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Output feedback stabilization of stochastic high-order planar nonlinear systems with stochastic inverse dynamics and output-constraint

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Abstract In this paper, we solve the output feedback control problem of stochastic high-order planar nonlinear systems with output constraint and stochastic integral input-to-state stability (SiISS) inverse dynamics. By employing a key coordinate transformation, a stochastic nonlinear system with output constraint and SiISS inverse dynamics is converted into an unconstrained system. By skillfully constructing an observer and adopting SiISS small-gain conditions, we develop a new output feedback control design and analysis method, and prove that all the closed-system signals are bounded almost surely, the output constraint is not violated almost surely, and the equilibrium point of the closed-loop system is stochastically asymptotically stable.

Keywords stochastic high-order planar nonlinear system, output constraint, stochastic inverse dynamics, output feedback stabilization, stochastic integral input-to-state stability (SiISS)

1 Introduction

Due to the intrinsic nonlinearity of high-order nonlinear systems and the potential nonexistence of the Jacobian linearization at the origin, the stabilization of high-order nonlinear systems has been recognized as a challenging problem. Fortunately, with the help of the adding a power integrator technique, these difficulties were successfully overcome by [1]. Subsequently, this method was applied to stochastic nonlinear systems and produced many interesting results. Regarding state feedback control design, the adaptive state feedback stabilization of stochastic high-order systems with nonlinear parameterization was first addressed in [2]. Subsequently, some issues, such as inverse optimal stabilization [3], adaptive control [4, 5], control of stochastic systems with a time-varying delay [6–8], and the stabilization problem for stochastic high-order switched nonlinear systems [9], were further addressed. Regarding output feedback control design, stochastic high-order nonlinear systems were first considered in [10]. Then, in [11,12], the output feedback stabilization was achieved under some weak assumptions on high-order nonlinearities as well as drift and diffusion terms. In [13], the output tracking problem for a benchmark stochastic high-order mechanical system was investigated. In [14], stochastic high-order nonlinear systems were further investigated using homogeneous domination and the sign function.

A more desirable objective than the conventional stabilization task is the stabilization of stochastic nonlinear systems with the pre-specified output constraint. This is necessary for the system operation security and performance specifications. By incorporating the barrier Lyapunov function (BLF) [15] into control design, the finite-time stabilization and the adaptive control problem for stochastic high-order nonlinear systems with output constraint were investigated in [16–18], respectively. Then, in [19], these results were extended to the systems with full-state constraints. These control methods, which were reported in [16–19], essentially depend on the information about the state of the entire system. In many practical applications, because of the difficulty of measuring all states, developing some new methods based on output feedback design is an interesting topic. In [20, 21], two explicit stabilizer design

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schemes were presented by merging a tan-type BLF into an output feedback control design. However, the radial unboundedness of the Lyapunov function constructed in [20, 21] cannot be ensured, and then the stochastic stability theorem cannot directly be applied to stability analysis. Recently, in [22], a system transformation method in [23] was developed and successfully solved this problem in stochastic high-order planar nonlinear systems. However, its obvious drawback is that stochastic inverse dynamics are neglected.

Unmodeled dynamics, which are one of the main causes of system instability, exist in many practical systems. To deal with these dynamics, input-to-state stability (ISS) in [24] and integral input-to-state stability (iISS) in [25] are regarded as the two most representative tools. Refs. [26, 27] extended ISS to stochastic systems and proposed the definition of stochastic input-to-state stability (SISS). Using this definition, a sufficient condition for SISS based on the Lyapunov function was formulated in [28]. By applying this condition, state feedback stabilization, global practical tracking control, and output feedback control for a stochastic system with SISS inverse dynamics, were studied in [29–31], respectively. Stochastic integral input-to-state stability (SiISS), a weaker concept than SISS, was proposed in [32] and refined in [33], in which the authors established two SiISS small-gain conditions. By applying the results obtained in [32, 33], some feedback control problems of stochastic systems with SiISS were studied in [34–37]. However, all these results are not applicable if an output constraint exists in the system, which provides great motivation for our research goal.

Due to the existence of stochastic noise, inherent nonlinearities of stochastic nonlinear systems, and invalidity of BLFs, the design of an output feedback controller and analysis methods to deal with the output constraint and stochastic inverse dynamics is a challenging and interesting problem. The objective of this paper is to design an output feedback controller for stochastic high-order planar nonlinear systems with output constraint and SiISS inverse dynamics. Our main contributions are summarized as follows:

(i) Our work is not an easy generalization of [22]. In fact, even without considering stochastic inverse dynamics, the results in our work are also new and more general than those in [22] since the powers of the studied system are greater or equal to one rather than some ratios of two odd numbers.

(ii) Based on a new state transformation function, the output-constrained system can be converted into an unconstrained form. For this unconstrained system, by using the adding a power integrator technique and introducing a delicate manipulation of sign function, an innovative state feedback controller is designed without employing some commonly-used BLFs. To implement the output feedback, a reduced-order nonsmooth observer equipped with a nonlinear gain function is constructed accordingly by taking into consideration the inherent nonlinearities of the system. Then, by combining the state feedback controller with this observer, an output feedback controller is constructed to ensure that all the closed-loop system signals are bounded almost surely, the output constraint is not violated almost surely, and the equilibrium point of the closed-loop system is stochastically asymptotically stable.

Notations. $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^n$ represent the set of all real numbers, the set of all nonnegative real numbers, and the real n-dimensional space, respectively. For any x, $|x|$ represents its norm. $\mathcal{C}^i$ is the set of functions with the ith continuous partial derivatives. $a \wedge b$ is the minimum of a and b. $\mathcal{K}$ is the set of all functions that are strictly increasing, continuous and vanishes at the origin. $\mathcal{K}_\infty$ is the set of all functions that are of class $\mathcal{K}$ and unbounded. For simplicity, a function $f(x(t))$ is usually defined as $f(x)$ or f. $\mathrm{Tr}\{A\}$ denotes the trace of matrix A. $\lceil\cdot\rfloor^\varrho = \mathrm{sgn}(\cdot)|\cdot|^\varrho$, ϱ is a positive constant, and $\mathrm{sgn}(\cdot)$ represents sign function.

2 Preliminaries and problem statement

2.1 Preliminaries

Consider the following stochastic nonlinear system:

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + g^{\mathrm{T}}(x(t))\mathrm{d}\omega, \ \forall t \geqslant 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state with the initial value $x(0) = x_0$, ω is an m-dimensional standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $g(x) : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ are local Lipschitz with $f(0) = 0$ and $g(0) = 0$. For any given $V(x(t)) \in \mathcal{C}^2$, define the differential operator

$$\mathcal{L}V = \frac{\partial V}{\partial x} + \frac{1}{2}\mathrm{Tr}\left\{g\frac{\partial^2 V}{\partial x^2}g^{\mathrm{T}}\right\}. \tag{2}$$

Definition 1 ([38]). The equilibrium $x=0$ of system (1) is (i) stochastically globally stable, if for any $\varepsilon>0$, there exists a class of $\mathcal{K}$ function $\lambda(\cdot)$ such that $P\{|x(t)|<\lambda(|x_0|)\}\geqslant 1-\varepsilon$, $\forall t\geqslant 0$, $x_0\in\mathbb{R}^n\backslash\{0\}$; or (ii) stochastically asymptotically stable, if it is stochastically globally stable and $P\{\lim_{t\to\infty}|x(t)|=0\}=1$, $x_0\in\mathbb{R}^n$.

Lemma 1 ([39]). Let $q_1>0$, $q_2>0$, and $a>0$ be real numbers. For any $x,y\in\mathbb{R}$, there holds $|x|^{q_1}|y|^{q_2}\leqslant (q_1/(q_1+q_2))a|x|^{q_1+q_2}+(q_2/(q_1+q_2))a^{-\frac{q_1}{q_2}}|y|^{q_1+q_2}$.

Lemma 2 ([40]). For a given continuous function $p(x,y)$ with $x\in\mathbb{R}^m$, $y\in\mathbb{R}^n$, there exist smooth functions $c(x)\geqslant 0$, $d(y)\geqslant 0$, $u(x)\geqslant 1$, and $v(y)\geqslant 1$ such that $|p(x,y)|\leqslant c(x)+d(y)$, $|p(x,y)|\leqslant u(x)v(y)$.

Lemma 3 ([14]). $p(x)=\lceil x\rceil^c$, $c\geqslant 2$ is a $\mathcal{C}^2$ function for $x\in\mathbb{R}$ and $\dot p(x)=c|x|^{c-1}$, $\ddot p(x)=c(c-1)\lceil x\rceil^{c-2}$.

Lemma 4 ([41]). Let $l=\frac{a}{b}\geqslant 1$ with $n\geqslant 1$ and $s>0$ be real values. For any $x,y\in\mathbb{R}$, $|\lceil x\rceil^l-\lceil y\rceil^l|\leqslant l(1+2^{l-2})(|x-y|^l+|x-y||y|^{l-1})$, $|\lceil x\rceil^{\frac{1}{l}}-\lceil y\rceil^{\frac{1}{l}}|\leqslant 2^{1-\frac{1}{l}}|x-y|^{\frac{1}{l}}$, $|x^l-y^l|\leqslant 2^{1-\frac{1}{b}}|\lceil x\rceil^a-\lceil y\rceil^a|^{\frac{1}{b}}$, $-\lceil x-y\rceil^s(\lceil x\rceil^l-\lceil y\rceil^l)\leqslant -2^{1-l}|x-y|^{l+s}$.

Lemma 5 ([41]). For a given continuous monotone function $p(x):[a,b]\to\mathbb{R}$ with $p(a)=0$, there holds $\int_a^b p(s)\mathrm{d}s\leqslant |p(b)|\cdot|b-a|$.

Lemma 6 ([39]). Let $l\geqslant 1$ be real numbers. For any $x_i\in\mathbb{R}$, $i=1,\ldots,n$, there holds $(\sum_{i=1}^n|x_i|)^l\leqslant b\sum_{i=1}^n|x_i|^l$, where $b=n^{l-1}$ if $l\geqslant 1$ and $b=1$ if $l<1$.

Lemma 7 ([38]). For the stochastic system (1), if there exists a $\mathcal{C}^2$ function $V(x)$, a nonnegative function $W(x)$, class $\mathcal{K}_\infty$ functions α_1 and α_2, constants $c_1>0$ and $c_2\geqslant 0$ such that $\alpha_1(|x|)\leqslant V(x)\leqslant\alpha_2(|x|)$, $\mathcal{L}V(x)\leqslant -c_1W(x)+c_2$. Then (i) there exists an almost surely unique solution on $[0,\infty)$. (ii) When $c_2=0$ and $W(x)$ is a continuous positive definite function, the equilibrium point $x=0$ is stochastically asymptotically stable.

2.2 Problem statement

This paper considers a class of stochastic high-order planar nonlinear systems,

$$\begin{cases}\mathrm{d}z_0=f_0(z_0,x_1)\mathrm{d}t+g_0^{\mathrm{T}}(z_0,x_1)\mathrm{d}\omega,\\ \mathrm{d}x_1=(\lceil x_2\rceil^{p_1}+f_1(z_0,x_1))\mathrm{d}t+g_1^{\mathrm{T}}(z_0,x_1)\mathrm{d}\omega,\\ \mathrm{d}x_2=(\lceil u\rceil^{p_2}+f_2(z_0,x))\mathrm{d}t+g_2^{\mathrm{T}}(z_0,x)\mathrm{d}\omega,\\ y=x_1,\end{cases}\tag{3}$$

with a symmetric output constraint

$$y(t)\in\Omega_y=\{y(t)\in\mathbb{R}:-k_{a_1}<y(t)<k_{a_1}\},\tag{4}$$

where $x=\bar x_2=[x_1,x_2]^{\mathrm{T}}\in\mathbb{R}^2$ is the system state with the initial value $x(0)=x_0$, $z_0=(z_1,\ldots,z_l)^{\mathrm{T}}\in\mathbb{R}^l$ is stochastic inverse dynamics with the initial value $z_0(0)=z_{01}$, $u(t)\in\mathbb{R}$ and $y(t)\in\mathbb{R}$ are control input and the measurable output, and $x_2(t)\in\mathbb{R}$ is unmeasurable. System powers $p_i\geqslant 1$, $i=1,2$, are two positive values. ω is an m-dimensional standard Wiener process defined on a complete probability space $(\Omega,\mathcal{F},P)$. For $i=1,2$, $f_i:\mathbb{R}^l\times\mathbb{R}^i\to\mathbb{R}$ and $g_i:\mathbb{R}^l\times\mathbb{R}^i\to\mathbb{R}^{m\times 1}$ are continuously differentiable functions with $f_i(0,0)=0$ and $g_i(0,0)=0$, and $f_0:\mathbb{R}^l\times\mathbb{R}\to\mathbb{R}^l$ and $g_0:\mathbb{R}^l\times\mathbb{R}\to\mathbb{R}^{m\times l}$ are locally Lipschitz continuous functions with $f_0(0,0)=0$ and $g_0(0,0)=0$. k_{a_1} is a given positive constant.

The objective of this paper is to design an output feedback controller for the system (3) such that all the closed-system signals are bounded almost surely, symmetric output constraint (4) is not violated almost surely, and the equilibrium point of the closed-loop system is stochastically asymptotically stable.

To achieve this objective, we need some assumptions.

Assumption 1. For $i=1,2$ and $g_1(z_0,x_1)\equiv 0$, there exist a nonnegative constant ϖ and some nonnegative smooth functions β_{1i}, β_{2i}, β_{23}, and β_{24} such that

$$|f_i(z_0,\bar x_i)|\leqslant\beta_{1i}(|z_0|)|z_0|^{r_i+\varpi}+\beta_{2i}(x_1)\sum_{j=1}^{i}|x_j|^{\frac{r_i+\varpi}{r_j}},\tag{5}$$

$$|g_2(z_0,x)\leqslant\beta_{23}(|z_0|)|z_0|^{\frac{2r_2+\varpi}{2}}+\beta_{24}(x_1)\sum_{j=1}^{2}|x_j|^{\frac{2r_2+\varpi}{2r_j}},\tag{6}$$

where $r_1 = 1$, $r_2 = \frac{r_1+\varpi}{p_1}$. Meanwhile, one of the following conditions should be satisfied:

$$\text{(i) } \frac{r_1}{r_2} = 1, \text{ (ii) } r_2 + \varpi \geqslant r_1 \text{ and } \frac{r_1}{r_2} \geqslant 2. \tag{7}$$

Assumption 2. The z_0-subsystem in system (3) is SiISS with x_1 being the input, along with a Lyapunov function $V_0(z_0) \in \mathcal{C}^2$ such that

$$\alpha_1(|z_0|) \leqslant V_0(z_0) \leqslant \alpha_2(|z_0|), \ \mathcal{L}V_0(z_0) \leqslant -\alpha_3(|z_0|) + \gamma(|x_1|), \tag{8}$$

where α_1, α_2, and γ are $\mathcal{K}_\infty$ functions and α_3 is a positive definite continuous function.

Remark 1. As shown in Assumption 1, the selection of ϖ makes the values of $\frac{r_2+\varpi}{r_j}$ and $\frac{2r_2+\varpi}{2r_j}$ to be in a certain interval $[\frac{1}{p_{i-1}}, 0)$ with $i, j = 1, 2$, and $\frac{1}{p_0} = 1$. Though Refs. [32, 36, 37] considered output feedback control for stochastic nonlinear systems with SiISS inverse dynamics, their assumptions on f_i and g_i, $i = 1, \ldots, n$, just depend on z_0 and x_1. More importantly, output constraint is also ignored in these studies. When $z_0 = 0$, Assumption 1 reduces to Assumption 1 in [22].

Assumption 2 implies that z_0-subsystem is characterized by SiISS. Compared with SISS, SiISS is a weaker condition since α_3 is only positive definite continuous instead of $\mathcal{K}_\infty$.

Remark 2. Compared with the existing result [22] on output feedback control for stochastic nonlinear systems with output constraint, system (3) is more general since the powers in this work only need some values to be greater than one rather than some fractional powers with both odd numerators and denominators. What's more, SiISS inverse dynamics are also not included in [22].

3 Main results

3.1 System transformation

Inspired by [22, 23], we first introduce an equivalent coordinate transformation

$$z_0(t) = z_0(t), \ \xi_1(t) = T(x_1(t)) = \tan\left(\frac{x_1(t)}{K_1}\right), \ \xi_2(t) = x_2(t), \tag{9}$$

where $K_1 = \frac{2k_{a_1}}{\pi}$. By the properties of tangent function, we have

$$x_1(t) = T^{-1}(\xi_1(t)) = K_1 \arctan(\xi_1(t)). \tag{10}$$

It is obvious that $x_1(t)$ is strictly increasing and smooth with respect to $\xi_1(t)$, and the following properties are obtained:

$$\begin{cases} x_1(t) \to -k_{a_1}, \ \text{when } \xi_1(t) \to -\infty; \\ x_1(t) \to k_{a_1}, \ \text{when } \xi_1(t) \to \infty. \end{cases} \tag{11}$$

Namely, for given initial state $y(0) = x_1(0) \in \Omega_y$, the symmetric output constraint (4) is not violated almost surely provided that $\xi_1(t)$ is bounded almost surely. From (3), (9), and Itô's formula, the constrained system (3) is converted into an unconstrained form:

$$\begin{cases} \mathrm{d}z_0 = \bar{f}_0(z_0, \xi_1)\mathrm{d}t + \bar{g}_0^{\mathrm{T}}(z_0, \xi_1)\mathrm{d}\omega, \\ \mathrm{d}\xi_1 = \left(H_{\xi_1}\lceil \xi_2 \rfloor^{p_1} + \bar{f}_1(z_0, \xi_1)\right)\mathrm{d}t + \bar{g}_1^{\mathrm{T}}(z_0, \xi_1)\mathrm{d}\omega, \\ \mathrm{d}\xi_2 = \left(\lceil u \rfloor^{p_2} + \bar{f}_2(z_0, \xi)\right)\mathrm{d}t + \bar{g}_2^{\mathrm{T}}(z_0, \xi)\mathrm{d}\omega, \end{cases} \tag{12}$$

where $\xi = (\xi_1, \xi_2)^{\mathrm{T}}$, $\bar{f}_0(z_0, \xi_1) = f_0(z_0, \xi_1)$, $\bar{g}_0(z_0, \xi_1) = g_0(z_0, \xi_1)$, $H_{\xi_1}(\xi_1) = \frac{1+\xi_1^2}{K_1}$, $\bar{f}_1(z_0, \xi_1) = H_{\xi_1} f_1(z_0, \xi_1)$, $\bar{g}_1^{\mathrm{T}}(z_0, \xi_1) = 0$, $\bar{f}_2(z_0, \xi) = f_2(z_0, \xi)$, and $\bar{g}_2^{\mathrm{T}}(z_0, \xi) = g_2(z_0, \xi)$.

3.2 State-feedback controller design

Before constructing a state-feedback controller for the system (12), we choose a constant $\lambda \geqslant 1$ such that $\max_{1\leqslant i\leqslant 2}\{\frac{r_i+\varpi}{\lambda}\} \leqslant r_1$ and set $\tau = r_1$.

Step 1. Set $z_1 = \lceil \xi_1 \rceil^{\frac{\tau}{r_1}}$, $\xi_1^* = 0$ and consider the Lyapunov function

$$V_1(\xi_1) = W_1(\xi_1) = \int_{\xi_1^*}^{\xi_1} \left\lceil \lceil s \rceil^{\frac{\tau}{r_1}} - \lceil \xi_1^* \rceil^{\frac{\tau}{r_1}} \right\rceil^{\frac{4\tau\lambda-\varpi-r_1}{\tau}} \mathrm{d}s. \tag{13}$$

By (2), (12), (13), and Itô's formula,

$$\mathcal{L}V_1 = H_{\xi_1} \lceil z_1 \rceil^{\frac{4\tau\lambda-\varpi-r_1}{\tau}} (\lceil \xi_2 \rceil^{p_1} - \lceil \xi_2^* \rceil^{p_1}) + \frac{\partial V_1}{\partial \xi_1} \bar{f}_1 + H_{\xi_1} \lceil z_1 \rceil^{\frac{4\tau\lambda-\varpi-r_1}{\tau}} \lceil \xi_2^* \rceil^{p_1}. \tag{14}$$

From the definition of tangent function, it follows that

$$|\arctan x| \leqslant |x|, \ \forall x \in \mathbb{R}. \tag{15}$$

Taking this into consideration, and using (10) and Assumption 2, we have

$$\mathcal{L}V_0 \leqslant -\alpha_3(|z_0|) + \gamma(K_1|\arctan \xi_1|) \leqslant -\alpha_3(|z_0|) + \tilde{\gamma}(|\xi_1|), \tag{16}$$

where $\tilde{\gamma}(|\xi_1|) = \gamma(K_1|\xi_1|)$. By (5), (10), (13), (15), and Lemmas 1 and 2,

$$\begin{aligned} \frac{\partial V_1}{\partial \xi_1} \bar{f}_1 &\leqslant H_{\xi_1} |z_1|^{\frac{4\tau\lambda-\varpi-r_1}{\tau}} \left(\beta_{11} |z_0|^{r_1+\varpi} + \beta_{21} |x_1|^{\frac{r_1+\varpi}{r_1}} \right) \\ &\leqslant H_{\xi_1} |z_1|^{\frac{4\tau\lambda-\varpi-r_1}{\tau}} \beta_{11} |z_0|^{r_1+\varpi} + H_{\xi_1} |z_1|^{\frac{4\tau\lambda-\varpi-r_1}{\tau}} \beta_{21} K_1^{\frac{r_1+\varpi}{r_1}} |\xi_1|^{\frac{r_1+\varpi}{r_1}} \\ &\leqslant a_{11}(\xi_1) |z_1|^{4\lambda} + \psi_1(|z_0|) |z_0|^{4\tau\lambda}, \end{aligned} \tag{17}$$

where a_{11} and ψ_1 are some nonnegative smooth functions. Choose the virtual controller

$$\xi_2^*(\xi_1) = -\left(\frac{a_{11}(\xi_1) + 2 + \nu_1(\xi_1)}{H_1} \right)^{\frac{1}{p_1}} \lceil z_1 \rceil^{\frac{r_2}{\tau}} \triangleq -\sigma_1(\xi_1) \lceil z_1 \rceil^{\frac{r_2}{\tau}}, \tag{18}$$

where $H_1 \triangleq \frac{1}{K_1} \leqslant \frac{1+\xi_1^2}{K_1} = H_{\xi_1}$ and ν_1 is a nonnegative smooth function to be determined. Substituting (17) and (18) into (14) and considering (16), we get

$$\begin{aligned} \mathcal{L}(V_1 + V_0) \leqslant\ & -2|z_1|^{4\lambda} - \nu_1(\xi_1)|z_1|^{4\lambda} + \psi_1(|z_0|)|z_0|^{4\tau\lambda} - \alpha_3(|z_0|) \\ & + \tilde{\gamma}(|\xi_1|) + H_{\xi_1} \lceil z_1 \rceil^{\frac{4\tau\lambda-\varpi-r_1}{\tau}} (\lceil \xi_2 \rceil^{p_1} - \lceil \xi_2^* \rceil^{p_1}). \end{aligned} \tag{19}$$

Step 2. Set

$$W_2(\xi) = \int_{\xi_2^*}^{\xi_2} \left\lceil \lceil s \rceil^{\frac{\tau}{r_2}} - \lceil \xi_2^* \rceil^{\frac{\tau}{r_2}} \right\rceil^{\frac{4\tau\lambda-\varpi-r_2}{\tau}} \mathrm{d}s, \tag{20}$$

and define $V_2(z_0, \xi) = V_0(z_0) + V_1(\xi_1) + W_2(\xi)$ and $z_2 = \lceil \xi_2 \rceil^{\frac{\tau}{r_2}} - \lceil \xi_2^* \rceil^{\frac{\tau}{r_2}}$. By (2), (12), (19), (20), and Itô's formula, then

$$\begin{aligned} \mathcal{L}V_2 \leqslant\ & -2|z_1|^{4\lambda} - \nu_1(\xi_1)|z_1|^{4\lambda} + \psi_1(|z_0|)|z_0|^{4\tau\lambda} + H_{\xi_1} \lceil z_1 \rceil^{\frac{4\tau\lambda-\varpi-r_1}{\tau}} (\lceil \xi_2 \rceil^{p_1} - \lceil \xi_2^* \rceil^{p_1}) - \alpha_3(|z_0|) \\ & + \tilde{\gamma}(|\xi_1|) + \frac{\partial W_2}{\partial \xi_1} (H_{\xi_1} \lceil \xi_2 \rceil^{p_1} + \bar{f}_1) + \frac{\partial W_2}{\partial \xi_2} (\lceil u \rceil^{p_2} + \bar{f}_2) + \frac{1}{2} \frac{\partial^2 W_2}{\partial \xi_2^2} \bar{g}_2^{\mathrm{T}} \bar{g}_2. \end{aligned} \tag{21}$$

By the definition of τ and λ, it is easy to get $\frac{4\tau\lambda-\varpi-r_2}{\tau} > 2$. By Lemma 3, there holds

$$\frac{\partial W_2}{\partial \xi_1} = -\int_{\xi_2^*}^{\xi_2} \left| \lceil s \rceil^{\frac{\tau}{r_2}} - \lceil \xi_2^* \rceil^{\frac{\tau}{r_2}} \right|^{\frac{4\tau\lambda-\varpi-r_2}{\tau}-1} \mathrm{d}s \cdot \left(\frac{4\tau\lambda - \varpi - r_2}{\tau} \right) \frac{\partial \lceil \xi_2^* \rceil^{\frac{\tau}{r_2}}}{\partial \xi_1}, \tag{22}$$

$$\frac{\partial W_2}{\partial \xi_2} = \lceil z_2 \rceil^{\frac{4\tau\lambda-\varpi-r_2}{\tau}}, \tag{23}$$

$$\frac{\partial^2 W_2}{\partial \xi_2^2} = \left(\frac{4\tau\lambda - \varpi - r_2}{\tau}\right)|z_2|^{\frac{4\tau\lambda-\varpi-r_2}{\tau}-1}\frac{\partial z_2}{\partial \xi_2}. \tag{24}$$

It follows from (5), (10), (15), (18), (22), and Lemmas 1, 2, and 4–6 that

$$\begin{aligned}
\frac{\partial W_2}{\partial \xi_1}\left(H_{\xi_1}\lceil \xi_2\rceil^{p_1} + \bar{f}_1\right) \leqslant\ & b_{11}(\xi_1)\left|\int_{\xi_2^*}^{\xi_2}\left|\lceil s\rceil^{\frac{\tau}{r_2}} - \lceil \xi_2^*\rceil^{\frac{\tau}{r_2}}\right|^{\frac{4\tau\lambda-\varpi-r_2}{\tau}-1}\mathrm{d}s\right|\left|\frac{\partial \lceil \xi_2^*\rceil^{\frac{\tau}{r_2}}}{\partial \xi_1}\right| \\
& \cdot\left(\beta_{11}|z_0|^{r_1+\varpi} + K_1^{\frac{r_1+\varpi}{r_1}}\beta_{21}|z_1|^{\frac{r_1+\varpi}{\tau}} + |\xi_2|^{p_1}\right) \\
\leqslant\ & b_{11}(\xi_1)\left|\int_{\xi_2^*}^{\xi_2}\left|\lceil s\rceil^{\frac{\tau}{r_2}} - \lceil \xi_2^*\rceil^{\frac{\tau}{r_2}}\right|^{\frac{4\tau\lambda-\varpi-r_2}{\tau}-1}\mathrm{d}s\right|\left|\frac{\partial \lceil \xi_2^*\rceil^{\frac{\tau}{r_2}}}{\partial \xi_1}\right| \\
& \cdot\left(\varrho_{11}(|z_0|)|z_0|^{r_1+\varpi} + \varrho_{12}(\xi_1)|z_1|^{\frac{r_1+\varpi}{\tau}} + \varrho_{13}(\xi_1)|z_2|^{\frac{r_1+\varpi}{\tau}}\right) \\
\leqslant\ & \frac{1}{4}|z_1|^{4\lambda} + a_{21}(\xi_1)|z_2|^{4\lambda} + \psi_{21}(|z_0|)|z_0|^{4\tau\lambda},
\end{aligned} \tag{25}$$

where b_{11}, a_{21}, ψ_{21}, ϱ_{11}, ϱ_{12}, and ϱ_{13} are some nonnegative smooth functions. By (5), (9), (10), (15), (18), (23), and Lemmas 1, 2, and 6,

$$\begin{aligned}
\frac{\partial W_2}{\partial \xi_2}\bar{f}_2 \leqslant\ & |z_2|^{\frac{4\tau\lambda-\varpi-r_2}{\tau}}\left(\beta_{12}|z_0|^{r_2+\varpi} + K_1^{\frac{r_2+\varpi}{r_1}}\beta_{22}|z_1|^{\frac{r_2+\varpi}{\tau}} + \beta_{22}|\xi_2|^{\frac{r_2+\varpi}{r_2}}\right) \\
\leqslant\ & |z_2|^{\frac{4\tau\lambda-\varpi-r_2}{\tau}}\left(\varrho_{21}(|z_0|)|z_0|^{r_2+\varpi} + \varrho_{22}(\xi_1)|z_1|^{\frac{r_2+\varpi}{\tau}} + \varrho_{23}(\xi_1)|z_2|^{\frac{r_2+\varpi}{\tau}}\right) \\
\leqslant\ & \frac{1}{4}|z_1|^{4\lambda} + a_{22}(\xi_1)|z_2|^{4\lambda} + \psi_{22}(|z_0|)|z_0|^{4\tau\lambda},
\end{aligned} \tag{26}$$

where a_{22}, ψ_{22}, ϱ_{21}, ϱ_{22}, and ϱ_{23} are some nonnegative smooth functions. From (6), (9), (10), (15), (18), (24), and Lemmas 1, 2, 4, and 6, it is clear that

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 W_2}{\partial \xi_2^2}\bar{g}_2^{\mathrm{T}}\bar{g}_2 \leqslant\ & \left(\frac{4\tau\lambda-\varpi-r_2}{\tau}\right)|z_2|^{\frac{4\tau\lambda-\varpi-r_2}{\tau}-1}\left|\frac{\partial z_2}{\partial \xi_2}\right|\Bigg(\beta_{23}|z_0|^{\frac{2r_2+\varpi}{2}} \\
& + K_1^{\frac{2r_2+\varpi}{2r_1}}\beta_{24}|z_1|^{\frac{2r_2+\varpi}{2\tau}} + \beta_{24}|\xi_2|^{\frac{2r_2+\varpi}{2r_2}}\Bigg)^2 \\
\leqslant\ & \left(\frac{4\tau\lambda-\varpi-r_2}{\tau}\right)|z_2|^{\frac{4\tau\lambda-\varpi-r_2}{\tau}-1}\left|\frac{\partial z_2}{\partial \xi_2}\right|\Bigg(\varrho_{31}(|z_0|)|z_0|^{2r_2+\varpi} \\
& + \varrho_{32}(\xi_1)|z_1|^{\frac{2r_2+\varpi}{\tau}} + \varrho_{33}(\xi_1)|z_2|^{\frac{2r_2+\varpi}{\tau}}\Bigg) \\
\leqslant\ & \frac{1}{4}|z_1|^{4\lambda} + a_{23}(\xi_1)|z_2|^{4\lambda} + \psi_{23}(|z_0|)|z_0|^{4\tau\lambda},
\end{aligned} \tag{27}$$

where a_{23}, ψ_{23}, ϱ_{31}, ϱ_{32}, and ϱ_{33} are some nonnegative smooth functions. Due to $\frac{r_2 p_1}{\tau} > 1$, by Lemma 4,

$$\begin{aligned}
\left(\lceil \xi_2\rceil^{p_1} - \lceil \xi_2^*\rceil^{p_1}\right) \leqslant\ & \left|\left\lceil\lceil \xi_2\rceil^{\frac{\tau}{r_2}}\right\rceil^{\frac{r_2p_1}{\tau}} - \left\lceil\lceil \xi_2^*\rceil^{\frac{\tau}{r_2}}\right\rceil^{\frac{r_2p_1}{\tau}}\right| \\
\leqslant\ & \frac{r_2p_1}{\tau}\left(1 + 2^{\frac{r_2p_1}{\tau}-2}\right)\left(|z_2|^{\frac{r_2p_1}{\tau}} + \left|\lceil \xi_2^*\rceil^{\frac{\tau}{r_2}}\right|^{\frac{r_2p_1}{\tau}-1}|z_2|\right),
\end{aligned} \tag{28}$$

which, together with (18), and Lemmas 1 and 2, implies that

$$\begin{aligned}
H_{\xi_1}\lceil z_1\rceil^{\frac{4\tau\lambda-\varpi-r_1}{\tau}}\left(\lceil \xi_2\rceil^{p_1} - \lceil \xi_2^*\rceil^{p_1}\right) \leqslant\ & b_{21}(\xi_1)|z_1|^{\frac{4\tau\lambda-\varpi-r_1}{\tau}}\left(|z_2|^{\frac{r_2p_1}{\tau}} + \left|\lceil \xi_2^*\rceil^{\frac{\tau}{r_2}}\right|^{\frac{r_2p_1}{\tau}-1}|z_2|\right) \\
\leqslant\ & \frac{1}{4}|z_1|^{4\lambda} + a_{24}(\xi_1)|z_2|^{4\lambda},
\end{aligned} \tag{29}$$

where b_{21} and a_{24} are some nonnegative smooth functions. Substituting (23), (25)–(27), and (29) into (21) yields

$$\mathcal{L}V_2 \leqslant -|z_1|^{4\lambda} + \psi_1(|z_0|)|z_0|^{4\tau\lambda} + \psi_2(|z_0|)|z_0|^{4\tau\lambda} + a_2(\xi_1)|z_2|^{4\lambda}$$

$$-\nu_1(\xi_1)|z_1|^{4\lambda}-\alpha_3(|z_0|)+\tilde{\gamma}(|\xi_1|)+|z_2|^{\frac{4\tau\lambda-\varpi-r_2}{\tau}}\lceil u\rfloor^{p_2}, \tag{30}$$

where $\psi_2=\sum_{i=1}^{3}\psi_{2i}$ and $a_2=\sum_{i=1}^{4}a_{2i}$. Choose the state-feedback controller

$$u(\xi)=-(1+a_2(\xi_1))^{\frac{1}{p_2}}\lceil z_2\rfloor^{\frac{r_3}{\tau}}\triangleq-\sigma_2(\xi_1)\lceil z_2\rfloor^{\frac{r_3}{\tau}}, \tag{31}$$

with $r_3=\frac{r_2+\varpi}{p_2}$. Putting (30) and (31) together yields

$$\mathcal{L}V_2\leqslant-|z_1|^{4\lambda}-|z_2|^{4\lambda}+\psi_1(|z_0|)|z_0|^{4\tau\lambda}+\psi_2(|z_0|)|z_0|^{4\tau\lambda}-\nu_1(\xi_1)|z_1|^{4\lambda}-\alpha_3(|z_0|)+\tilde{\gamma}(|\xi_1|). \tag{32}$$

3.3 Output feedback controller design

Since $x_2=\xi_2$ is unmeasurable, a reduced-order observer is constructed as follows:

$$\dot{z}=-L(\xi_1)\left(H_{\xi_1}\lceil z+S(\xi_1)\rfloor^{\frac{r_1+\varpi}{\tau}}+\bar{f}_1\right), \tag{33}$$

where $S(\xi_1)$ is a smooth function with $S(0)=0$ and $\frac{\partial S(\xi_1)}{\partial\xi_1}=L(\xi_1)>0$ which will be defined later. Let $\hat{\xi}_2$ be the estimate of ξ_2 with $\lceil\hat{\xi}_2\rfloor^{\frac{\tau}{r_2}}=z+S(\xi_1)$ and $e=\lceil\xi_2\rfloor^{\frac{\tau}{r_2}}-\lceil\hat{\xi}_2\rfloor^{\frac{\tau}{r_2}}$. It follows from (12), (33), Lemma 3, and Itô's formula

$$\begin{aligned}\mathrm{d}e=&\left(\frac{\tau}{r_2}|\xi_2|^{\frac{\tau}{r_2}-1}(\lceil u\rfloor^{p_2}+\bar{f}_2)+\frac{\tau}{2r_2}\left(\frac{\tau}{r_2}-1\right)\lceil\xi_2\rfloor^{\frac{\tau}{r_2}-2}\bar{g}_2^{\mathrm{T}}\bar{g}_2\right.\\&\left.-LH_{\xi_1}\left(\lceil\xi_2\rfloor^{p_1}-\lceil\hat{\xi}_2\rfloor^{p_1}\right)\right)\mathrm{d}t+\frac{\tau}{r_2}|\xi_2|^{\frac{\tau}{r_2}-1}\bar{g}_2^{\mathrm{T}}\mathrm{d}\omega.\end{aligned} \tag{34}$$

Choosing $W_3(e)=\frac{\tau}{4\tau\lambda-\varpi}|e|^{\frac{4\tau\lambda-\varpi}{\tau}}$, and using (2), (34), and Lemma 3, we obtain

$$\begin{aligned}\mathcal{L}W_3=&\lceil e\rfloor^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}\left(\frac{\tau}{r_2}|\xi_2|^{\frac{\tau}{r_2}-1}(\lceil u\rfloor^{p_2}+\bar{f}_2)+\frac{\tau}{2r_2}\left(\frac{\tau}{r_2}-1\right)\lceil\xi_2\rfloor^{\frac{\tau}{r_2}-2}\bar{g}_2^{\mathrm{T}}\bar{g}_2\right.\\&\left.-LH_{\xi_1}\left(\lceil\xi_2\rfloor^{p_1}-\lceil\hat{\xi}_2\rfloor^{p_1}\right)\right)+\frac{4\tau\lambda-\varpi-\tau}{2\tau}|e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}-1}\left(\frac{\tau}{r_2}|\xi_2|^{\frac{\tau}{r_2}-1}\right)^2\bar{g}_2^{\mathrm{T}}\bar{g}_2.\end{aligned} \tag{35}$$

The following propositions are used to estimate the right-hand side of (35), whose proofs are in Appendix A.

Proposition 1. It is easy to show that

$$-\lceil e\rfloor^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}(\lceil\xi_2\rfloor^{p_1}-\lceil\hat{\xi}_2\rfloor^{p_1})\leqslant-\frac{|e|^{4\lambda}}{2^{\varpi}}. \tag{36}$$

Proposition 2. There exist nonnegative smooth functions $M_{11}(\xi_1)$, $N_{11}(\xi_1)$, and $\psi_{31}(|z_0|)$ such that

$$\begin{aligned}&\frac{\tau}{r_2}\lceil e\rfloor^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}|\xi_2|^{\frac{\tau}{r_2}-1}(\lceil u\rfloor^{p_2}+\bar{f}_2)\\&\leqslant\frac{1}{4}|z_1|^{4\lambda}+\frac{1}{16}|z_2|^{4\lambda}+M_{11}(\xi_1)|e|^{4\lambda}+N_{11}(\xi_1)|e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}|u^{p_2}|^{\frac{\tau+\varpi}{r_2+\varpi}}+\psi_{31}(|z_0|)|z_0|^{4\tau\lambda}.\end{aligned} \tag{37}$$

Proposition 3. There exist some nonnegative smooth functions $M_{12}(\xi_1)$ and $\psi_{32}(|z_0|)$ such that

$$\begin{aligned}&\lceil e\rfloor^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}\frac{\tau}{2r_2}\left(\frac{\tau}{r_2}-1\right)\lceil\xi_2\rfloor^{\frac{\tau}{r_2}-2}\bar{g}_2^{\mathrm{T}}\bar{g}_2\\&\leqslant\frac{1}{4}|z_1|^{4\lambda}+\frac{1}{16}|z_2|^{4\lambda}+M_{12}(\xi_1)|e|^{4\lambda}+\psi_{32}(|z_0|)|z_0|^{4\tau\lambda}.\end{aligned} \tag{38}$$

Proposition 4. There exist some nonnegative smooth functions $M_{13}(\xi_1)$ and $\psi_{33}(|z_0|)$ such that

$$\frac{4\tau\lambda-\varpi-\tau}{2\tau}|e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}-1}\left(\frac{\tau}{r_2}|\xi_2|^{\frac{\tau}{r_2}-1}\right)^2\bar{g}_2^{\mathrm{T}}\bar{g}_2$$

$$\leqslant \frac{1}{4}|z_1|^{4\lambda} + \frac{1}{16}|z_2|^{4\lambda} + M_{13}(\xi_1)|e|^{4\lambda} + \psi_{33}(|z_0|)|z_0|^{4\tau\lambda}. \tag{39}$$

With the help of Propositions 1–4, Eq. (35) can be rewritten as

$$\begin{aligned}\mathcal{L}W_3 \leqslant\ & \frac{3}{4}|z_1|^{4\lambda} + \frac{3}{16}|z_2|^{4\lambda} + \sum_{i=1}^{3} M_{1i}(\xi_1)|e|^{4\lambda} + \psi_3(|z_0|)|z_0|^{4\tau\lambda} \\ & + N_{11}(\xi_1)|e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}|u^{p_2}|^{\frac{\tau+\varpi}{r_2+\varpi}} - ML(\xi_1)|e|^{4\lambda},\end{aligned} \tag{40}$$

where $\psi_3 = \sum_{i=1}^{3}\psi_{3i}$ and $M = \frac{1}{2^{\varpi}K_1}$.

Because $x_2 = \xi_2$ is unmeasurable, the state feedback controller u in (31) is unavailable. By the certainty equivalence principle, we obtain an implementable output feedback controller by replacing ξ_2 with $\hat{\xi}_2$ generated from (33),

$$u(\xi_1, \hat{\xi}_2) = -\sigma_2(\xi_1)\left\lceil \lceil \hat{\xi}_2 \rfloor^{\frac{\tau}{r_2}} - \lceil \xi_2^* \rfloor^{\frac{\tau}{r_2}} \right\rfloor^{\frac{r_3}{\tau}}. \tag{41}$$

From (41), Lemma 6, and the definition of e and z_2, it follows that

$$\begin{aligned}|u^{p_2}|^{\frac{\tau+\varpi}{r_2+\varpi}} &= \left|\sigma_2^{p_2}\left\lceil \lceil \hat{\xi}_2 \rfloor^{\frac{\tau}{r_2}} - \lceil \xi_2^* \rfloor^{\frac{\tau}{r_2}} \right\rfloor^{\frac{r_2+\varpi}{\tau}}\right|^{\frac{\tau+\varpi}{r_2+\varpi}} \\ &\leqslant |\sigma_2|^{\frac{p_2(\tau+\varpi)}{r_2+\varpi}}\left|\lceil \hat{\xi}_2 \rfloor^{\frac{\tau}{r_2}} - \lceil \xi_2 \rfloor^{\frac{\tau}{r_2}} + \lceil \xi_2 \rfloor^{\frac{\tau}{r_2}} - \lceil \xi_2^* \rfloor^{\frac{\tau}{r_2}}\right|^{\frac{\tau+\varpi}{\tau}} \\ &\leqslant 2^{\frac{\tau+\varpi}{\tau}-1}|\sigma_2|^{\frac{p_2(\tau+\varpi)}{r_2+\varpi}}\left(|e|^{\frac{\tau+\varpi}{\tau}} + |z_2|^{\frac{\tau+\varpi}{\tau}}\right).\end{aligned} \tag{42}$$

By (42) and Lemmas 1 and 2, we obtain

$$\begin{aligned}N_{11}(\xi_1)|e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}|u^{p_2}|^{\frac{\tau+\varpi}{r_2+\varpi}} &\leqslant \bar{N}_{11}(\xi_1)|e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}\left(|e|^{\frac{\tau+\varpi}{\tau}} + |z_2|^{\frac{\tau+\varpi}{\tau}}\right) \\ &\leqslant \frac{1}{16}|z_2|^{4\lambda} + M_{14}(\xi_1)|e|^{4\lambda},\end{aligned} \tag{43}$$

where $\bar{N}_{11}$ and M_{14} are some nonnegative smooth functions. Substituting (43) into (40) leads to

$$\mathcal{L}W_3 \leqslant \frac{3}{4}|z_1|^{4\lambda} + \frac{1}{4}|z_2|^{4\lambda} + \sum_{i=1}^{4} M_{1i}(\xi_1)|e|^{4\lambda} + \psi_3(|z_0|)|z_0|^{4\tau\lambda} - ML(\xi_1)|e|^{4\lambda}. \tag{44}$$

Under the new controller $u(\xi_1, \hat{\xi}_2)$ rather than u in (31), Eq. (32) is no longer valid. Instead,

$$\begin{aligned}\mathcal{L}V_2 \leqslant\ & -|z_1|^{4\lambda} + \psi_1(|z_0|)|z_0|^{4\tau\lambda} + \psi_2(|z_0|)|z_0|^{4\tau\lambda} + a_2(\xi_1)|z_2|^{4\lambda} \\ & -\nu_1(\xi_1)|z_1|^{4\lambda} - \alpha_3(|z_0|) + \tilde{\gamma}(|\xi_1|) + |z_2|^{\frac{4\tau\lambda-\varpi-r_2}{\tau}}\lceil u(\xi_1, \hat{\xi}_2)\rfloor^{p_2} \\ \leqslant\ & -|z_1|^{4\lambda} - |z_2|^{4\lambda} + |z_2|^{\frac{4\lambda\tau-\varpi-r_2}{\tau}}\sigma_2^{p_2}\left|\lceil z_2 \rfloor^{\frac{r_2+\varpi}{\tau}} - \lceil z_2 - e \rfloor^{\frac{r_2+\varpi}{\tau}}\right| \\ & +\psi_1(|z_0|)|z_0|^{4\tau\lambda} + \psi_2(|z_0|)|z_0|^{4\tau\lambda} - \nu_1(\xi_1)|z_1|^{4\lambda} - \alpha_3(|z_0|) + \tilde{\gamma}(|\xi_1|).\end{aligned} \tag{45}$$

Due to $\frac{r_2+\varpi}{\tau} > 1$, by Lemma 4,

$$\left|\lceil z_2 \rfloor^{\frac{r_2+\varpi}{\tau}} - \lceil z_2 - e \rfloor^{\frac{r_2+\varpi}{\tau}}\right| \leqslant \frac{r_2+\varpi}{\tau}\left(1 + 2^{\frac{r_2+\varpi}{\tau}-2}\right)|e|\left(|e|^{\frac{\varpi+r_2}{\tau}-1} + |z_2|^{\frac{\varpi+r_2}{\tau}-1}\right). \tag{46}$$

Thus, it follows from (46) and Lemmas 1 and 2 that

$$\begin{aligned}|z_2|^{\frac{4\lambda\tau-\varpi-r_2}{\tau}}\sigma_2^{p_2}\left|\lceil z_2 \rfloor^{\frac{r_2+\varpi}{\tau}} - \lceil z_2 - e \rfloor^{\frac{r_2+\varpi}{\tau}}\right| &\leqslant |z_2|^{\frac{4\lambda\tau-\varpi-r_2}{\tau}}\bar{N}_{12}(\xi_1)|e|\left(|e|^{\frac{\varpi+r_2}{\tau}-1} + |z_2|^{\frac{\varpi+r_2}{\tau}-1}\right) \\ &\leqslant \frac{1}{2}|z_2|^{4\lambda} + M_{15}(\xi_1)|e|^{4\lambda},\end{aligned} \tag{47}$$

where $\bar{N}_{12}$ and M_{15} are some nonnegative smooth function. Substituting (47) into (45) leads to

$$\begin{aligned}\mathcal{L}V_2 \leqslant\ & -|z_1|^{4\lambda} - \frac{1}{2}|z_2|^{4\lambda} + M_{15}(\xi_1)|e|^{4\lambda} + \psi_1(|z_0|)|z_0|^{4\tau\lambda} \\ & +\psi_2(|z_0|)|z_0|^{4\tau\lambda} - \nu_1(\xi_1)|z_1|^{4\lambda} - \alpha_3(|z_0|) + \tilde{\gamma}(|\xi_1|). \end{aligned} \tag{48}$$

From (44) and (48), one obtains

$$\begin{aligned}\mathcal{L}(V_2 + W_3) \leqslant\ & -\frac{1}{4}|z_1|^{4\lambda} - \frac{1}{4}|z_2|^{4\lambda} + \left(\sum_{i=1}^{5} M_{1i}(\xi_1) - ML(\xi_1)\right)|e|^{4\lambda} \\ & +\psi(|z_0|)|z_0|^{4\tau\lambda} - \nu_1(\xi_1)|z_1|^{4\lambda} - \alpha_3(|z_0|) + \tilde{\gamma}(|\xi_1|), \end{aligned} \tag{49}$$

where $\psi = \sum_{i=1}^{3} \psi_i$. Setting

$$V(\Xi) = V_2(z_0, \xi) + W_3(e), \tag{50}$$

where $\Xi = (z_0, \xi_1, \xi_2, e)^{\mathrm{T}}$, and choosing $M_1(\xi_1) \geqslant \sum_{i=1}^{5} M_{1i}(\xi_1)$, we get

$$S(\xi_1) = \int_0^{\xi_1} \frac{M_1(s)}{M}\mathrm{d}s + \frac{\bar{m}}{M}\xi_1, \tag{51}$$

where $\bar{m}$ is a known positive constant. By the definition of the observer gain $S(\xi_1)$ with $\frac{\partial S(\xi_1)}{\partial \xi_1} = L(\xi_1) > 0$, it is easy to deduce that $L(\xi_1) = \frac{M_1(\xi_1)}{M} + \frac{\bar{m}}{M} \geqslant \frac{\sum_{i=1}^{5} M_{1i}(\xi_1)}{M} + \frac{\bar{m}}{M}$, and then

$$\mathcal{L}V \leqslant -\bar{m}|e|^{4\lambda} - \frac{1}{4}|z_1|^{4\lambda} - \frac{1}{4}|z_2|^{4\lambda} + \psi(|z_0|)|z_0|^{4\tau\lambda} - \nu_1(\xi_1)|z_1|^{4\lambda} - \alpha_3(|z_0|) + \tilde{\gamma}(|\xi_1|). \tag{52}$$

3.4 Stability and constraint analysis

Theorem 1. For the system (3), if Assumptions 1 and 2 hold with

$$\limsup_{s\to 0^+} \frac{\psi(s)s^{4\tau\lambda}}{\alpha_3(s)} < \infty,\ \limsup_{s\to \infty} \frac{\psi(s)s^{4\tau\lambda}}{\alpha_3(s)} < \infty,\ \limsup_{s\to 0^+} \frac{\tilde{\gamma}(s)}{s^{4\tau\lambda}} < \infty, \tag{53}$$

then there exists an output feedback controller such that for any initial value $y(0) \in \Omega_y$:

(i) The closed-loop system consisting of (3), (4), (9), (33), and (41) has an almost surely unique solution on $[0, \infty)$;

(ii) All the closed-loop system signals are bounded almost surely, and the symmetric output constraint (4) is not violated almost surely;

(iii) The equilibrium point of the closed-loop system is stochastically asymptotically stable.

Proof. (i) The proof is divided into three parts.

Part I: From (53), there exists a positive constant κ such that

$$\psi(s)s^{4\tau\lambda} \leqslant \kappa\alpha_3(s),\ \forall s \geqslant 0. \tag{54}$$

By setting the Lyapunov function $V_{z_0}(z_0) = \kappa V_0(z_0)$ and using (16) and (54),

$$\mathcal{L}V_{z_0} \leqslant -\kappa\alpha_3(|z_0|) + \kappa\tilde{\gamma}(|\xi_1|) \leqslant -\psi(|z_0|)z_0^{4\tau\lambda} + \kappa\tilde{\gamma}(|\xi_1|). \tag{55}$$

Setting $\bar{V}(\Xi) = V(\Xi) + V_{z_0}(z_0)$ and considering (52) and (55), it is clear that

$$\begin{aligned}\mathcal{L}\bar{V} \leqslant\ & -\bar{m}|e|^{4\lambda} - \frac{1}{4}|z_1|^{4\lambda} - \frac{1}{4}|z_2|^{4\lambda} + \psi(z_0)|z_0|^{4\tau\lambda} + \kappa\tilde{\gamma}(|\xi_1|) \\ & -\nu_1(\xi_1)|z_1|^{4\lambda} - \alpha_3(|z_0|) + \tilde{\gamma}(|\xi_1|) - \psi(z_0)z_0^{4\tau\lambda} \\ =\ & -\alpha_3(|z_0|) + (\kappa+1)\tilde{\gamma}(|\xi_1|) - \nu_1(\xi_1)|\xi_1|^{4\tau\lambda} - \bar{m}|e|^{4\lambda} - \frac{1}{4}|z_1|^{4\lambda} - \frac{1}{4}|z_2|^{4\lambda}. \end{aligned} \tag{56}$$

From (53), there exists a nonnegative smooth function $\nu_1(\xi_1)$ such that

$$(\kappa+1)\tilde{\gamma}(|\xi_1|) \leqslant \nu_1(\xi_1)\xi_1^{4\tau\lambda}. \tag{57}$$

Substituting (57) into (56) yields

$$\mathcal{L}\bar{V} \leqslant -\alpha_3(|z_0|) - \bar{m}|e|^{4\lambda} - \frac{1}{4}|z_1|^{4\lambda} - \frac{1}{4}|z_2|^{4\lambda}. \tag{58}$$

Part II: We prove that $\lceil u \rceil^{p_2}$ in (41) satisfies the local Lipschitz condition. By (41) and Lemma 3,

$$\begin{aligned} \frac{\partial \lceil u(\xi_1, \hat{\xi}_2) \rceil^{p_2}}{\partial \xi_1} &= -\frac{\partial \sigma_2^{p_2}(\xi_1)}{\partial \xi_1} \left\lceil \lceil \hat{\xi}_2 \rceil^{\frac{\tau}{r_2}} - \lceil \xi_2^* \rceil^{\frac{\tau}{r_2}} \right\rceil^{\frac{r_3 p_2}{\tau}} - \frac{r_3 p_2}{\tau} \sigma_2^{p_2}(\xi_1) \left| \lceil \hat{\xi}_2 \rceil^{\frac{\tau}{r_2}} - \lceil \xi_2^* \rceil^{\frac{\tau}{r_2}} \right|^{\frac{r_3 p_2}{\tau} - 1} \\ &\quad \cdot \left(L(\xi_1) + \frac{\partial \sigma_1^{\frac{\tau}{r_2}}(\xi_1)}{\partial \xi_1} \lceil \xi_1 \rceil^{\frac{\tau}{r_1}} + \frac{\tau}{r_1} \sigma_1^{\frac{\tau}{r_2}}(\xi_1) |\xi_1|^{\frac{\tau}{r_1} - 1} \right), \\ \frac{\partial \lceil u(\xi_1, \hat{\xi}_2) \rceil^{p_2}}{\partial \hat{\xi}_2} &= -\frac{r_3 p_2}{r_2} \sigma_2^{p_2}(\xi_1) \left| \lceil \hat{\xi}_2 \rceil^{\frac{\tau}{r_2}} - \lceil \xi_2^* \rceil^{\frac{\tau}{r_2}} \right|^{\frac{r_3 p_2}{\tau} - 1} |\hat{\xi}_2|^{\frac{\tau}{r_2} - 1}. \end{aligned} \tag{59}$$

From (7), (41), and the definition of τ and r_3, it is obvious that $\frac{r_3 p_2}{\tau} - 1 \geqslant 0$ and $\frac{\tau}{r_i} - 1 \geqslant 0$, $i = 1, 2$. Since $\sigma_i(\xi_1)$, $i = 1, 2$, and $L(\xi_1)$ are some smooth functions, by (59), $\frac{\partial \lceil u(\xi_1, \hat{\xi}_2) \rceil^{p_2}}{\partial \xi_1}$ and $\frac{\partial \lceil u(\xi_1, \hat{\xi}_2) \rceil^{p_2}}{\partial \hat{\xi}_2}$ are continuous, and then $\lceil u(\xi_1, \hat{\xi}_2) \rceil^{p_2}$ is $\mathcal{C}^1$. Because f_1, f_2, and g_2 are some locally Lipschitz functions, the locally Lipschitz condition of the closed-loop system (3), (4), (9), (33), and (41) is verified.

Part III: Firstly, we show that for $i = 1, 2$,

$$c_{i1} |\xi_i - \xi_i^*|^{\frac{4\tau\lambda - \varpi}{r_i}} \leqslant W_i(\bar{\xi}_i) \leqslant c_{i2} |z_i|^{\frac{4\tau\lambda - \varpi}{\tau}}, \tag{60}$$

where c_{i1} and c_{i2} are some positive constants. By Lemmas 4 and 5, we deduce that

$$W_i(\bar{\xi}_i) \leqslant |z_i|^{\frac{4\tau\lambda - \varpi - r_i}{\tau}} \left| \left\lceil \lceil \xi_i \rceil^{\frac{\tau}{r_i}} \right\rceil^{\frac{r_i}{\tau}} - \left\lceil \lceil \xi_i^* \rceil^{\frac{\tau}{r_i}} \right\rceil^{\frac{r_i}{\tau}} \right| \leqslant c_{i2} |z_i|^{\frac{4\tau\lambda - \varpi}{\tau}}, \tag{61}$$

where $c_{i2} = 2^{1 - \frac{r_i}{\tau}}$. The left-hand side of (60) will be proven by considering two cases.

Case (I): When $\xi_i^* \leqslant \xi_i$, $i = 1, 2$, there are three situations.

(a) If $0 \leqslant \xi_i^* \leqslant \xi_i$, by using $||x| - |y||^p \leqslant ||x|^p - |y|^p|$ for $p \geqslant 1$,

$$\begin{aligned} W_i(\bar{\xi}_i) &= \int_{\xi_i^*}^{\xi_i} \left\lceil \lceil s \rceil^{\frac{\tau}{r_i}} - \lceil \xi_i^* \rceil^{\frac{\tau}{r_i}} \right\rceil^{\frac{4\tau\lambda - \varpi - r_i}{\tau}} \mathrm{d}s \\ &= \int_{\xi_i^*}^{\xi_i} \left(s^{\frac{\tau}{r_i}} - \xi_i^{*\frac{\tau}{r_i}} \right)^{\frac{4\tau\lambda - \varpi - r_i}{\tau}} \mathrm{d}s \\ &\geqslant \int_{\xi_i^*}^{\xi_i} (s - \xi_i^*)^{\frac{4\tau\lambda - \varpi - r_i}{r_i}} \mathrm{d}s \\ &= \frac{r_i}{4\tau\lambda - \varpi} (\xi_i - \xi_i^*)^{\frac{4\tau\lambda - \varpi}{r_i}}. \end{aligned} \tag{62}$$

(b) If $\xi_i^* \leqslant \xi_i \leqslant 0$, similar to (62), there holds

$$\begin{aligned} W_i(\bar{\xi}_i) &= \int_{\xi_i^*}^{\xi_i} \left\lceil \lceil s \rceil^{\frac{\tau}{r_i}} - \lceil \xi_i^* \rceil^{\frac{\tau}{r_i}} \right\rceil^{\frac{4\tau\lambda - \varpi - r_i}{\tau}} \mathrm{d}s \\ &= \int_{\xi_i^*}^{\xi_i} \left(-(-s)^{\frac{\tau}{r_i}} + (-\xi_i^*)^{\frac{\tau}{r_i}} \right)^{\frac{4\tau\lambda - \varpi - r_i}{\tau}} \mathrm{d}s \\ &\geqslant \int_{\xi_i^*}^{\xi_i} (s - \xi_i^*)^{\frac{4\tau\lambda - \varpi - r_i}{r_i}} \mathrm{d}s \\ &\geqslant \frac{r_i}{4\tau\lambda - \varpi} (\xi_i - \xi_i^*)^{\frac{4\tau\lambda - \varpi}{r_i}}. \end{aligned} \tag{63}$$

(c) If $\xi_i^* \leqslant 0 \leqslant \xi_i$, by Lemma 6, there holds

$$W_i(\bar{\xi}_i) = \int_{\xi_i^*}^{0} \left\lceil \lceil s \rceil^{\frac{\tau}{r_i}} - \lceil \xi_i^* \rceil^{\frac{\tau}{r_i}} \right\rceil^{\frac{4\tau\lambda - \varpi - r_i}{\tau}} \mathrm{d}s + \int_{0}^{\xi_i} \left\lceil \lceil s \rceil^{\frac{\tau}{r_i}} - \lceil \xi_i^* \rceil^{\frac{\tau}{r_i}} \right\rceil^{\frac{4\tau\lambda - \varpi - r_i}{\tau}} \mathrm{d}s$$

$$
\begin{aligned}
&= \int_{\xi_i^*}^{0} \left(-(-s)^{\frac{\tau}{r_i}} + (-\xi_i^*)^{\frac{\tau}{r_i}}\right)^{\frac{4\tau\lambda-\varpi-r_i}{\tau}} \mathrm{d}s + \int_0^{\xi_i} \left(s^{\frac{\tau}{r_i}} + (-\xi_i^*)^{\frac{\tau}{r_i}}\right)^{\frac{4\tau\lambda-\varpi-r_i}{\tau}} \mathrm{d}s \\
&\geqslant \frac{2^{\frac{4\tau\lambda-\varpi-r_i}{\tau}}}{2^{\frac{4\tau\lambda-\varpi-r_i}{r_i}}} \int_{\xi_i^*}^{0} (s-\xi_i^*)^{\frac{4\tau\lambda-\varpi-r_i}{r_i}} \mathrm{d}s + \frac{2^{\frac{4\tau\lambda-\varpi-r_i}{\tau}}}{2^{\frac{4\tau\lambda-\varpi-r_i}{r_i}}} \int_0^{\xi_i} (s-\xi_i^*)^{\frac{4\tau\lambda-\varpi-r_i}{r_i}} \mathrm{d}s \\
&\geqslant \frac{r_i}{4\tau\lambda-\varpi} \cdot \frac{2^{\frac{4\tau\lambda-\varpi-r_i}{\tau}}}{2^{\frac{4\tau\lambda-\varpi-r_i}{r_i}}} (\xi_i-\xi_i^*)^{\frac{4\tau\lambda-\varpi}{r_i}}. \qquad (64)
\end{aligned}
$$

In conclusion, one has

$$W_i(\bar{\xi}_i) \geqslant c_{i1}|\xi_i-\xi_i^*|^{\frac{4\tau\lambda-\varpi}{r_i}}, \qquad (65)$$

where $c_{i1} = \frac{r_i}{4\tau\lambda-\varpi} \cdot \frac{2^{\frac{4\tau\lambda-\varpi-r_i}{\tau}}}{2^{\frac{4\tau\lambda-\varpi-r_i}{r_i}}}$.

Case (II): Similarly, when $\xi_i \leqslant \xi_i^*$, $i=1,2$, there holds

$$W_i(\bar{\xi}_i) \geqslant c_{i1}|\xi_i-\xi_i^*|^{\frac{4\tau\lambda-\varpi}{r_i}}. \qquad (66)$$

Combining (61) with (65) and (66), Eq. (60) holds.

From Assumption 2, the definitions of $V_0(z_0)$ and $V_{z_0}(z_0)$, it follows that

$$(\kappa+1)\alpha_1(|z_0|) \leqslant V_0(z_0) + V_{z_0}(z_0) \leqslant (\kappa+1)\alpha_2(|z_0|). \qquad (67)$$

By (60), (67), and the definitions of $W_3(e)$ and $\bar{V}(\Xi)$, we know that $\bar{V}(\Xi)$ is positive definite and radially unbounded, and there exist some $\mathcal{K}_\infty$ functions $\bar{\alpha}_1(\Xi)$ and $\bar{\alpha}_2(\Xi)$ such that

$$\bar{\alpha}_1(|\Xi|) \leqslant \bar{V}(\Xi) \leqslant \bar{\alpha}_2(|\Xi|). \qquad (68)$$

Clearly, $V_2(z_0,\xi)$, $W_3(e)$, and $V_{z_0}(z_0)$ are $\mathcal{C}^2$, so is $\bar{V}(\Xi)$. By Parts I–III and Lemma 7, the closed-loop system (3), (4), (9), (33), and (41) has an almost surely unique solution on $[0,\infty)$.

(ii) Define the stopping time $T_k = \inf\{t \geqslant 0 : |\Xi(t)| \geqslant k\}$. By (58) and Itô's formula,

$$E\{\bar{V}(\Xi(T_k \wedge t))\} = \bar{V}(\Xi(0)) + E\int_0^{T_k\wedge t} \mathcal{L}\bar{V}(\Xi(s))\mathrm{d}s \leqslant \bar{V}(\Xi(0)). \qquad (69)$$

From (68) and (69), it follows that

$$
\begin{aligned}
\bar{V}(\Xi(0)) &\geqslant E\{\bar{V}(\Xi(T_k\wedge t))\} \\
&\geqslant \int_{\{\sup_{0\leqslant s\leqslant t}|\Xi(s)|>k\}} \bar{V}(\Xi(T_k\wedge t))\mathrm{d}P \\
&= \int_{\{\sup_{0\leqslant s\leqslant t}|\Xi(s)|>k\}} \bar{V}(\Xi(T_k))\mathrm{d}P \\
&\geqslant P\left\{\sup_{0\leqslant s\leqslant t}|\Xi(s)|>k\right\} \inf_{|\Xi|\geqslant k} \bar{\alpha}_1(|\Xi|),\ \forall t>0. \qquad (70)
\end{aligned}
$$

Thus, it is easy to show that for $\forall t>0$,

$$P\left\{\sup_{0\leqslant s\leqslant t}|\Xi(s)|>k\right\} \leqslant \frac{\bar{V}(\Xi(0))}{\inf_{|\Xi|\geqslant k}\bar{\alpha}_1(|\Xi|)}, \qquad (71)$$

$$P\left\{\sup_{0\leqslant s\leqslant t}|\Xi(s)|<k\right\} \geqslant 1 - \frac{\bar{V}(\Xi(0))}{\inf_{|\Xi|\geqslant k}\bar{\alpha}_1(|\Xi|)}. \qquad (72)$$

Setting $k\to\infty$ first and then $t\to\infty$, by the monotone convergence theorem and the radial unboundedness of $\bar{\alpha}_1(|\Xi|)$, one obtains $P\{\sup_{t\geqslant 0}|\Xi(t)|<\infty\}=1$. Therefore, $z_0(t)$, $\xi_1(t)$, $\xi_2(t)$, and $e(t)$ are bounded almost surely, so are $x_1(t)$, $x_2(t)$, and $\hat{\xi}_2(t)$. By the definition of $\xi_2^*(t)$ and $u(t)$, the almost sure boundedness of $\xi_2^*(t)$ and $u(t)$ can be ensured.

Since $\xi_1(x_1)$ is bounded almost surely, by (11), the constraint (4) is not violated almost surely for any $y(0) \in \Omega_y$.

(iii) From (58), (68), and Lemma 7, we can derive that $P\{\lim_{t\to\infty}(|z_0(t)| + |\xi_1(t)| + |\xi_2(t)| + |e(t)|) = 0\} = 1$. By the equivalent coordinate transformation, the equilibrium point of the closed-loop system is stochastically asymptotically stable.

Remark 3. The significance of Theorem 1 is that a systematic approach to continuous output feedback control is successfully developed to achieve the pre-specified symmetric output constraint for stochastic high-order planar nonlinear systems with SiISS inverse dynamics.

4 A simulation example

Consider a stochastic nonlinear system

$$\begin{cases} \mathrm{d}z_0 = f_0(z_0, x_1)\mathrm{d}t + g_0^{\mathrm{T}}(z_0, x_1)\mathrm{d}\omega, \\ \mathrm{d}x_1 = (\lceil x_2 \rceil^{p_1} + f_1(z_0, x_1))\mathrm{d}t + g_1^{\mathrm{T}}(z_0, x_1)\mathrm{d}\omega, \\ \mathrm{d}x_2 = (\lceil u \rceil^{p_2} + f_2(z_0, x))\mathrm{d}t + g_2^{\mathrm{T}}(z_0, x)\mathrm{d}\omega, \\ y = x_1, \end{cases} \tag{73}$$

with output constraint $y(t) \in \Omega_y = \{y(t) \in \mathbb{R}^n : -1.6 < y(t) < 1.6\}$, $\forall t \geqslant 0$, where $p_1 = 1$, $p_2 = \frac{15}{4}$, $f_0 = -\frac{2z_0}{1+z_0^4} + \frac{5}{256}z_0x_1^4$, $g_0 = \sqrt{z_0^2 + \frac{1}{128}z_0^2x_1^4}$, $f_1 = \frac{1}{16}\sin(x_1)x_1 + \frac{1}{12}\frac{z_0}{1+z_0^4}$, $f_2 = 0$, $g_1 = 0$, $g_2 = \frac{1}{9}x_1\cos(x_2) + \frac{z_0}{1+z_0^4}$. It is easy to verify that Assumption 1 holds by choosing $r_1 = 1$, $\varpi = 0$, $r_2 = 1$, and $r_3 = \frac{4}{15}$. By introducing

$$\xi_1 = T(x_1) = \tan\left(\frac{x_1}{K_1}\right),\ \xi_2 = x_2, \tag{74}$$

where $K_1 = \frac{3.2}{\pi}$, Eq. (73) can be rewritten as

$$\begin{cases} \mathrm{d}z_0 = \bar{f}_0(z_0, \xi_1)\mathrm{d}t + \bar{g}_0^{\mathrm{T}}(z_0, \xi_1)\mathrm{d}\omega, \\ \mathrm{d}\xi_1 = \left(H_{\xi_1}\lceil \xi_2 \rceil^{p_1} + \bar{f}_1(z_0, \xi_1)\right)\mathrm{d}t + \bar{g}_1^{\mathrm{T}}(z_0, \xi_1)\mathrm{d}\omega, \\ \mathrm{d}\xi_2 = \left(\lceil u \rceil^{p_2} + \bar{f}_2(z_0, \xi)\right)\mathrm{d}t + \bar{g}_2^{\mathrm{T}}(z_0, \xi)\mathrm{d}\omega, \end{cases} \tag{75}$$

where $\xi = (\xi_1, \xi_2)^{\mathrm{T}}$, $\bar{f}_0 = f_0$, $\bar{g}_0 = g_0$, $H_{\xi_1} = \frac{1+\xi_1^2}{K_1}$, $\bar{f}_1 = H_{\xi_1}f_1$, $\bar{f}_2 = f_2$, $\bar{g}_1 = g_1$, $\bar{g}_2 = g_2$. For z_0-subsystem of system (73), by setting $V_0 = \ln(1 + z_0^4)$, one can verify that $\mathcal{L}V_0 \leqslant -\frac{2z_0^4}{1+z_0^4} + \frac{1}{16}K_1^4\xi_1^4$. Assumption 2 is satisfied by choosing $\alpha_3(s) = \frac{s^4}{1+s^4}$ and $\gamma(s) = \frac{1}{16}K_1^4s^4$.

Following the same procedure as in Section 3, one obtains the output feedback controller

$$\dot{z} = -L(\xi_1)(H_{\xi_1}(z + S(\xi_1)) + H_{\xi_1}f_1), \tag{76}$$

$$\hat{\xi}_2 = z + S(\xi_1),\ e = \xi_2 - \hat{\xi}_2, \tag{77}$$

$$u = -\sigma_2(\xi_1)\lceil \hat{\xi}_2 - \xi_2^* \rceil^{\frac{4}{15}}, \tag{78}$$

where $\sigma_2^{\frac{15}{4}} = a_2 + 1$, $a_2 = \frac{2048}{5625\pi^4} + \frac{64}{25\pi^2} + \frac{3}{4} + 3H_{\xi_1}|\frac{\partial\xi_2^*}{\partial\xi_1}| + \frac{27}{8}(H_{\xi_1}|\frac{\partial\xi_2^*}{\partial\xi_1}|(\sigma_1 + \frac{1}{5\pi}))^{\frac{4}{3}} + \frac{9}{8}H_{\xi_1}^{\frac{4}{3}} + \frac{27}{4}H_{\xi_1}^4$, $\sigma_1 = \frac{3.2(a_{11}(\xi_1)+2+\frac{8192}{625\pi^4})}{\pi}$, $a_{11}(\xi_1) = \frac{1}{5\pi}H_{\xi_1} + \frac{1}{16}H_{\xi_1}^{\frac{4}{3}}$, $L(\xi_1) = 269705 + 13.647\xi_1^2 + 65.2492\xi_1^{\frac{8}{3}} + 64.1972\xi_1^{\frac{32}{9}} + 6.2\xi_1^4 + 5.3\xi_1^6 + 32784\xi_1^8 + 4.4425\xi_1^{\frac{88}{9}} + 5.9\xi_1^{\frac{128}{27}} + 17.112\xi_1^{\frac{16}{3}} + 68924\xi_1^{\frac{32}{3}} + 59.319\xi_1^{\frac{128}{9}} + 1509\xi_1^{16} + 636\xi_1^{24} + 5.832\xi_1^{\frac{352}{9}}$, $S(\xi_1) = 269705\xi_1 + 4.549\xi_1^3 + 17.8\xi_1^{\frac{11}{3}} + 14.2\xi_1^{\frac{41}{9}} + 1.24\xi_1^5 + 0.76\xi_1^7 + 3643\xi_1^9 + 0.44\xi_1^{\frac{97}{9}} + 1.1\xi_1^{\frac{155}{27}} + 0.1\xi_1^{\frac{19}{3}} + 5908\xi_1^{\frac{35}{3}} + 3.9\xi_1^{\frac{137}{9}} + 88.8\xi_1^{17} + 25.44\xi_1^{25} + 0.009\xi_1^{\frac{361}{9}}$.

By choosing the initial value $(z_0(0), x_1(0), x_2(0)) = (0.4, -0.9, 0.5)$, Figure 1 clearly shows the response of the closed-loop system.

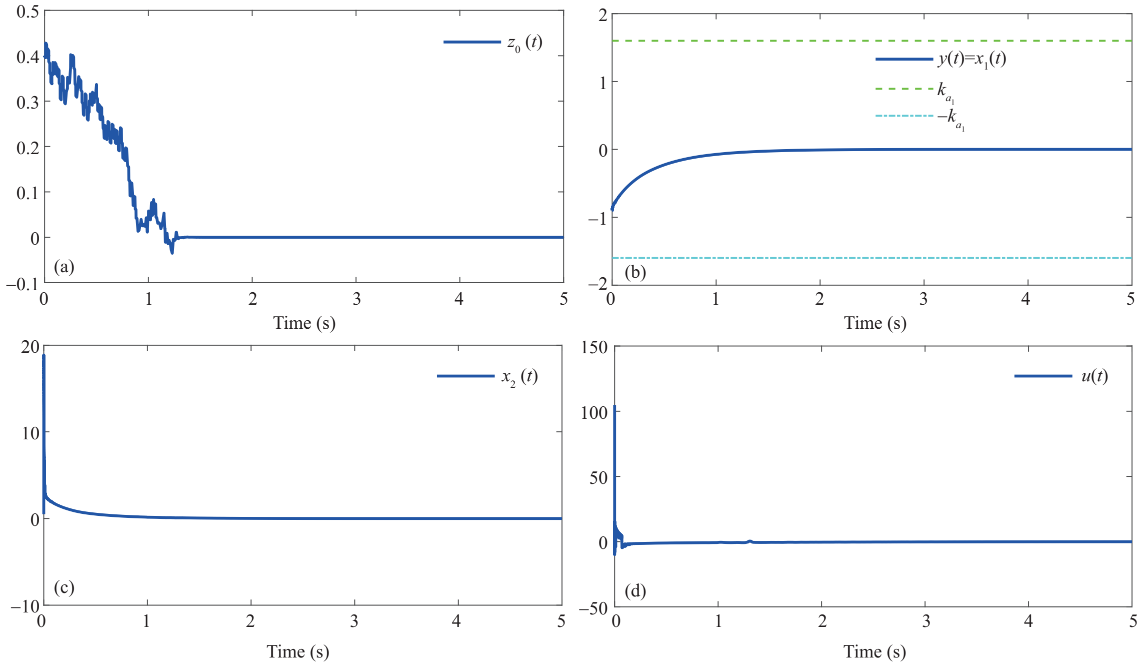

Figure 1 (Color online) The phase curves of (a) $z_0(t)$, (b) $x_1(t)$, (c) $x_2(t)$, and (d) $u(t)$.

5 Conclusion

In this paper, we study output feedback stabilization of stochastic systems with output constraint and Si-ISS inverse dynamics. However, the existing work only discusses the planar case, and the implementation of an output feedback controller for n-dimensional stochastic state/output-constrained systems needs to be further explored.

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References

1 Lin W, Qian C J. Adding one power integrator: a tool for global stabilization of high-order lower-triangular systems. Syst Control Lett, 2000, 39: 339–351

2 Xie X J, Tian J. Adaptive state-feedback stabilization of high-order stochastic systems with nonlinear parameterization. Automatica, 2009, 45: 126–133

3 Li W Q, Xie X J. Inverse optimal stabilization for stochastic nonlinear systems whose linearizations are not stabilizable. Automatica, 2009, 45: 498–503

4 Li W Q, Jing Y W, Zhang S Y. Adaptive state-feedback stabilization for a large class of high-order stochastic nonlinear systems. Automatica, 2011, 47: 819–828

5 Li W Q, Liu X H, Zhang S Y. Further results on adaptive state-feedback stabilization for stochastic high-order nonlinear systems. Automatica, 2012, 48: 1667–1675

6 Xie X J, Liu L. A homogeneous domination approach to state feedback of stochastic high-order nonlinear systems with time-varying delay. IEEE Trans Automat Contr, 2013, 58: 494–499

7 Zhao C R, Xie X J. Global stabilization of stochastic high-order feedforward nonlinear systems with time-varying delay. Automatica, 2014, 50: 203–210

8 Xue L R, Zhang T L, Zhang W H, et al. Global adaptive stabilization and tracking control for high-order stochastic nonlinear systems with time-varying delays. IEEE Trans Automat Contr, 2018, 63: 2928–2943

9 Niu B, Liu M, Li A. Global adaptive stabilization of stochastic high-order switched nonlinear non-lower triangular systems. Syst Control Lett, 2020, 136: 104596

10 Xie X J, Li W Q. Output-feedback control of a class of high-order stochastic nonlinear systems. Int J Control, 2009, 82: 1692–1705

11 Li W Q, Jing Y W, Zhang S Y. Output-feedback stabilization for stochastic nonlinear systems whose linearizations are not stabilizable. Automatica, 2010, 46: 752–760

12 Li W Q, Xie X J, Zhang S Y. Output-feedback stabilization of stochastic high-order nonlinear systems under weaker conditions. SIAM J Control Optim, 2011, 49: 1262–1282

13 Xie X J, Duan N. Output tracking of high-order stochastic nonlinear systems with application to benchmark mechanical system. IEEE Trans Automat Contr, 2010, 55: 1197–1202

14 Xie X J, Duan N, Zhao C R. A combined homogeneous domination and sign function approach to output-feedback stabilization of stochastic high-order nonlinear systems. IEEE Trans Automat Contr, 2014, 59: 1303–1309

15 Tee K P, Ge S S, Tay E H. Barrier Lyapunov functions for the control of output-constrained nonlinear systems. Automatica, 2009, 45: 918–927

16 Fang L D, Ma L, Ding S H, et al. Finite-time stabilization for a class of high-order stochastic nonlinear systems with an output constraint. Appl Math Computation, 2019, 358: 63–79

17 Fang L D, Ma L, Ding S H, et al. Finite-time stabilization of high-order stochastic nonlinear systems with asymmetric output constraints. IEEE Trans Syst Man Cybern Syst, 2021, 51: 7201–7213
18 Fang L D, Ding S H, Park J H, et al. Adaptive fuzzy control for nontriangular stochastic high-order nonlinear systems subject to asymmetric output constraints. IEEE Trans Cybern, 2022, 52: 1280–1291
19 Cui R H, Xie X J. Adaptive state-feedback stabilization of state-constrained stochastic high-order nonlinear systems. Sci China Inf Sci, 2021, 64: 200203
20 Fang L D, Ding S H, Park J H, et al. Adaptive fuzzy output-feedback control design for a class of p-norm stochastic nonlinear systems with output constraints. IEEE Trans Circuits Syst I, 2021, 68: 2626–2638
21 Fang L D, Ma L, Ding S H. Finite-time fuzzy output-feedback control for p-norm stochastic nonlinear systems with output constraints. AIMS Mathematics, 2021, 6: 2244–2267
22 Cui R H, Xie X J. Output feedback stabilization of stochastic planar nonlinear systems with output constraint. Automatica, 2022, 143: 110471
23 Bechlioulis C P, Rovithakis G A. Robust adaptive control of feedback linearizable MIMO nonlinear systems with prescribed performance. IEEE Trans Automat Contr, 2008, 53: 2090–2099
24 Sontag E D. Smooth stabilization implies coprime factorization. IEEE Trans Automat Contr, 1989, 34: 435–443
25 Sontag E D. Comments on integral variants of ISS. Syst Control Lett, 1998, 34: 93–100
26 Tsinias J. Stochastic input-to-state stability and applications to global feedback stabilization. Int J Control, 1998, 71: 907–930
27 Tang C, Basar T. Stochastic stability of singularly perturbed nonlinear systems. In: Proceedings of the 40th IEEE Conference on Decision and Control, 2001. 399–404
28 Liu S J, Zhang J F, Jiang Z P. Decentralized adaptive output-feedback stabilization for large-scale stochastic nonlinear systems. Automatica, 2007, 43: 238–251
29 Xie X J, Tian J. State-feedback stabilization for high-order stochastic nonlinear systems with stochastic inverse dynamics. Intl J Robust Nonlinear, 2007, 17: 1343–1362
30 Xue L R, Zhang W H, Xie X J. Global practical tracking for stochastic time-delay nonlinear systems with SISS-like inverse dynamics. Sci China Inf Sci, 2017, 60: 122201
31 Zhao C R, Xie X J. Output feedback stabilization using small-gain method and reduced-order observer for stochastic nonlinear systems. IEEE Trans Automat Contr, 2013, 58: 523–529
32 Yu X, Xie X J. Output feedback regulation of stochastic nonlinear systems with stochastic iISS inverse dynamics. IEEE Trans Automat Contr, 2010, 55: 304–320
33 Yu X, Xie X J, Duan N. Small-gain control method for stochastic nonlinear systems with stochastic iISS inverse dynamics. Automatica, 2010, 46: 1790–1798
34 Duan N, Xie X J, Yu X. State feedback stabilization of stochastic nonlinear systems with SiISS inverse dynamics. Intl J Robust Nonlinear, 2011, 21: 1903–1919
35 Xie X J, Duan N, Yu X. State-feedback control of high-order stochastic nonlinear systems with SiISS inverse dynamics. IEEE Trans Automat Contr, 2011, 56: 1921–1926
36 Yu X, Xie X J, Wu Y Q. Further results on output-feedback regulation of stochastic nonlinear systems with SiISS inverse dynamics. Int J Control, 2010, 83: 2140–2152
37 Duan N, Yu X, Xie X J. Output feedback control using small-gain conditions for stochastic nonlinear systems with SiISS inverse dynamics. Int J Control, 2011, 84: 47–56
38 Deng H, Krstic M, Williams R J. Stabilization of stochastic nonlinear systems driven by noise of unknown covariance. IEEE Trans Automat Contr, 2001, 46: 1237–1253
39 Qian C J, Lin W. Non-Lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization. Syst Control Lett, 2001, 42: 185–200
40 Lin W, Qian C J. Adaptive control of nonlinearly parameterized systems: the smooth feedback case. IEEE Trans Automat Contr, 2002, 47: 1249–1266
41 Sun Z Y, Zhang X H, Xie X J. Global continuous output-feedback stabilization for a class of high-order nonlinear systems with multiple time delays. J Franklin Institute, 2014, 351: 4334–4356

Appendix A

Proof of Proposition 1. By Lemma 4 and the definition of e,

$$\begin{aligned} -\lceil e\rfloor^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}\left(\lceil\xi_2\rfloor^{p_1}-\lceil\hat{\xi}_2\rfloor^{p_1}\right) &= -\lceil e\rfloor^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}\left(\left\lceil\lceil\xi_2\rfloor^{\frac{\tau}{r_2}}\right\rfloor^{\frac{r_1+\varpi}{\tau}}-\left\lceil\lceil\hat{\xi}_2\rfloor^{\frac{\tau}{r_2}}\right\rfloor^{\frac{r_1+\varpi}{\tau}}\right)\\ &= -\lceil e\rfloor^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}\left(\left\lceil\lceil\xi_2\rfloor^{\frac{\tau}{r_2}}\right\rfloor^{\frac{r_1+\varpi}{\tau}}-\left\lceil\lceil\xi_2\rfloor^{\frac{\tau}{r_2}}-e\right\rfloor^{\frac{r_1+\varpi}{\tau}}\right)\\ &\leqslant -\frac{|e|^{4\lambda}}{2^{\varpi}}. \end{aligned} \tag{A1}$$

Proof of Proposition 2. From (5), (9), (10), (15), (18), and Lemmas 1, 2, and 6, it follows that

$$\begin{aligned} &\frac{\tau}{r_2}\lceil e\rfloor^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}|\xi_2|^{\frac{\tau}{r_2}-1}(\lceil u\rfloor^{p_2}+\bar{f}_2)\\ &\leqslant \frac{\tau}{r_2}|e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}|\xi_2|^{\frac{\tau}{r_2}-1}\left(|u|^{p_2}+\beta_{12}|z_0|^{r_2+\varpi}+K_1^{\frac{r_2+\varpi}{r_1}}\beta_{22}|z_1|^{\frac{r_2+\varpi}{\tau}}+\beta_{22}|\xi_2|^{\frac{r_2+\varpi}{r_2}}\right)\\ &\leqslant |e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}\left(k_{21}(\xi_1)|u^{p_2}|^{\frac{\tau+\varpi}{r_2+\varpi}}+k_{22}(|z_0|)|z_0|^{\tau+\varpi}+k_{23}(\xi_1)|z_1|^{\frac{\tau+\varpi}{\tau}}+k_{24}(\xi_1)|z_2|^{\frac{\tau+\varpi}{\tau}}\right)\\ &\leqslant \frac{1}{4}|z_1|^{4\lambda}+\frac{1}{16}|z_2|^{4\tau\lambda}+M_{11}(\xi_1)|e|^{4\lambda}+N_{11}(\xi_1)|e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}|u^{p_2}|^{\frac{\tau+\varpi}{r_2+\varpi}}+\psi_{31}(|z_0|)|z_0|^{4\tau\lambda}, \end{aligned} \tag{A2}$$

where k_{21}, k_{22}, k_{23}, k_{24}, M_{11}, N_{11}, and ψ_{31} are some nonnegative smooth functions.

Proof of Proposition 3. By (6), (9), (10), (15), (18), and Lemmas 1, 2, and 6,

$$\begin{aligned}
&\lceil e \rceil^{\frac{4\tau\lambda-\varpi-\tau}{\tau}} \frac{\tau}{2r_2}\left(\frac{\tau}{2r_2}-1\right)\lceil \xi_2 \rceil^{\frac{\tau}{r_2}-2}\bar{g}_2^{\mathrm{T}}\bar{g}_2 \\
&\leqslant \frac{\tau}{2r_2}\left(\frac{\tau}{2r_2}-1\right)|e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}|\xi_2|^{\frac{\tau}{r_2}-2}\left(\beta_{23}|z_0|^{\frac{2r_2+\varpi}{2}}+K_1^{\frac{2r_2+\varpi}{2r_1}}\beta_{24}|z_1|^{\frac{2r_2+\varpi}{2\tau}}+\beta_{24}|\xi_2|^{\frac{2r_2+\varpi}{2r_2}}\right)^2 \\
&\leqslant |e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}}\left(k_{31}(|z_0|)|z_0|^{\tau+\varpi}+k_{32}(\xi_1)|z_1|^{\frac{\tau+\varpi}{\tau}}+k_{33}(\xi_1)|z_2|^{\frac{\tau+\varpi}{\tau}}\right) \\
&\leqslant \frac{1}{4}|z_1|^{4\lambda}+\frac{1}{16}|z_2|^{4\lambda}+M_{12}(\xi_1)|e|^{4\lambda}+\psi_{32}(|z_0|)|z_0|^{4\tau\lambda},
\end{aligned} \tag{A3}$$

where k_{31}, k_{32}, k_{33}, M_{12}, and ψ_{32} are some nonnegative smooth functions.

Proof of Proposition 4. From (6), (9), (10), (15), (18), and Lemmas 1, 2, and 6, it follows that

$$\begin{aligned}
&\frac{4\tau\lambda-\varpi-\tau}{2\tau}|e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}-1}\left(\frac{\tau}{r_2}|\xi_2|^{\frac{\tau}{r_2}-1}\right)^2\bar{g}_2^{\mathrm{T}}\bar{g}_2 \\
&\leqslant \frac{\tau(4\tau\lambda-\varpi-\tau)}{2r_2^2}|e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}-1}\left(|\xi_2|^{\frac{2(\tau-r_2)}{r_2}}\left(\beta_{23}|z_0|^{\frac{2r_2+\varpi}{2}}+K_1^{\frac{2r_2+\varpi}{2r_1}}\beta_{24}|z_1|^{\frac{2r_2+\varpi}{2\tau}}+\beta_{24}|\xi_2|^{\frac{2r_2+\varpi}{2r_2}}\right)^2\right) \\
&\leqslant |e|^{\frac{4\tau\lambda-\varpi-\tau}{\tau}-1}\left(k_{41}(|z_0|)|z_0|^{2\tau+\varpi}+k_{42}(\xi_1)|z_1|^{\frac{2\tau+\varpi}{\tau}}+k_{43}(\xi_1)|z_2|^{\frac{2\tau+\varpi}{\tau}}\right) \\
&\leqslant \frac{1}{4}|z_1|^{4\lambda}+\frac{1}{16}|z_2|^{4\lambda}+M_{13}(\xi_1)|e|^{4\lambda}+\psi_{33}(|z_0|)|z_0|^{4\tau\lambda},
\end{aligned} \tag{A4}$$

where k_{41}, k_{42}, k_{43}, M_{13}, and ψ_{33} are some nonnegative smooth functions.