without the PE condition. This indicates that the constructed adaptive control system is essentially different from the existing high-gain observer-based results.

The design procedure of the new adaptive control scheme contains four main steps:

(1) estimation of the system unknown parameters and the output derivatives up to the n^* order;

(2) estimation of the reference output derivatives up to the n^* order;

(3) specification of an analytical adaptive control law; and

(4) analysis of the system performance.

To show the basic control ideas, we first present the design details for the system (2) with $n^* = 1$. Then, we give the general design procedure for the system (2) with a general relative degree.

3.2.1 *Design for systems with relative degree one*

For the system (2) with $n^* = n - m = 1$, the usual MRAC system needs to know the first-order derivative of the reference output. However, the new adaptive control system designed below no longer needs the derivative of $y^*(t)$ while still ensuring exponential or practical output tracking.

Step 1: Estimation of θ_p^* and $\dot{y}(t)$. Recall the parametrized model (5). To obtain an estimate of θ_p^*, we use a standard gradient algorithm designed as

$$\dot{\theta}_p(t) = -\frac{\Gamma\phi(t)\epsilon(t)}{m^2(t)}, \tag{35}$$

where $\phi(t), \epsilon(t)$, and $m(t)$ have the same meanings with those in Section 2. We will use $\theta_p(t)$ to estimate $\dot{y}(t)$.

From (5), we introduce a signal $\hat{y}(t)$ defined as

$$\hat{y}(t)=\theta_p^{\mathrm{T}}(t)\phi(t)+\frac{\Lambda_{n-1}(s)}{\Lambda_e(s)}[y](t). \tag{36}$$

Based on $\hat{y}(t)$, we construct an estimate of $\dot{y}(t)$, defined as $\hat{\dot{y}}(t)$, by

$$\hat{\dot{y}}(t)=\dot{\hat{y}}(t)=-\frac{\Gamma\epsilon(t)\phi^{\mathrm{T}}(t)\phi(t)}{m^2(t)}+\theta_p^{\mathrm{T}}(t)\varphi_1(t)+\frac{s\Lambda_{n-1}(s)}{\Lambda_e(s)}[y](t), \tag{37}$$

where

$$\varphi_1(t)=\dot{\phi}(t)=\left[\frac{s}{\Lambda_e(s)}[u](t),\frac{s^2}{\Lambda_e(s)}[u](t),\ldots,\frac{s^{m+1}}{\Lambda_e(s)}[u](t),\frac{s}{\Lambda_e(s)}[y](t),\frac{s^2}{\Lambda_e(s)}[y](t),\ldots,\frac{s^n}{\Lambda_e(s)}[y](t)\right]^{\mathrm{T}}. \tag{38}$$

From (36)–(38), one can verify that $\hat{y}(t)$ and $\hat{\dot{y}}(t)$ are known.

Step 2: Estimation of $\dot{y}^*(t)$. Instead of directly estimating $\dot{y}^*(t)$ by a standard high-gain differential observer, we first estimate two differential signals $\dot{\bar{e}}(t)$ and $\dot{\epsilon}(t)$ by the following two high-gain observers:

$$\dot{x}_0(t)=x_1(t)+\frac{\alpha_0}{\delta_1}(\bar{e}(t)-x_0(t)),\ \dot{x}_1(t)=\frac{\alpha_1}{\delta_1^2}(\bar{e}(t)-x_0(t)), \tag{39}$$

and

$$\dot{z}_0(t)=z_1(t)+\frac{\beta_0}{\delta_2}(\epsilon(t)-z_0(t)),\ \dot{z}_1(t)=\frac{\beta_1}{\delta_2^2}(\epsilon(t)-z_0(t)), \tag{40}$$

where $s^2+\alpha_0 s+\alpha_1$ and $s^2+\beta_0 s+\beta_1$ are Hurwitz polynomials with respect to s, and δ_i, $i=1,2$, are two chosen positive constants. Based on the differential observation theory in [32], we see that $x_1(t)$ and $z_1(t)$ are the estimates of $\dot{\bar{e}}(t)$ and $\dot{\epsilon}(t)$, respectively. Using $x_1(t)$, $z_1(t)$, and $\hat{\dot{y}}(t)$, we construct an estimate of $\dot{y}^*(t)$ as

$$\hat{\dot{y}}^*(t)=\hat{\dot{y}}(t)-x_1(t)-z_1(t). \tag{41}$$

Since $\hat{\dot{y}}(t),x_1(t)$, and $z_1(t)$ are all known, the signal $\hat{\dot{y}}^*(t)$ defined in (41) is surely known.

Remark 3. When estimating some unknown derivative of a signal, the existing results mainly employed a differential observer to directly obtain an estimate of the derivative. However, the high-gain observation theory in [32] indicates that the observation error does not converge to zero. The reason is clarified in Remark 2. In this paper, different from the existing results, we propose an indirect estimation strategy to estimate $\dot{y}^*(t)$, as defined in (41). We will show that this indirect estimation strategy is crucial to ensure exponential or practical output tracking for any bounded reference output.

Step 3: Specification of the adaptive control law. Motivated by the usual MRAC law, we design the adaptive control law as

$$u(t)=\eta_1^{\mathrm{T}}(t)\omega_1(t)+\eta_2^{\mathrm{T}}(t)\omega_2(t)+\eta_{20}(t)y(t)+\eta_3(t)\xi(t), \tag{42}$$

where $\omega_i(t)$, $i=1,2,20$, have the same meanings as those in (13) for $n^*=1$. With $\eta_3(t)=\frac{1}{\hat{z}_m(t)}$, $\eta_1(t),\eta_2(t)$, and $\eta_{20}(t)$ are obtained from the following equation:

$$\eta_1^{\mathrm{T}}(t)a(\lambda)\hat{P}(\lambda,\hat{p})+(\eta_2^{\mathrm{T}}(t)a(\lambda)+\eta_{20}(t)\Lambda_c(\lambda))\hat{Z}(\lambda,\hat{z})=\Lambda_c(\lambda)(\hat{P}(\lambda,\hat{p})-\eta_3(t)\hat{Z}(\lambda,\hat{z})X_m(\lambda)) \tag{43}$$

for $X_m(\lambda)=\lambda+\gamma_0$ with γ_0 being a chosen positive constant. Similar to (14), Eq. (43) always has a solution $\{\eta_1,\eta_2,\eta_{20},\eta_3\}$. The estimates $\hat{p}$ and $\hat{z}$ in (43) are obtained from the parameter update law (35). Moreover,

$$\xi(t)=\hat{\dot{y}}^*(t)+\gamma_0 y^*(t) \tag{44}$$

can be regarded as an estimate of the reference input signal.

For $\eta_i(t)$, $i=1,2,20,3$, in (42), the following lemma clarifies the existence of their ideal values.

Lemma 2. For the given polynomial $X_m(s) = s + \lambda_0$, there exist constant parameters $\eta_1^* \in \mathbb{R}^{n-1}$, $\eta_2^* \in \mathbb{R}^{n-1}$, $\eta_{20}^* \in \mathbb{R}$, and $\eta_3^* = \frac{1}{z_m} \in \mathbb{R}$ such that

$$\eta_1^{*\mathrm{T}} a(s)P(s) + (\eta_2^{*\mathrm{T}} a(s) + \eta_{20}^* \Lambda_c(s))Z(s) = \Lambda_c(s)(P(s) - \eta_3^* Z(s)X_m(s)) \tag{45}$$

with $a(s) = [1, s, \ldots, s^{n-2}]$ and $\Lambda_c(s)$ is a monic stable polynomial of degree $n-1$.

To prove (45), the key step is to construct a linear equation with respect to $\{\eta_1^*, \eta_2^*, \eta_{20}^*, \eta_3^*\}$. Then, it is straightforward to obtain the solution from the linear equation regardless if $P(s)$ and $Z(s)$ are coprime or not. The procedure of the linear equation construction is similar to Lemma A1 in Appendix A. For further details on the equation construction, the readers are referred to [6].

Step 4: System performance analysis. Now, we provide the following two theorems that clarify some stronger tracking capabilities of the new adaptive control system.

Theorem 2. Consider the system (2) with $n^* = 1$. Suppose that Assumptions (A1)–(A4) hold and $\phi(t)$ defined in (7) is persistently exciting. Then, the parameter update law (35) ensures $\lim_{t\to\infty} \theta_p(t) = \theta_p^*$ exponentially. Moreover, there exist constants $\delta_i^* > 0$, $i = 1, 2$, such that, if $\delta_i < \delta_i^*$, the adaptive control law (42) ensures closed-loop stability and

$$\lim_{t\to\infty} \bar{e}(t) = \lim_{t\to\infty} \dot{\bar{e}}(t) = 0, \text{ exponentially.} \tag{46}$$

Proof. It follows from Lemma A3 in Appendix A that $\theta_p(t)$ converges to θ_p^* exponentially under the PE condition on $\phi(t)$. Let $\eta^* = [\eta_1^*, \eta_2^*, \eta_{20}^*, \eta_3^*]^{\mathrm{T}}$ and $\eta(t) = [\eta_1^{\mathrm{T}}(t), \eta_2^{\mathrm{T}}(t), \eta_{20}(t), \eta_3(t)]^{\mathrm{T}}$. Then, from the matching equation (43), one can verify that $\eta(t)$ converges to the ideal value η^* if $\theta_p(t)$ converges to θ_p^*. In other words, $\lim_{t\to\infty}(\theta_p(t) - \theta_p^*) = \lim_{t\to\infty}(\eta(t) - \eta^*) = 0$ exponentially.

Now, we construct three steps to prove that all closed-loop signals are bounded and Eq. (46) holds. Firstly, we show that all signals are bounded and asymptotic output tracking is achieved by assuming $\dot{\bar{e}}(t)$ and $\dot{\epsilon}(t)$ are available. Secondly, we show $\bar{e}(t)$, $\dot{\bar{e}}$, $\epsilon(t)$, and $\dot{\epsilon}(t)$ all converge to zero exponentially by assuming that $\dot{\bar{e}}(t)$ and $\dot{\epsilon}(t)$ are available. Finally, based on the well-known separation principle of the high-gain observation theory, it is straightforward to obtain the results of Theorem 2. The details are as follows.

Given that $\dot{\bar{e}}(t)$ and $\dot{\epsilon}(t)$ are available for control, we have the following analysis. In this case, $x_1 = \dot{\bar{e}}(t) = \dot{y}(t) - \dot{y}^*(t)$ and $\dot{\epsilon}(t) = \dot{\hat{y}}(t) - \dot{y}(t)$. From (41), we have $\hat{\dot{y}}^*(t) = \dot{\hat{y}}(t) - \dot{\bar{e}}(t) - \dot{\epsilon}(t) = \dot{y}^*(t)$ which implies that $\hat{\dot{y}}^*(t)$ in (44) is identically equal to $\dot{y}^*(t)$, and $\xi(t) = \dot{y}^*(t) + \gamma_0 y^*(t)$. Thus, the adaptive control law (42) can be seen as a standard indirect MRAC law with $\xi(t)$ as the reference input. The usual indirect MRAC theory in [5, 6] indicates that all closed-loop signals are bounded and asymptotic output tracking is achieved.

Based on (23), we obtain that the closed-loop system satisfies the following equation:

$$\eta_3^* X_m(s)[\bar{e}](t) = \tilde{\eta}^{\mathrm{T}}(t)\Phi(t) + \varepsilon_0(t), \tag{47}$$

where $\tilde{\eta}(t) = \eta(t) - \eta^*$, $\varepsilon_0(t)$ is an exponentially decaying signal, and

$$\Phi(t) = [\omega_1^{\mathrm{T}}(t), \omega_2^{\mathrm{T}}(t), y(t), \xi(t)]^{\mathrm{T}}. \tag{48}$$

Based on the fact that $\tilde{\eta}(t)$ converges to zero exponentially and all closed-loop signals are bounded, we conclude that $\bar{e}(t)$ converges to zero exponentially. In addition, with $X_m(s) = s + \gamma_0$, we have $\eta_3^* \dot{\bar{e}}(t) = -\eta_3^* \gamma_0 \bar{e}(t) + \tilde{\eta}^{\mathrm{T}}(t)\Phi(t) + \varepsilon_0(t)$. Since $\eta_3^* = \frac{1}{z_m} \neq 0$, the above equation implies that $\dot{\bar{e}}(t)$ converges to zero exponentially.

From (5) and (8), we see that $\epsilon(t) = \tilde{\theta}_p^{\mathrm{T}}(t)\phi(t)$. It follows from $\lim_{t\to\infty} \tilde{\theta}_p(t) = 0$ exponentially that $\epsilon(t)$ converges to zero exponentially. Additionally, combining (5) and (37) yields

$$\dot{\epsilon}(t) = -\frac{\Gamma\epsilon(t)\phi^{\mathrm{T}}(t)\phi(t)}{m^2(t)} + \tilde{\theta}_p^{\mathrm{T}}(t)\varphi_1(t) \tag{49}$$

which implies that $\dot{\epsilon}(t)$ converges to zero exponentially.

When using the high-gain observers (39) and (40), the well-known separation principle allows us to separate the stability analysis into two steps. First, we analyze the adaptive control law (42) with $\dot{\bar{e}}(t)$ and $\dot{\epsilon}(t)$ being available ensures $\bar{e}(t)$ and $\dot{\bar{e}}(t)$ converge to zero exponentially. This property has been verified in Step 2. Then, Theorem 6 in Appendix A indicates that, by replacing $\dot{\bar{e}}(t)$ and $\dot{\epsilon}(t)$ with their estimates $x_1(t)$ and $z_1(t)$, the adaptive control law (42) can recover the system performance if the observer gains are sufficiently high. Specifically, there exist small positive constants $\delta_i^* > 0$, $i = 1, 2$, such that if $\delta_i < \delta_i^*$, the adaptive control law (42) ensures closed-loop stability and $\lim_{t\to\infty} \bar{e}(t) = \lim_{t\to\infty} \dot{\bar{e}}(t) = 0$ exponentially.

Theorem 2 demonstrates that the designed adaptive control system has a stronger tracking property: although $\dot{y}^*(t)$ is unknown, the proposed adaptive control law (42) still ensures $\dot{y}(t) - \dot{y}^*(t)$ converges to zero exponentially fast under the PE condition. If we directly estimate $\dot{y}^*(t)$ by the high-gain observer, the exponential convergence of the error $\dot{y}(t) - \dot{y}^*(t)$ cannot be achieved. This is because the mismatch between $\dot{y}^*(t)$ and its estimate generated by the high-gain observer cannot be eliminated even under the PE condition. This also implies that the indirect estimation of $\dot{y}^*(t)$ is a crucial step to achieve exponential convergence of the errors $\bar{e}(t)$ and $\dot{\bar{e}}(t)$.

The following theorem clarifies the closed-loop system performance for the system (2) without the PE condition.

Theorem 3. Consider the system (2) with $n^* = 1$. Suppose that Assumptions (A1)–(A4) hold and $\ddot{y}^*(t) \in L^\infty$. Then, for any given $\mu > 0$, there exist $T > 0$ and $\delta_i^* > 0$, $i = 1, 2$, dependent on μ, such that, if $\delta_i \leqslant \delta_i^*$, the adaptive control law (42) with the parameter update law (35) ensures closed-loop stability and

$$|\bar{e}(t)| \leqslant \mu, \ |\dot{\bar{e}}(t)| \leqslant \mu, \ \forall t \geqslant T. \tag{50}$$

Proof. Given that $\dot{\bar{e}}(t)$ and $\dot{\epsilon}(t)$ are available for control, the proof of Theorem 2 has verified that the adaptive control law (42) ensures all closed-loop signals are bounded and $\bar{e}(t)$ converges to zero asymptotically. Next, we show $\epsilon(t), \dot{\epsilon}(t)$, and $\dot{\bar{e}}(t)$ all converge to zero asymptotically under the condition that $\dot{\bar{e}}(t)$ and $\dot{\epsilon}(t)$ are available.

Ignoring the effect of the exponentially decaying terms, we derive from (47) that $\dot{\bar{e}}(t) = \frac{z_m s}{s+\lambda_0}[\tilde{\eta}^{\rm T}\Phi](t)$. Using (24), we have

$$\dot{\bar{e}}(t) = H(s)\frac{z_m s}{s+\lambda_0} s[\tilde{\eta}^{\rm T}\Phi](t) + sK(s)[e](t) = H(s)\frac{z_m s}{s+\lambda_0}[\dot{\tilde{\eta}}^{\rm T}\Phi + \tilde{\eta}^{\rm T}\dot{\Phi}](t) + sK(s)[e](t). \tag{51}$$

It follows from (15) that $\dot{\tilde{\eta}} \in L^\infty$. Moreover, based on the definitions of $\omega_i(t)$, $i = 1, 2$, below (13), we see that $\dot{\omega}_i(t) \in L^\infty$. From (5) and (38), we get $\dot{y}(t) \in L^\infty$. As verified in the proof of Theorem 2, $\xi(t) = \dot{y}^*(t) + \gamma_0 y^*(t)$ when $\dot{\bar{e}}(t)$ and $\dot{\epsilon}(t)$ are available. Thus, under the condition that $\ddot{y}^*(t) \in L^\infty$, it follows from (48) that $\dot{\Phi}(t) \in L^\infty$. Together with all closed-loop signals being bounded, we have $\dot{\tilde{\eta}}^{\rm T}\Phi + \tilde{\eta}^{\rm T}\dot{\Phi} \in L^\infty$ and $\frac{z_m s}{s+\lambda_0}[\dot{\tilde{\eta}}^{\rm T}\Phi + \tilde{\eta}^{\rm T}\dot{\Phi}](t) \in L^\infty$. Thus, referring to the convergence analysis below (28), we obtain that $\lim_{t\to\infty} \dot{\bar{e}}(t) = 0$.

From (49), we see that $\dot{\epsilon}(t) \in L^\infty$. Together with the property $\frac{\epsilon(t)}{m(t)} \in L^2$ shown in Lemma 1, it follows that $\lim_{t\to\infty} \epsilon(t) = 0$. With $\epsilon(t) = \tilde{\theta}_p^{\rm T}(t)\phi(t)$ and using (24) again, we have

$$\dot{\epsilon}(t) = H(s)s^2[\tilde{\theta}_p^{\rm T}\phi](t) + sK(s)[\tilde{\theta}_p^{\rm T}\phi](t) = H(s)s^2[\tilde{\theta}_p^{\rm T}\phi](t) + sK(s)[\epsilon](t).$$

Note that $s^2[\tilde{\theta}_p^{\rm T}\phi](t) = \ddot{\theta}_p^{\rm T}(t)\phi(t) + 2\dot{\theta}_p^{\rm T}(t)\dot{\phi}(t) + \tilde{\theta}_p^{\rm T}(t)\ddot{\phi}(t)$. From (5), (35), and (38), one can verify that $\ddot{\theta}_p(t)$, $\dot{\theta}_p(t)$, $\phi(t)$, and $\dot{\phi}(t)$ are all bounded. Moreover, under the condition that $\ddot{y}^* \in L^\infty$, it follows from (38) and (42) that $\ddot{\phi}(t) \in L^\infty$. Thus, $s^2[\tilde{\theta}_p^{\rm T}\phi](t) \in L^\infty$. Then, we refer to the convergence analysis below (28) again, and obtain that $\lim_{t\to\infty} \dot{\epsilon}(t) = 0$.

Until now, we have proven that $\bar{e}(t)$, $\dot{\bar{e}}(t)$, $\epsilon(t)$, and $\dot{\epsilon}(t)$ all converge to zero asymptotically under the condition that $\dot{\bar{e}}(t)$ and $\dot{\epsilon}(t)$ are available. Theorem 6 in Appendix A indicates that, if the feedback signals $\dot{\bar{e}}(t)$ and $\dot{\epsilon}(t)$ are replaced by their estimates $x_1(t)$ and $z_1(t)$, the system performance can be recovered. Moreover, if $\bar{e}(t)$, $\dot{\bar{e}}(t)$, $\epsilon(t)$, and $\dot{\epsilon}(t)$ all converge to zero asymptotically, the adaptive control law (42) ensures that Eq. (50) holds.

So far, we have shown that, for any given bounded reference output $y^*(t)$ with its derivative being unknown, the developed adaptive control law (42) ensures closed-loop stability and higher-order output tracking properties for the system (2) with $n^* = 1$, as demonstrated in Theorems 2 and 3.

3.2.2 *Design for systems with a general relative degree*

Now, we start to address the general relative degree case. Based on the design procedure for the case of $n^*=1$, we see that the key technical issue is to design some auxiliary signals (e.g., $x_1(t), z_1(t)$, and $\hat{y}(t)$ for the case of $n^*=1$) and use them to construct the estimates of the reference output derivatives up to the n^* order. Especially, some of the designed auxiliary signals (e.g., $x_1(t), z_1(t)$ for the case of $n^*=1$) should converge to zero exponentially so that exponential output tracking could be achieved under the PE condition. Moreover, in the absence of the PE condition, practical output tracking could be achieved based on the high-gain observation theory.

The design procedure of the general relative degree case also contains four steps.

Step 1: Estimation of θ_p^* and $y^{(i)}(t),\ i=1,\ldots,n^*$. For the general relative degree case, we still use the update law (35) to obtain an estimate of θ_p^*.

To construct the estimates of $y^{(i)}(t),\ i=1,\ldots,n^*$, we first give the following lemma demonstrating a key property of $\hat{y}^{(i)}(t)$, $i=1,\ldots,n^*$.

Lemma 3. For $\hat{y}(t)$ defined in (36), its derivatives $\hat{y}^{(i)}(t)$, $i=1,\ldots,n^*$, can be expressed by using $y^{(j)}(t)$, $j=0,\ldots,i-1$, $\frac{s^k}{\Lambda_e(s)}[u](t)$, $k=1+m,\ldots,i+m$, $\theta_p(t)$, $\epsilon(t)$, $\phi(t)$, and $y(t)$.

Proof. We prove this lemma by mathematical induction. From (37), we see that $\dot{\hat{y}}(t)$ can be expressed by $y(t)$, $\theta_p(t)$, $\epsilon(t)$, $\phi(t)$, $y(t)$, and $\frac{s^{m+1}}{\Lambda_e(s)}[u](t)$. This indicates that Lemma 3 holds for $i=1$. Suppose that Lemma 3 holds for $i<n^*$. Then, we can express $\hat{y}^{(n^*-1)}(t)$ as

$$\hat{y}^{(n^*-1)}=H_{n^*-1}\left(y,\dot{y},\ldots,y^{(n^*-2)},\frac{s^{1+m}}{\Lambda_e(s)}[u],\ldots,\frac{s^{n^*-1+m}}{\Lambda_e(s)}[u],\theta_p,\epsilon,\phi\right),$$

where H_{n^*-1} is a known and smooth mapping with respect to its variables. From (35), we see that $\dot{\theta}_p$ can be expressed by ϕ and ϵ. Moreover, based on the fact $\dot{\epsilon}=\dot{\hat{y}}-\dot{y}$, it follows from (37) that $\dot{\epsilon}$ can be expressed by ϵ,ϕ,θ_p,y, and $\dot{y}$. Eq. (38) indicates that $\dot{\phi}$ can be expressed by ϕ,y, and $\frac{s^{m+1}}{\Lambda_e(s)}[u]$. Thus, based on all of the variables in H_{n^*-1}, one can derive the derivative of $\hat{y}^{(n^*-1)}$ can be expressed by $y^{(j)}(t)$, $j=0,\ldots,n^*-1$, $\frac{s^k}{\Lambda_e(s)}[u](t)$, $k=1+m,\ldots,n$, $\theta_p(t)$, $\epsilon(t)$, $\phi(t)$, and $y(t)$. In other words, Lemma 3 holds for $i=n^*$.

Based on Lemma 3, we express $\hat{y}^{(i)}(t)$, $i=1,2,\ldots,n^*$, as

$$\hat{y}^{(i)}=H_i\left(y,\dot{y},\ldots,y^{(i-1)},\frac{s^{1+m}}{\Lambda_e(s)}[u],\ldots,\frac{s^{i+m}}{\Lambda_e(s)}[u],\theta_p,\epsilon,\phi\right). \tag{52}$$

where H_i, $i=1,\ldots,n^*$, are known and smooth mappings with respect to its variables. Note that the variables $\dot{y}$,…,$y^{(i-1)}$ in H_i are all unknown. Then, $\hat{y}^{(i)}$, $i=2,\ldots,n^*$, are unavailable. Nevertheless, based on (52), we construct an estimate of $\ddot{y}(t)$ as

$$\hat{\ddot{y}}=H_2\left(y,\hat{\dot{y}},\frac{s^{1+m}}{\Lambda_e(s)}[u],\frac{s^{2+m}}{\Lambda_e(s)}[u],\theta_p,\epsilon,\phi\right), \tag{53}$$

where $\hat{\dot{y}}$ is defined in (37). Since y, $\hat{\dot{y}}$, $\frac{s^{1+m}}{\Lambda_e(s)}[u]$, $\frac{s^{2+m}}{\Lambda_e(s)}[u]$, θ_p, ϵ, and ϕ are all known, the estimate $\hat{\ddot{y}}$ defined in (53) is surely known. Using $\hat{\dot{y}}$ and $\hat{\ddot{y}}$ defined in (37) and (53), respectively, we construct an estimate of $\dddot{y}(t)$ as $\widehat{\dddot{y}}=H_3(y,\hat{\dot{y}},\hat{\ddot{y}},\frac{s^{1+m}}{\Lambda_e(s)}[u],\frac{s^{2+m}}{\Lambda_e(s)}[u],\frac{s^{3+m}}{\Lambda_e(s)}[u],\theta_p,\epsilon,\phi)$. Following the above estimation procedure, we can sequentially construct an estimate of $y^{(i)}(t)$ which has the form

$$\widehat{y^{(i)}}=H_i\left(y,\hat{\dot{y}},\hat{\ddot{y}},\ldots,\widehat{y^{(i-1)}},\frac{s^{1+m}}{\Lambda_e(s)}[u],\ldots,\frac{s^{i+m}}{\Lambda_e(s)}[u],\theta_p,\epsilon,\phi\right) \tag{54}$$

for $i=1,\ldots,n^*$. Considering the mappings H_i are known and the variables of H_i in (54) are known, we conclude that the signals $\widehat{y^{(i)}}$ defined in (54) are known.

Step 2: Estimation of $y^{*(i)}(t),\ i=1,\ldots,n^*$. As clarified in Remarks 2 and 3, if we directly estimate $y^{*(i)}(t)$, there will exist a mismatch between $y^{*(i)}(t)$ and its estimate, and the mismatch cannot be eliminated even under the PE condition. To overcome this drawback of direct estimation, this paper proposes an indirect estimation strategy to estimate $y^{*(i)}(t),\ i=1,\ldots,n^*$. Specifically, we first estimate the differential signals $\bar{e}^{(i)}(t)$ and $\epsilon^{(i)}(t)$, $i=1,\ldots,n^*$, by the following high-gain observers:

$$\begin{aligned}
&\dot{x}_0(t)=x_1(t)+\frac{\alpha_0}{\delta_1}(\bar{e}(t)-x_0(t)),\\
&\vdots\\
&\dot{x}_{n^*-1}(t)=x_{n^*}(t)+\frac{\alpha_{n^*-1}}{\delta_1^{n^*}}(\bar{e}(t)-x_0(t)),\\
&\dot{x}_{n^*}(t)=\frac{\alpha_{n^*}}{\delta_1^{n^*+1}}(\bar{e}(t)-x_0(t)),
\end{aligned}\tag{55}$$

and

$$\begin{aligned}
&\dot{z}_0(t)=z_1(t)+\frac{\beta_0}{\delta_2}(\epsilon(t)-z_0(t)),\\
&\vdots\\
&\dot{z}_{n^*-1}(t)=z_{n^*}(t)+\frac{\beta_{n^*-1}}{\delta_2^{n^*}}(\epsilon(t)-z_0(t)),\\
&\dot{z}_{n^*}(t)=\frac{\beta_{n^*}}{\delta_2^{n^*+1}}(\epsilon(t)-z_0(t)),
\end{aligned}\tag{56}$$

where α_i and β_i are chosen constants such that $s^{n^*+1}+\alpha_0 s^{n^*}+\cdots+\alpha_{n^*-1}s+\alpha_{n^*}$ and $s^{n^*+1}+\beta_0 s^{n^*}+\cdots+\alpha_{n^*-1}s+\beta_{n^*}$ are Hurwitz polynomials with respect to s, and δ_i, $i=1,2$, are two chosen positive constants. Based on the differential observation theory in [32], we see that $x_i(t)$ and $z_i(t)$ are the estimates of $\bar{e}^{(i)}(t)$ and $\epsilon^{(i)}(t)$, respectively. Using $x_i(t)$, $z_i(t)$, and $\widehat{y^{(i)}}(t)$, we construct the estimates of $y^{*(i)}(t)$ as

$$\widehat{y^{*(i)}}(t)=\widehat{y^{(i)}}(t)-x_i(t)-z_i(t),\ i=1,\ldots,n^*.\tag{57}$$

Since $\widehat{y^{(i)}}(t), x_i(t)$, and $z_i(t)$ are all known, the estimates $\widehat{y^{*(i)}}(t)$, $i=1,\ldots,n^*$, defined in (57) are also known.

Step 3: Specification of an adaptive control law. The adaptive control law for the general relative degree case is designed as

$$u(t)=\eta_1^{\mathrm{T}}(t)\omega_1(t)+\eta_2^{\mathrm{T}}(t)\omega_2(t)+\eta_{20}(t)y(t)+\eta_3(t)\xi(t),\tag{58}$$

where $\omega_i(t)$, $i=1,2,20$, have the same meaning as those in (13). The parameters η_i, $i=1,2,20$, are also obtained from (43) with $\eta_3(t)=\frac{1}{\hat{z}_m(t)}$ and $X_m(\lambda)=\lambda^{n^*}+\gamma_{n^*-1}\lambda^{n^*-1}+\cdots+\gamma_1\lambda_1+\lambda_0$, where γ_i, $i=0,\ldots,n^*-1$, are constants to be designed such that $X_m(\lambda)$ is a Hurwitz polynomial with respect to λ. Particularly, the signal $\xi(t)$ is of the form

$$\xi(t)=\widehat{y^{*(n^*)}}(t)+\gamma_{n^*-1}\widehat{y^{*(n^*-1)}}(t)+\cdots+\gamma_1\hat{\dot{y}}^*(t)+\gamma_0 y^*(t),$$

where $\widehat{y^{*(i)}}(t)$, $i=1,\ldots,n^*$, are defined in (57).

Step 4: System performance analysis. For the system (2) with a general relative degree, we give the following theorem.

Theorem 4. Consider the system (2) with general relative degree n^* ($1\leqslant n^*\leqslant n$). Suppose that Assumptions (A1)–(A4) hold and $\phi(t)$ defined in (7) is persistently exciting. Then, the parameter update law (35) ensures $\lim_{t\to\infty}\theta_p(t)=\theta_p^*$ exponentially. Moreover, there exist constants $\delta_i^*>0$, $i=1,2$, such that, if $\delta_i<\delta_i^*$, the adaptive control law (58) ensures closed-loop stability and

$$\lim_{t\to\infty}\bar{e}^{(i)}(t)=0,\ i=0,1,\ldots,n^*,\ \text{exponentially}.\tag{59}$$

Proof. It is straightforward to obtain $\lim_{t\to\infty}\theta_p(t)=\theta_p^*$ exponentially under $\phi(t)$ is of PE.

Now, we prove that all closed-loop signals are bounded and Eq. (59) holds. Similar to the case of $n^*=1$, we first show that all signals are bounded and asymptotic output tracking is achieved by assuming $\bar{e}^{(i)}(t)$

and $\epsilon^{(i)}(t)$, $i = 1, \ldots, n^*$, are available. Secondly, we show $\bar{e}^{(i)}(t)$ and $\epsilon^{(i)}(t)$, $i = 1, \ldots, n^*$, all converge to zero exponentially by assuming $\bar{e}^{(i)}(t)$ and $\epsilon^{(i)}(t)$, $i = 1, \ldots, n^*$, are available. Finally, based on the well-known separation principle of the high-gain observation theory, it is straightforward to obtain the results of Theorem 4. The details are as follows.

Given that $\bar{e}^{(i)}(t)$ and $\epsilon^{(i)}(t)$, $i = 1, \ldots, n^*$, are available for control, there is no need to use the high-gain observers (55) and (56), and the signals $x_i(t)$ and $z_i(t)$ used in (57) satisfy

$$x_i(t) = \bar{e}^{(i)}(t),\ z_i(t) = \epsilon^{(i)}(t),\ i = 1, \ldots, n^*. \tag{60}$$

From (37) and the fact that $\dot{\epsilon}(t) = \dot{\hat{y}}(t) - \dot{y}(t)$, we obtain $\dot{y}(t) = \dot{\hat{y}}(t) - \dot{\epsilon}(t)$ which means $\dot{y}(t)$ is known if $\dot{\epsilon}(t)$ is available. Thus, it follows from (53) that $\widehat{\dot{y}}(t) = \ddot{\hat{y}}(t)$. In a similar fashion, one can further verify that, if $\epsilon^{(i)}(t)$, $i = 1, \ldots, n^*$, are available, then

$$\widehat{y^{(i)}}(t) = \hat{y}^{(i)}(t),\ i = 1, \ldots, n^*. \tag{61}$$

Combining (57), (60), and (61) yields that $\widehat{y^{*(i)}}(t) = \hat{y}^{(i)}(t) - (y^{(i)}(t) - y^{*(i)}(t)) - (\hat{y}^{(i)}(t) - y^{(i)}(t)) = y^{*(i)}(t)$ which implies that $\widehat{y^{*(i)}}(t)$ is identically equal to $y^{*(i)}(t)$. In other words, if $\bar{e}^{(i)}(t)$ and $\epsilon^{(i)}(t)$, $i = 1, \ldots, n^*$, are available, then the adaptive control law (58) can be seen as a standard indirect MRAC law with $\xi(t)$ as the reference input. The rest of the proof is quite similar to that of Theorem 2, and thus, omitted here.

Theorem 5. Consider the system (2) with general relative degree n^* ($1 \leqslant n^* \leqslant n$). Suppose that Assumptions (A1)–(A4) hold and the $(n^* + 1)$-order derivative of $y^*(t)$ is bounded. Then, for any given $\mu > 0$, there exist $T > 0$ and $\delta_i^* > 0$, $i = 1, 2$, dependent on μ, such that, if $\delta_i \leqslant \delta_i^*$, the adaptive control law (58) with the parameter update law (35) ensures closed-loop stability and

$$|\bar{e}^{(i)}(t)| \leqslant \mu,\ \forall t \geqslant T,\ i = 0, 1, \ldots, n^*. \tag{62}$$

Proof. The proof of Theorem 4 has verified that, if $\bar{e}^{(i)}(t)$ and $\epsilon^{(i)}(t)$, $i = 1, \ldots, n^*$, are available, the adaptive control law (58) can be seen as a standard indirect MRAC law with $\xi(t)$ as the reference input. Then, based on Theorem 1, we see that all closed-loop signals are bounded and $\bar{e}^{(i)}(t)$, $i = 1, \ldots, n^* - 1$, converge to zero asymptotically. Next, we need to prove $\bar{e}^{(n^*)}(t)$ and $\epsilon^{(i)}(t)$, $i = 1, \ldots, n^*$, all converge to zero asymptotically under the conditions that $y^{*(n^*+1)}(t) \in L^\infty$ and $\bar{e}^{(i)}(t), \epsilon^{(i)}(t)$, $i = 1, \ldots, n^*$, are available. Actually, referring to the analysis in the proof of Theorem 2, it is not difficult to show $\bar{e}^{(n^*)}(t)$ and $\epsilon^{(i)}(t)$, $i = 1, \ldots, n^*$, all converge to zero asymptotically. The rest of the proof is similar to that of Theorem 2, and thus, omitted here.

So far, by Theorem 1, we have answered the higher-order tracking error convergence question proposed in Section 2. More importantly, by using the higher-order tracking properties in Theorem 1, we constructed a new adaptive control system which has some stronger tracking capabilities, as demonstrated in Theorems 4 and 5.

4 Simulation study

In this section, we present two examples with simulation results to demonstrate the design procedure and verify the validity of Theorems 1 and 5, respectively.

4.1 Simulation for a linearized aircraft model

Simulation model. To verify Theorem 1, we consider a linearized lateral dynamics model of the DC-8 aircraft with an aileron as the actuator ([34])

$$\dot{x}(t) = Ax(t) + B\delta_a(t),\ y(t) = Cx(t), \tag{63}$$

where $x = [\beta, p, \phi, r]^{\mathrm{T}} \in \mathbb{R}^4$ is the system state vector with β, p, ϕ, r representing the side-slip angle, roll rate, roll angle, and yaw rate, respectively, $y = \beta$ is the system output, and $\delta_a \in \mathbb{R}$ is the aileron servos' angle as the input.

As shown in [34], at an altitude of 33000 ft, Mach number 0.84, and nominal forward speed 825 ft/s, the DC-8 aircraft lateral-perturbation dynamics matrices are

$$A = \begin{bmatrix} -0.0869 & 0 & 0.0390 & -1 \\ -4.424 & -1.184 & 0 & 0.335 \\ 1 & 1 & 0 & 0 \\ 2.148 & -0.021 & 0 & -0.228 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 4.120 \\ 0 \\ 0.125 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}. \tag{64}$$

From (64), we write its input-output description as

$$\begin{aligned} &\beta(s) = G(s)\delta_a(s), \ G(s) = \frac{Z(s)}{P(s)}, \\ &Z(s) = -0.125s^2 + 0.0922s + 0.0383, \ P(s) = s^4 + 1.4989s^3 + 2.5477s^2 + 2.8327s + 0.0113. \end{aligned} \tag{65}$$

Apparently, the relative degree of the model (63) is two. From (65), one can verify that there exists an unstable zero at $s = 1.077$, and thus, the system (63) is the non-minimum phase system. It is assumed that all system parameters are unknown. Considering that the system is the non-minimum phase system, we employ the indirect APPC method to illustrate the validity of Theorem 1.

Parametrized model and parameter adaptation. From (6), (7), and (65), we obtain

$$\begin{aligned} &\theta_p^* = [0.0383, 0.0922, -0.125, 0, -0.0113, -2.8327, -2.5477, -1.4989]^{\mathrm{T}}, \\ &\phi(t) = \left[\frac{1}{\Lambda_e(s)}[u](t), \frac{s}{\Lambda_e(s)}[u](t), \frac{s^2}{\Lambda_e(s)}[u](t), \frac{s^3}{\Lambda_e(s)}[u](t), \frac{1}{\Lambda_e(s)}[y](t), \frac{s}{\Lambda_e(s)}[y](t), \frac{s^2}{\Lambda_e(s)}[y](t), \frac{s^3}{\Lambda_e(s)}[y](t)\right]^{\mathrm{T}} \end{aligned}$$

with $\Lambda_e(s) = s^4 + 4.7s^3 + 5.9s^2 + 2.5s + 0.3$. Then, the estimated vector of the system unknown parameters is defined as $\theta_p(t) = [\hat{z}_0(t), \hat{z}_1(t), \hat{z}_2(t), \hat{z}_3(t) - \hat{p}_0(t), -\hat{p}_1(t), -\hat{p}_2(t), -\hat{p}_3(t)]^{\mathrm{T}}$.

The parameters in $\theta_p(t)$ are updated by the parameter update law (9) with $\Gamma = 0.5I$, $\theta_0 = 50\%\theta_p^*$, and $\kappa = 1$. Note that $\theta_0 = 50\%\theta_p^*$ means that the initial estimate is set as 50% of the true value.

Reference signal and the APPC law specification. The reference signal is chosen as

$$y_m(t) = 0.5\sin(0.1t) + 0.5\cos(0.1t) \ \text{deg}$$

with $Q_m(s) = s^2 + 0.01$. One can verify that $Q_m(s)[y](t) = 0$ which satisfies the condition (4).

The APPC law is of the form (16). We choose $A^* = s^9 + 3.758s^8 + 8.1046s^7 + 12.1766s^6 + 11.4751s^5 + 6.6086s^4 + 2.3306s^3 + 0.4924s^2 + 0.0574s + 0.0028$. Note that $\hat{P}(s, \hat{p}) = s^4 + \hat{p}_3 s^3 + \hat{p}_2 s^2 + \hat{p}_1 s + \hat{p}_0$ and $\hat{Z}(s, \hat{z}) = \hat{z}_3 s^3 + \hat{z}_2 s^2 + \hat{z}_1 s + \hat{z}_0$. Together with $Q_m(s) = s^2 + 0.01$, we solve the Diophantine equation (12) to derive a solution $\{C(s), D(s)\}$, based on which we calculate the parameters $\psi_1(t), \psi_2(t)$, and $\psi_3(t)$ with $\Lambda_d(s) = s^5 + 3.6s^4 + 3.91s^3 + 1.536s^2 + 0.238s + 0.012$.

System responses. The initial states are chosen as

$$[\beta(0), p(0), \phi(0), r(0)]^{\mathrm{T}} = [0.2°, -0.1°/\mathrm{s}, 0.2°, 0.3°/\mathrm{s}]^{\mathrm{T}}.$$

Employing the APPC law (16) to the simulation model (63), the system responses are as follows. Figure 1 shows the aircraft side-slip angle β tracks the reference output y_m, and at the same time $\dot{\beta}$ tracks $\dot{y}_m$. From Figure 1, we see that the desired output tracking is achieved and at the same time the first-order derivative tracking is also ensured. Figures 2 and 3 are the trajectories of the actuator (the aileron servos' angle) and the parameter adaptation, respectively. As shown in Figure 3, without persistent excitation, the parameter estimates may not converge to their nominal values. However, the closed-loop stability, asymptotic output tracking, and asymptotic output's derivative tracking are still achieved.

4.2 Simulation for a numerical model

Simulation model. To verify the validity of Theorem 5, we consider a numerical system model:

$$y(s) = G(s)u(s), \ G(s) = \frac{Z(s)}{P(s)}, \tag{66}$$

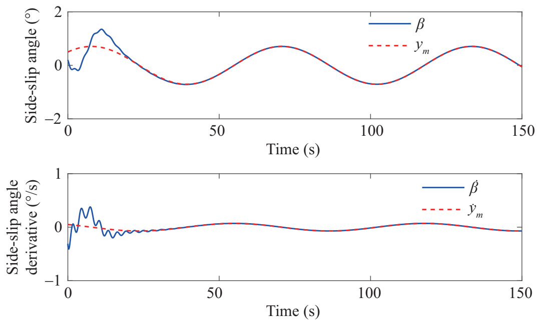

Figure 1 (Color online) Signal β tracks y_m, and $\dot{\beta}$ tracks $\dot{y}_m$.

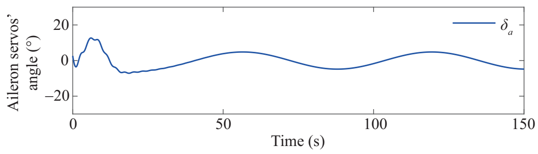

Figure 2 (Color online) Trajectory of the aileron servos' angle.

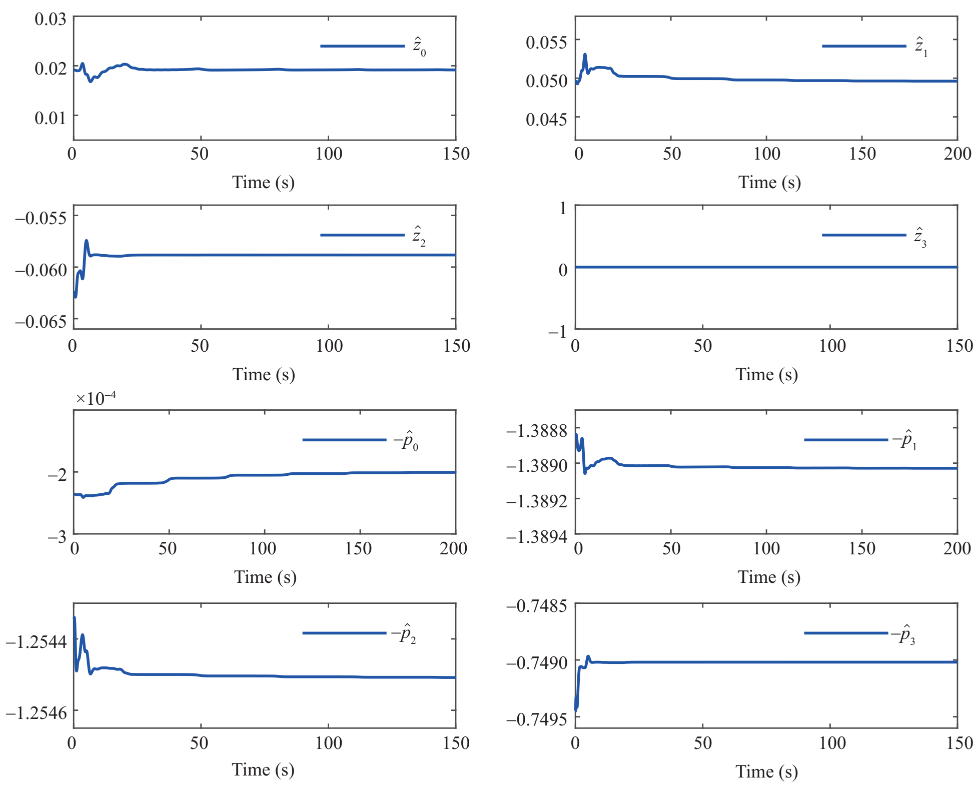

Figure 3 (Color online) Trajectories of the estimated parameters $\hat{z}, \hat{p}$.

where u and y are the input and output, respectively, and $Z(s) = 5,\ P(s) = s^2 - 1.3s + 0.4$. One can verify that this model has unstable poles at $s = 0.5$ and $s = 0.8$. In this simulation, it is assumed that the coefficients of $Z(s)$ and $P(s)$ are all unknown, the relative degree n^* is known, and the order n of $P(s)$ is known.

Parametrized model and parameter adaptation. From (6), (7), and (66), we obtain $\theta_p^* = [5, -0.4, 1.3]^{\mathrm{T}}$ and $\phi(t) = [\frac{1}{\Lambda_e(s)}[u](t), \frac{1}{\Lambda_e(s)}[y](t), \frac{s}{\Lambda_e(s)}[y](t)]^{\mathrm{T}}$ with $\Lambda_e = s^2 + 0.7s + 0.1$. The estimated vector of the system unknown parameters is defined as $\theta_p(t) = [\hat{z}_0(t), -\hat{p}_0(t), -\hat{p}_1(t)]^{\mathrm{T}}$. For this case, the parameters in $\theta_p(t)$ are updated by the parameter update law (35) with $\Gamma = I$ and $\kappa = 1$. The initial estimate for θ_p^* is also set as $\theta_p(0) = 80\%\theta_p^*$.

Reference signal and the adaptive control law specification. The reference signal is chosen as $y^*(t) = 0.5\sin(0.6t) + 0.5\cos(0.4t) + 2$. In this simulation, it is assumed that the reference output $y^*(t)$ is measurable, however, its analytical expression is unknown, that is, its any order derivatives are all unknown. From (58), the adaptive control law is designed as

$$u(t) = \eta_1^{\mathrm{T}}(t)\omega_1(t) + \eta_2^{\mathrm{T}}(t)\omega_2(t) + \eta_{20}(t)y(t) + \eta_3(t)\xi(t), \tag{67}$$

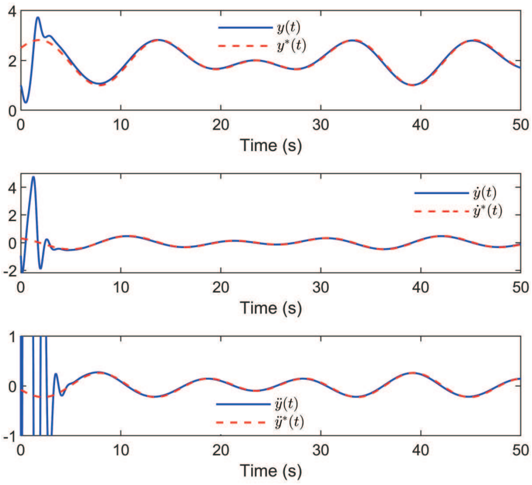

Figure 4 (Color online) Trajectories of the output and its derivative.

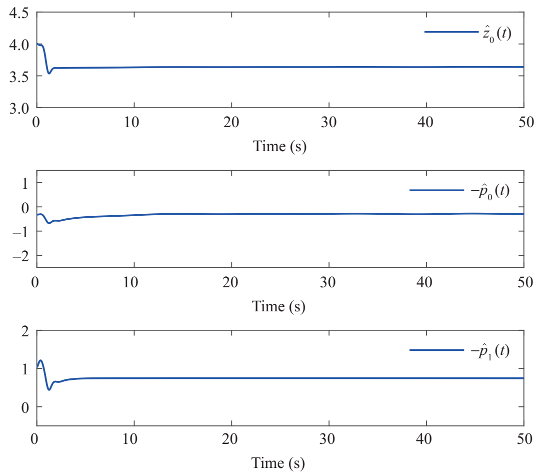

Figure 5 (Color online) Trajectories of the estimated parameters $\hat{z}_0(t)$, $-\hat{p}_0(t)$, $-\hat{p}_1(t)$.

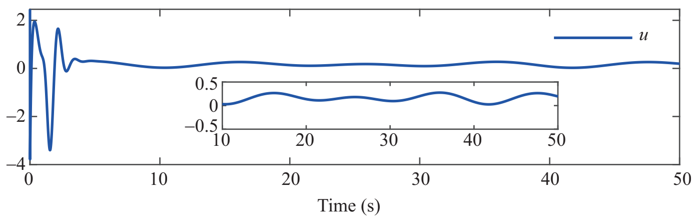

Figure 6 (Color online) Trajectory of the system control input.

where $\omega_1(t) = \frac{a(s)}{\Lambda_c(s)}[u](t) \in \mathbb{R}$, $\omega_2(t) = \frac{a(s)}{\Lambda_c(s)}[y](t) \in \mathbb{R}$ with $a(s) = 1$ and $\Lambda_c = s + 1$. To construct $\xi(t)$, we choose $X_m(s) = s^2 + 9s + 20$. Then, $\xi(t)$ is specified as $\xi(t) = \widehat{\ddot{y}}^*(t) + 9\widehat{\dot{y}}^*(t) + 20y^*(t)$, where $\widehat{\dot{y}}^*(t) = \widehat{\dot{y}}(t) - x_1(t) - z_1(t)$, $\widehat{\ddot{y}}^*(t) = \widehat{\ddot{y}}(t) - x_2(t) - z_2(t)$, $\widehat{\dot{y}}(t), \widehat{\ddot{y}}(t)$ are generated from (37) and (53), and $x_i(t), z_i(t)$, $i = 1, 2$, are generated from the high-gain differential observers (55) and (56) with $\delta_1 = 0.03$, $\delta_2 = 0.01$, $\alpha_0 = 0.3$, $\alpha_1 = 0.03$, $\alpha_2 = 0.001$, $\beta_0 = 0.15$, $\beta_1 = 0.0066$, $\beta_2 = 8 \times 10^{-5}$

System responses. The initial state is $[y(0), \dot{y}(0), \ddot{y}(0)]^{\mathrm{T}} = [1, -1, 10.6]^{\mathrm{T}}$. Applying the adaptive control law (67) to the simulation model (66), the system responses are as follows. Figure 4 shows the output $y(t)$ tracks the reference output $y^*(t)$, and at the same time $\dot{y}(t)$ tracks $\dot{y}^*(t)$ and $\ddot{y}(t)$ tracks $\ddot{y}^*(t)$. From Figure 4, we see that the adaptive control law (67) ensures the desired output tracking performance and at the same time the first and second-order derivatives tracking is ensured. Figures 5 and 6 are the trajectories of the parameter adaptation and the control input, respectively. Similar to the APPC simulation case, the parameter estimates may also not converge to their nominal values, however, the desired system performance (closed-loop stability and higher-order tracking) is still achieved.

It should be noted that the construction of $\xi(t)$ is crucial in the adaptive control law (67), and depends on $x_i(t), z_i(t)$, $i = 1, 2$, $\widehat{\dot{y}}(t)$, and $\widehat{\ddot{y}}(t)$. Thus, we present the following figures. Figures 7(a) and (b) are the trajectories of $x_i(t)$ and $z_i(t)$, $i = 0, 1, 2$, generated by the high-gain observers. Figures 8(a) and (b) present the trajectories of $\widehat{\dot{y}}(t), \widehat{\ddot{y}}(t)$ and $\widehat{\dot{y}}^*(t), \widehat{\ddot{y}}^*(t)$, respectively. Based on the responses shown in Figures 7 and 8, we see that the indirect estimation strategy is effective for estimating $\dot{y}^*$ and $\ddot{y}^*$.

5 Concluding remarks

In this paper, the new higher-order tracking properties $e^{(i)}(t)$, $i = 0, 1, \ldots, n-1$, have been demonstrated for the usual indirect MRAC and APPC schemes under some standard design conditions. Such properties are closer to the nominal control case, where the tracking error and its derivatives up to the (n^*-1)-order converge to zero exponentially. Based on these higher-order properties, a new indirect adaptive control framework is developed, where the derivatives of the reference signal are unknown, but similar higher-

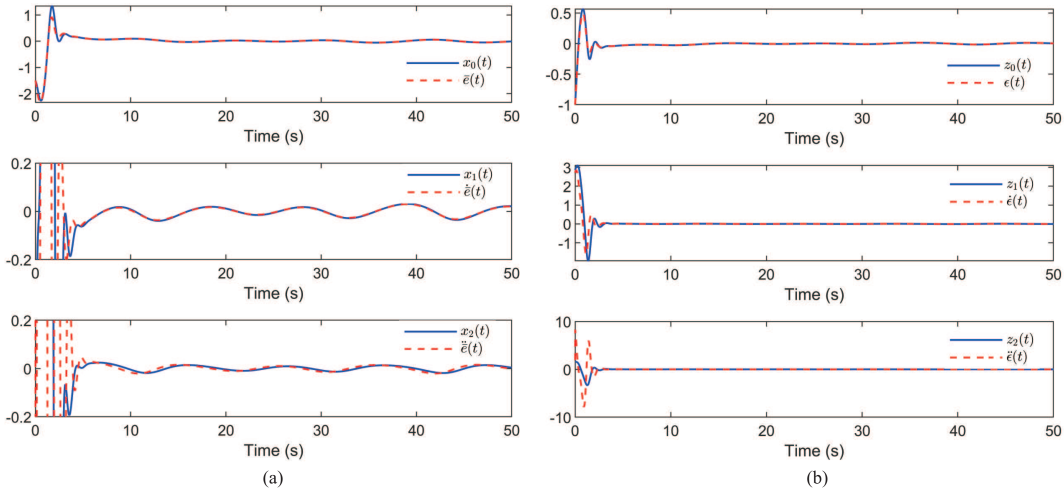

Figure 7 (Color online) Trajectories of the high-gain differential observers (a) $x_i(t), i = 0, 1, 2$ and (b) $z_i(t), i = 0, 1, 2$.

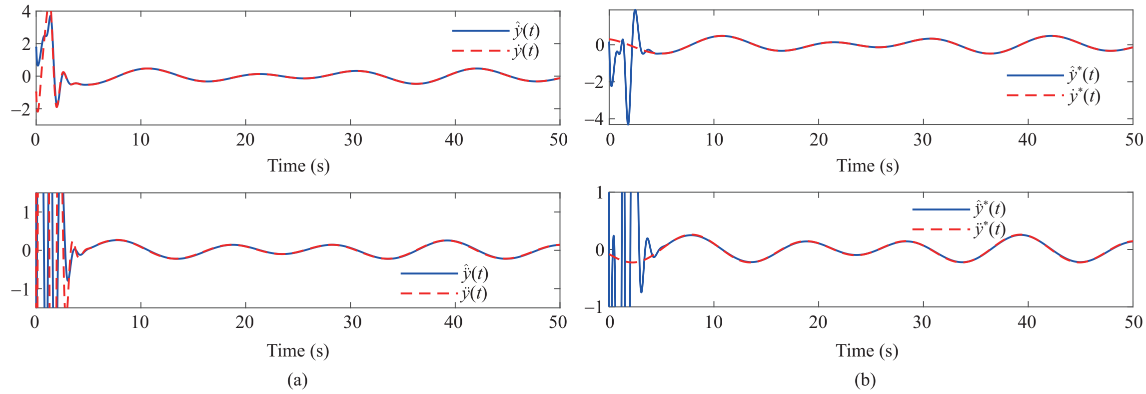

Figure 8 (Color online) Trajectories of (a) the constructed system output derivative and (b) the constructed reference signal derivative.

order tracking properties are still obtained. In particular, an indirect estimation strategy is proposed to estimate the derivatives of the reference signal, which is essential to ensure higher-order exponential convergence of the output tracking error. The results obtained in this paper encourage the authors to explore whether the nonlinear adaptive control systems covering feedback linearizable systems and strict-feedback nonlinear systems also have such higher-order properties, which deserve further studies.

Acknowledgements This work was supported in part by National Natural Science Foundation of China (Grant Nos. 62322304, 61925303, 62173323, 62003277, 62088101, U20B2073), Foundation (Grant No. 2019-JCJQ-ZD-049), and Beijing Institute of Technology Research Fund Program for Young Scholars.

References

1 Morse A S. Global stability of parameter-adaptive control systems. IEEE Trans Automat Contr, 1980, 25: 433–439
2 Goodwin G C, Mayne D Q. A parameter estimation perspective of continuous time model reference adaptive control. Automatica, 1987, 23: 57–70
3 Narendra K S, Annaswamy A M. Stable Adaptive Systems. Englewood Cliffs: Prentice-Hall, 1989
4 Sastry S, Bodson M. Adaptive Control: Stability, Convergence, and Robustness. Englewood Cliffs: Prentice-Hall, 1989
5 Ioannou P A, Sun J. Robust Adaptive Control. Englewood Cliffs: Prentice-Hall, 1996
6 Tao G. Adaptive Control Design and Analysis. Hoboken: Wiley, 2003
7 Tao G. Multivariable adaptive control: a survey. Automatica, 2014, 50: 2737–2764
8 Tao G, Song G. Higher order tracking properties of model reference adaptive control systems. IEEE Trans Automat Contr, 2018, 63: 3912–3918
9 Wen L Y, Tao G, Song G. Higher-order tracking properties of nonlinear adaptive control systems. Syst Control Lett, 2020, 145: 104781
10 Tao G, Wen L Y. Higher-order tracking properties of adaptive backstepping control systems. Automatica, 2023, 153: 111019
11 Krstic M, Kanellakopoulos I, Kokotovic P V. Nonlinear and Adaptive Control Design. New York: John Wiley & Sons, 1995
12 Zhu Y, Krstic M, Su H Y. Delay-adaptive control for linear systems with distributed input delays. Automatica, 2020, 116: 108902
13 Roy S, Baldi S, Ioannou P A. An adaptive control framework for underactuated switched euler-lagrange systems. IEEE Trans Automat Contr, 2022, 67: 4202–4209

14 Ortega R, Nikiforov V, Gerasimov D. On modified parameter estimators for identification and adaptive control. A unified framework and some new schemes. Annu Rev Control, 2020, 50: 278–293
15 Wang C L, Wen C Y, Hu Q L. Event-triggered adaptive control for a class of nonlinear systems with unknown control direction and sensor faults. IEEE Trans Automat Contr, 2019, 65: 763–770
16 He S D, Dai S L, Dong C. Adaptive synchronization control of uncertain multiple USVs with prescribed performance and preserved connectivity. Sci China Inf Sci, 2022, 65: 199201
17 Fu J J, Lv Y Z, Yu W W. Robust adaptive time-varying region tracking control of multi-robot systems. Sci China Inf Sci, 2023, 66: 159202
18 Wang L, Sun W, Wu Y Q. Adaptive asymptotic tracking control for stochastic nonlinear systems with unknown backlash-like hysteresis. J Syst Sci Complex, 2022, 35: 1824–1838
19 Ning P J, Hua C C, Meng R. Adaptive control for a class of nonlinear time-delay system based on the fully actuated system approaches. J Syst Sci Complex, 2022, 35: 522–534
20 Gaudio J E, Annaswamy A M, Lavretsky E, et al. Parameter estimation in adaptive control of time-varying systems under a range of excitation conditions. IEEE Trans Automat Contr, 2022, 67: 5440–5447
21 Culbertson P, Slotine J J, Schwager M. Decentralized adaptive control for collaborative manipulation of rigid bodies. IEEE Trans Robot, 2021, 37: 1906–1920
22 Van M, Ge S S. Adaptive fuzzy integral sliding-mode control for robust fault-tolerant control of robot manipulators with disturbance observer. IEEE Trans Fuzzy Syst, 2020, 29: 1284–1296
23 Guay M, Moshksar E, Dochain D. A constrained extremum-seeking control approach. Intl J Robust Nonlinear, 2015, 25: 3132–3153
24 Liu Y J, Lu S, Tong S C, et al. Adaptive control-based barrier Lyapunov functions for a class of stochastic nonlinear systems with full state constraints. Automatica, 2018, 87: 83–93
25 Hou Z S, Xiong S S. On model-free adaptive control and its stability analysis. IEEE Trans Automat Contr, 2019, 64: 4555–4569
26 Oliveira T R, Rodrigues V H P, Fridman L. Generalized model reference adaptive control by means of global HOSM differentiators. IEEE Trans Automat Contr, 2018, 64: 2053–2060
27 Yang X F, Zhang J, Fridman E. Periodic averaging of discrete-time systems: a time-delay approach. IEEE Trans Automat Contr, 2022. doi: 10.1109/tac.2022.3209496
28 Zhang Y J, Zhang J F. A quantized output feedback MRAC scheme for discrete-time linear systems. Automatica, 2022, 145: 110575
29 Wang L, Kellett C M. Adaptive semiglobal nonlinear output regulation: an extended-state observer approach. IEEE Trans Automat Contr, 2020, 65: 2670–2677
30 Zhang Y J, Tao G, Chen M, et al. A matrix decomposition based adaptive control scheme for a class of MIMO non-canonical approximation systems. Automatica, 2019, 103: 490–502
31 Zhang Y J, Zhang J F, Liu X K. Matrix decomposition-based adaptive control of noncanonical form MIMO DT nonlinear systems. IEEE Trans Automat Contr, 2022, 67: 4330–4337
32 Khalil H K. Nonlinear Systems. 3rd ed. New Jersey: Prentice Hall, 2002
33 Song G, Tao G. New higher-order convergence properties for multivariable model reference adaptive control systems. In: Proceedings of the American Control Conference, Philadelphia, 2019. 1090–1095
34 Schmidt L V. Introduction to Aircraft Flight Dynamics. Reston: American Institute of Aeronautics and Astronautics, Inc., 1998

Appendix A

This section introduces three lemmas, a definition, and a theorem that are used in the proofs of Theorem 1, Theorem 2, Theorem 4, and Theorem 5. More details of these results can be seen in [6, 32].

Lemma A1 ([6]). For an indirect MRAC system with $\omega_1(t) = \frac{a(s)}{\Lambda_c(s)}[u](t) \in \mathbb{R}^{n-1}$, $\omega_2(t) = \frac{a(s)}{\Lambda_c(s)}[y](t) \in \mathbb{R}^{n-1}$ with $a(s) = [1, s, \ldots, s^{n-2}]$ and $\Lambda_c(s)$ a monic stable polynomial of degree $n-1$, there exist constant parameters $\theta_1^*, \theta_2^*, \theta_{20}^*, \theta_3^*$ such that

$$\theta_1^{*\mathrm{T}} a(s)P(s) + (\theta_2^{*\mathrm{T}} a(s) + \theta_{20}^* \Lambda_c(s))Z(s) = \Lambda_c(s)(P(s) - \theta_3^* Z(s)P_m(s)).$$

Lemma A2 ([6]). For an indirect APPC system with $Q_m(s)$ below (4) and $A^*(s)$ being any given stable monic polynomial of degree $2n + n_q - 1$, there exists a unique solution $\{C(s), D(s)\}$ to the equation

$$C(s)Q_m(s)P(s) + D(s)Z(s) = A^*(s).$$

Moreover, $C(s)$ is a monic polynomial of degree $n-1$ and the degree of $D(s)$ is not larger than $n_q + n - 1$.

Lemma A3 ([6]). Consider the system (5). If the signal $\phi(t)$ is persistently exciting, then $\lim_{t\to\infty} \theta_p(t) = \theta_p^*$ exponentially.

Definition 1 ([6]). A bounded vector signal $x(t) \in \mathbb{R}^q$, $q \geqslant 1$, is persistent excitation if there exist δ_0 and α_0 such that $\int_\sigma^{\sigma+\delta_0} x(t)x^{\mathrm{T}}(t)\mathrm{d}t \geqslant \alpha_0 I, \forall \sigma \geqslant t_0$.

Theorem 6 ([32]). Consider the closed-loop system of the plant (A1) and (A2) and the output feedback controller (A4)–(A5). Suppose the origin of (A3) is asymptotically stable and $\mathcal{R}$ is its region of attraction. Let $\mathcal{S}$ be any compact set in the interior of $\mathcal{R}$ and $\mathcal{Q}$ be any compact subset of $\mathcal{R}^\rho$. Then,

(1) given any $\mu > 0$, there exists $\varepsilon_1^* > 0$, dependent on μ, such that, for every $0 < \varepsilon \leqslant \varepsilon_1^*$, the solutions of the closed-loop system, starting in $\mathcal{S} \times \mathcal{Q}$, satisfy $\|\mathcal{X}(t) - \mathcal{X}_r(t)\| \leqslant \mu, \forall t \geqslant 0$, where $\mathcal{X}_r$ is the solution of (A3), starting at $\mathcal{X}(0)$;

(2) if the origin of (A3) is exponentially stable and that $f(\mathcal{X})$ is continuously differentiable in some neighborhood of $\mathcal{X} = 0$, then there exists $\varepsilon_2^* > 0$ such that, for every $0 < \varepsilon \leqslant \varepsilon_2^*$, the origin of the closed-loop system is exponentially stable and $\mathcal{S} \times \mathcal{Q}$ is a subset of its region of attraction.

The system information in Theorem 6 is given as follows. The closed-loop system (A1) and (A2) is

$$\dot{x} = Ax + B\phi(x, z, u), \ \dot{z} = \psi(x, z, u), \tag{A1}$$

$$y = Cx, \ \zeta = q(x, z), \tag{A2}$$

where $u \in \mathbb{R}^p$ is the control input, $y \in \mathbb{R}^m$ and $\zeta \in \mathbb{R}^s$ are measured outputs, and $x \in \mathbb{R}^\rho$ and $z \in \mathbb{R}^\ell$ constitute the state vector. The matrices $A \in \mathbb{R}^{\rho\times\rho}$, $B \in \mathbb{R}^{\rho\times m}$, and $C \in \mathbb{R}^{m\times\rho}$ are given by

$$
\begin{aligned}
A &= \text{ block diag}\left[A_1,\ldots,A_m\right],\ A_i = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{\rho_i\times\rho_i}, \\
B &= \text{block diag}\left[B_1,\ldots,B_m\right],\ B_i = [0,0,\ldots,0,1]^{\mathrm{T}}_{\rho_i\times 1}, \\
C &= \text{ block diag}\left[C_1,\ldots,C_m\right],\ C_i = [1,0,\ldots,0]_{1\times\rho_i},
\end{aligned}
$$

where $1 \leqslant i \leqslant m$ and $\rho = \rho_1 + \cdots + \rho_m$. The functions ϕ, ψ, and q are locally Lipschitz for $(x,z,u) \in D_x \times D_z \times \mathbb{R}^p$, where $D_x \subset \mathbb{R}^\rho$ and $D_z \subset \mathbb{R}^s$ are domains that contain their respective origins. Moreover, $\phi(0,0,0) = 0$, $\psi(0,0,0) = 0$, and $q(0,0) = 0$. The state feedback controller is of the form $u = \gamma(\vartheta, x, \zeta)$, $\dot{\vartheta} = \Gamma(\vartheta, x, \zeta)$, where γ and Γ are locally Lipschitz functions and globally bounded functions of x. Moreover, $\gamma(0,0,0) = 0$ and $\Gamma(0,0,0) = 0$. For convenience, we write the closed-loop system under state feedback as

$$\dot{\mathcal{X}} = f(\mathcal{X}), \tag{A3}$$

where $\mathcal{X} = (x, z, \vartheta)$. The output feedback controller is

$$\dot{\vartheta} = \Gamma(\vartheta, \hat{x}, \zeta),\ u = \gamma(\vartheta, \hat{x}, \zeta), \tag{A4}$$

where $\hat{x}$ is generated by the high-gain observer

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{x}, \zeta, u) + H(y - C\hat{x}). \tag{A5}$$

The observer gain H is chosen as $H =$ block diag$\left[H_1,\ldots,H_m\right]$, $H_i = [\frac{\alpha_1^i}{\varepsilon}, \ldots, \frac{\alpha_{\rho_i-1}^i}{\varepsilon^{\rho_i-1}}, \frac{\alpha_{\rho_i}^i}{\varepsilon^{\rho_i}}]^{\mathrm{T}}_{\rho_i\times 1}$, where ε is a positive constant to be specified and α_j^i are chosen constants such that $s^{\rho_i} + \alpha_1^i s^{\rho_i-1} + \cdots + \alpha_{\rho_i-1}^i s + \alpha_{\rho_i}^i = 0$ are Hurwitz. The function $\phi_0(x,\zeta,u)$ is a nominal model of $\phi(x,z,u)$, which is locally Lipschitz and globally bounded in x. Moreover, $\phi_0(0,0,0) = 0$.