SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

April 2024, Vol. 67, Iss. 4, 142201:1–142201:21
https://doi.org/10.1007/s11432-022-3766-3

A model reduction approach for discrete-time linear time-variant systems with delayed inputs

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Received 12 August 2022/Revised 27 October 2022/Accepted 23 April 2023/Published online 8 March 2024

Abstract A model reduction approach is presented for discrete-time linear time-variant input-delayed systems. According to this proposed approach, a dynamical variable is constructed by taking advantage of the current state and historical information of input. It is revealed that the behavior of this dynamical variable is governed by a discrete-time linear delay-free system. It is worth noting that the presented variable transformation does not require the system matrix to be invertible. Based on the reduced delay-free models, stabilizing control laws can be easily obtained for the original delayed system. For the case with a single input delay, the constructed variable is an exact prediction for the future state, and thus the stabilizing control law could be designed by replacing the future state with its prediction. Finally, three discrete-time periodic systems with delayed input are employed to illustrate how to utilize the presented model reduction approaches.

Keywords reduction approach, predictor-based feedback, time-varying systems, delayed input, periodic system

1 Introduction

Time delays appear in practical engineering plants with aftereffects. Some familiar examples include the manufacturing process [1] and the internal combustion engine [2]. In addition, owing to the information transmission for feedback control, delays may arise in control inputs [3]. Owing to the wide applications, systems with delayed states and/or delayed inputs have been attracting persistent attention from many researchers [4–6].

A state transformation was proposed by Kwon and Pearson [7] to develop controllers for linear time-invariant (LTI) systems with delayed control. It was revealed in [7] that the dynamics of the new variable is governed by an input-delay-free system. Further, a control law was designed to stabilize this input-delayed system with the aid of the reduced delay-free system. In 1982, the approach in [7] was generalized by Artstein to general continuous-time LTI (CT-LTI) input-delayed systems in [8], where the approach was named reduction approaches. In [9], a class of feedback control laws using past information of the control input was developed for LTI systems with general input-delays. According to the result in [9], the resultant closed-loop system possesses a preassigned finite spectrum in the complex plane. It was pointed out by Artstein in [8] that the reduction-type scheme was implicitly utilized in the finite-spectrum approach in [9].

Since 1982, the model reduction method presented by Artstein has been widely utilized, and has been named Artstein's reduction method [10, 11], or Artstein reduction [12]. In [13], the model reduction approach was applied to CT-LTI input-delayed systems subjected to parameter uncertainties and the feedback gain in the reduction-based control law was given by feasible conditions of a group of linear matrix inequalities. In [14], a state transformation as presented in [7, 8] was utilized to design a control law to robustly stabilize continuous-time uncertain systems with a time-variant input delay by adopting the linear matrix inequality (LMI) technique. Such a state transformation was also used in [15, 16] to

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design robust stabilizing controllers of uncertain CT-LTI systems with one unknown input delay. Based on the variable change in [8] for CT-LTI systems with multiple input delays, a recursive method was presented in [17] to establish controllers of blocked linear systems with delayed inputs, delayed outputs, and delayed interconnections among these blocks.

The Artstein's reduction method has also been utilized in discrete-time input-delayed systems. In [11], the Artstein's reduction method was utilized to stabilize an uncertain discrete-time LTI system with one time-variant input-delay. A state transformation was firstly constructed by using the bounds of the input delay, and then the LMI technique was used to solve the control gains. The reduction method presented by Artstein was generalized in [18] to transform discrete-time LTI systems with multiple input-delays into delay-free systems. The reduction method in [18] was also utilized in [19] to design an observer-based controller of discrete-time LTI plants with multiple input-delays. Obviously, system matrices are required to be invertible when the reduction methods in [11,18] are utilized. Recently, the model reduction technique was also utilized to design anti-disturbance control laws for a discrete-time LTI system with a delayed input and a delay-free input in [20].

In the aforementioned results on model reduction approaches, the involved systems mainly include LTI systems with input-delays. Recently, model reduction approaches were investigated for linear time-variant (LTV) systems with single constant input-delays in [21], where two state transformation methods were presented to convert delayed systems into delay-free systems. In the first method, the state transition matrix obtained by setting the input to zero needs to be utilized. In the second method, the transition matrix was not utilized under the condition of the periodicity of the system matrix. The model reduction approach was further investigated in [22] for an LTV system with one input-delay by using Floquet theory. The key of the approach in [22] is to reduce the considered time-variant system to a new system with a time-invariant system matrix.

In the results of the aforementioned studies, the state transformation generated by taking advantage of current states and past information of control input plays a key role. In [23], a predictor to exactly predict a certain future state was utilized to construct a control law for continuous-time LTI input-delayed systems with a single constant delay. The application of this control law results in a closed-loop system with a finite spectrum. The prediction scheme in [23] was also applied in [24] to design dynamical control laws in the form of observer-based state feedback in LTI systems with uncertain time-variant delays in inputs and outputs. In [25], exact predictor-based feedbacks were designed for multi-input LTI systems with distinct input-delays. Actually, there also exist state transformations in the control law. In [26], the prediction scheme for continuous-time LTI systems with single delays was utilized to establish controllers for a type of input-delayed system with a stochastic delay. Compared with the results in [13, 16], a significant difference of the schemes in [23–25] lies in the fact that the state transformation is constructed based on the zero-input response (not simply the state) plus past information of control input, and the change variable is the exact prediction of delayed systems. The discrete counterpart of the scheme in [23] was investigated in [27], and a prediction scheme was also given. The method in [27] does not require invertibility of the system matrix, which overcomes the shortcoming of that in [11]. It is easily found that time-delay systems can also be converted into ordinary systems without delays when the state transformation is defined according to the prediction schemes in [23, 27]. Such a fact implies that the predictor-based feedback should be viewed as a reduction-type scheme.

In the present paper, discrete-time LTV systems with multiple input-delays are investigated. We aim to develop a model reduction approach to convert such a class of delayed systems into time-variant systems without delays. The motivation for such an investigation stems from two aspects. On one hand, time-variant discrete-time and continuous-time systems should play equally important roles. However, there seems a lack of results on the model reduction approach for discrete-time LTV systems with multiple input delays. On the other hand, the system matrix is required to be invertible for some existing results on model reduction for discrete-time LTI systems. If the result in this current paper is adopted in discrete-time LTI systems with multiple input-delays, the invertibility restriction for system matrices can be removed.

Throughout this paper, for two integers $a \leqslant b$, the set $\{a, a+1, \dots, b\}$ is defined as $\mathbb{I}[a, b]$.

2 Problem formulation and preliminaries

In this paper, two types of discrete-time linear time-variant (DT-LTV) delayed systems are investigated. The first type of systems is in the form of

$$x(t+1) = A(t)x(t) + \sum_{i=0}^{N} B_i(t)u(t-h_i), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^r$ is the control input, $h_i, i \in \mathbb{I}[0,N]$ are known input delays, and $A(t) \in \mathbb{R}^{n\times n}$ and $B_i(t) \in \mathbb{R}^{n\times r}, i \in \mathbb{I}[1,N]$ are matrix sequences with proper dimensions. Without loss of generality, the delayed inputs can be assumed to be ordered such that $0 = h_0 < h_1 < h_2 < \cdots < h_N = h$. The second type of systems takes the following form:

$$x(t+1) = A(t)x(t) + \sum_{i=1}^{r} b_i(t)u_i(t-h_i), \tag{2}$$

where $u_i(t) \in \mathbb{R}, i \in \mathbb{I}[1,r]$ are the scalar inputs, and $b_i(t) \in \mathbb{R}^n, i \in \mathbb{I}[1,r]$. Similarly, for the system (2) the scalar delayed inputs are also assumed to be ordered such that $1 \leqslant h_1 \leqslant h_2 \leqslant \cdots \leqslant h_r = h$. For the system (1), there are multiple delays in the same control input. Moreover, these delays are the same for all the input channels. For the system (2), the input delays in different input channels may be distinct. For the system (2), let

$$B(t) = \begin{bmatrix} b_1(t) & b_2(t) & \cdots & b_r(t) \end{bmatrix} \in \mathbb{R}^{n\times r}, \quad u(t) = \begin{bmatrix} u_1(t) & u_2(t) & \cdots & u_r(t) \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^r. \tag{3}$$

If all the input-delays $h_i, i \in \mathbb{I}[1,r]$ are equal to h, the system (2) is degraded to the following system with a single input delay

$$x(t+1) = A(t)x(t) + B(t)u(t-h). \tag{4}$$

For LTI input-delayed systems, the celebrated model reduction approach [8] and the finite spectrum assignment approach [9] can be used to transform them into delay-free systems. For LTI systems with single input-delays, the control laws designed by these two approaches possess the structure of predictor-based feedback. For this case, the predictor is utilized to predict a certain future state of the original system. The concept of finite spectrum is not appropriate for LTV systems. In this paper, we aim to generalize the model reduction approach in [8] to DT-LTV input-delayed systems. In addition, feedback control laws are also designed to stabilize the delayed systems (1) and (2) with developed model reduction methods as tools.

In the sequel, the concept and properties of transition matrices are introduced for DT-LTV systems. For details, one can refer to [28].

Definition 1. For the DT-LTV system

$$x(t+1) = A(t)x(t) + B(t)u(t) \tag{5}$$

with the state $x(t) \in \mathbb{R}^n$ and the input $u(t) \in \mathbb{R}^r$, a matrix sequence $\Phi(k,l)$ is called its transition matrix if $\Phi(k,l)$ satisfies

$$\begin{cases} \Phi(k+1,l) = A(k)\Phi(k,l), \\ \Phi(k,k) = I, \end{cases}$$

for any integers k and l.

According to Definition 1, it is easily known that $\Phi(l+1,l) = A(l)$, and

$$\Phi(k,l) = A(k-1)A(k-2)\cdots A(l), \ k > l. \tag{6}$$

In addition, the system matrix $A(t)$ and transition matrix $\Phi(k,l)$ satisfy the commutativity-like law. This is the following conclusion.

Lemma 1. For the DT-LTV system (5), let $\Phi(k,l)$ be its transition matrix. Then, for $k > l$ there holds

$$\Phi(k,l) = \Phi(k,l+1)A(l).$$

With transition matrices as tools, the state response of the system (5) could be explicitly expressed.

Lemma 2. For the system (5), let $\Phi(k,l)$ be its transition matrix, and x_0 be the initial value of $x(t)$ at $t=t_0$. Then, the state response of this system is explicitly given as

$$x(t)=\Phi(t,t_0)x_0+\sum_{i=t_0}^{t-1}\Phi(t,i+1)B(i)u(i). \tag{7}$$

At the end of this section, a formula for the determinant of partitioned matrices is provided in the following lemma. This result will be used in the sequel three sections.

Lemma 3. Let A and D be two square matrices, and B and C be matrices with appropriate dimensions. If D is invertible, then there holds

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix}=\det D\det\left(A-BD^{-1}C\right).$$

3 The case with a single input delay

In this section, the system (4) with a single input-delay $h>0$ is investigated. It has been known that the dynamic of an h-ahead step predictor of a discrete-time linear time-invariant system (DT-LTI) with a single input-delay can be characterized by a delay-free system. Inspired by this result, the exact prediction scheme is taken into consideration for this time-varying system (4).

By using Lemma 2, the exact expression for a certain future state $x(t+h)$ is expressed as

$$x(t+h)=\Phi(t+h,t)x(t)+\sum_{i=t}^{t+h-1}\Phi(t+h,i+1)B(i)u(i-h) \tag{8}$$

by utilizing the current state and historical information of the control input. The right-hand side in (8) is available at time instant t. For the notational convenience, let $\chi_h(t)$ denote it. That is to say, $\chi_h(t)$ is the exact estimation for the future state $x(t+h)$ of (4). By simple calculation, it is deduced that

$$\chi_h(t)=\Phi(t+h,t)x(t)+\sum_{j=t-h}^{t-1}\Phi(t+h,h+j+1)B(h+j)u(j). \tag{9}$$

For the variable transformation in (9), the following conclusion is obtained.

Theorem 1. For the discrete-time LTV delayed system (4), construct a variable change in (9). Then, the dynamic equation of $\chi_h(t)$ can be characterized by

$$\chi_h(t+1)=A(t+h)\chi_h(t)+B(t+h)u(t). \tag{10}$$

Proof. By using the system equation in (4), for the variable $\chi_h(t)$ in (9) it is deduced that

$$\begin{aligned}
\chi_h(t+1) &= \Phi(t+1+h,t+1)x(t+1)+\sum_{j=t+1-h}^{t}\Phi(t+1+h,h+j+1)B(h+j)u(j)\\
&= \Phi(t+1+h,t+1)\left[A(t)x(t)+B(t)u(t-h)\right]\\
&\quad+\sum_{j=t+1-h}^{t}\Phi(t+1+h,h+j+1)B(h+j)u(j)\\
&= \Phi(t+1+h,t+1)A(t)x(t)+\Phi(t+1+h,t+1)B(t)u(t-h)\\
&\quad+\sum_{j=t+1-h}^{t}\Phi(t+1+h,h+j+1)B(h+j)u(j).
\end{aligned}$$

By using the property in Lemma 1 for transition matrices, from the previous expression we have

$$\chi_h(t+1) = \Phi(t+1+h,t)x(t)+\Phi(t+1+h,t+1)B(t)u(t-h)$$

$$
\begin{aligned}
&+\sum_{j=t-h}^{t-1}\Phi(t+1+h,h+j+1)B(h+j)u(j)\\
&+B(t+h)u(t)-\Phi(t+1+h,t+1)B(t)u(t-h)\\
=\ &\Phi(t+1+h,t)x(t)+\sum_{j=t-h}^{t-1}\Phi(t+1+h,h+j+1)B(h+j)u(j)+B(t+h)u(t).
\end{aligned}
$$

By Definition 1, from the previous relation it is deduced that

$$
\begin{aligned}
\chi_h(t+1) &= A(t+h)\Phi(t+h,t)x(t)+A(t+h)\sum_{j=t-h}^{t-1}\Phi(t+h,h+j+1)B(h+j)u(j)+B(t+h)u(t)\\
&= A(t+h)\left[\Phi(t+h,t)x(t)+\sum_{j=t-h}^{t-1}\Phi(t+h,h+j+1)B(h+j)u(j)\right]+B(t+h)u(t)\\
&= A(t+h)\chi_h(t)+B(t+h)u(t).
\end{aligned}
$$

This is the conclusion of this theorem.

It is revealed in Theorem 1 that the input-delayed system (4) is converted into the delay-free system (10) under the variable change (9). According to this result in Theorem 1, a controller can be established to stabilize the delayed system (4). If its corresponding delay-free system (5) has a stabilizing law $u(t) = K(t)x(t)$, then under the control law $u(t) = K(t+h)\chi_h(t)$ the closed-loop system of (10) is obtained as

$$\chi_h(t+1) = [A(t+h)+B(t+h)K(t+h)]\,\chi_h(t), \tag{11}$$

which is asymptotically stable. In this case, $\lim_{t\to\infty}\chi_h(t) = 0$, and thus $\lim_{t\to\infty}u(t) = 0$. With these two relations, it follows from (9) that

$$\lim_{t\to\infty}\Phi(t+h,t)x(t) = 0. \tag{12}$$

If the system matrix $A(t)$ is invertible and bounded for each t, then $\Phi(t+h,t)$ is invertible and bounded for each t. Thus, it is immediately obtained from (12) that $\lim_{t\to\infty}x(t) = 0$. This is the result of the following theorem.

Theorem 2. For the DT-LTV system (4) with an input-delay, it is assumed that the system matrix $A(t)$ is invertible and bounded for each t. If $u(t) = K(t)x(t)$ is a stabilizing controller of its corresponding delay-free system (5), then the control law

$$u(t) = K(t+h)\left[\Phi(t+h,t)x(t)+\sum_{j=t-h}^{t-1}\Phi(t+h,h+j+1)B(h+j)u(j)\right] \tag{13}$$

can stabilize the delayed system (4).

It should be pointed out that the term in the square bracket of the control law (13) is the exact estimation of the future state $x(t+h)$ of the system (4). In this sense, the control law (13) possesses the structure of predictor-based state feedback. Such a control law was also provided in Chapter 8 of [29]. On the other hand, it seems difficult to prove the result of Theorem 2 for the case where $A(t)$ is not invertible for each t. Nevertheless, it should be conjectured that this conclusion still holds for this case. At the end of this section, we attempt to confirm that the result of Theorem 2 is true for time-invariant systems without the requirement for the invertibility of the system matrix.

When the system (4) is time-invariant, it can be simply written as

$$x(t+1) = Ax(t)+Bu(t-h). \tag{14}$$

According to Theorem 1, a variable transformation for this system can be given as

$$\chi_h(t) = A^h x(t)+\sum_{j=t-h}^{t-1}A^{t-j-1}Bu(j), \tag{15}$$

and the dynamic equation of $\chi_h(t)$ is

$$\chi_h(t+1) = A\chi_h(t) + Bu(t).$$

According to this model, a control law for (14) can be designed as

$$u(t) = K\left[A^h x(t) + \sum_{j=t-h}^{t-1} A^{t-j-1}Bu(j)\right]. \tag{16}$$

Such a predictor-based feedback was utilized in [27]. In addition, this control law was also utilized as a basis to attenuate external disturbances in [30]. Since the considered system (14) is a linear time-invariant system, the spectrum of the resultant closed-loop system can be explored. Such investigation was not conducted in [27,30]. A possible reason is that it is a bit difficult to deal with the case where the system matrix A is singular. In the following theorem, the spectrum property of the closed-loop system under the control law (16) is characterized.

Theorem 3. For the discrete-time LTI delayed system (14), under the control law (16) the characteristic equation of the resultant closed-loop system is $\det(zI - A - BK) = 0$.

Proof. Let $x(z)$ and $u(z)$ denote the Z-transformations of $x(t)$ and $u(t)$, respectively. Taking Z-transformation for the delayed system (14) gives

$$zx(z) = Ax(z) + z^{-h}Bu(z). \tag{17}$$

In addition, it is easily derived that the control law (16) can be equivalently expressed as

$$u(t) = KA^h x(t) + K\sum_{j=0}^{h-1} A^j Bu(t-j-1). \tag{18}$$

By carrying out Z-transformation for both sides of (18), it is derived that

$$u(z) = KA^h x(z) + K\sum_{j=0}^{h-1} z^{-j-1}A^j Bu(z). \tag{19}$$

Let

$$M_1(z) = \begin{bmatrix} zI_n - A & -z^{-h}B \\ KA^h & -I_r + K\sum_{j=0}^{h-1} z^{-j-1}A^j B \end{bmatrix}. \tag{20}$$

By combining (17) and (19), the following matrix-vector form can be obtained:

$$M_1(z)\begin{bmatrix} x^{\mathrm{T}}(z) & u^{\mathrm{T}}(z) \end{bmatrix}^{\mathrm{T}} = 0.$$

Thus, the characteristic polynomial $\Delta_1(z)$ of the closed-loop system for the considered system (14) under the control law (16) is obtained as

$$\Delta_1(z) = \det M_1(z).$$

Let

$$M(z) = \begin{bmatrix} I_n & 0_{n\times r} \\ K\sum_{j=0}^{h-1} z^{h-1-j}A^j & I_r \end{bmatrix}, \tag{21}$$

and notice that

$$\begin{aligned} & K\sum_{j=0}^{h-1} z^{h-1-j}A^j\,(zI_n - A) \\ &= K\left(A^{h-1} + zA^{h-2} + \cdots + z^{h-2}A + z^{h-1}I_n\right)(zI_n - A) \\ &= K\left(z^h I_n - A^h\right). \end{aligned} \tag{22}$$

Then, it is derived that

$$M(z)M_1(z) = \begin{bmatrix} zI_n - A & -z^{-h}B \\ K\left(z^h I_n - A^h\right) + KA^h & -I_r \end{bmatrix} = \begin{bmatrix} zI_n - A & -z^{-h}B \\ z^h K & -I_r \end{bmatrix},$$

from which, by using Lemma 3, we have

$$\begin{aligned} &\det\left[M(z)M_1(z)\right] \\ &\quad= \det(-I_r)\det\left[zI_n - A - z^{-h}B(I_r)^{-1}\left(z^h K\right)\right] \\ &\quad= (-1)^r \det\left(zI_n - A - BK\right). \end{aligned}$$

In addition, it is obvious that $\det M(z) = 1$. Combining this with the previous relation gives $\Delta_1(z) = (-1)^r \det(zI_n - A - BK)$. From this, the conclusion of this theorem can be immediately obtained.

According to Theorem 3, if K is a stabilizing feedback gain for the delay-free system $x(t+1) = Ax(t) + Bu(t)$, then the control law (16) can stabilize the delayed system (14). It should be pointed out that this conclusion does not require the invertibility of the system matrix A.

4 The case with multiple input delays

DT-LTV systems with general multiple input delays are investigated in this section. The considered system is in the form of (1). Similarly to the case in Section 3, the aim in this section is to transform the system (1) into a system without delays. In Section 3, it has been revealed that the dynamics of the ahead prediction is a delay-free system. For this reason, we firstly investigate the h-ahead prediction for the system (1).

By using Lemma 2, the future state $x(t+h)$ for the system (1) is expressed as

$$\begin{aligned} x(t+h) &= \Phi(t+h,t)x(t) + \sum_{j=t}^{t+h-1} \Phi(t+h,j+1) \sum_{i=0}^{N} B_i(j)u(j-h_i) \\ &= \Phi(t+h,t)x(t) + \sum_{i=0}^{N} \sum_{j=t}^{t+h-1} \Phi(t+h,j+1)B_i(j)u(j-h_i). \end{aligned}$$

Let $k = j - h_i$. In view of $h_0 = 0$, by the previous relation it is deduced that

$$\begin{aligned} x(t+h) &= \Phi(t+h,t)x(t) + \sum_{i=0}^{N} \sum_{k=t-h_i}^{t+h-1-h_i} \Phi(t+h,h_i+k+1)B_i(h_i+k)u(k) \\ &= \Phi(t+h,t)x(t) + \sum_{i=1}^{N} \sum_{k=t-h_i}^{t-1} \Phi(t+h,h_i+k+1)B_i(h_i+k)u(k) \\ &\quad + \sum_{i=1}^{N-1} \sum_{k=t}^{t+h-1-h_i} \Phi(t+h,h_i+k+1)B_i(h_i+k)u(k) + \sum_{k=t}^{t+h-1} \Phi(t+h,k+1)B_0(k)u(k). \end{aligned} \tag{23}$$

In the previous expression, the last two terms cannot be available at the current time instant t. Therefore, we cannot construct a new variable in terms of the true prediction of the state $x(t+h)$. Alternatively, the last two terms in (23) are deleted, and the following variable transformation is constructed

$$\xi_h(t) = \Phi(t+h,t)x(t) + \sum_{i=1}^{N} \sum_{j=t-h_i}^{t-1} \Phi(t+h,h_i+j+1)B_i(h_i+j)u(j). \tag{24}$$

For this new variable $\xi_h(t)$, the result in the following theorem is obtained to describe its dynamics.

Theorem 4. Given the DT-LTV delayed system (1), construct a variable $\xi_h(t)$ in (24). Then, the dynamic equation of $\xi_h(t)$ can be characterized by

$$\xi_h(t+1) = A(t+h)\xi_h(t) + B(t+h)u(t), \tag{25}$$

with

$$B(t+h)=\sum_{i=0}^{N}\Phi(t+1+h,h_i+t+1)B_i(h_i+t). \tag{26}$$

Proof. By using the system equation (1), for the variable $\xi_h(t)$ in (24) it is deduced that

$$\begin{aligned}\xi_h(t+1) &= \Phi(t+1+h,t+1)x(t+1)+\sum_{i=1}^{N}\sum_{j=t+1-h_i}^{t}\Phi(t+1+h,h_i+j+1)B_i(h_i+j)u(j)\\ &= \Phi(t+1+h,t+1)\left[A(t)x(t)+\sum_{i=0}^{N}B_i(t)u(t-h_i)\right]\\ &\quad+\sum_{i=1}^{N}\sum_{j=t+1-h_i}^{t}\Phi(t+1+h,h_i+j+1)B_i(h_i+j)u(j)\\ &= \Phi(t+1+h,t+1)A(t)x(t)+\Phi(t+1+h,t+1)\sum_{i=0}^{N}B_i(t)u(t-h_i)\\ &\quad+\sum_{i=1}^{N}\sum_{j=t+1-h_i}^{t}\Phi(t+1+h,h_i+j+1)B_i(h_i+j)u(j).\end{aligned}$$

By using Lemma 1, in view of $h_0=0$ it follows from the above relation that

$$\begin{aligned}\xi_h(t+1) &= \Phi(t+1+h,t)x(t)+\Phi(t+1+h,t+1)\sum_{i=0}^{N}B_i(t)u(t-h_i)\\ &\quad+\sum_{i=1}^{N}\sum_{j=t-h_i}^{t-1}\Phi(t+1+h,h_i+j+1)B_i(h_i+j)u(j)\\ &\quad+\sum_{i=1}^{N}\Phi(t+1+h,h_i+t+1)B_i(h_i+t)u(t)\\ &\quad-\sum_{i=1}^{N}\Phi(t+1+h,t+1)B_i(t)u(t-h_i)\\ &= \Phi(t+1+h,t)x(t)+\Phi(t+1+h,t+1)B_0(t)u(t)\\ &\quad+\sum_{i=1}^{N}\sum_{j=t-h_i}^{t-1}\Phi(t+1+h,h_i+j+1)B_i(h_i+j)u(j)\\ &\quad+\sum_{i=1}^{N}\Phi(t+1+h,h_i+t+1)B_i(h_i+t)u(t).\end{aligned} \tag{27}$$

From Definition 1, it is derived from (27) that

$$\begin{aligned}\xi_h(t+1) &= A(t+h)\Phi(t+h,t)x(t)+A(t+h)\sum_{i=1}^{N}\sum_{j=t-h_i}^{t-1}\Phi(t+h,h_i+j+1)B_i(h_i+j)u(j)\\ &\quad+\sum_{i=0}^{N}\Phi(t+1+h,h_i+t+1)B_i(h_i+t)u(t).\end{aligned} \tag{28}$$

By defining the time-varying matrix $B(t+h)$ as in (26), the expression in (28) is rewritten as

$$\xi_h(t+1)=A(t+h)\left[\Phi(t+h,t)x(t)+\sum_{i=1}^{N}\sum_{j=t-h_i}^{t-1}\Phi(t+h,h_i+j+1)B_i(h_i+j)u(j)\right]+B(t+h)u(t),$$

which is the system (25) without input delays. The proof is thus completed.

In Theorem 4, a model reduction approach has been presented to transform the time-varying discrete-time input-delayed system into a delay-free system by constructing a new variable in terms of the state and historical information of control input. Different from the system with one input-delay, the defined variable $\xi_h(t)$ is not the prediction for the future state of (1).

From the result in Theorem 4, the stabilization problem for the delayed system (1) may be solved based on its reduced delay-free system. According to Theorem 4, its corresponding delay-free system is

$$x(t+1) = A(t)x(t) + B(t)u(t), \tag{29}$$

where $B(t)$ is obtained by (26). It should be pointed out that the time-variant matrix $B(t)$ is related to all the input delays $h_i, i \in \mathbb{I}[1, N]$. Now, assume that its corresponding delay-free system (29) is stabilized by controller $u(t) = K(t)x(t)$. Then, the control law $u(t) = K(t+h)\xi_h(t)$ can stabilize the system (25). With this, similar to the case with a single input delay the following conclusion is easily obtained.

Theorem 5. For the DT-LTV system (1) with multiple input-delays, its corresponding delay-free system is given in (29), where the time-variant input matrix $B(t)$ is provided in (26). If the system matrix $A(t)$ is invertible and bounded for each t, and $u(t) = K(t)x(t)$ is a stabilizing controller of the system (29), then under the control law

$$u(t) = K(t+h)\left[\Phi(t+h,t)x(t) + \sum_{i=1}^{N}\sum_{j=t-h_i}^{t-1}\Phi(t+h,h_i+j+1)B_i(h_i+j)u(j)\right], \tag{30}$$

the resultant closed-loop system is stable.

Similarly to the case in Section 3, it is a bit difficult to prove the conclusion of Theorem 5 without the requirement of invertibility of the system matrix $A(t)$. At the end of this section, we make much effort to provide complete results for time-invariant systems.

The result in Theorem 4 is applicable for DT-LTI systems with multiple input-delays in the following form:

$$x(t+1) = Ax(t) + \sum_{i=0}^{N} B_i u(t-h_i). \tag{31}$$

Similar to the system (1), the delayed inputs are ordered such that $0 = h_0 < h_1 < h_2 < \cdots < h_N = h$. For this time-invariant system, there hold

$$\Phi(t+h, h_i+j+1) = A^{t+h-h_i-j-1},$$

and

$$\Phi(t+1+h, h_i+t+1) = A^{h-h_i}.$$

Thus, from Theorem 4 a conclusion can be deduced as a corollary.

Corollary 1. For the LTI delayed system (31), construct the following variable:

$$\xi_h(t) = A^h x(t) + \sum_{i=1}^{N}\sum_{j=t-h_i}^{t-1} A^{t+h-h_i-j-1}B_i u(j). \tag{32}$$

Then, the system (31) is transformed into the delay-free system

$$\xi_h(t+1) = A\xi_h(t) + \left(\sum_{i=0}^{N} A^{h-h_i}B_i\right)u(t). \tag{33}$$

Remark 1. A model reduction is also considered for the DT-LTI system (31) in [18], where the variable transformation is constructed as

$$\zeta(t) = x(t) + \sum_{i=1}^{N}\sum_{j=1}^{h_i} A^{j-h_i-1}B_i u(t-j),$$

and the delayed system (31) is transformed into

$$\zeta(t+1) = Az(t) + \left(\sum_{i=0}^{N} A^{-h_i} B_i\right) u(t).$$

Obviously, the approach in [18] needs a restriction that the system matrix A of the system (31) is invertible. However, such a restriction can be avoided if Corollary 1 is applied. This fact illustrates the advantages of the model reduction approach presented in the current paper.

Based on the resulting delay-free system (33) and the variable transformation (32) in Corollary 1, the following control law can be designed:

$$u(t) = K\left[A^h x(t) + \sum_{i=1}^{N} \sum_{j=t-h_i}^{t-1} A^{t+h-h_i-j-1} B_i u(j)\right] \tag{34}$$

for the delayed system (31). The spectrum of the closed-loop system under this control law is characterized in the following theorem.

Theorem 6. The characteristic equation of the closed-loop system for the DT-LTI delayed system (31) under control law (34) is

$$\Delta_2(z) = \det\left(zI_n - A - \sum_{i=0}^{N} A^{h-h_i} B_i K\right) = 0. \tag{35}$$

Proof. For a time sequence $f(t)$, its Z-transformation is defined as $\mathcal{Z}(f(t))$. For the sake of notational simplicity, let $x(z) = \mathcal{Z}(x(t))$ and $u(z) = \mathcal{Z}(u(t))$ for the system (31). By taking Z-transformation for both sides of (31), we have

$$zx(z) = Ax(z) + \sum_{i=1}^{N} z^{-h_i} B_i u\left(z\right). \tag{36}$$

It is easily known that the control law (34) can be equivalently written as

$$u(t) = KA^h x(t) + \sum_{i=1}^{N} \sum_{j=0}^{h_i-1} KA^{h-h_i+j} B_i u(t-j-1),$$

from which we have

$$u(z) = KA^h x(z) + \sum_{i=1}^{N} \sum_{j=0}^{h_i-1} z^{-j-1} KA^{h-h_i+j} B_i u(z). \tag{37}$$

By combining (36) and (37), the following matrix-vector form is obtained

$$\begin{bmatrix} zI_n - A & -\sum_{i=0}^{N} z^{-h_i} B_i \\ KA^h & K\sum_{i=1}^{N}\sum_{j=0}^{h_i-1} z^{-j-1} A^{h-h_i+j} B_i - I_r \end{bmatrix} \begin{bmatrix} x(z) \\ u(z) \end{bmatrix} = 0. \tag{38}$$

Let

$$M_2(z) = \begin{bmatrix} zI_n - A & -\sum_{i=0}^{N} z^{-h_i} B_i \\ KA^h & K\sum_{i=1}^{N}\sum_{j=0}^{h_i-1} z^{-j-1} A^{h-h_i+j} B_i - I_r \end{bmatrix}. \tag{39}$$

Then, it follows from (38) that the characteristic polynomial of the closed-loop system for the considered system (31) under the controller (34) is $\Delta_2(z) = \det M_2(z)$.

It is easy to derive that

$$K\sum_{i=1}^{N}\sum_{j=0}^{h_i-1} z^{-j-1} A^{h-h_i+j} B_i = K\sum_{i=1}^{N}\sum_{j=0}^{h_i-1} z^{j-h_i} A^{h-j-1} B_i.$$

With this, it is derived that

$$
\begin{aligned}
& -\left(K\sum_{j=0}^{h-1} z^{h-1-j}A^{j}\right)\left(\sum_{i=0}^{N} z^{-h_i}B_i\right) - I_r + K\sum_{i=1}^{N}\sum_{j=0}^{h_i-1} z^{-j-1}A^{h-h_i+j}B_i \\
& = -\left(K\sum_{j=0}^{h-1} z^{j}A^{h-1-j}\right)\left(\sum_{i=0}^{N} z^{-h_i}B_i\right) - I_r + K\sum_{i=1}^{N}\sum_{j=0}^{h_i-1} z^{-j-1}A^{h-h_i+j}B_i \\
& = -K\sum_{i=0}^{N}\sum_{j=0}^{h-1} z^{j-h_i}A^{h-1-j}B_i + K\sum_{i=1}^{N}\sum_{j=0}^{h_i-1} z^{j-h_i}A^{h-j-1}B_i - I_r \\
& = -K\sum_{i=0}^{N}\sum_{j=h_i}^{h-1} z^{j-h_i}A^{h-j-1}B_i - I_r.
\end{aligned}
$$

In view of this relation and (22), pre-multiplying $M_2(z)$ in (39) by $M(z)$ in (21) yields

$$
\begin{aligned}
M(z)M_2(z) &= \begin{bmatrix} zI_n - A & -\sum_{i=0}^{N} B_i z^{-h_i} \\ K\left(z^h I_n - A^h\right) + KA^h & -K\sum_{i=0}^{N}\sum_{j=h_i}^{h-1} z^{j-h_i}A^{h-j-1}B_i - I_r \end{bmatrix} \\
&= \begin{bmatrix} zI_n - A & -\sum_{i=0}^{N} B_i z^{-h_i} \\ z^h K & -K\sum_{i=0}^{N}\sum_{j=h_i}^{h-1} z^{j-h_i}A^{h-j-1}B_i - I_r \end{bmatrix}.
\end{aligned} \tag{40}
$$

Further, by calculation it is obtained that

$$
\begin{aligned}
& (zI_n - A)\sum_{i=0}^{N}\sum_{j=h_i}^{h-1} z^{j-h_i-h}A^{h-j-1}B_i - \sum_{i=0}^{N} z^{-h_i}B_i \\
& = \sum_{i=0}^{N}\sum_{j=h_i}^{h-1} z^{j-h_i-h+1}A^{h-j-1}B_i - \sum_{i=0}^{N}\sum_{j=h_i}^{h-1} z^{j-h_i-h}A^{h-j}B_i - \sum_{i=0}^{N} z^{-h_i}B_i \\
& = \sum_{i=0}^{N} z^{-h_i}B_i - \sum_{i=0}^{N} z^{-h}A^{h-h_i}B_i - \sum_{i=0}^{N} z^{-h_i}B_i \\
& = -\sum_{i=0}^{N} z^{-h}A^{h-h_i}B_i,
\end{aligned}
$$

and

$$
\left(z^h K\right)\sum_{i=0}^{N}\sum_{j=h_i}^{h-1} z^{j-h_i-h}A^{h-j-1}B_i - K\sum_{i=0}^{N}\sum_{j=h_i}^{h-1} z^{j-h_i}A^{h-j-1}B_i - I_r = -I_r.
$$

In view of these two relations, if the following matrix $R(z)$ is constructed

$$
R(z) = \begin{bmatrix} I_n & \sum_{i=0}^{N}\sum_{j=h_i}^{h-1} A^{h-j-1}B_i z^{j-h_i-h} \\ 0_{r\times n} & I_r \end{bmatrix}, \tag{41}
$$

then it follows from (40) that

$$
M(z)M_2(z)R(z) = \begin{bmatrix} zI_n - A & -\sum_{i=0}^{N} z^{-h}A^{h-h_i}B_i \\ z^h K & -I_r \end{bmatrix},
$$

from which, by Lemma 3, we have

$$
\det\left[M(z)M_2(z)R(z)\right]
$$

$$= \det(-I_r)\det\left[zI_n - A + \sum_{i=0}^{N} z^{-h}A^{h-h_i}B_i\left(-I_r\right)^{-1}\left(z^hK\right)\right]$$
$$= (-1)^r \det\left(zI_n - A - \sum_{i=0}^{N} A^{h-h_i}B_iK\right). \tag{42}$$

In addition, owing to the structure of the matrices $M(z)$ and $R(z)$ it is easily known that $\det M(z) = 1$ and $\det R(z) = 1$. With these two relations, the conclusion of this theorem follows from (42). By now, the proof of this theorem is completed.

According to Theorem 6, if K is a stabilizing feedback gain for the delay-free system (33), then the control law (34) can stabilize the system (31) with multiple input delays. Owing to the result of Theorem 6, it has been confirmed that the designed control law (34) is also applicable to the case where the system matrix A in (33) is not invertible.

5 The case with distinct input delays

In this section, we investigate the system (2) with distinct input-delays. Similarly to the cases in Sections 3 and 4, we also aim to transform the system (2) into an LTV delay-free system. An analogous treatment is adopted for this purpose.

From Lemma 2, the future value $x(t+h)$ of the state at the time instant $t+h$ for the system (2) is given as

$$x(t+h) = \Phi(t+h,t)x(t) + \sum_{j=t}^{t+h-1}\Phi(t+h,j+1)\sum_{i=1}^{r}b_i(j)u_i(j-h_i)$$
$$= \Phi(t+h,t)x(t) + \sum_{i=1}^{r}\sum_{j=t}^{t+h-1}\Phi(t+h,j+1)b_i(j)u_i(j-h_i).$$

Let $k = j - h_i$. Then, the above expression is calculated as

$$x(t+h) = \Phi(t+h,t)x(t) + \sum_{i=1}^{r}\sum_{k=t-h_i}^{t+h-1-h_i}\Phi(t+h,h_i+k+1)b_i(h_i+k)u_i(k)$$
$$= \Phi(t+h,t)x(t) + \sum_{i=1}^{r}\sum_{k=t-h_i}^{t-1}\Phi(t+h,h_i+k+1)b_i(h_i+k)u_i(k)$$
$$+\sum_{i=1}^{r-1}\sum_{k=t}^{t+h-1-h_i}\Phi(t+h,h_i+k+1)b_i(h_i+k)u_i(k). \tag{43}$$

In (43), the last term is not available at the current time instant t, and thus cannot be utilized to construct control laws. Therefore, we consider an alternative variable transformation

$$\xi_h(t) = \Phi(t+h,t)x(t) + \sum_{i=1}^{r}\sum_{j=t-h_i}^{t-1}\Phi(t+h,h_i+j+1)b_i(h_i+j)u_i(j), \tag{44}$$

which is obtained by deleting the last term from (43). In the following theorem, the dynamics of this new variable $\xi_h(t)$ in (44) is provided.

Theorem 7. For the LTV delayed system (2) with distinct input-delays, construct the variable change $\xi_h(t)$ in (44) and let

$$\mathring{B}(t+h) = \left[\, \beta_1(t+h)\ \ \beta_2(t+h)\ \cdots\ \beta_r(t+h) \,\right] \tag{45}$$

with

$$\beta_i(t+h) = \Phi(t+1+h, h_i+t+1)b_i(h_i+t). \tag{46}$$

Then, the evolution of $\xi_h(t)$ is governed by

$$\xi_h(t+1) = A(t+h)\xi_h(t) + \mathring{B}(t+h)u(t). \tag{47}$$

Proof. For the variable $\xi_h(t)$ in (44) it is deduced from (2) that

$$\begin{aligned}\xi_h(t+1) &= \Phi(t+1+h,t+1)x(t+1)+\sum_{i=1}^{r}\sum_{j=t+1-h_i}^{t}\Phi(t+1+h,h_i+j+1)b_i(h_i+j)u_i(j)\\ &= \Phi(t+1+h,t+1)\left[A(t)x(t)+\sum_{i=1}^{r}b_i(t)u_i(t-h_i)\right]\\ &\quad+\sum_{i=1}^{r}\sum_{j=t+1-h_i}^{t}\Phi(t+1+h,h_i+j+1)b_i(h_i+j)u_i(j)\\ &= \Phi(t+1+h,t+1)A(t)x(t)+\Phi(t+1+h,t+1)\sum_{i=1}^{r}b_i(t)u_i(t-h_i)\\ &\quad+\sum_{i=1}^{r}\sum_{j=t+1-h_i}^{t}\Phi(t+1+h,h_i+j+1)b_i(h_i+j)u_i(j).\end{aligned}$$

By Lemma 1, it is derived from the above relation that

$$\begin{aligned}\xi_h(t+1) &= \Phi(t+1+h,t)x(t)+\Phi(t+1+h,t+1)\sum_{i=1}^{r}b_i(t)u_i(t-h_i)\\ &\quad+\sum_{i=1}^{r}\sum_{j=t-h_i}^{t-1}\Phi(t+1+h,h_i+j+1)b_i(h_i+j)u_i(j)\\ &\quad+\sum_{i=1}^{r}\Phi(t+1+h,h_i+t+1)b_i(h_i+t)u_i(t)\\ &\quad-\sum_{i=1}^{r}\Phi(t+1+h,t+1)b_i(t)u_i(t-h_i)\\ &= \Phi(t+1+h,t)x(t)+\sum_{i=1}^{r}\sum_{j=t-h_i}^{t-1}\Phi(t+1+h,h_i+j+1)b_i(h_i+j)u_i(j)\\ &\quad+\sum_{i=1}^{r}\Phi(t+1+h,h_i+t+1)b_i(h_i+t)u_i(t).\end{aligned}\tag{48}$$

By Definition 1, it is obtained from (48) that

$$\begin{aligned}\xi_h(t+1) &= A(t+h)\Phi(t+h,t)x(t)+A(t+h)\sum_{i=1}^{r}\sum_{j=t-h_i}^{t-1}\Phi(t+h,h_i+j+1)b_i(h_i+j)u_i(j)\\ &\quad+\sum_{i=1}^{r}\Phi(t+1+h,h_i+t+1)b_i(h_i+t)u_i(t).\end{aligned}\tag{49}$$

With the time-variant matrix $\mathring{B}(t+h)$ in (45), the expression in (49) is written as

$$\xi_h(t+1)=A(t+h)\left[\Phi(t+h,t)x(t)+\sum_{i=1}^{r}\sum_{j=t-h_i}^{t-1}\Phi(t+h,h_i+j+1)B_i(h_i+j)u(j)\right]+\mathring{B}(t+h)u(t),$$

which is the delay-free system (47). The proof is thus completed.

In Theorem 7, a model reduction technique is developed to transform a DT-LTV system with distinct input-delays into a delay-free system. Similarly to the system (1) with multiple input-delays, the constructed variable $\xi_h(t)$ is not the prediction for the future value of the state at the time instant $t+h$.

Based on the conclusion of Theorem 7, the stabilization problem for the delayed system (2) can be solved. According to Theorem 7, its corresponding delay-free system is

$$x(t+1)=A(t)x(t)+\mathring{B}(t)u(t),\tag{50}$$

where $\mathring{B}(t)$ is obtained by (45). Analogously to the case with multiple input-delays, the time-variant matrix $\mathring{B}(t)$ is related to all the input delays $h_i, i \in \mathbb{I}[1,r]$. Similarly, it is assumed that the delay-free system (50) is stabilized by controller $u(t) = K(t)x(t)$. Then, the control law $u(t) = K(t+h)\xi_h(t)$ can stabilize the system (47). Similarly to the treatment in Section 3, the following conclusion is immediately provided by summarizing the previous statement.

Theorem 8. For the discrete-time LTV input-delayed system (2) with distinct input-delays, its corresponding delay-free system is given in (50), where the time-variant input matrix $\mathring{B}(t)$ is provided in (45). If the system matrix $A(t)$ is invertible and bounded for each t, and $u(t) = K(t)x(t)$ is a stabilizing controller of the system (50), then under the control law

$$u(t) = K(t+h)\left[\Phi(t+h,t)x(t) + \sum_{i=1}^{r}\sum_{j=t-h_i}^{t-1}\Phi(t+h,h_i+j+1)b_i(h_i+j)u_i(j)\right], \tag{51}$$

with $u(t)$ defined as in (3), the closed-loop system composed of (2) and (51) is stable.

The conclusions of Theorem 7 can be utilized to design stabilizing laws for DT-LTI systems with distinct input-delays in the following form:

$$x(t+1) = Ax(t) + \sum_{i=1}^{r} b_i u_i(t-h_i), \tag{52}$$

where $x(t) \in \mathbb{R}^n$ is the state of this system, and $u_i(t) \in \mathbb{R}, i \in \mathbb{I}[1,r]$ are the control inputs in the r channels. Similarly to the system (2), the delayed inputs are ordered so that $1 \leqslant h_1 \leqslant h_2 \leqslant \cdots \leqslant h_r = h$. For this time-invariant system, there hold

$$\Phi(t+h,h_i+j+1) = A^{t+h-h_i-j-1},$$

and

$$\Phi(t+1+h,h_i+t+1) = A^{h-h_i}.$$

Thus, by Theorem 7 we have the following corollary.

Corollary 2. For the LTI delayed system (52), construct the following variable:

$$\xi_h(t) = A^h x(t) + \sum_{i=1}^{r}\sum_{j=t-h_i}^{t-1} A^{t+h-h_i-j-1} b_i u_i(j). \tag{53}$$

Then, the dynamic of the variable $\xi_h(t)$ can be captured by

$$\xi_h(t+1) = A\xi_h(t) + \mathring{B}u(t), \tag{54}$$

with

$$\mathring{B} = [\, A^{h-h_1}b_1 \;\; A^{h-h_2}b_2 \;\; \cdots \;\; A^{h-h_{r-1}}b_{r-1} \;\; b_r \,], \quad u(t) = [\, u_1(t) \;\; u_2(t) \;\; \cdots \;\; u_r(t) \,]^{\mathrm{T}}. \tag{55}$$

Based on the resulting delay-free system (54) and the variable transformation (53) in Corollary 2, the following control law can be designed for the system (52):

$$u_k(t) = K_k\left[A^h x(t) + \sum_{i=1}^{r}\sum_{j=t-h_i}^{t-1} A^h A^{t+h-h_i-j-1} b_i u_i(j)\right],\ k \in \mathbb{I}[1,r], \tag{56}$$

with $K_k \in \mathbb{R}^{1\times n}, k \in \mathbb{I}[1,r]$. For this time-invariant system, the characteristic equation of the closed-loop system is provided in the following theorem.

Theorem 9. For the DT-LTI delayed system (52), the characteristic equation of the closed-loop system under control law (56) is

$$\Delta_3(z) = \det\left[zI_n - A - \left(\sum_{i=1}^{r} A^{h-h_i} b_i K_i\right)\right] = 0. \tag{57}$$

Proof. For a time sequence $f(t)$, let $\mathcal{Z}(f(t))$ denote its Z-transformation. For the sake of notational simplicity, let $x(z) = \mathcal{Z}(x(t))$ and $u_i(z) = \mathcal{Z}(u_i(t)), i \in \mathbb{I}[1,r]$ for the system (52). By taking Z-transformation for both sides of (52), one has

$$zx(z) = Ax(z) + \sum_{i=1}^{r} z^{-h_i} b_i u_i(z). \tag{58}$$

It is easily known that the control law (56) can be equivalently written as

$$u_k(t) = K_k A^h x(t) + \sum_{i=1}^{r} \sum_{j=0}^{h_i-1} K_k A^{h-h_i+j} b_i u_i(t-j-1),$$

from which we have

$$u_k(z) = K_k A^h x(z) + \sum_{i=1}^{r} \sum_{j=0}^{h_i-1} z^{-j-1} K_k A^{h-h_i+j} b_i u_k(z). \tag{59}$$

Let

$$M_3(z) = \begin{bmatrix} zI_n - A & -z^{-h_1}b_1 & -z^{-h_2}b_2 & \cdots & -z^{-h_r}b_r \\ K_1A^h & K_1\sum_{j=0}^{h_1-1} z^{-j-1}A^{h-h_1+j}b_1 - I & K_1\sum_{j=0}^{h_2-1} z^{-j-1}A^{h-h_2+j}b_2 & \cdots & K_1\sum_{j=0}^{h_r-1} z^{-j-1}A^{h-h_r+j}b_r \\ K_2A^h & K_2\sum_{j=0}^{h_1-1} z^{-j-1}A^{h-h_1+j}b_1 & K_2\sum_{j=0}^{h_2-1} z^{-j-1}A^{h-h_2+j}b_2 - I & \cdots & K_2\sum_{j=0}^{h_r-1} z^{-j-1}A^{h-h_r+j}b_r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K_rA^h & K_r\sum_{j=0}^{h_1-1} z^{-j-1}A^{h-h_1+j}b_1 & K_r\sum_{j=0}^{h_2-1} z^{-j-1}A^{h-h_2+j}b_2 & \cdots & K_r\sum_{j=0}^{h_r-1} z^{-j-1}A^{h-h_r+j}b_r - I \end{bmatrix}.$$

By combining (58) and (59), the following matrix-vector form is obtained:

$$M_3(z)[\, x_1^{\mathrm{T}}(z) \ u_1^{\mathrm{T}}(z) \ u_2^{\mathrm{T}}(z) \ \cdots \ u_r^{\mathrm{T}}(z) \,]^{\mathrm{T}} = 0. \tag{60}$$

Then, it follows from (60) that the characteristic polynomial of the closed-loop system for the considered system (52) under the controller (56) is $\Delta_3(z) = \det M_3(z)$.

Let

$$G(z) = \begin{bmatrix} I_n & & & & \\ K_1\sum_{j=0}^{h-1} z^{h-1-j}A^j & I & & & \\ K_2\sum_{j=0}^{h-1} z^{h-1-j}A^j & 0 & I & & \\ \cdots & \cdots & \cdots & & \\ K_r\sum_{j=0}^{h-1} z^{h-1-j}A^j & 0 & 0 & \cdots & I \end{bmatrix}. \tag{61}$$

Similarly to (22), it is easily obtained that

$$K_k \sum_{j=0}^{h-1} z^{h-1-j} A^j (zI_n - A) = K_k \left(z^h I_n - A^h\right). \tag{62}$$

In addition, it is easy to derive that

$$K_k \sum_{j=0}^{h_i-1} z^{-j-1} A^{h-h_i+j} b_i = K_k \sum_{j=0}^{h_i-1} z^{j-h_i} A^{h-j-1} b_i.$$

With this, we have

$$\begin{aligned}
&-K_k\sum_{j=0}^{h-1}z^jA^{h-1-j}b_iz^{-h_i}+K_k\sum_{j=0}^{h_i-1}A^{h-h_i+j}B_iz^{-j-1}\\
&=-K_k\sum_{j=0}^{h-1}z^{j-h_i}A^{h-1-j}b_i+K_k\sum_{j=0}^{h_i-1}z^{j-h_i}A^{h-j-1}b_i\\
&=-K_k\sum_{j=h_i}^{h-1}z^{j-h_i}A^{h-j-1}b_i.
\end{aligned}$$

In view of this relation and (62), pre-multiplying $M_3(z)$ in (60) by $G(z)$ in (61) yields

$$G(z)M_3(z)=\begin{bmatrix}
zI_n-A & -z^{j-h_1}b_1 & -z^{-h_2}b_2 & \cdots & -z^{-h_r}b_r\\
z^hK_1 & -I-K_1\sum_{j=h_1}^{h-1}z^{j-h_1}A^{h-1-j}b_1 & -K_1\sum_{j=h_2}^{h-1-j}z^{j-h_2}A^{h-1-j}b_2 & \cdots & -K_1\sum_{j=h_r}^{h-1}z^{j-h_r}A^{h-1-j}b_r\\
z^hK_2 & -K_2\sum_{j=h_1}^{h-1}z^{j-h_1}A^{h-1-j}b_1 & -I-K_2\sum_{j=h_2}^{h-1}z^{j-h_2}A^{h-1-j}b_2 & \cdots & -K_2\sum_{j=h_2}^{h-1}z^{j-h_r}A^{h-1-j}b_r\\
\cdots & \cdots & \cdots & & \cdots\\
z^hK_r & -K_r\sum_{j=h_1}^{h-1}z^{j-h_1}A^{h-1-j}b_1 & -K_r\sum_{j=h_2}^{h-1}z^{j-h_2}A^{h-1-j}b_2 & \cdots & -I-K_r\sum_{j=h_r}^{h-1}z^{j-h_r}A^{h-1-j}b_r
\end{bmatrix}. \tag{63}$$

Furthermore, by calculation it can be obtained that

$$\begin{aligned}
&(zI_n-A)\sum_{j=h_i}^{h-1}z^{j-h_i-h}A^{h-j-1}b_i-z^{-h_i}b_i\\
&=\sum_{j=h_i}^{h-1}z^{j-h_i-h+1}A^{h-j-1}b_i-\sum_{j=h_i}^{h-1}z^{j-h_i-h}A^{h-j}b_i-z^{-h_i}b_i\\
&=z^{-h_i}b_i-z^{-h}A^{h-h_i}b_i-z^{-h_i}b_i\\
&=-z^{-h}A^{h-h_i}b_i,
\end{aligned}$$

and

$$z^hK_k\sum_{j=h_i}^{h-1}z^{j-h_i-h}A^{h-j-1}b_i-K_k\sum_{j=h_i}^{h-1}z^{j-h_i}A^{h-j-1}b_i=0.$$

In view of these two relations, if the following matrix $H(z)$ is constructed:

$$H(z)=\begin{bmatrix}
I_n & \sum_{j=h_1}^{h-1}z^{j-h_1-h}A^{h-j-1}b_1 & \sum_{j=h_2}^{h-1}z^{j-h_2-h}A^{h-j-1}b_2 & \cdots & \sum_{j=h_r}^{h-1}z^{j-h_r-h}A^{h-j-1}b_r\\
0_{1\times n} & 1 & & & \\
0_{1\times n} & 0 & 1 & & \\
\cdots & \cdots & & \ddots & \\
0_{1\times n} & 0 & 0 & \cdots & 1
\end{bmatrix},$$

then it follows from (63) that

$$G(z)M_3(z)H(z) = \begin{bmatrix} zI_n - A & -z^{-h}A^{h-h_1}b_1 & -z^{-h}A^{h-h_2}b_2 & \cdots & -z^{-h}A^{h-h_r}b_r \\ z^hK_1 & -1 & 0 & \cdots & 0 \\ z^hK_2 & 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ z^hK_r & 0 & 0 & \cdots & -1 \end{bmatrix},$$

from which, by Lemma 3, we have

$$\begin{aligned} &\det\left[G(z)M_3(z)H(z)\right] \\ &= \det(-I_r)\det\left[zI_n - A - \sum_{i=1}^{r} z^{-h}A^{h-h_i}b_i\left(z^hK_i\right)\right] \\ &= \det(-I_r)\det\left[zI_n - A - \left(\sum_{i=1}^{r} A^{h-h_i}b_iK_i\right)\right]. \end{aligned} \tag{64}$$

In addition, owing to the structure of the matrices $G(z)$ and $H(z)$ it is easily known that $\det G(z) = 1$ and $\det H(z) = 1$. With these two relations, the conclusion of this theorem follows from (64). By now, the proof of this theorem is completed.

For the control law (56), let

$$K = \begin{bmatrix} K_1^{\mathrm{T}} & K_2^{\mathrm{T}} & \cdots & K_r^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}. \tag{65}$$

Then, by using the notation $\mathring{B}$ in (55) it is clear that

$$\sum_{i=1}^{r} A^{h-h_i}b_iK_i = \begin{bmatrix} A^{h-h_1}b_1 & A^{h-h_2}b_2 & \cdots & A^{h-h_{r-1}}b_{r-1} & b_r \end{bmatrix} K = \mathring{B}K.$$

With this, it is known from Theorem 9 that if K is a stabilizing feedback gain for the delay-free system (54), then the control law (56) can stabilize the system (52) with distinct input delays. Due to the result of Theorem 9, it has been confirmed that the designed control law (56) is also applicable when the system matrix A in (54) is not invertible.

Remark 2. In [25], a predictor-based approach was presented to develop control laws for continuous-time systems with distinct input-delays. These controllers need to be obtained by the backstepping method. It can be clearly observed that the method in the present paper is completely different from that in [25]. In the present paper, the design of the controllers does not involve iteration, and the control law is given in an explicit form.

On model reduction approaches for DT-LTV systems, many results have existed. In [22], the Floquet theory was utilized to transform an input-delayed system into a system without delays. Similarly to the case of continuous-time case, the key to this method is to change a periodic system into an LTI system. An evident shortcoming of the approach in [22] is that the system matrix is required to be invertible. In addition, the approach in [22] is not applicable to systems with multiple input-delays.

6 Numerical examples

Three examples are employed in this section to demonstrate the effectiveness of the developed model reduction approaches for discrete-time LTV input-delayed systems. For the sake of simplicity, the considered time-varying systems are three periodic systems.

Example 1. Consider a DT-LTV system in the form of (4) with periodic state and input matrices. For this system, for any integer $t \geqslant 0$ there hold

$$A(t+3) = A(t), \quad B(t+3) = B(t).$$

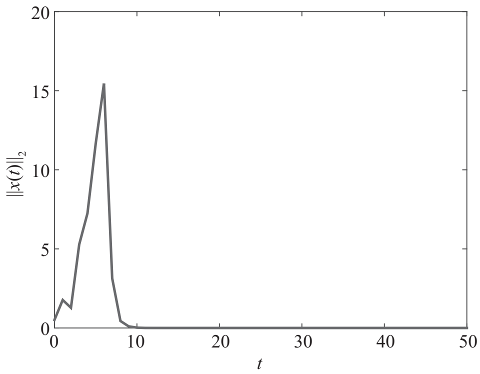

Figure 1 The state norms under the control law (13).

The system matrices and input matrices are as follows:

$$A(0) = \begin{bmatrix} -3 & 2 \\ -3 & 3 \end{bmatrix},\ A(1) = \begin{bmatrix} -1 & 2 \\ 0.5 & 0 \end{bmatrix},\ A(2) = \begin{bmatrix} 1 & 2 \\ 2.5 & 3 \end{bmatrix},$$

$$B(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B(1) = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix},\ B(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

the input delay is $h = 5$, and the period of this system is 3. For this time-varying system, the predictor-based feedback law (13) is utilized. In this control law, the periodic feedback gain $K(t)$ is designed to ensure the stability of the closed-loop system of its corresponding delay-free system. Since the original system is periodic, naturally the gain $K(t)$ is designed to possess periodicity, that is, the gain matrix $K(t)$ satisfies $K(t+3) = K(t)$. Using the LMIs technique in [31], a 3-periodic stabilizing state feedback gain $K(t)$ can be obtained with

$$K(0) = \begin{bmatrix} 3.0001 & -2.91 \end{bmatrix},\ \ K(1) = \begin{bmatrix} 1.0043 & -2.5631 \end{bmatrix},\ \ K(2) = \begin{bmatrix} -2.2217 & -2.8165 \end{bmatrix}.$$

When the control law (13) with the designed gain $K(t)$ is utilized for the system (4), the evolution curve of the norm of the state of the closed-loop system is as shown in Figure 1, where the initial state is

$$x(0) = \begin{bmatrix} 0.5 & 0.1 \end{bmatrix}^{\mathrm{T}}.$$

According to Figure 1, the periodic input-delayed system in this example is stabilized by the designed predictor-based controller (13).

Example 2. Consider a discrete-time system with period 2 in the form of (1). For this system, for any integer $t \geqslant 0$ there hold

$$A(t+2) = A(t),\quad B_1(t+2) = B_1(t),\quad B_2(t+2) = B_2(t).$$

The system matrices and the input matrices are as follows:

$$A(0) = \begin{bmatrix} -1.4 & 0.95 \\ -1.2 & -1.44 \end{bmatrix},\ A(1) = \begin{bmatrix} -1.2 & 0.65 \\ 1.41 & 0.94 \end{bmatrix},$$

$$B_1(0) = \begin{bmatrix} -1 \\ 0.8 \end{bmatrix},\ \ B_1(1) = \begin{bmatrix} 0.4 \\ -0.5 \end{bmatrix},\ \ B_2(0) = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix},\ \ B_2(1) = \begin{bmatrix} -0.3 \\ -0.1 \end{bmatrix},$$

and the input delays are $h_1 = 2$ and $h_2 = 5$. For this time-varying system, the control law (30) is utilized. In this control law, the feedback gain $K(t)$ should be designed to make the closed-loop system of its delay-free system (29) with $B(t)$ given in (26) stable. According to the expression of (26), it can be calculated that

$$B(t+2) = B(t)$$

holds for any integer $t \geqslant 0$, and

$$B(0) = \begin{bmatrix} 2.6140 \\ -5.4262 \end{bmatrix}, \quad B(1) = \begin{bmatrix} -2.1293 \\ 0.2990 \end{bmatrix}.$$

In addition, since the corresponding delay-free system is periodic, the feedback gain $K(t)$ could be designed to be also periodic and satisfy $K(t+2) = K(t)$. By using the LMIs technique in [31], a 2-periodic stabilizing state feedback gain $K(t)$ can be obtained with

$$K(0) = \begin{bmatrix} 0.3035 & 1.3434 \end{bmatrix}, \quad K(1) = \begin{bmatrix} 0.1075 & 0.4960 \end{bmatrix}.$$

A simulation is performed when the control law (30) with the designed gain $K(t)$ is applied to this example system. The initial value of the state in this simulation is

$$x(0) = \begin{bmatrix} -0.3787 & -0.1225 \end{bmatrix}^{\mathrm{T}}.$$

The evolution curve of the norm of the state of the closed-loop system is depicted in Figure 2. It is observed that the discrete-time periodic system (1) with multiple input-delays is stabilized when the designed control law (30) is applied.

Example 3. Consider a discrete-time system with period 2 in the form of (2). For this system, for any integer $t \geqslant 0$ there hold

$$A(t+2) = A(t), \quad b_1(t+2) = b_1(t), \quad b_2(t+2) = b_2(t).$$

The system matrices and the input matrices are as follows:

$$A(0) = \begin{bmatrix} -1.4 & 0.95 \\ -1.2 & -1.44 \end{bmatrix}, \; A(1) = \begin{bmatrix} -1.2 & 0.65 \\ 1.41 & 0.94 \end{bmatrix},$$

$$b_1(0) = \begin{bmatrix} -1 \\ 0.8 \end{bmatrix}, \quad b_1(1) = \begin{bmatrix} 0.4 \\ -0.5 \end{bmatrix}, \quad b_2(0) = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}, \quad b_2(1) = \begin{bmatrix} -0.3 \\ -0.1 \end{bmatrix},$$

and the input delays are $h_1 = 1$ and $h_2 = 3$. For this time-varying system, the control law (51) is utilized. In this control law, the feedback gain $K(t)$ should be designed to make the closed-loop system of its delay-free system (50) with $\mathring{B}(t)$ given in (45) stable. According to the expression of (45), it can be calculated that

$$\mathring{B}(t+2) = \mathring{B}(t)$$

holds for any integer $t \geqslant 0$, and

$$\mathring{B}(0) = \begin{bmatrix} 1.3980 & -0.3 \\ -1.2338 & -0.1 \end{bmatrix}, \quad \mathring{B}(1) = \begin{bmatrix} -3.0331 & 1 \\ -1.1165 & 0.2 \end{bmatrix}.$$

In addition, since the corresponding delay-free system is periodic, the feedback gain $K(t)$ could be designed to be also periodic and satisfy $K(t+2) = K(t)$. By using the LMIs technique in [31], a 2-periodic stabilizing state feedback gain $K(t)$ can be obtained with

$$K(0) = \begin{bmatrix} -1.8043 & -3.1968 \\ -4.0728 & -10.6463 \end{bmatrix}, \quad K(1) = \begin{bmatrix} 1.0648 & 0.4255 \\ 0.9621 & 4.1497 \end{bmatrix}.$$

A simulation is performed when the control law (51) with the designed gain $K(t)$ is applied to this example system. The initial value of the state in this simulation is

$$x(0) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}^{\mathrm{T}}.$$

The evolution curve of the norm of the state of the closed-loop system is depicted in Figure 3. It is observed that the discrete-time periodic system (2) with multiple input-delays is stabilized when the designed control law (51) is applied.

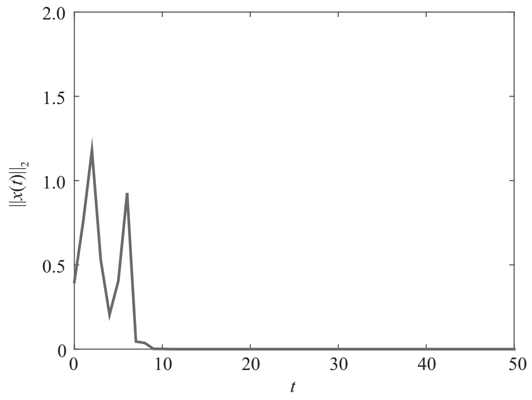

Figure 2 The state norms under the control law (30).

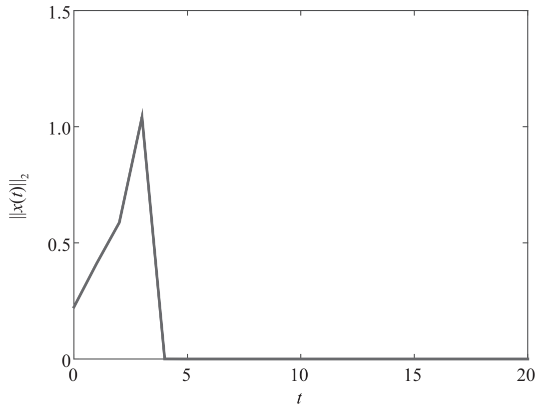

Figure 3 The state norms under the control law (51).

7 Conclusion

In this paper, model reduction approaches are presented to convert DT-LTV input-delayed systems into delay-free systems. To realize such a transformation, a variable is constructed in terms of historical information of control input. For DT-LTV systems with single input-delays, the constructed variable is in fact the exact prediction of a certain future state. However, such a conclusion is not true for systems with multiple input-delays or distinct input-delays. When the reduction approach is applied to a system with a single input-delay, the corresponding delay-free system can be obtained by a simple copy of its original system. However, for DT-LTV systems with multiple input-delays, input matrices of resulting delay-free systems need to be calculated by an expression related to transition matrices. With the aid of the model reduction approaches, stabilizing control laws of discrete-time LTV input-delayed systems can be designed by taking full advantage of the simple structure of the reduced delay-free systems. Moreover, stabilizing control laws of DT-LTV systems with single input-delays can be established to possess the structure of predictor-based state feedback laws. In particular, for time-invariant discrete-time systems, it is rigorously proven that the stability of the resulting closed-loop systems can be guaranteed without the invertibility of the system matrix by using the proposed model reduction approach.

Acknowledgements This work was supported by Shenzhen Science and Technology Program (Grant Nos. RCJC20210609104400-005, KQTD20210811090146075), HIT Wuhu Robot Technology Research Institute (Grant No. HIT-CXY-CMP2-IARU-21-01), Science Center Program of National Natural Science Foundation of China (Grant No. 62188101), National Natural Science Foundation of China (Grant No. 62173112), and Joint Funds of the National Natural Science Foundation of China (Grant No. U2013203).

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