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Special Topic: Analysis and Control of Stochastic Systems

Fixed-time stabilization of output-constrained stochastic high-order nonlinear systems

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Abstract In this study, the fixed-time stabilization problem of stochastic high-order nonlinear systems with output constraint and high-order and low-order nonlinearities is addressed. A new coordinate transformation is employed to directly convert output-constrained stochastic systems into an equivalent unconstrained form. By fully extracting the characteristics of system nonlinearities and using the stochastic fixed-time stability theorem, a new design and analysis method is constructed to guarantee that the trivial solution of the closed-loop system is stochastically fixed-time stable while fulfilling the output constraint.

Keywords stochastic high-order nonlinear system, output constraint, high-order and low-order nonlinearity, fixed-time stabilization, finite-time stabilization

1 Introduction

To ensure safety and meet performance specifications, the output/state of nonlinear systems needs to be constrained. Violation of the output/state constraints during operation may result in performance reduction, unexpected danger, or system breakdown [1]. In the past three decades, many tools have been proposed to handle output/state constraints, the most representative of which is the barrier Lyapunov function (BLF) first proposed in [2, 3].

Control design and stability analysis of stochastic high-order nonlinear systems are significantly challenging in the nonlinear control community due to the uncontrollability and nonfeedback linearizability of Jacobian linearization. In [4], a key technique called adding power integrators was introduced to solve this problem. Subsequently, many important achievements have been made in this field [5–14].

However, Refs. [5–14] only addressed the asymptotic stabilization problem. In comparison with asymptotic stabilization, finite-time stabilization has some distinct features such as fast response and high accuracy; thus, it is more desirable than asymptotic stabilization in many engineering applications. Due to these benefits, Refs. [15, 16] introduced the concept of stochastic finite-time stability and developed the corresponding Lyapunov criteria. On this basis, Refs. [17, 18] and [19] investigated the finite-time stabilization problem for stochastic strict-feedback and high-order systems, respectively. Ref. [20] proposed a generalized finite-time stability theorem for stochastic nonlinear systems that relaxes the constraint on infinitesimal generators and reveals the important role of white-noise in the finite-time stabilization of stochastic systems. For stochastic nonlinear systems with finite-time stochastic input-to-state stability (FT-SISS) inverse dynamics, Refs. [21,22], [23], and [24] established finite-time control design strategies for stochastic strict-feedback, high-order systems, and low-order nonlinear systems, respectively. Recently, a weaker concept of stochastic inverse dynamics called finite-time stochastic integral input-to-state stability (FT-SiISS) was introduced in [25]. Despite significant progress in pure finite-time stabilization problem, few results have been obtained in analyzing finite-time stabilization problems for stochastic systems with output constraint. In the only results, Refs. [26, 27] employed the fractional-type BLF to directly handle the output constraint in the finite-time stabilization task. However, these methods are only suitable for a quite limited class of stochastic high-order systems because the nonlinear terms of the system must satisfy the low-order growth condition. By exploring the properties of these nonlinearities, an essential

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breakthrough was achieved by [28], where a finite-time stabilizer is delicately constructed for such systems with high-order and low-order nonlinearities.

Because the settling time of [26–28] depends on the initial condition, the unavailability of the exact initial condition hinders the application of these control schemes in [26–28]. Recently, a new definition of fixed-time stability and its Lyapunov condition were proposed in [29] and further refined in [30]. Subsequently, several studies of stochastic fixed-time control have been conducted. Specifically, by combining with the fuzzy logic technique, Ref. [31] solved the fixed-time stabilization problem of stochastic interconnected systems, Ref. [32] investigated global fixed-time stabilization of stochastic super-twisting systems, Refs. [33] and [34] designed fixed-time event-triggered controllers of stochastic nontriangular and nonstrict-feedback systems, Ref. [35] investigated the fixed-time consensus for stochastic multi-agent systems, and Ref. [36] proposed the stochastic fixed-time control strategy for stochastic switching systems. However, Refs. [31–36] could not guarantee fixed-time convergence of stochastic systems while fulfilling the output constraint.

Based on the aforementioned studies, we can not help but ask a significant question: Can we design a fixed-time state-feedback controller for output-constrained stochastic high-order nonlinear systems with high-order and low-order nonlinearities?

This study provides a satisfactory answer to this problem. Because the system model investigated in this study includes high-order powers, high-order and low-order nonlinearities, and output constraint, the controllers in the aforementioned studies are no longer applicable. In this study, a novel coordinate transformation is employed to directly convert the output-constrained stochastic high-order nonlinear system into an equivalent unconstrained one. For the transformed system, by fully extracting the characteristics of system nonlinearities and using the stochastic fixed-time stability theorem, we construct a state-feedback stabilizer to ensure that the state of the closed-loop system is stochastically fixed-time stable while guaranteeing the achievement of the prespecified output constraint.

This paper is organized as follows. Section 2 states the investigated problem. Section 3 discusses the main results. Section 4 presents a simulation example. Section 5 concludes this paper.

2 Problem statement and preliminaries

2.1 Problem statement

Consider the following class of stochastic high-order nonlinear systems:

$$\begin{aligned} &\mathrm{d}x_i(t) = \big(\lceil x_{i+1}(t)\rceil^{p_i} + f_i(\bar{x}_i(t))\big)\mathrm{d}t + g_i(\bar{x}_i(t))\mathrm{d}\omega,\ i=1,\dots,n-1,\\ &\mathrm{d}x_n(t) = \big(\lceil u(t)\rceil^{p_n} + f_n(x(t))\big)\mathrm{d}t + g_n(x(t))\mathrm{d}\omega,\\ &y(t) = x_1(t), \end{aligned} \tag{1}$$

with the symmetric output constraint

$$y(t) \in \Omega_y = \{y(t)\in\mathbb{R} : -\epsilon_l < y(t) < \epsilon_l\}, \tag{2}$$

where $u(t)\in\mathbb{R}$ and $y(t)\in\mathbb{R}$ are control input and system output, respectively. $\bar{x}_i(t) = (x_1(t),\dots,x_i(t))^{\mathrm{T}} \in \mathbb{R}^i$, $i=1,\dots,n-1$, $\bar{x}_n(t) = x(t)\in\mathbb{R}^n$ is the measurable state with the initial value $x(0)=x_0$, and ω is an r-dimensional standard Wiener process defined on a complete probability space (Ω, F, P) with a filtration $\{F_t\}_{t\geqslant 0}$ satisfying the usual condition. For $i=1,\dots,n$, $p_i\geqslant 1$ is called the high order of the system, and $f_i(\bar{x}_i):\mathbb{R}^i\to\mathbb{R}$ and $g_i(\bar{x}_i):\mathbb{R}^i\to\mathbb{R}^{1\times r}$ are some of the continuous functions that satisfy $f_i(0)=0$ and $g_i(0)=0$, respectively. ϵ_l is a given positive constant. $\lceil\cdot\rceil^{\varrho} = \mathrm{sgn}(\cdot)|\cdot|^{\varrho}$, ϱ is a positive constant, and $\mathrm{sgn}(\cdot)$ represents a sign function.

For system (1) with the given output constraint (2), the control objective is to design a state-feedback controller such that the trivial solution of the closed-loop system is stochastically fixed-time stable while ensuring that Eq. (2) is satisfied. To achieve this aim, we need a key assumption.

Assumption 1. There exist the constants $\varpi_n \leqslant \varpi_{n-1} \leqslant \cdots \leqslant \varpi_1 \in (-\frac{1}{\sum_{j=1}^{n} p_1\cdots p_{j-1}}, 0)$, $0\leqslant \gamma_n \leqslant \gamma_{n-1} \leqslant \cdots \leqslant \gamma_1$ and the known smooth functions $f_{ij}(\bar{x}_i)$, $g_{ij}(\bar{x}_i)$, $i=1,\dots,n$, $j=1,2$, such that

$$|f_i(\bar{x}_i)| \leqslant f_{i1}(\bar{x}_i)\sum_{j=1}^{i}|x_j|^{\frac{r_i+\varpi_i}{r_j}} + f_{i2}(\bar{x}_i)\sum_{j=1}^{i}|x_j|^{\frac{h_i+\gamma_i}{h_j}},$$

$$|g_i(\bar{x}_i)| \leqslant g_{i1}(\bar{x}_i)\sum_{j=1}^{i}|x_j|^{\frac{2r_i+\varpi_i}{2r_j}} + g_{i2}(\bar{x}_i)\sum_{j=1}^{i}|x_j|^{\frac{2h_i+\gamma_i}{2h_j}}, \tag{3}$$

where $r_1 = h_1 = 1$, $r_{j+1} = \frac{r_j+\varpi_j}{p_j}$, $h_{j+1} = \frac{h_j+\gamma_j}{p_j}$, $j = 1,\ldots,n$. From Assumption 1, we determine that the low-orders $\frac{r_i+\varpi_i}{r_j}$ and $\frac{2r_i+\varpi_i}{2r_j}$ and the high-orders $\frac{h_i+\gamma_i}{h_j}$ and $\frac{2h_i+\gamma_i}{2h_j}$ can take all the values in $(0, \frac{1}{p_j\cdots p_{i-1}}]$ and $[\frac{1}{p_j\cdots p_{i-1}},\infty)$, respectively.

Remark 1. A significant feature of Assumption 1 is the flexibility to choose the monotonic parameters ϖ_i and γ_i, which enables Assumption 1 to include different assumptions in the existing work on state-feedback stabilization of stochastic systems with output constraint. Specifically, by setting $\gamma_i = 0$, Assumption 1 reduces to Assumption 1 in [26,27]. If $p_i \in \mathbb{R}^{\geqslant 1}_{\text{odd}} = \{\frac{p}{q} \in \mathbb{R}|p \geqslant q$ and p, q are positive odd integers$\}$, then Assumption 1 is the same as Assumption 1 in [28].

Remark 2. Several practical output-constrained systems can be modeled by (1) with (2), for example, the under-actuated, weakly coupled, unstable mechanical system in [5]:

$$\begin{aligned}
&\mathrm{d}x_1(t) = x_2(t)\mathrm{d}t,\\
&\mathrm{d}x_2(t) = \Big(x_3(t)^3 + \frac{49}{50}\sin(x_1(t))\Big)\mathrm{d}t,\\
&\mathrm{d}x_3(t) = x_4(t)\mathrm{d}t,\\
&\mathrm{d}x_4(t) = \Big(u(t) + 15x_1(t) + \frac{113}{10}x_3(t) + 15x_3(t)^3\Big)\mathrm{d}t - \frac{1}{8}(10x_1(t) + x_3(t))\mathrm{d}\omega,\\
&y(t) = x_1(t),
\end{aligned} \tag{4}$$

with an output constraint $y(t) \in \Omega_y = \{y(t) \in \mathbb{R} : -\frac{\pi}{4} < y(t) < \frac{\pi}{4}\}$. Obviously, Eq. (4) is a special form of (1) with $p_1 = p_3 = p_4 = 1$, $p_2 = 3$, $\varpi_i = -\frac{1}{12} \in (-\frac{1}{8}, 0)$, $\gamma_i = 0$, $i = 1,\ldots,4$, $r_1 = 1$, $r_2 = \frac{r_1+\varpi_1}{p_1} = \frac{11}{12}$, $r_3 = \frac{r_2+\varpi_2}{p_2} = \frac{5}{18}$, $r_4 = \frac{r_3+\varpi_3}{p_3} = \frac{7}{36}$, $r_5 = \frac{r_4+\varpi_4}{p_4} = \frac{1}{9}$, $h_1 = 1$, $h_2 = \frac{h_1+\gamma_1}{p_1} = 1$, $h_3 = \frac{h_2+\gamma_2}{p_2} = \frac{1}{3}$, $h_4 = \frac{h_3+\gamma_3}{p_3} = \frac{1}{3}$, $h_5 = \frac{h_4+\gamma_4}{p_4} = \frac{1}{3}$.

We can easily determine that $f_1(x_1) = 0$, $g_1(x_1) = 0$, $|f_2(\bar{x}_2)| = \frac{49}{50}|\sin x_1| \leqslant \frac{49}{50}(|x_1|^{\frac{5}{6}} + |x_1|)$, $g_2(\bar{x}_2) = 0$, $f_3(\bar{x}_3) = 0$, $g_3(\bar{x}_3) = 0$, $|f_4(\bar{x}_4)| = |15x_1 + \frac{113}{10}x_3 + 15x_3^3| \leqslant (\frac{263}{10} + 15x_1^2 + 15x_3^2)(|x_1|^{\frac{1}{9}} + |x_3|)$, $|g_4(\bar{x}_4)| = |\frac{1}{8}(10x_1 + x_3)| \leqslant (\frac{11}{8} + \frac{5}{4}x_1^2)(|x_1|^{\frac{11}{72}} + |x_3|)$; thus, Assumption 1 is satisfied.

2.2 Preliminaries

To derive the main results, we recall some technical lemmas. Consider the following stochastic nonlinear system:

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}\omega,\ \forall t \geqslant 0, \tag{5}$$

with the initial value $x(0) = x_0$. $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $g(x) : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are continuous function vectors and matrices, respectively. ω is an r-dimensional standard Wiener process.

Definition 1 ([20])**.** The trivial solution of system (5) is said to be stochastically finite-time stable for any initial value $x_0 \in \mathbb{R}^n$, if there exists a solution $x(t, x_0)$ in system (5) and the following statements hold.

(i) Stable in probability: For any $\varepsilon \in (0,1)$ and $\lambda > 0$, if there exists a $\delta(\epsilon,\lambda) > 0$ such that $P\{|x(t,x_0)| < \lambda(|x_0|), \forall t > 0\} \geqslant 1 - \varepsilon$ whenever $|x_0| < \delta$.

(ii) Finite-time attractiveness in probability: For any initial value $x_0 \in \mathbb{R}^n \setminus \{0\}$, the first hitting time $\tau_{x_0} = \inf\{t \geqslant 0 : x(t,x_0) = 0\}$, which is also called the stochastic setting time, is finite almost surely, that is, $P(\tau_{x_0} < \infty) = 1$. Furthermore, $x(t + \tau_{x_0}, x_0) = 0$, a.s., $\forall t \geqslant 0$.

Definition 2 ([29])**.** The trivial solution of system (5) is said to be stochastically fixed-time stable if the trivial solution is stochastically finite-time stable and $E(\sigma(x_0) \leqslant T_0$, $\forall x_0 \in \mathbb{R}^n \setminus \{0\}$, where T_0 is a positive constant independent of the initial values.

Lemma 1 ([10])**.** If $q > 0$, $p \geqslant 1$ is odd, then $||c| - |d||^p \leqslant ||c|^p - |d|^p|$, $|c + d|^p \leqslant 2^{p-1}|c^p + d^p|$, $|\lceil c\rceil^{\frac{q}{p}} - \lceil d\rceil^{\frac{q}{p}}| \leqslant 2^{\frac{p-1}{p}}|\lceil c\rceil^q - \lceil d\rceil^q|^{\frac{1}{p}}$ hold for any $c \in \mathbb{R}$, $d \in \mathbb{R}$.

Lemma 2 ([17])**.** Suppose that there exists a nonnegative radially unbounded function $V(x) \in \mathcal{C}^2$. If $\mathcal{L}V \leqslant 0$, then Eq. (5) has a continuous solution on $[0,\infty)$ for any initial data.

Lemma 3 ([10]). For given $m > 0$ and any $a_i \in \mathbb{R}$, $i = 1, \ldots, n$, there holds $(|a_1| + \cdots + |a_n|)^m \leqslant d_m(|a_1|^m + \cdots + |a_n|^m)$, where $d_m = n^{m-1}$ if $m \geqslant 1$ and $d_m = 1$ if $m < 1$.

Lemma 4 ([30]). For system (5), suppose that there exist a function $V \in \mathcal{C}^2$, class $\mathcal{K}_\infty$ functions α_1, α_2, and constants $c_1 > 0$, $c_2 > 0$, $0 < p < 1$, $q > 1$ such that

$$\alpha_1(|x|) \leqslant V(x) \leqslant \alpha_2(|x|),\ \mathcal{L}V(x) \leqslant -c_1 V(x)^p - c_2 V(x)^q.$$

Then, the trivial solution of system (5) is stochastically fixed-time stable.

Lemma 5 ([10]). For a given continuous function $q(x, y)$, there exist smooth functions $q_1(x) \geqslant 0$, $q_2(y) \geqslant 0$, $q_3(x) \geqslant 1$, and $q_4(y) \geqslant 1$ such that $|q(x, y)| \leqslant q_1(x) + q_2(y)$, $|q(x, y)| \leqslant q_3(x) q_4(y)$.

Lemma 6 ([10]). The function $y(x) = \lceil x \rfloor^a$, $a \geqslant 2$ is $\mathcal{C}^2$ for $x \in \mathbb{R}$, $\dot{y}(x) = a|x|^{a-1}$, and $\ddot{y}(x) = a(a-1)\lceil x \rfloor^{a-2}$.

Lemma 7 ([10]). For $m > 0$, $n > 0$, $a > 0$, and any $c \in \mathbb{R}$, $d \in \mathbb{R}$, there holds $|c|^m |d|^n \leqslant a|c|^{m+n} + (\frac{n}{m+n})(\frac{m+n}{m})^{\frac{-m}{n}} a^{\frac{-m}{n}} |d|^{m+n}$.

Lemma 8 ([10]). For a given function $p : [a, b] \to \mathbb{R}$, $b > a$ with $p(a) = 0$, there holds $|\int_a^b p(x)\mathrm{d}x| \leqslant |p(b)|\,|b - a|$.

3 Main results

3.1 System transformation

In this subsection, we first introduce the following equivalent coordinate transformation:

$$x_1(t) = \lambda_1 \arctan(\xi_1(t)),\ x_i(t) = \xi_i(t),\ i = 2, \ldots, n, \tag{6}$$

where $\lambda_1 = \frac{2\epsilon_l}{\pi}$. Notably, $x_1(t) = \lambda_1 \arctan(\xi_1(t))$ meets the following properties:

$$\begin{cases} x_1(t) \to -\epsilon_l, & \text{when } \xi_1(t) \to -\infty, \\ x_1(t) \to \epsilon_l, & \text{when } \xi_1(t) \to \infty. \end{cases} \tag{7}$$

Hence, the constraint (2) is not transgressed almost surely if $\xi_1(t)$ is almost surely bounded. Using (6), we can obtain the following unconstrained systems:

$$\begin{aligned} \mathrm{d}\xi_1 &= \big(D_1(\xi_1)\lceil \xi_2 \rfloor^{p_1} + \bar{f}_1(\xi_1)\big)\mathrm{d}t + \bar{g}_1(\xi_1)\mathrm{d}\omega, \\ \mathrm{d}\xi_i &= \big(\lceil \xi_{i+1} \rfloor^{p_i} + \bar{f}_i(\bar{\xi}_i)\big) + \bar{g}_i(\bar{\xi}_i)\mathrm{d}\omega,\ i = 2, \ldots, n-1, \\ \mathrm{d}\xi_n &= \big(\lceil u \rfloor^{p_n} + \bar{f}_n(\xi)\big)\mathrm{d}t + \bar{g}_n(\xi)\mathrm{d}\omega, \end{aligned} \tag{8}$$

where $D_1(\xi_1) = \frac{\partial \xi_1}{\partial x_1} = \frac{1+\xi_1^2}{\lambda_1}$, $\bar{f}_1(\xi_1) = D_1(\xi_1) f_1(\xi_1) + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2} g_1^{\mathrm{T}}(\xi_1) g_1(\xi_1)$, $\bar{g}_1(\xi_1) = D_1(\xi_1) g_1(\xi_1)$, $\bar{f}_j(\bar{\xi}_j) = f_j(\bar{\xi}_j)$, and $\bar{g}_j(\bar{\xi}_j) = g_j(\bar{\xi}_j)$, $j = 2, \ldots, n$.

3.2 Control design

To facilitate control design, we introduce a proposition, whose proof is given in Appendix A.

Proposition 1. For $k = 1, \ldots, n$, we find the known nonnegative smooth functions $h_{k1}(\bar{\xi}_k)$ and $\eta_{k1}(\bar{\xi}_k)$ such that

$$|\bar{f}_k(\bar{\xi}_k)| \leqslant h_{k1}(\bar{\xi}_k) \sum_{j=1}^{k} |\xi_j|^{\frac{r_k + \varpi_k}{r_j}}, \tag{9}$$

$$|\bar{g}_k(\bar{\xi}_k)| \leqslant \eta_{k1}(\bar{\xi}_k) \sum_{j=1}^{k} |\xi_j|^{\frac{2r_k + \varpi_k}{2r_j}}. \tag{10}$$

Then, we introduce the following key transformations:

$$z_j = \lceil \xi_j \rfloor^{\frac{\mu}{r_j}} - \lceil \xi_j^* \rfloor^{\frac{\mu}{r_j}},\ j = 1, \ldots, k,$$

$$\xi_1^* = 0,$$
$$\xi_k^*(\bar{\xi}_{k-1}) = -\phi_{k-1}(\bar{\xi}_{k-1})^{\frac{r_k}{\mu}} \lceil z_{k-1} \rfloor^{\frac{r_k}{\mu}},\ k = 2, \ldots, n+1, \tag{11}$$

where $\mu \geqslant \max_{1 \leqslant i \leqslant n}\{2r_i, r_i + \varpi_i\}$ and ϕ_j, $j = 1, \ldots, n$, are nonnegative smooth functions to be determined. With the aid of (11), we formulate the following proposition, whose proof is given in Appendix B.

Proposition 2. For system (8), we find the nonnegative smooth functions $\phi_1, \ldots, \phi_n$ and the following state-feedback controller:

$$u(\xi) = \xi_{n+1}^*(\xi) = -\phi_n(\xi)^{\frac{r_{n+1}}{\mu}} \lceil z_n \rfloor^{\frac{r_{n+1}}{\mu}}, \tag{12}$$

such that the Lyapunov function $V_n(\xi) = \sum_{k=1}^n W_k(\bar{\xi}_k)$ satisfies

$$\mathcal{L}V_n \leqslant -\frac{1}{4}\sum_{j=1}^n |z_j|^4 - \frac{1}{4}\sum_{j=1}^n |z_j|^{\frac{16}{3}}, \tag{13}$$

where

$$W_k(\bar{\xi}_k) = \int_{\xi_k^*}^{\xi_k} \left\lceil \lceil s \rfloor^{\frac{\mu}{r_k}} - \lceil \xi_k^* \rfloor^{\frac{\mu}{r_k}} \right\rfloor^{\frac{4\mu - r_k - \varpi_k}{\mu}} \mathrm{d}s. \tag{14}$$

3.3 Fixed-time stability and constraint analysis

We state the main results.

Theorem 1. For system (1) with the constraint (2), if Assumption 1 holds with $\varpi_1 = \cdots = \varpi_n$, then a state-feedback controller (12) guarantees that for any initial value $(x_1(0), \ldots, x_n(0)) \in \Omega_y \times \mathbb{R}^{n-1}$,

(i) The closed-loop system (1), (6), and (12) has a continuous solution on $[0, \infty)$;

(ii) All of the signals are almost surely bounded and the constraint (2) is achieved almost surely;

(iii) The trivial solution of the closed-loop system is stochastically fixed-time stable.

Proof. (i) We first prove that $V_n(\xi)$ is $\mathcal{C}^2$ and radially unbounded by two parts.

Part I. From $V_n(\xi) = \sum_{k=1}^n W_k(\bar{\xi}_k)$ and the definitions of μ and r_i, it follows that $W_k(\bar{\xi}_k)$ in (14) is $\mathcal{C}^2$, so is $V_n(\xi)$.

Part II. We next prove that, for $i = 1, \ldots, n$,

$$c_{i1}|\xi_i - \xi_i^*|^{\frac{4\mu - \varpi_i}{r_i}} \leqslant W_i(\bar{\xi}_i) \leqslant c_{i2}|z_i|^{\frac{4\mu - \varpi_i}{\mu}}, \tag{15}$$

where $c_{i1} = \frac{r_i \cdot 2^{\frac{4\mu - r_i - \varpi_i}{\mu}}}{(4\mu - \varpi_i)\cdot 2^{\frac{4\mu - r_i - \varpi_i}{r_i}}}$ and $c_{i2} = 2^{1 - \frac{r_i}{\mu}}$. Based on Lemma 1, we derive the following expression:

$$W_i(\bar{\xi}_i) \leqslant |\xi_i - \xi_i^*||z_i|^{\frac{4\mu - r_i - \varpi_i}{\mu}} \leqslant 2^{1 - \frac{r_i}{\mu}}|z_i|^{\frac{4\mu - \varpi_i}{\mu}}. \tag{16}$$

Let us prove the left-hand side of (16) through two cases.

Case 1. $\xi_i^* \leqslant \xi_i$.

(1) If $0 \leqslant \xi_i^* \leqslant \xi_i$, then, based on Lemma 1, we derive the following expression:

$$\begin{aligned} W_i(\bar{\xi}_i) &= \int_{\xi_i^*}^{\xi_i} \left\lceil \lceil s \rfloor^{\frac{\mu}{r_i}} - \lceil \xi_i^* \rfloor^{\frac{\mu}{r_i}} \right\rfloor^{\frac{4\mu - r_i - \varpi_i}{\mu}} \mathrm{d}s \\ &= \int_{\xi_i^*}^{\xi_i} \left(s^{\frac{\mu}{r_i}} - \xi_i^{*\frac{\mu}{r_i}} \right)^{\frac{4\mu - r_i - \varpi_i}{\mu}} \mathrm{d}s \\ &\geqslant \int_{\xi_i^*}^{\xi_i} (s - \xi_i^*)^{\frac{4\mu - r_i - \varpi_i}{r_i}} \mathrm{d}s. \end{aligned} \tag{17}$$

(2) If $\xi_i^* \leqslant \xi_i \leqslant 0$, then, similar to (17), we derive the following expression:

$$W_i(\bar{\xi}_i) = \int_{\xi_i^*}^{\xi_i} \left\lceil \lceil s \rfloor^{\frac{\mu}{r_i}} - \lceil \xi_i^* \rfloor^{\frac{\mu}{r_i}} \right\rfloor^{\frac{4\mu - r_i - \varpi_i}{\mu}} \mathrm{d}s$$

$$
\begin{aligned}
&= \int_{\xi_i^*}^{\xi_i} \left(-(-s)^{\frac{\mu}{r_i}} + (-\xi_i^*)^{\frac{\mu}{r_i}} \right)^{\frac{4\mu - r_i - \varpi_i}{\mu}} \mathrm{d}s \\
&\geqslant \int_{\xi_i^*}^{\xi_i} (s - \xi_i^*)^{\frac{4\mu - r_i - \varpi_i}{r_i}} \mathrm{d}s. \qquad (18)
\end{aligned}
$$

(3) If $\xi_i^* \leqslant 0 \leqslant \xi_i$, then, based on Lemma 1, we derive the following expression:

$$
\begin{aligned}
W_i(\bar{\xi}_i) &= \int_{\xi_i^*}^{0} \left(-(-s)^{\frac{\mu}{r_i}} + (-\xi_i^*)^{\frac{\mu}{r_i}} \right)^{\frac{4\mu - r_i - \varpi_i}{\mu}} \mathrm{d}s + \int_0^{\xi_i} \left(s^{\frac{\mu}{r_i}} + (-\xi_i^*)^{\frac{\mu}{r_i}} \right)^{\frac{4\mu - r_i - \varpi_i}{\mu}} \mathrm{d}s \\
&\geqslant \frac{2^{\frac{4\mu - r_i - \varpi_i}{\mu}}}{2^{\frac{4\mu - r_i - \varpi_i}{r_i}}} \int_{\xi_i^*}^{\xi_i} (s - \xi_i^*)^{\frac{4\mu - r_i - \varpi_i}{r_i}} \mathrm{d}s. \qquad (19)
\end{aligned}
$$

By combining (17)–(19), we determine that, when $\xi_i^* \leqslant \xi_i$,

$$
W_i(\bar{\xi}_i) \geqslant \frac{2^{\frac{4\mu - r_i - \varpi_i}{\mu}}}{2^{\frac{4\mu - r_i - \varpi_i}{r_i}}} \int_{\xi_i^*}^{\xi_i} (s - \xi_i^*)^{\frac{4\mu - r_i - \varpi_i}{r_i}} \mathrm{d}s = \frac{r_i \cdot 2^{\frac{4\mu - r_i - \varpi_i}{\mu}}}{(4\mu - \varpi_i) \cdot 2^{\frac{4\mu - r_i - \varpi_i}{r_i}}} (\xi_i - \xi_i^*)^{\frac{4\mu - \varpi_i}{r_i}}. \qquad (20)
$$

Case 2. When $\xi_i^* \geqslant \xi_i$, using the same analysis process as (17)–(19), we derive the following expression:

$$
W_i(\bar{\xi}_i) \geqslant \frac{r_i \cdot 2^{\frac{4\mu - r_i - \varpi_i}{\mu}}}{(4\mu - \varpi_i) \cdot 2^{\frac{4\mu - r_i - \varpi_i}{r_i}}} (\xi_i^* - \xi_i)^{\frac{4\mu - \varpi_i}{r_i}}. \qquad (21)
$$

Based on (20) and (21), the left-hand side of (16) holds. Then, using (16), we derive the following expression:

$$
\alpha_1(|\xi|) \leqslant V_n(\xi) \leqslant \beta_1(|\xi|), \qquad (22)
$$

where $\alpha_1(|\xi|) = \sum_{i=1}^n c_{i1} |\xi_i - \xi_i^*|^{\frac{4\mu - \varpi_i}{r_i}}$ and $\beta_1(|\xi|) = \sum_{i=1}^n c_{i2} |z_i|^{\frac{4\mu - \varpi_i}{\mu}}$. Hence, the radial unboundedness of $V_n(\xi)$ can be obtained. From (13), (22), and Lemma 2, we can conclude that the closed-loop system (1), (6), and (12) has a continuous solution on $[0, \infty)$.

(ii) For any $k \in \{2, 3, 4, \ldots\}$, let the stopping time $\sigma_k = \inf\{t \geqslant 0 : |\xi(t)| \geqslant k\}$. Using (13) and the following Itô's formula, we derive the following expression:

$$
\mathrm{E}\{V_n(\xi(\sigma_k \wedge t))\} = V_n(\xi(0)) + \mathrm{E} \int_0^{\sigma_k \wedge t} \mathcal{L} V_n(\xi(s)) \mathrm{d}s \leqslant V_n(\xi(0)). \qquad (23)
$$

For any $t \geqslant 0$ and $k > 0$, based on the continuity of $\xi(t)$ and the definition of σ_k, $\{\sup_{0 \leqslant s \leqslant t} |\xi(s)| > k\} \in \{\sigma_k \leqslant t\}$, which, together with (22), means that

$$
\begin{aligned}
\mathrm{E}\{V_n(\xi(\sigma_k \wedge t))\} &\geqslant \int_{\{\sup_{0 \leqslant s \leqslant t} |\xi(s)| > k\}} V_n(\xi(\sigma_k \wedge t)) \mathrm{d}P \\
&= \int_{\{\sup_{0 \leqslant s \leqslant t} |\xi(s)| > k\}} V_n(\xi(\sigma_k)) \mathrm{d}P \\
&\geqslant P\left\{ \sup_{0 \leqslant s \leqslant t} |\xi(s)| > k \right\} \inf_{|\xi| \geqslant k} V_n(\xi) \\
&\geqslant P\left\{ \sup_{0 \leqslant s \leqslant t} |\xi(s)| > k \right\} \inf_{|\xi| \geqslant k} \alpha_1(|\xi|), \ \forall t > 0, \ \forall t > 0. \qquad (24)
\end{aligned}
$$

Substituting (24) into (23) leads to

$$
P\left\{ \sup_{0 \leqslant s \leqslant t} |\xi(s)| > k \right\} \leqslant \frac{V_n(\xi(0))}{\inf_{|\xi| \geqslant k} \alpha_1(|\xi|)}, \ \forall t > 0. \qquad (25)
$$

Setting $k \to \infty$ and $t \to \infty$, and using the radial unboundedness of $\alpha_1(|\xi|)$, we derive $P\{\sup_{t \geqslant 0} |\xi(t)| < \infty\} = 1$. Thus, $\xi(t), \xi_1(t), \xi_2(t), \ldots, \xi_n(t)$ are almost surely bounded, so are $x_1(t), x_2(t), \ldots, x_n(t)$. Keeping this in mind and using the definitions of $\xi_2^*(t), \ldots, \xi_n^*(t)$ and $u(t)$, we can recursively prove the almost

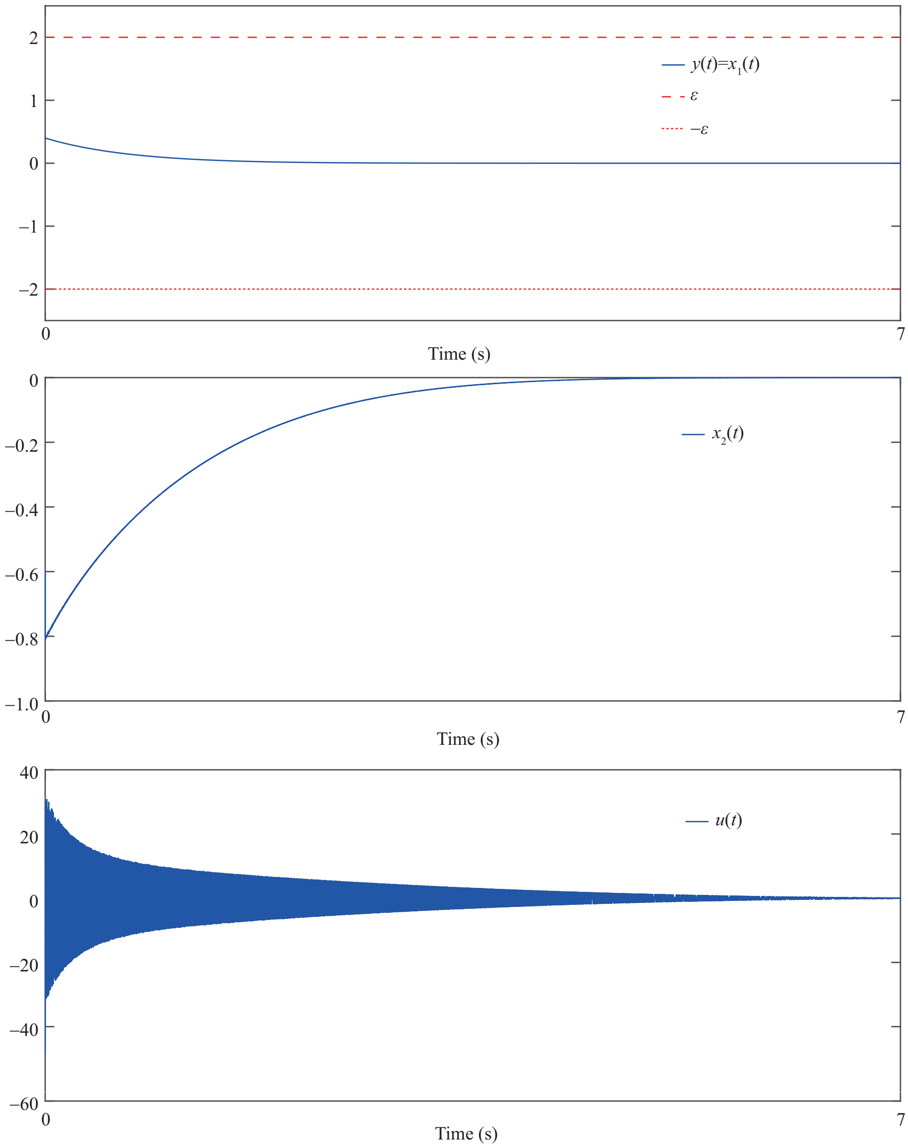

Figure 1 (Color online) Responses of the closed-loop system with $(x_1(0), x_2(0)) = (0.4, -0.6)$.

sure boundedness of $\xi_2^*(t), \ldots, \xi_n^*(t)$, $u(t)$. Based on (7) and the almost sure boundedness of $\xi_1(t)$, the constraint (2) is achieved almost surely.

(iii) Based on (16), Lemma 3, and $\varpi_1 = \cdots = \varpi_n$, we derive the following expression:

$$V_n(\xi)^{\frac{4\mu}{4\mu-\varpi_n}} \leqslant c_1 \left(\sum_{j=1}^{n} |z_j|^{\frac{4\mu-\varpi_j}{\mu}} \right)^{\frac{4\mu}{4\mu-\varpi_n}} \leqslant c_1 \sum_{j=1}^{n} |z_j|^4,$$
$$V_n(\xi)^{\frac{16\mu}{3(4\mu-\varpi_n)}} \leqslant c_2 \left(\sum_{j=1}^{n} |z_j|^{\frac{4\mu-\varpi_j}{\mu}} \right)^{\frac{16\mu}{3(4\mu-\varpi_n)}} \leqslant c_3 \sum_{j=1}^{n} |z_j|^{\frac{16}{3}}, \tag{26}$$

where $c_1 = (\max_{1\leqslant j\leqslant n}\{c_{j2}\})^{\frac{4\mu}{4\mu-\varpi_n}}$, $c_2 = (\max_{1\leqslant j\leqslant n}\{c_{j2}\})^{\frac{16\mu}{3(4\mu-\varpi_n)}}$, and $c_3 = n^{\frac{16\mu}{3(4\mu-\varpi_n)}-1} c_2$. Substituting (26) into (13) leads to

$$\mathcal{L}V_n(\xi) \leqslant -c(V_n(\xi)^{\frac{4\mu}{4\mu-\varpi_n}} + V_n(\xi)^{\frac{16\mu}{3(4\mu-\varpi_n)}}), \tag{27}$$

where $c = \min\{\frac{1}{4c_1}, \frac{1}{4c_3}\}$. Because $0 < \frac{4\mu}{4\mu-\varpi_n} < 1$ and $\frac{16\mu}{3(4\mu-\varpi_n)} > 1$, based on (27) and Lemma 4, the trivial solution of the closed-loop system is stochastically fixed-time stable.

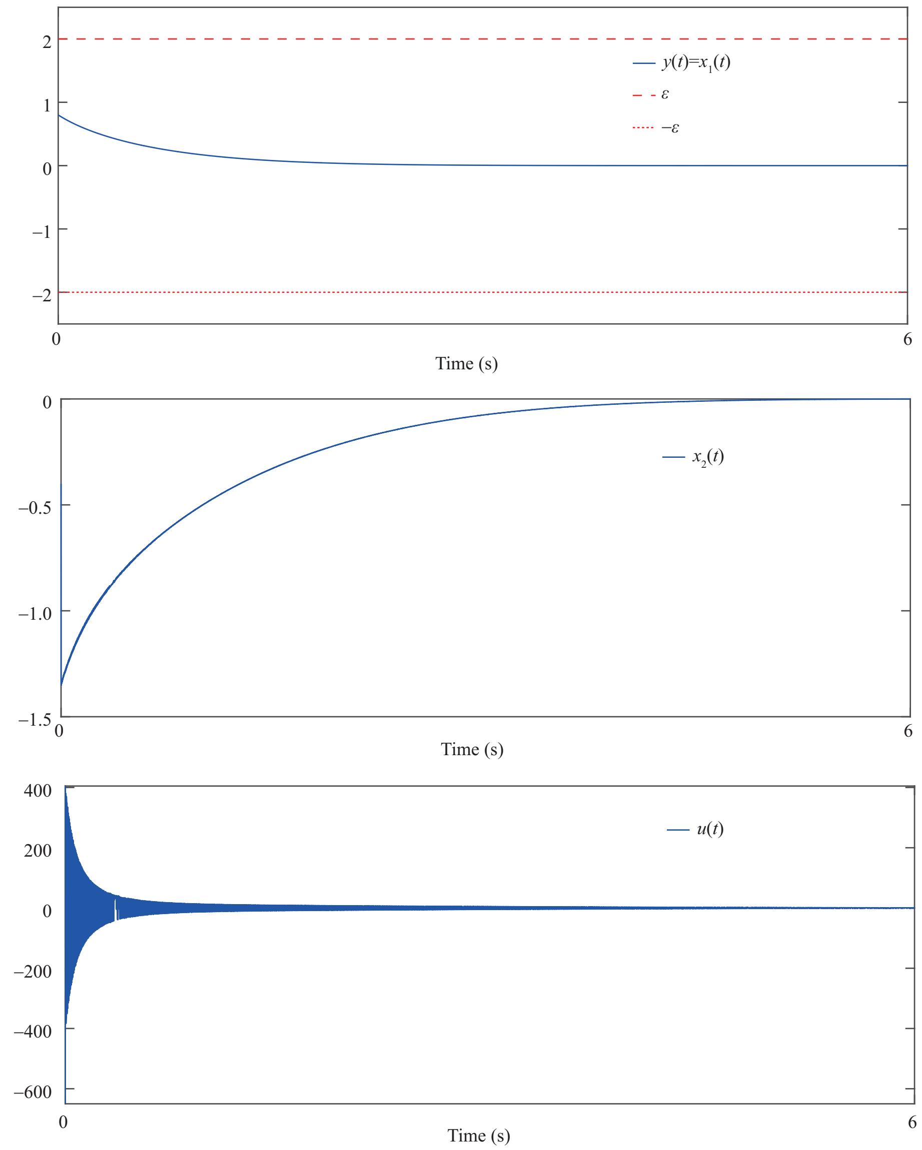

Figure 2 (Color online) Responses of the closed-loop system with $(x_1(0), x_2(0)) = (0.8, -0.5)$.

4 Simulation example

Consider the following system:

$$\begin{aligned} & \mathrm{d}x_1 = (\lceil x_2 \rceil^{p_1} + f_1(x_1))\,\mathrm{d}t + g_1(x_1)\mathrm{d}\omega, \\ & \mathrm{d}x_2 = (\lceil u \rceil^{p_2} + f_2(\bar{x}_2))\,\mathrm{d}t + g_2(\bar{x}_2)\mathrm{d}\omega, \\ & y = x_1, \end{aligned} \tag{28}$$

with the constraint

$$y \in \Omega_y = \{y \in \mathbb{R} : -2 < y < 2\}, \tag{29}$$

where $p_1 = \frac{5}{3}$, $p_2 = 3$, $f_1 = \frac{1}{4}x_1^{\frac{10}{11}}$, $g_1 = 0$, $f_2 = \frac{1}{4}\lceil x_2 \rceil^{\frac{5}{6}} + \frac{1}{2}x_2^{\frac{5}{3}}$, and $g_2 = \frac{1}{8}\lceil x_1 \rceil^{\frac{1}{2}}$. By choosing $r_1 = 1$, $h_1 = 1$, $\varpi_1 = \varpi_2 = -\frac{1}{11} \in (-\frac{3}{8}, 0)$, and $\gamma_1 = \gamma_2 = \frac{2}{3}$, we obtain $r_2 = \frac{r_1+\varpi_1}{p_1} = \frac{6}{11}$, $r_3 = \frac{r_2+\varpi_2}{p_2} = \frac{5}{33}$, $h_2 = \frac{h_1+\gamma_1}{p_1} = 1$, $h_3 = \frac{h_2+\gamma_2}{p_2} = \frac{5}{9}$, $f_1 \leqslant \frac{1}{4}|x_1|^{\frac{10}{11}}$, $f_2 \leqslant \frac{1}{4}|x_2|^{\frac{5}{6}} + \frac{1}{2}|x_2|^{\frac{5}{3}}$, and $g_2 \leqslant \frac{1}{8}|x_1|^{\frac{1}{2}}$. Hence, Assumption 1 holds.

Based on

$$\xi_1 = \tan\left(\frac{x_1}{\lambda_1}\right),\ \xi_2 = x_2, \tag{30}$$

where $\lambda_1 = \frac{4}{\pi}$, Eq. (28) can be rewritten as follows:

$$\begin{aligned} \mathrm{d}\xi_1 &= \big(D_1(\xi_1)\lceil\xi_2\rceil^{p_1} + \bar{f}_1(\xi_1)\big)\,\mathrm{d}t + \bar{g}_1(\xi_1)\mathrm{d}\omega, \\ \mathrm{d}\xi_2 &= \big(\lceil u\rceil^{p_2} + \bar{f}_2(\bar{\xi}_2)\big)\,\mathrm{d}t + \bar{g}_2(\bar{\xi}_2)\mathrm{d}\omega, \end{aligned} \tag{31}$$

where $D_1 = \frac{\pi(1+\xi_1^2)}{4}$, $\bar{f}_1 = D_1 f_1 + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2} g_1^{\mathrm{T}} g_1$, $\bar{g}_1 = D_1 g_1$, $\bar{f}_2 = f_2$, and $\bar{g}_2 = g_2$. Based on $\mu = 2$, $V_1 = \int_{\xi_1^*}^{\xi_1} \lceil \lceil s\rceil^{\frac{\varsigma}{r_1}} - \lceil \xi_1^*\rceil^{\frac{\varsigma}{r_1}} \rceil^{\frac{4\mu - r_1 - \varpi_1}{\varsigma}} \mathrm{d}s$ with $\xi_1^* = 0$, $z_1 = \lceil \xi_1\rceil^2$, and $\xi_2^* = -(\frac{452(\frac{1787}{1197} + \frac{1}{4} z_1^{\frac{4}{3}} + \frac{304}{1243} z_1)}{355})^{\frac{3}{5}} z_1 \triangleq -\beta_1 z_1$ guarantee that $\mathcal{L}V_1 \leqslant -\frac{5}{4} z_1^4 - \frac{1}{4} z_1^{\frac{16}{3}} + D_1 \lceil z_1\rceil^{\frac{39}{11}} (\lceil \xi_2\rceil^{\frac{5}{3}} - \lceil \xi_2^*\rceil^{\frac{5}{3}})$. By setting $z_2 = \lceil \xi_2\rceil^{\frac{11}{3}} - \lceil \xi_2^*\rceil^{\frac{11}{3}}$ and $V_2 = V_1 + \int_{\xi_2^*}^{\xi_2} \lceil \lceil s\rceil^{\frac{\mu}{r_2}} - \lceil \xi_2^*\rceil^{\frac{\mu}{r_2}} \rceil^{\frac{4\mu - r_2 - \varpi_2}{\mu}} \mathrm{d}s$, the controller

$$u = -\Big(\frac{1}{4} + \phi + \frac{1}{4} z_2^{\frac{4}{3}}\Big)^{\frac{1}{3}} \lceil z_2\rceil^{\frac{5}{66}} \tag{32}$$

leads to $\mathcal{L}V_2 \leqslant -\frac{1}{4}\sum_{j=1}^2 z_j^4 - \frac{1}{4}\sum_{j=1}^2 z_j^{\frac{16}{3}}$, where $\beta_2 = \frac{1493}{533}\beta_1^{\frac{6}{5}}(\frac{2}{3}\xi_1^{\frac{5}{3}} + \frac{608}{1243}\xi_1)\xi_1^2 + 2\beta_1^{\frac{11}{5}}\xi_1$ and $\phi = \frac{4131}{385}(D_1\beta_1^{\frac{5}{3}}\beta_2)^{\frac{88}{67}} + \frac{13993}{1938}\beta_2^{\frac{8}{7}} + \frac{751}{324}(D_1\beta_2)^{\frac{88}{67}} + (\frac{3}{4} + \frac{1}{2}\xi_2^2) + \frac{756}{835}(\frac{1}{4}\beta_1^{\frac{5}{6}} + \frac{1}{2}\beta_1^{\frac{5}{6}}\xi_2^{\frac{5}{6}})^{\frac{88}{83}} + \frac{747}{8099} + \frac{115}{1021}\beta_1^{\frac{704}{183}} + \frac{181255}{8} D_1^{\frac{44}{3}}$.

Figures 1 and 2 illustrate the responses of the closed-loop system (28)–(30) and (32) with the mean of 10 sample sizes and two different initial states, that is, $(x_1(0), x_2(0)) = (0.4, -0.6)$ and $(x_1(0), x_2(0)) = (0.8, -0.5)$. Stochastic fixed-time stabilization can be achieved with the expectation of the settling time being less than $T_{\max} = \frac{2}{4^{\frac{3+\bar{p}}{4}}(1 - \frac{3+\bar{p}}{4})} + \frac{2(2^{\frac{5+\bar{p}}{4}} - 1)}{4^{\frac{5+\bar{p}}{4}}(\frac{5+\bar{p}}{4} - 1)} = 168.4$, which is independent of any initial value.

5 Conclusion

This paper studies the fixed-time state feedback control problem of stochastic high-order nonlinear systems with output constraint and high-order and low-order nonlinearities.

Some critical challenges remain unresolved: (i) For the stochastic system with output constraint in [35], the method of designing a fixed-time output feedback controller needs to be clarified. (ii) An important issue corresponding to FT-SiISS in [37] is how to define the fixed-time stochastic integral input-to-state stability.

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Appendix A Proof of Proposition 1

By (6),

$$|x_1|^{\eta} \ = \ |\lambda_1 \arctan(\xi_1)|^{\eta} \leqslant \lambda_1^{\eta}|\xi_1|^{\eta}, \tag{A1}$$

where $\eta \in \{\frac{r_1+\varpi_1}{r_1}, \frac{h_1+\gamma_1}{r_1}, \frac{2r_1+\varpi_1}{2r_1}, \frac{2h_1+\gamma_1}{2r_1}\}$. By (3), (A1), and Lemmas 3 and 5,

$$\begin{aligned}\bar{f}_1(\xi_1) \ &\leqslant \ |D_1(\xi_1)||f_1(\xi_1)| + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2}|g_1(\xi_1)|^2 \\ &\leqslant \ |D_1(\xi_1)|\left|f_{11}(\xi_1)|x_1|^{\frac{r_1+\varpi_1}{r_1}} + f_{12}(\xi_1)|x_1|^{\frac{h_1+\gamma_1}{h_1}}\right| \\ &\quad + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2}\left|g_{11}(\xi_1)|x_1|^{\frac{2r_1+\varpi_1}{2r_1}} + g_{12}(\xi_1)|x_1|^{\frac{2h_1+\gamma_1}{2h_1}}\right|^2 \\ &\leqslant \ \bar{f}_{11}(\xi_1)|\xi_1|^{\frac{r_1+\varpi_1}{r_1}} + \bar{f}_{12}(\xi_1)|\xi_1|^{\frac{h_1+\gamma_1}{h_1}},\end{aligned} \tag{A2}$$

where $\bar{f}_{11}(\xi_1)$ and $\bar{f}_{12}(\xi_1)$ are some known nonnegative smooth functions. Similar to (A2), one can find some nonnegative smooth functions $\bar{g}_{11}(\xi_1)$ and $\bar{g}_{12}(\xi_1)$ such that

$$\bar{g}_1(\xi_1) \ \leqslant \ \bar{g}_{11}(\xi_1)|\xi_1|^{\frac{2r_1+\varpi_1}{2r_1}} + \bar{g}_{12}(\xi_1)|\xi_1|^{\frac{2h_1+\gamma_1}{2h_1}}. \tag{A3}$$

For $i = 2, \ldots, n$, it follows from (3), (A1), and Lemmas 3 and 5 that

$$|\bar{f}_i(\bar{\xi}_i)| \ \leqslant \ \bar{f}_{i1}(\bar{\xi}_i)\sum_{j=1}^{i}|\xi_j|^{\frac{r_i+\varpi_i}{r_j}} + \bar{f}_{i2}(\bar{\xi}_i)\sum_{j=1}^{i}|\xi_j|^{\frac{h_i+\gamma_i}{h_j}},$$

$$|\bar{g}_i(\bar{\xi}_i)| \leqslant \bar{g}_{i1}(\bar{\xi}_i)\sum_{j=1}^{i}|\xi_j|^{\frac{2r_i+\varpi_i}{2r_j}} + \bar{g}_{i2}(\bar{\xi}_i)\sum_{j=1}^{i}|\xi_j|^{\frac{2h_i+\gamma_i}{2h_j}}, \tag{A4}$$

where $\bar{f}_{i1}(\bar{\xi}_i)$, $\bar{f}_{i2}(\bar{\xi}_i)$, $\bar{g}_{i1}(\bar{\xi}_i)$, and $\bar{g}_{i2}(\bar{\xi}_i)$ are some known nonnegative smooth functions. By $\varpi_n \leqslant \varpi_{n-1} \leqslant \cdots \leqslant \varpi_1 \in (-\frac{1}{\sum_{l=1}^{n} p_1\cdots p_{l-1}}, 0)$ and $0 \leqslant \gamma_n \leqslant \gamma_{n-1} \leqslant \cdots \leqslant \gamma_1$ in Assumption 1, one can deduce that $\frac{r_i+\varpi_i}{r_j} \leqslant \frac{1}{p_j\cdots p_{i-1}} \leqslant \frac{h_i+\gamma_i}{h_j}$. Using this fact, (A1), (A4), and Lemma 5, one has

$$\begin{aligned}|\bar{f}_i(\bar{\xi}_i)| &\leqslant \bar{f}_{i1}(\bar{\xi}_i)\sum_{j=1}^{i}|\xi_j|^{\frac{r_i+\varpi_i}{r_j}} + \bar{f}_{i2}(\bar{\xi}_i)\sum_{j=1}^{i}|\xi_j|^{\frac{h_i+\gamma_i}{h_j}}\\ &= \sum_{j=1}^{i}\left(\bar{f}_{i1}(\bar{\xi}_i) + \bar{f}_{i2}(\bar{\xi}_i)|\xi_j|^{\frac{h_i+\gamma_i}{h_j}-\frac{r_i+\varpi_i}{r_j}}\right)|\xi_j|^{\frac{r_i+\varpi_i}{r_j}}\\ &\leqslant h_{i1}(\bar{\xi}_i)\sum_{j=1}^{i}|\xi_j|^{\frac{r_i+\varpi_i}{r_j}},\end{aligned} \tag{A5}$$

where $h_{i1}(\bar{\xi}_i)$ is a known nonnegative smooth function. Similarly, there exists a known nonnegative smooth function $\eta_{i1}(\bar{\xi}_i)$ such that

$$\begin{aligned}|\bar{g}_i(\bar{\xi}_i)| &\leqslant \bar{g}_{i1}(\bar{\xi}_i)\sum_{j=1}^{i}|\xi_j|^{\frac{2r_i+\varpi_i}{2r_j}} + \bar{g}_{i2}(\bar{\xi}_i)\sum_{j=1}^{i}|\xi_j|^{\frac{2h_i+\gamma_i}{2h_j}}\\ &= \sum_{j=1}^{i}\left(\bar{g}_{i1}(\bar{\xi}_i) + \bar{g}_{i2}(\bar{\xi}_i)|\xi_j|^{\frac{2h_i+\gamma_i}{2h_j}-\frac{2r_i+\varpi_i}{2r_j}}\right)|\xi_j|^{\frac{2r_i+\varpi_i}{2r_j}}\\ &\leqslant \eta_{i1}(\bar{\xi}_i)\sum_{j=1}^{i}|\xi_j|^{\frac{2r_i+\varpi_i}{2r_j}}.\end{aligned} \tag{A6}$$

Appendix B Proof of Proposition 2

We prove the proposition by induction.

Step 1: Set $z_1 = \lceil\xi_1\rceil^{\frac{\mu}{r_1}}$ and the Lyapunov function $V_1(\xi_1) = W_1(\xi_1) = \frac{r_1}{4\mu-\varpi_1}|\xi_1|^{\frac{4\mu-\varpi_1}{r_1}}$. By (8), Lemma 6, and Itô's formula,

$$\begin{aligned}\mathcal{L}V_1 \leqslant\ & D_1(\xi_1)\lceil z_1\rceil^{\frac{4\mu-r_1-\varpi_1}{\mu}}\left(\lceil\xi_2\rceil^{p_1} - \lceil\xi_2^*\rceil^{p_1} + \lceil\xi_2^*\rceil^{p_1}\right) + \lceil z_1\rceil^{\frac{4\mu-r_1-\varpi_1}{\mu}}\bar{f}_1(\xi_1)\\ &+\frac{4\mu-r_1-\varpi_1}{2r_1}|z_1|^{\frac{4\mu-2r_1-\varpi_1}{\mu}}\bar{g}_1^{\mathrm{T}}(\xi_1)\bar{g}_1(\xi_1).\end{aligned} \tag{B1}$$

From (9) and Lemmas 5 and 7, it follows that

$$\lceil z_1\rceil^{\frac{4\mu-r_1-\varpi_1}{\mu}}\bar{f}_1(\xi_1) \leqslant h_{11}(\xi_1)|z_1|^{\frac{4\mu-r_1-\varpi_1}{\mu}}|z_1|^{\frac{r_1+\varpi_1}{\mu}} = \beta_{11}(\xi_1)|z_1|^4, \tag{B2}$$

where $\beta_{11}(\xi_1)$ is a nonnegative smooth function. Using (10) and Lemmas 3, 5, and 7, one can find a nonnegative smooth function $\beta_{12}(\xi_1)$ such that

$$\begin{aligned}\frac{4\mu-r_1-\varpi_1}{2r_1}|z_1|^{\frac{4\mu-2r_1-\varpi_1}{\mu}}\bar{g}_1^{\mathrm{T}}(\xi_1)\bar{g}_1(\xi_1) &\leqslant \frac{4\mu-r_1-\varpi_1}{2r_1}|z_1|^{\frac{4\mu-2r_1-\varpi_1}{\mu}}\left(\eta_{11}(\xi_1)|z_1|^{\frac{2r_1+\varpi_1}{2\mu}}\right)^2\\ &\leqslant \frac{4\mu-r_1-\varpi_1}{2r_1}\eta_{11}(\xi_1)^2|z_1|^{\frac{4\mu-2r_1-\varpi_1}{\mu}}|z_1|^{\frac{2r_1+\varpi_1}{\mu}}\\ &\leqslant \beta_{12}(\xi_1)|z_1|^4.\end{aligned} \tag{B3}$$

Substituting (B2), (B3), and the virtual controller

$$\xi_2^*(\xi_1) = -\left(\frac{n-1+\frac{1}{4}+\frac{1}{4}z_1^{\frac{4}{3}}+\beta_1(\xi_1)}{D}\right)^{\frac{1}{p_1}}\lceil z_1\rceil^{\frac{r_2}{\mu}} \triangleq -\phi_1(\xi_1)^{\frac{r_2}{\mu}}\lceil z_1\rceil^{\frac{r_2}{\mu}} \tag{B4}$$

into (B1) yields

$$\begin{aligned}\mathcal{L}V_1 &\leqslant D_1(\xi_1)\lceil z_1\rceil^{\frac{4\mu-r_1-\varpi_1}{\mu}}\left(\lceil\xi_2\rceil^{p_1} - \lceil\xi_2^*\rceil^{p_1} + \lceil\xi_2^*\rceil^{p_1}\right) + \beta_1(\xi_1)|z_1|^4\\ &\leqslant -\left(n-1+\frac{1}{4}\right)|z_1|^4 + \lceil z_1\rceil^{\frac{4\mu-r_1-\varpi_1}{\mu}}\left(\lceil\xi_2\rceil^{p_1} - \lceil\xi_2^*\rceil^{p_1}\right) - \frac{1}{4}|z_1|^{\frac{16}{3}},\end{aligned} \tag{B5}$$

where $D = \frac{1}{\lambda_1}$ and $\beta_1(\xi_1) = \beta_{11}(\xi_1) + \beta_{12}(\xi_1)$.

Step 2: Set $V_2(\bar{\xi}_2) = V_1(\xi_1) + W_2(\bar{\xi}_2) = V_1(\xi_1) + \int_{\xi_2^*}^{\xi_2}\lceil\lceil s\rceil^{\frac{\mu}{r_2}} - \lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}\rceil^{\frac{4\mu-r_2-\varpi_2}{\mu}}\mathrm{d}s$ and $z_2 = \lceil\xi_2\rceil^{\frac{\mu}{r_2}} - \lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}$. Applying (8), (B5), and Itô's formula, one gets

$$\begin{aligned}\mathcal{L}V_2 =\ & \mathcal{L}V_1 + \frac{\partial W_2}{\partial\xi_1}(D_1(\xi_1)\lceil\xi_2\rceil^{p_1} + \bar{f}_1(\xi_1)) + \frac{\partial W_2}{\partial\xi_2}(\lceil\xi_3\rceil^{p_2} + \bar{f}_2(\bar{\xi}_2)) + \frac{1}{2}\frac{\partial^2 W_2}{\partial\xi_1^2}\bar{g}_1^{\mathrm{T}}(\xi_1)\bar{g}_1(\xi_1)\\ &+\frac{\partial^2 W_2}{\partial\xi_1\partial\xi_2}\bar{g}_1^{\mathrm{T}}(\xi_1)\bar{g}_2(\bar{\xi}_2) + \frac{1}{2}\frac{\partial^2 W_2}{\partial\xi_2^2}\bar{g}_2^{\mathrm{T}}(\bar{\xi}_2)\bar{g}_2(\bar{\xi}_2)\end{aligned}$$

$$\begin{aligned}&\leqslant -\Big(n-1+\frac{1}{4}\Big)|z_1|^4+\lceil z_1\rceil^{\frac{4\mu-r_1-\varpi_1}{\mu}}\left(\lceil\xi_2\rceil^{p_1}-\lceil\xi_2^*\rceil^{p_1}\right)-\frac{1}{4}|z_1|^{\frac{16}{3}}+\frac{\partial W_2}{\partial\xi_1}(D_1(\xi_1)\lceil\xi_2\rceil^{p_1}+\bar f_1(\xi_1))\\&\quad+\frac{\partial W_2}{\partial\xi_2}(\lceil\xi_3\rceil^{p_2}+\bar f_2(\bar\xi_2))+\frac{1}{2}\frac{\partial^2W_2}{\partial\xi_1^2}\bar g_1^{\rm T}(\xi_1)\bar g_1(\xi_1)+\frac{\partial^2W_2}{\partial\xi_1\partial\xi_2}\bar g_1^{\rm T}(\xi_1)\bar g_2(\bar\xi_2)\\&\quad+\frac{1}{2}\frac{\partial^2W_2}{\partial\xi_2^2}\bar g_2^{\rm T}(\bar\xi_2)\bar g_2(\bar\xi_2).\end{aligned}\tag{B6}$$

By the definition of μ and Lemma 6,

$$\frac{\partial W_2}{\partial\xi_1}=-\frac{4\mu-\varpi_2-r_2}{\mu}\frac{\partial\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}}{\partial\xi_1}\int_{\xi_2^*}^{\xi_2}\left|\lceil s\rceil^{\frac{\mu}{r_2}}-\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}\right|^{\frac{4\mu-\varpi_2-r_2-\mu}{\mu}}{\rm d}s,\tag{B7}$$

$$\frac{\partial W_2}{\partial\xi_2}=\lceil z_2\rceil^{\frac{4\mu-\varpi_2-r_2}{\mu}},\tag{B8}$$

$$\begin{aligned}\frac{\partial^2W_2}{\partial\xi_1^2}=&-\left(\frac{4\mu-\varpi_2-r_2}{\mu}\right)\left(\frac{\partial^2\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}}{\partial\xi_1^2}\right)\int_{\xi_2^*}^{\xi_2}\left|\lceil s\rceil^{\frac{\mu}{r_2}}-\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}\right|^{\frac{4\mu-\varpi_2-r_2-\mu}{\mu}}{\rm d}s+\left(\frac{4\mu-\varpi_2-r_2}{\mu}\right)\\&\cdot\left(\frac{4\mu-\varpi_2-r_2-\mu}{\mu}\right)\left(\frac{\partial\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}}{\partial\xi_1}\right)^2\int_{\xi_2^*}^{\xi_2}\left\lceil\lceil s\rceil^{\frac{\mu}{r_2}}-\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}\right\rceil^{\frac{4\mu-\varpi_2-r_2-2\mu}{\mu}}{\rm d}s,\end{aligned}\tag{B9}$$

$$\frac{\partial^2W_2}{\partial\xi_2\partial\xi_1}=\frac{4\mu-\varpi_2-r_2}{\mu}|z_2|^{\frac{4\mu-\varpi_2-r_2-\mu}{\mu}}\frac{\partial z_2}{\partial\xi_1},\tag{B10}$$

$$\frac{\partial^2W_2}{\partial\xi_2^2}=\frac{4\mu-\varpi_2-r_2}{\mu}|z_2|^{\frac{4\mu-\varpi_2-r_2-\mu}{\mu}}\frac{\partial z_2}{\partial\xi_2}.\tag{B11}$$

It follows from (9), (B7), and Lemmas 1, 3, 5, 7, and 8 that

$$\begin{aligned}\frac{\partial W_2}{\partial\xi_1}(D_1(\xi_1)\lceil\xi_2\rceil^{p_1}+\bar f_1(\xi_1))\leqslant&\left|\frac{4\mu-\varpi_2-r_2}{\mu}\frac{\partial\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}}{\partial\xi_1}\int_{\xi_2^*}^{\xi_2}\left|\lceil s\rceil^{\frac{\mu}{r_2}}-\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}\right|^{\frac{4\mu-\varpi_2-r_2-\mu}{\mu}}{\rm d}s\right|\\&\cdot\left(D_1(\xi_1)|\xi_2|^{p_1}+h_1(\xi_1)|\xi_1|^{\frac{r_1+\varpi_1}{r_1}}\right)\\\leqslant&\left|\frac{4\mu-\varpi_2-r_2}{\mu}\frac{\partial\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}}{\partial\xi_1}|\xi_2-\xi_2^*||z_2|^{\frac{4\mu-\varpi_2-r_2-\mu}{\mu}}\right|\\&\cdot\Big(D_1(\xi_1)|z_2+\phi_1z_1|^{\frac{r_1+\varpi_1}{\mu}}+h_{11}(\xi_1)|z_1|^{\frac{r_1+\varpi_1}{\mu}}\Big)\\\leqslant&\frac{1}{6}|z_1|^4+\beta_{21}(\bar\xi_2)|z_2|^4,\end{aligned}\tag{B12}$$

where $\beta_{21}(\bar\xi_2)$ is a nonnegative smooth function. Similar to (B12), it is clear that

$$\begin{aligned}\frac{\partial W_2}{\partial\xi_2}\bar f_2(\bar\xi_2)\leqslant&|z_2|^{\frac{4\mu-\varpi_2-r_2}{\mu}}\left(h_{21}(\bar\xi_2)\sum_{j=1}^2|\xi_j|^{\frac{r_2+\varpi_2}{r_j}}\right)\\\leqslant&|z_2|^{\frac{4\mu-\varpi_2-r_2}{\mu}}\left(h_{21}(\bar\xi_2)\Big(|z_1|^{\frac{r_2+\varpi_2}{\mu}}+|z_2+\phi_1z_1|^{\frac{r_2+\varpi_2}{\mu}}\Big)\right)\\\leqslant&\frac{1}{6}|z_1|^4+\beta_{22}(\bar\xi_2)|z_2|^4,\end{aligned}\tag{B13}$$

where $\beta_{22}(\bar\xi_2)$ is a nonnegative smooth function. From (10), (B4), (B10), and Lemmas 1, 3, 5, and 7,

$$\begin{aligned}\frac{\partial^2W_2}{\partial\xi_2\partial\xi_1}\bar g_2^{\rm T}(\bar\xi_2)\bar g_1(\xi_1)\leqslant&\left|\frac{4\mu-\varpi_2-r_2}{\mu}\frac{\partial z_2}{\partial\xi_1}|z_2|^{\frac{4\mu-\varpi_2-r_2-\mu}{\mu}}\right|\left(\eta_{21}(\bar\xi_2)\sum_{j=1}^2|\xi_j|^{\frac{2r_2+\varpi_2}{2r_j}}\right)\left(\eta_{11}(\xi_1)|\xi_1|^{\frac{2r_1+\varpi_1}{2r_1}}\right)\\\leqslant&\left|\frac{4\mu-\varpi_2-r_2}{\mu}\frac{\partial z_2}{\partial\xi_1}|z_2|^{\frac{4\mu-\varpi_2-r_2-\mu}{\mu}}\right|\varrho_1(\bar\xi_2)\sum_{j=1}^2|z_j|^{\frac{2r_2+\varpi_2+2r_1+\varpi_1}{2\mu}}\\\leqslant&\frac{1}{6}|z_1|^4+\beta_{23}(\bar\xi_2)|z_2|^4,\end{aligned}\tag{B14}$$

where $\varrho_1(\bar\xi_2)$ and $\beta_{23}(\bar\xi_2)$ are some nonnegative smooth functions. By (10), (B11), and Lemmas 1, 3, 5, and 7, one has

$$\begin{aligned}\frac{1}{2}\frac{\partial^2W_2}{\partial\xi_2^2}\bar g_2^{\rm T}(\bar\xi_2)\bar g_2(\bar\xi_2)\leqslant&\left|\frac{4\mu-\varpi_2-r_2}{2\mu}|z_2|^{\frac{4\mu-\varpi_2-r_2-\mu}{\mu}}\frac{\partial z_2}{\partial\xi_2}\right|\left(\eta_{21}(\bar\xi_2)\sum_{j=1}^2|\xi_j|^{\frac{2r_2+\varpi_2}{2r_j}}\right)^2\\\leqslant&\left|\frac{4\mu-\varpi_2-r_2}{2r_2}|z_2|^{\frac{4\mu-\varpi_2-r_2-\mu}{\mu}}|\xi_2|^{\frac{\mu-r_2}{r_2}}\right|\Bigg(\eta_{21}(\bar\xi_2)\Big(|z_1|^{\frac{2r_2+\varpi_2}{2\mu}}\\&+|z_2+\phi_1z_1|^{\frac{2r_2+\varpi_2}{2\mu}}\Big)\Bigg)^2\\\leqslant&\frac{1}{6}|z_1|^4+\beta_{24}(\bar\xi_2)|z_2|^4,\end{aligned}\tag{B15}$$

where $\beta_{24}(\bar\xi_2)$ is a nonnegative smooth function. By (10), (B4), (B9), and Lemmas 1, 3, 5, 7, and 8,

$$\frac{1}{2}\frac{\partial^2W_2}{\partial\xi_1^2}\bar g_1^{\rm T}(\xi_1)\bar g_1(\xi_1)\leqslant\frac{1}{2}\left|\left(\frac{4\mu-\varpi_2-r_2}{\mu}\right)\left(\frac{\partial^2\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}}{\partial\xi_1^2}\right)\int_{\xi_2^*}^{\xi_2}\left|\lceil s\rceil^{\frac{\mu}{r_2}}-\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}\right|^{\frac{4\mu-\varpi_2-r_2-\mu}{\mu}}{\rm d}s\right.$$

$$
\begin{aligned}
&+\left(\frac{\partial\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}}{\partial\xi_1}\right)^2\left(\frac{4\mu-\varpi_2-r_2-\mu}{\mu}\right)\int_{\xi_2^*}^{\xi_2}\left\lceil\lceil s\rceil^{\frac{\mu}{r_2}}-\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}\right\rceil^{\frac{4\mu-\varpi_2-r_2-2\mu}{\mu}}\mathrm{d}s\\
&\cdot\left(\frac{4\mu-\varpi_2-r_2}{\mu}\right)\Bigg|\left(\eta_{11}(\xi_1)|\xi_1|^{\frac{2r_1+\varpi_1}{2r_1}}\right)^2\\
\leqslant\ &\frac{1}{2}\Bigg|\left(\frac{4\mu-\varpi_2-r_2}{\mu}\right)\left(\frac{\partial^2\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}}{\partial\xi_1^2}\right)|\xi_2-\xi_2^*||z_2|^{\frac{4\mu-\varpi_2-r_2-\mu}{\mu}}\\
&+\left(\frac{\partial\lceil\xi_2^*\rceil^{\frac{\mu}{r_2}}}{\partial\xi_1}\right)^2\left(\frac{4\mu-\varpi_2-r_2-\mu}{\mu}\right)|\xi_2-\xi_2^*||z_2|^{\frac{4\mu-\varpi_2-r_2-2\mu}{\mu}}\\
&\cdot\left(\frac{4\mu-\varpi_2-r_2}{\mu}\right)\Bigg|\left(\eta_{11}(\xi_1)|z_1|^{\frac{2r_1+\varpi_1}{2\mu}}\right)^2\\
\leqslant\ &\frac{1}{6}|z_1|^4+\beta_{25}(\bar{\xi}_2)|z_2|^4,
\end{aligned}\tag{B16}
$$

where $\beta_{25}(\bar{\xi}_2)$ is a nonnegative smooth function. Since $\frac{p_1r_2}{\mu}\leqslant 1$, by Lemmas 1, 5, and 7, there is a nonnegative smooth function $\beta_{26}(\bar{\xi}_2)$ such that

$$
\begin{aligned}
D_1(\xi_1)\lceil z_1\rceil^{\frac{4\mu-\varpi_1-r_1}{\mu}}\left(\lceil\xi_2\rceil^{p_1}-\lceil\xi_2^*\rceil^{p_1}\right)\ &\leqslant\ 2^{1-\frac{\varpi_1+r_1}{\mu}}D_1|z_1|^{\frac{4\mu-\varpi_1-r_1}{\mu}}|z_2|^{\frac{\varpi_1+r_1}{\mu}}\\
&\leqslant\ \frac{1}{6}|z_1|^4+\beta_{26}(\bar{\xi}_2)|z_2|^4.
\end{aligned}\tag{B17}
$$

Substituting (B12)–(B17) into (B6) leads to

$$
\mathcal{L}V_2\ \leqslant\ -\left(n-2+\frac{1}{4}\right)|z_1|^4+\beta_2(\bar{\xi}_2)|z_2|^4+\lceil z_2\rceil^{\frac{4\mu-\varpi_2-r_2}{\mu}}\left(\lceil\xi_3\rceil^{p_2}-\lceil\xi_3^*\rceil^{p_2}+\lceil\xi_3^*\rceil^{p_2}\right)-\frac{1}{4}|z_1|^{\frac{16}{3}},\tag{B18}
$$

where $\beta_2(\bar{\xi}_2)=\sum_{j=1}^{6}\beta_{2j}(\bar{\xi}_2)$. Then, one can design the virtual controller

$$
\xi_3^*(\bar{\xi}_2)=-\left(n-2+\frac{1}{4}+\frac{1}{4}z_2^{\frac{4}{3}}+\beta_2(\bar{\xi}_2)\right)^{\frac{1}{p_2}}\lceil z_2\rceil^{\frac{r_3}{\mu}}\triangleq-\phi_2^{\frac{r_3}{\mu}}(\bar{\xi}_2)\lceil z_2\rceil^{\frac{r_3}{\mu}},\tag{B19}
$$

such that

$$
\mathcal{L}V_2\leqslant-\left(n-2+\frac{1}{4}\right)\sum_{j=1}^{2}|z_j|^4+\lceil z_2\rceil^{\frac{4\mu-r_2-\varpi_2}{\mu}}\left(\lceil\xi_3\rceil^{p_2}-\lceil\xi_3^*\rceil^{p_2}\right)-\frac{1}{4}\sum_{j=1}^{2}|z_j|^{\frac{16}{3}}.\tag{B20}
$$

Inductive Step ($3\leqslant k\leqslant n$): Suppose that at Step $k-1$, there exist a Lyapunov function $V_{k-1}(\bar{\xi}_{k-1})$ and a series of virtual controllers $\xi_2^*(\xi_1),\ldots,\xi_k^*(\bar{\xi}_{k-1})$ with the following form:

$$
\begin{aligned}
&\xi_2^*(\xi_1)=-\phi_1^{\frac{r_2}{\mu}}(\xi_1)\lceil z_1\rceil^{\frac{r_2}{\mu}},\\
&\vdots\\
&\xi_k^*(\bar{\xi}_{k-1})=-\phi_{k-1}^{\frac{r_k}{\mu}}(\bar{\xi}_{k-1})\lceil z_{k-1}\rceil^{\frac{r_k}{\mu}},\\
&z_j=\lceil\xi_j\rceil^{\frac{\mu}{r_j}}-\lceil\xi_j^*\rceil^{\frac{\mu}{r_j}},\ j=1,\ldots,k-1,
\end{aligned}\tag{B21}
$$

such that

$$
\mathcal{L}V_{k-1}\leqslant-\left(n-(k-1)+\frac{1}{4}\right)\sum_{j=1}^{k-1}|z_j|^4+\lceil z_{k-1}\rceil^{\frac{4\mu-r_{k-1}-\varpi_{k-1}}{\mu}}\left(\lceil\xi_k\rceil^{p_{k-1}}-\lceil\xi_k^*\rceil^{p_{k-1}}\right)-\frac{1}{4}\sum_{j=1}^{k-1}|z_j|^{\frac{16}{3}}.\tag{B22}
$$

We next prove that Eq. (B22) still holds at Step k.

Setting $z_k=\lceil\xi_k\rceil^{\frac{\mu}{r_k}}-\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}$, we can choose the k-th Lyapunov function $V_k(\bar{\xi}_k)=V_{k-1}(\bar{\xi}_{k-1})+W_k(\bar{\xi}_k)=V_{k-1}(\bar{\xi}_k)+\int_{\xi_k^*}^{\xi_k}\lceil\lceil s\rceil^{\frac{\mu}{r_k}}-\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}\rceil^{\frac{4\mu-r_k-\varpi_k}{\mu}}\mathrm{d}s$. By (8), (B22), and Itô's formula,

$$
\begin{aligned}
\mathcal{L}V_k\ \leqslant\ &\mathcal{L}V_{k-1}+\frac{\partial W_k}{\partial\xi_1}\left(D_1\lceil\xi_2\rceil^{p_1}+\bar{f}_1(\xi_1)\right)+\sum_{j=2}^{k}\frac{\partial W_k}{\partial\xi_j}\left(\lceil\xi_{j+1}\rceil^{p_j}+\bar{f}_j(\bar{\xi}_j)\right)+\frac{1}{2}\sum_{i,j=1}^{k}\frac{\partial^2W_k}{\partial\xi_i\partial\xi_j}\bar{g}_i^{\mathrm{T}}(\bar{\xi}_i)\bar{g}_j(\bar{\xi}_j)\\
\leqslant\ &-\left(n-(k-1)+\frac{1}{4}\right)\sum_{j=1}^{k-1}|z_j|^4+\lceil z_{k-1}\rceil^{\frac{4\mu-r_{k-1}-\varpi_{k-1}}{\mu}}\left(\lceil\xi_k\rceil^{p_{k-1}}-\lceil\xi_k^*\rceil^{p_{k-1}}\right)-\frac{1}{4}\sum_{j=1}^{k-1}|z_j|^{\frac{16}{3}}\\
&+\frac{\partial W_k}{\partial\xi_1}\left(D_1\lceil\xi_2\rceil^{p_1}+\bar{f}_1(\xi_1)\right)+\sum_{j=2}^{k}\frac{\partial W_k}{\partial\xi_j}\left(\lceil\xi_{j+1}\rceil^{p_j}+\bar{f}_j(\bar{\xi}_j)\right)+\frac{1}{2}\sum_{i,j=1}^{k}\frac{\partial^2W_k}{\partial\xi_i\partial\xi_j}\bar{g}_i^{\mathrm{T}}(\bar{\xi}_i)\bar{g}_j(\bar{\xi}_j),
\end{aligned}\tag{B23}
$$

For $i,j=1,\ldots,k-1$, a simple calculation yields

$$
\frac{\partial W_k}{\partial\xi_j}=-\frac{4\mu-\varpi_k-r_k}{\mu}\frac{\partial\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}}{\partial\xi_j}\int_{\xi_k^*}^{\xi_k}\left|\lceil s\rceil^{\frac{\mu}{r_k}}-\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}\right|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\mathrm{d}s,\tag{B24}
$$

$$\frac{\partial W_k}{\partial \xi_k} = \lceil z_k \rceil^{\frac{4\mu-\varpi_k-r_k}{\mu}}, \tag{B25}$$

$$\frac{\partial^2 W_k}{\partial \xi_i \partial \xi_j} = -\frac{4\mu-\varpi_k-r_k}{\mu}\left(\frac{\partial^2 \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}}{\partial \xi_i \partial \xi_j}\right)\int_{\xi_k^*}^{\xi_k}\left|\lceil s \rceil^{\frac{\mu}{r_k}} - \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}\right|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}} \mathrm{d}s + \left(\frac{4\mu-\varpi_k-r_k}{\mu}\right)$$
$$\cdot \frac{\partial \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}}{\partial \xi_i}\frac{\partial \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}}{\partial \xi_j}\left(\frac{4\mu-\varpi_k-r_k-\mu}{\mu}\right)\int_{\xi_k^*}^{\xi_k}\left\lceil \lceil s \rceil^{\frac{\mu}{r_k}} - \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}\right\rceil^{\frac{4\mu-\varpi_k-r_k-2\mu}{\mu}} \mathrm{d}s, \tag{B26}$$

$$\frac{\partial^2 W_k}{\partial \xi_k \partial \xi_j} = \frac{4\mu-\varpi_k-r_k}{\mu}|z_k|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\frac{\partial z_k}{\partial \xi_j}, \tag{B27}$$

$$\frac{\partial^2 W_k}{\partial \xi_k^2} = \frac{4\mu-\varpi_k-r_k}{\mu}|z_k|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\frac{\partial z_k}{\partial \xi_k}. \tag{B28}$$

Since $\xi_k^*(\bar{\xi}_k) = -\phi_{k-1}^{\frac{r_k}{\mu}}(\bar{\xi}_{k-1})\lceil z_{k-1} \rceil^{\frac{r_k}{\mu}}$ and $z_{k-1} = \lceil \xi_{k-1} \rceil^{\frac{\mu}{r_{k-1}}} - \lceil \xi_{k-1}^* \rceil^{\frac{\mu}{r_{k-1}}}$, $k = 2, \ldots, n$, one gets

$$\begin{aligned}\lceil \xi_k^*(\bar{\xi}_k) \rceil^{\frac{\mu}{r_k}} &= -\phi_{k-1}\lceil \xi_{k-1} \rceil^{\frac{\mu}{r_{k-1}}} - \phi_{k-1}\phi_{k-2}z_{k-2}\\ &= -\phi_{k-1}\lceil \xi_{k-1} \rceil^{\frac{\mu}{r_{k-1}}} - \phi_{k-1}\phi_{k-2}\lceil \xi_{k-2} \rceil^{\frac{\mu}{r_{k-2}}} - \cdots - \phi_{k-1}\phi_{k-2}\cdots\phi_1\lceil \xi_1 \rceil^{\frac{\mu}{r_1}}.\end{aligned} \tag{B29}$$

Hence, we derive that

$$\begin{aligned}\frac{\partial \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}}{\partial \xi_j} = &-\frac{\partial \phi_{k-1}}{\partial \xi_j}\lceil \xi_{k-1} \rceil^{\frac{\mu}{r_{k-1}}} - \frac{\partial(\phi_{k-1}\phi_{k-2})}{\partial \xi_j}\lceil \xi_{k-2} \rceil^{\frac{\mu}{r_{k-2}}} - \frac{\partial(\phi_{k-1}\phi_{k-2}\cdots\phi_j)}{\partial \xi_j}\lceil \xi_j \rceil^{\frac{\mu}{r_j}}\\ &-\phi_{k-1}\phi_{k-2}\cdots\phi_j\frac{\mu}{r_j}|\xi_j|^{\frac{\mu-r_j}{r_j}} - \cdots - \frac{\partial(\phi_{k-1}\phi_{k-2}\cdots\phi_1)}{\partial \xi_j}\lceil \xi_1 \rceil^{\frac{\mu}{r_1}},\end{aligned} \tag{B30}$$

$$\begin{aligned}\frac{\partial^2 \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}}{\partial \xi_j^2} = &-\frac{\partial^2 \phi_{k-1}}{\partial \xi_j^2}\lceil \xi_{k-1} \rceil^{\frac{\mu}{r_{k-1}}} - \cdots - \phi_{k-1}\phi_{k-2}\cdots\phi_j\frac{\mu(\mu-r_j)}{r_j^2}\lceil \xi_j \rceil^{\frac{\mu-2r_j}{r_j}}\\ &-\frac{\partial(\phi_{k-1}\phi_{k-2}\cdots\phi_j)}{\partial \xi_j}\frac{\mu}{r_j}|\xi_j|^{\frac{\mu-r_j}{r_j}} - \cdots - \frac{\partial(\phi_{k-1}\phi_{k-2}\cdots\phi_1)}{\partial \xi_j^2}\lceil \xi_1 \rceil^{\frac{\mu}{r_1}}.\end{aligned} \tag{B31}$$

By the definition of μ, we know that the zero-division problem of $\frac{\partial^2 \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}}{\partial \xi_j^2}$ cannot occur, which indicates that $W_k(\bar{\xi}_k)$ is $\mathcal{C}^2$.

For $i, j = 1, \ldots, k-1$, by (B29) and Lemmas 5 and 6, there are known nonnegative smooth functions $\hat{\psi}_{k1}(\bar{\xi}_{k-1})$ and $\hat{\psi}_{k2}(\bar{\xi}_{k-1})$ such that

$$\begin{aligned}\left|\frac{\partial \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}}{\partial \xi_j}\right| = &\left| -\frac{\partial \phi_{k-1}}{\partial \xi_j}\lceil \xi_{k-1} \rceil^{\frac{\mu}{r_{k-1}}} - \frac{\partial(\phi_{k-1}\phi_{k-2})}{\partial \xi_j}\lceil \xi_{k-2} \rceil^{\frac{\mu}{r_{k-2}}} - \frac{\partial(\phi_{k-1}\phi_{k-2}\cdots\phi_j)}{\partial \xi_j}\lceil \xi_j \rceil^{\frac{\mu}{r_j}}\right.\\ &\left.-\phi_{k-1}\phi_{k-2}\cdots\phi_j\frac{\partial \lceil \xi_j \rceil^{\frac{\mu}{r_j}}}{\partial \xi_j} - \cdots - \frac{\partial(\phi_{k-1}\phi_{k-2}\cdots\phi_1)}{\partial \xi_j}\lceil \xi_1 \rceil^{\frac{\mu}{r_1}}\right|\\ \leqslant\ &\hat{\psi}_{k1}(\bar{\xi}_{k-1})\sum_{l=1}^{k-1}|z_l|^{\frac{\mu-r_j}{\mu}},\end{aligned} \tag{B32}$$

$$\left|\frac{\partial^2 \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}}{\partial \xi_i \partial \xi_j}\right| \leqslant \hat{\psi}_{k2}(\bar{\xi}_{k-1})\sum_{l=1}^{k-1}|z_l|^{\frac{\mu-r_j-r_i}{\mu}}. \tag{B33}$$

By (9), (B21), (B24), (B29), and Lemmas 1, 3, 5, 7, and 8, there is a nonnegative smooth function $\beta_{k1}(\bar{\xi}_k)$ such that

$$\begin{aligned}&\frac{\partial W_k}{\partial \xi_1}(D_1\lceil \xi_2 \rceil^{p_1} + \bar{f}_1(\xi_1)) + \sum_{j=2}^{k-1}\frac{\partial W_k}{\partial \xi_j}(\lceil \xi_{j+1} \rceil^{p_j} + \bar{f}_j(\bar{\xi}_j))\\ \leqslant &\left|\frac{4\mu-\varpi_k-r_k}{\mu}\frac{\partial \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}}{\partial \xi_j}\int_{\xi_k^*}^{\xi_k}\left|\lceil s \rceil^{\frac{\mu}{r_k}} - \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}\right|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\mathrm{d}s\right| \cdot \left|D_1\lceil \xi_2 \rceil^{p_1} + h_{11}(\xi_1)|\xi_1|^{\frac{r_1+\varpi_1}{r_1}}\right|\\ &+\sum_{j=2}^{k-1}\left|\frac{4\mu-\varpi_k-r_k}{\mu}\frac{\partial \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}}{\partial \xi_j}\int_{\xi_k^*}^{\xi_k}\left|\lceil s \rceil^{\frac{\mu}{r_k}} - \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}\right|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\mathrm{d}s\right| \cdot \left||\xi_{j+1}|^{p_j} + h_{j1}(\bar{\xi}_j)\sum_{l=1}^{j}|\xi_l|^{\frac{r_j+\varpi_j}{r_l}}\right|\\ \leqslant &\left|\frac{4\mu-\varpi_k-r_k}{\mu}\frac{\partial \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}}{\partial \xi_j}|\xi_k-\xi_k^*||z_k|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\right| \cdot \left|D_1|z_2+\phi_1 z_1|^{\frac{\varpi_1+r_1}{\mu}} + h_{11}(\xi_1)|\xi_1|^{\frac{r_1+\varpi_1}{r_1}}\right|\\ &+\sum_{j=2}^{k-1}\left|\frac{4\mu-\varpi_k-r_k}{\mu}\frac{\partial \lceil \xi_k^* \rceil^{\frac{\mu}{r_k}}}{\partial \xi_j}|\xi_k-\xi_k^*||z_k|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\right| \cdot \left||z_{j+1}+\phi_j z_j|^{\frac{\varpi_j+r_j}{\mu}} + h_{j1}(\bar{\xi}_j)\sum_{l=1}^{j}|\xi_l|^{\frac{r_j+\varpi_j}{r_l}}\right|\\ \leqslant &\frac{1}{7}\sum_{j=1}^{k-1}|z_j|^4 + \beta_{k1}(\bar{\xi}_k)|z_k|^4.\end{aligned} \tag{B34}$$

Similarly, it can be deduced that

$$\frac{\partial W_k}{\partial \xi_k}\bar{f}_k(\bar{\xi}_k) \leqslant |z_k|^{\frac{4\mu-\varpi_k-r_k}{\mu}}\left(h_{k1}(\bar{\xi}_k)\sum_{j=1}^{k}|\xi_j|^{\frac{r_k+\varpi_k}{r_j}}\right)$$

$$\leqslant \frac{1}{7}\sum_{j=1}^{k-1}|z_j|^4+\beta_{k2}(\bar{\xi}_k)|z_k|^4, \tag{B35}$$

where $\beta_{k2}(\bar{\xi}_k)$ is a nonnegative smooth function. By (10), (B21), (B27), and Lemmas 1, 3, 5, and 7,

$$\begin{aligned}\sum_{j=1}^{k-1}\frac{\partial^2 W_k}{\partial\xi_k\partial\xi_j}\bar{g}_k^{\mathrm{T}}(\bar{\xi}_k)\bar{g}_j(\bar{\xi}_j) &\leqslant \sum_{j=1}^{k-1}\left|\left(\frac{4\mu-\varpi_k-r_k}{\mu}\right)|z_k|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\frac{\partial z_k}{\partial\xi_j}\right|\left(\eta_{k1}(\bar{\xi}_k)\sum_{j=1}^{k}|z_j+\phi_{j-1}z_{j-1}|^{\frac{2r_k+\varpi_k}{2\mu}}\right)\\ &\quad\cdot\left(\eta_{j1}(\bar{\xi}_j)\sum_{l=1}^{j}|\xi_l|^{\frac{2r_j+\varpi_j}{2r_l}}\right)\\ &\leqslant \sum_{j=1}^{k-1}\left|\left(\frac{4\mu-\varpi_k-r_k}{\mu}\right)|z_k|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\frac{\partial z_k}{\partial\xi_j}\right|\bar{\eta}_{k1}(\bar{\xi}_k)\sum_{j=1}^{k}|z_j|^{\frac{2r_k+\varpi_k+2r_j+\varpi_j}{2\mu}}\\ &\leqslant \frac{1}{7}\sum_{j=1}^{k-1}|z_j|^4+\beta_{k3}(\bar{\xi}_k)|z_k|^4,\end{aligned} \tag{B36}$$

where $\bar{\eta}_{k1}(\bar{\xi}_k)$ and $\beta_{k3}(\bar{\xi}_k)$ are some nonnegative smooth functions. One can deduce from (10), (B21), (B28), and Lemmas 1, 3, 5, and 7 that

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 W_k}{\partial\xi_k^2}\bar{g}_k^{\mathrm{T}}(\bar{\xi}_k)\bar{g}_k(\bar{\xi}_k) &\leqslant \left|\left(\frac{4\mu-\varpi_k-r_k}{2\mu}\right)|z_k|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\frac{\partial z_k}{\partial\xi_k}\right|\left(\eta_{k1}(\bar{\xi}_k)\sum_{j=1}^{k}|\xi_j|^{\frac{2r_k+\varpi_k}{2r_j}}\right)^2\\ &\leqslant \left|\left(\frac{4\mu-\varpi_k-r_k}{2r_k}\right)|z_k|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}|\xi_k|^{\frac{\mu-r_k}{r_k}}\right|\left(\eta_{k1}(\bar{\xi}_k)\sum_{j=1}^{k}|z_j+\phi_{j-1}z_{j-1}|^{\frac{2r_k+\varpi_k}{2\mu}}\right)^2\\ &\leqslant \frac{1}{7}\sum_{j=1}^{k-1}|z_j|^4+\beta_{k4}(\bar{\xi}_k)|z_k|^4,\end{aligned} \tag{B37}$$

where $\beta_{k4}(\bar{\xi}_k)$ is a nonnegative smooth function. By (10), (B21), (B26), (B32), (B33), and Lemmas 1, 3, 5, 7, and 8, there is a nonnegative smooth function $\beta_{k5}(\bar{\xi}_k)$ such that

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{k-1}\frac{\partial^2 W_k}{\partial\xi_j^2}\bar{g}_j^{\mathrm{T}}(\bar{\xi}_j)\bar{g}_j(\bar{\xi}_j) &\leqslant \frac{1}{2}\sum_{j=1}^{k-1}\left|\int_{\xi_k^*}^{\xi_k}\left\lceil s\right\rceil^{\frac{\mu}{r_k}}-\left\lceil\xi_k^*\right\rceil^{\frac{\mu}{r_k}}\right|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\mathrm{d}s\left(\frac{\partial^2\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}}{\partial\xi_j^2}\right)\left(\frac{4\mu-\varpi_k-r_k}{\mu}\right)\\ &\quad+\left(\frac{\partial\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}}{\partial\xi_j}\right)^2\left(\frac{4\mu-\varpi_k-r_k-\mu}{\mu}\right)\int_{\xi_k^*}^{\xi_k}\left\lceil\lceil s\rceil^{\frac{\mu}{r_k}}-\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}\right\rceil^{\frac{4\mu-\varpi_k-r_k-2\mu}{\mu}}\mathrm{d}s\\ &\quad\cdot\left(\frac{4\mu-\varpi_k-r_k}{\mu}\right)\left|\left(\eta_{j1}(\bar{\xi}_j)\sum_{l=1}^{j}|\xi_l|^{\frac{2r_j+\varpi_j}{2r_l}}\right)^2\right.\\ &\leqslant \frac{1}{2}\sum_{j=1}^{k-1}\left||\xi_k-\xi_k^*||z_k|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\left(\frac{\partial^2\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}}{\partial\xi_j^2}\right)\left(\frac{4\mu-\varpi_k-r_k}{\mu}\right)\right.\\ &\quad+\left(\frac{\partial\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}}{\partial\xi_j}\right)^2\left(\frac{4\mu-\varpi_k-r_k-\mu}{\mu}\right)|\xi_k-\xi_k^*||z_k|^{\frac{4\mu-\varpi_k-r_k-2\mu}{\mu}}\\ &\quad\cdot\left(\frac{4\mu-\varpi_k-r_k}{\mu}\right)\left|\left(\eta_{j1}(\bar{\xi}_j)\sum_{l=1}^{j}|z_l+\phi_{l-1}z_{l-1}|^{\frac{2r_j+\varpi_j}{2\mu}}\right)^2\right.\\ &\leqslant \frac{1}{7}\sum_{j=1}^{k-1}|z_j|^4+\beta_{k5}(\bar{\xi}_k)|z_k|^4.\end{aligned} \tag{B38}$$

Similar to (B38), one can find a nonnegative smooth function $\beta_{k6}(\bar{\xi}_k)$ such that

$$\begin{aligned}&\sum_{l_1,l_2=1,l_1\neq l_2}^{k-1}\frac{\partial^2 W_k}{\partial\xi_{l_1}\partial\xi_{l_2}}\bar{g}_{l_1}(\bar{\xi}_{l_1})^{\mathrm{T}}\bar{g}_{l_2}(\bar{\xi}_{l_2})\\ &\leqslant \sum_{l_1,l_2=1,l_1\neq l_2}^{k-1}\left|\int_{\xi_k^*}^{\xi_k}\left\lceil s\right\rceil^{\frac{\mu}{r_k}}-\left\lceil\xi_k^*\right\rceil^{\frac{\mu}{r_k}}\right|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\mathrm{d}s\left(\frac{\partial^2\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}}{\partial\xi_{l_1}\partial\xi_{l_2}}\right)\left(\frac{4\mu-\varpi_k-r_k}{\mu}\right)\\ &\quad+\left(\frac{\partial\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}}{\partial\xi_{l_1}}\right)\left(\frac{\partial\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}}{\partial\xi_{l_2}}\right)\left(\frac{4\mu-\varpi_k-r_k-\mu}{\mu}\right)\int_{\xi_k^*}^{\xi_k}\left\lceil\lceil s\rceil^{\frac{\mu}{r_k}}-\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}\right\rceil^{\frac{4\mu-\varpi_k-r_k-2\mu}{\mu}}\mathrm{d}s\\ &\quad\cdot\left(\frac{4\mu-\varpi_k-r_k}{\mu}\right)\left|\left(\eta_{l_11}(\bar{\xi}_{l_1})\sum_{j=1}^{l_1}|\xi_j|^{\frac{2r_{l_1}+\varpi_{l_1}}{2r_j}}\right)\left(\eta_{l_21}(\bar{\xi}_{l_2})\sum_{l=1}^{l_2}|\xi_l|^{\frac{2r_{l_2}+\varpi_{l_2}}{2r_l}}\right)\right.\\ &\leqslant \sum_{l_1,l_2=1,l_1\neq l_2}^{k-1}\left||\xi_k-\xi_k^*||z_k|^{\frac{4\mu-\varpi_k-r_k-\mu}{\mu}}\left(\frac{\partial^2\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}}{\partial\xi_{l_1}\partial\xi_{l_2}}\right)\left(\frac{4\mu-\varpi_k-r_k}{\mu}\right)\right.\\ &\quad+\left(\frac{\partial\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}}{\partial\xi_{l_1}}\right)\left(\frac{\partial\lceil\xi_k^*\rceil^{\frac{\mu}{r_k}}}{\partial\xi_{l_2}}\right)\left(\frac{4\mu-\varpi_k-r_k-\mu}{\mu}\right)|\xi_k-\xi_k^*||z_k|^{\frac{4\mu-\varpi_k-r_k-2\mu}{\mu}}\end{aligned}$$

$$\cdot\left(\frac{4\mu-\varpi_k-r_k}{\mu}\right)\Bigg|\left(\eta_{l_1 1}(\bar{\xi}_{l_1})\sum_{j=1}^{l_1}|z_j+\phi_{j-1}z_{j-1}|^{\frac{2r_{l_1}+\varpi_{l_1}}{2\mu}}\right)\left(\eta_{l_2 1}(\bar{\xi}_{l_2})\sum_{l=1}^{l_2}|z_l+\phi_{l-1}z_{l-1}|^{\frac{2r_{l_2}+\varpi_{l_2}}{2\mu}}\right)$$
$$\leqslant\frac{1}{7}\sum_{j=1}^{k-1}|z_j|^4+\beta_{k6}(\bar{\xi}_k)|z_k|^4. \tag{B39}$$

By Lemmas 1, 5, and 7, it is not hard to obtain

$$\begin{aligned}\lceil z_{k-1}\rceil^{\frac{4\mu-\varpi_{k-1}-r_{k-1}}{\mu}}\left(\lceil\xi_k\rceil^{p_{k-1}}-\lceil\xi_k^*\rceil^{p_{k-1}}\right) &\leqslant 2^{1-\frac{\varpi_{k-1}+r_{k-1}}{\mu}}|z_{k-1}|^{\frac{4\mu-\varpi_{k-1}-r_{k-1}}{\mu}}|z_k|^{\frac{\varpi_{k-1}+r_{k-1}}{\mu}}\\ &\leqslant\frac{1}{7}\sum_{j=1}^{k-1}|z_j|^4+\beta_{k7}(\bar{\xi}_k)|z_k|^4,\end{aligned} \tag{B40}$$

where $\beta_{k7}(\bar{\xi}_k)$ is a nonnegative smooth function. Substituting (B34)–(B40) into (B23) yields

$$\begin{aligned}\mathcal{L}V_k \leqslant& -\left(n-k+\frac{1}{4}\right)\sum_{j=1}^{k-1}|z_j|^4+\lceil z_k\rceil^{\frac{4\mu-r_k-\varpi_k}{\mu}}\left(\lceil\xi_{k+1}\rceil^{p_k}-\lceil\xi_{k+1}^*\rceil^{p_k}+\lceil\xi_{k+1}^*\rceil^{p_k}\right)\\ &+\beta_k(\bar{\xi}_k)|z_k|^4-\frac{1}{4}\sum_{j=1}^{k-1}|z_j|^{\frac{16}{3}},\end{aligned} \tag{B41}$$

where $\beta_k(\bar{\xi}_k)=\sum_{j=1}^{7}\beta_{kj}(\bar{\xi}_k)$. Substituting the virtual controller

$$\xi_{k+1}^*=-\left(n-k+\frac{1}{4}+\frac{1}{4}z_k^{\frac{4}{3}}+\beta_k(\bar{\xi}_k)\right)^{\frac{1}{p_k}}\lceil z_k\rceil^{\frac{r_{k+1}}{\mu}}\triangleq-\phi_k^{\frac{r_{k+1}}{\mu}}(\bar{\xi}_k)\lceil z_k\rceil^{\frac{r_{k+1}}{\mu}} \tag{B42}$$

into (B41) leads to

$$\mathcal{L}V_k\leqslant-\sum_{j=1}^{k}\left(n-k+\frac{1}{4}\right)|z_j|^4-\sum_{j=1}^{k}\frac{1}{4}|z_j|^{\frac{16}{3}}+\lceil z_k\rceil^{\frac{4\mu-r_k-\varpi_k}{\mu}}\left(\lceil\xi_{k+1}\rceil^{p_k}-\lceil\xi_{k+1}^*\rceil^{p_k}\right). \tag{B43}$$

Hence, Eq. (B22) still holds at Step k.

At the last step, setting $V_n(\xi)=V_{n-1}(\bar{\xi}_{n-1})+W_n(\xi)$, Eq. (12) leads to (13).