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A FAS approach for stabilization of generalized chained forms: part 2. Continuous control laws

Guang-Ren DUAN[1,2]

[1]*Shenzhen Key Laboratory of Control Theory and Intelligent Systems, Southern University of Science and Technology, Shenzhen* 518055*, China;*
[2]*Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin* 150001*, China*

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Abstract In this paper, continuous time-varying stabilizing controllers for the type of general nonholonomic systems proposed and treated in part 1 are designed using the fully actuated system (FAS) approach. The key step is to differentiate the first scalar equation, and by control of the obtained second-order scalar system, a proportional plus integral feedback form for the first control variable is obtained. With the solution to this designed second-order scalar system, the rest equations in the nonholonomic system form an independent time-varying subsystem which is then handled by the FAS approach. The overall designed controller contains an almost arbitrarily chosen design parameter, and is proven to guarantee the uniformly and globally exponential stability of the closed-loop system. The proposed approach is simple and effective, and is demonstrated with a practical example of ship control.

Keywords nonholonomic systems, feedback stabilization, fully actuated systems, Lyapunov stability, continuous controllers

1 Introduction

This paper is a continuation of [1], which investigates the stabilization of a type of nonholonomic systems using the fully actuated system (FAS) approach.

Ever since the celebrated work of Brockett [2] in 1983, nonholonomic systems have attracted a great deal of attention and have been extensively studied by numerous researchers. Among these systems, two types of chained forms, which are natural extensions of the Brockett's two examples proposed in [2], have been crucially investigated, since they arise from many application backgrounds, including car-type vehicles, mobile robots, surface vessels, underwater vehicles, and spacecraft [3,4]. It has been shown that the Brockett's chained forms are in fact some kinds of canonical forms for a wide class of nonholonomic systems, since many nonholonomic systems can be represented by models in these chained forms or are feedback equivalent to these chained forms [5,6].

As revealed by Brockett [2], for a nonholonomic system, there does not exist a smooth time-invariant stabilizing controller, or there may not even exist a continuous time-invariant one. Therefore, in the literature, two attempts have been generally made in the stabilization of nonholonomic systems, one is using discontinuous feedback controllers [7–14], and the other is utilizing continuous, but time-varying, feedback controllers. Discontinuous stabilizing controllers are more natural and are relatively easier to design, but they often result in nonsmooth system responses. Comparatively, the type of continuous time-varying stabilizing controllers for nonholonomic systems is preferable [15–21]. For time-varying controller designs, global asymptotical stability with exponential convergence is achieved in [22] about any desired configuration by using a nonsmooth, time-varying feedback control law. In [23], a recursive technique is proposed which appears to be an extension of the popular integrator backstepping idea to the tracking of nonholonomic control systems. For the design of continuous time-varying controllers for the Brockett's second chained form, the approach proposed in [15] is wise and convenient, which employs an extended

Email: duangr@sustech.edu.cn, g.r.duan@hit.edu.cn

σ-process and converts the problem into the stabilization of controllable time-varying linear systems with the time-varying terms decaying exponentially and therefore being allowed to be neglected in the design.

Most of the reported results for control of nonlinear systems, including nonholonomic ones, are based on the general state-space approach. Parallel to the state-space approach, recently, the FAS approach for control system analysis and design has been introduced, which is originated from the two series of studies [24–26] and [27–36], and has been extended to more complicated systems (see [37–42]). Furthermore, the approach has been successfully applied to solve the stabilization of some nonholonomic systems [3, 4, 43]. Particularly, a generalized type of nonholonomic systems has been recently proposed in [1], and has been treated with the FAS approach.

The generalized type of nonholonomic systems proposed in [1] is an extension of the Brockett's chained forms, which involves two sets of nonlinear time-varying terms and a set of "selective" variables, namely, ρ_i, $i = 1, 2, \ldots, n-1$. The set of "selective" variables takes the value of either x_0, the first state variable in the system, or u_0, the first control variable in the system. Therefore, instead of representing a single nonlinear system, this model proposed in [1] really represents a set of 2^{n-1} systems. In the case that the two sets of the nonlinear terms are set to 0 and 1, respectively, the type of systems reduces to the Brockett's first chained form if all ρ_i's are chosen to be x_0, and reduces to the Brockett's second chained form if all ρ_i's are chosen to be u_0. Under the condition that the first scalar state variable x_0 is restricted to be nonzero, with the help of the well-known σ-process, a strict-feedback system (SFS) model for the second subsystem is firstly obtained, and is then converted into a global FAS. Based on the obtained FAS, a discontinuous controller is designed, which drives all the states of the designed system to zero exponentially provided that the initial value of the first scalar subsystem is restricted to be nonzero.

In this paper, the stabilization of the general type of nonholonomic systems proposed and studied in [1] is reconsidered, but using continuous time-varying controllers. Different from [1, 15], the first control variable u_0 is designed through differentiating the first scalar subsystem. Consequently, the partial controller is given in a "proportional plus integral" form and is thus a time-varying one. Then, as in [1, 15], by introducing a "σ-process"-like transformation, the second subsystem, which is formed by all the system equations but the first one, is transformed into an almost SFS. The obtained almost SFS is actually an SFS with each equation having an additional nonlinear term containing an exponentially decaying multiplier. Fortunately, it is proven that under very mild conditions, a uniformly globally and exponentially (UGE) stabilizing controller for the SFS also UGE stabilizes the converted almost SFS. In other words, the series of additional terms in the almost SFS in general do not affect the stabilization of the converted system and can be neglected. Hence, the problem is again turned into the control of an SFS of exactly the same form as the one obtained in [1] for the case of discontinuous controller design. Further, by converting the SFS into a FAS, a UGE stabilizing controller for the original nonholonomic system is then derived. The practical example treated in [1] is again treated with the proposed continuous stabilizing controller. It is worth mentioning that, since the controller contains an integral term, the designed control system eventually admits such a feature that the arbitrary constant disturbance existing in the first scalar subsystem is automatically and completely decoupled.

In the subsequent sections, the n-dimensional vector space and the matrix space of dimension $m \times n$ are defined as $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$, respectively, and the set of all nonnegative scalars is defined as $\mathbb{R}^+$. I_n denotes the identity matrix of order n. For a matrix $A \in \mathbb{R}^{m\times n}$, the notation $\|A\|$ represents its spectral norm. Moreover, for $x, x_i \in \mathbb{R}^m$ and $A_i \in \mathbb{R}^{m\times m}$, $i = 0, 1, 2, \ldots, n$, the following standard symbols for the FAS approach are used in the paper:

$$x^{(0\sim n)} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(n)} \end{bmatrix}, \; x_{i\sim j} = \begin{bmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_j \end{bmatrix}, \; i \leqslant j,$$

$$x_{i\sim j}^{(0\sim n)} = \begin{bmatrix} x_i^{(0\sim n)} \\ x_{i+1}^{(0\sim n)} \\ \vdots \\ x_j^{(0\sim n)} \end{bmatrix}, \; x_k^{(n_k)}|_{k=i\sim j} = \begin{bmatrix} x_i^{(n_i)} \\ x_{i+1}^{(n_{i+1})} \\ \vdots \\ x_j^{(n_j)} \end{bmatrix}, \; i \leqslant j,$$

$$A_{0\sim n} = \begin{bmatrix} A_0 & A_1 & \cdots & A_n \end{bmatrix}, \ \Phi\left(A_{0\sim n}\right) = \begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -A_0 & -A_1 & \cdots & -A_n \end{bmatrix}.$$

The paper consists of 7 sections. Section 2 gives the type of general nonholonomic systems to be stabilized, and the problem is further converted into a problem of stabilizing an almost SFS in Section 3. Analysis results on stabilizability of the obtained almost SFS are presented in Section 4. Section 5 further presents the general FAS approach. Application of the proposed approach to a ship control is provided in Section 6, followed by some concluding remarks in Section 7.

2 Problem statement

In this paper, the stabilization of the following general type of nonholonomic systems, which is introduced in [1], is considered

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_2\hat{\varphi}_1(\cdot) + \rho_1 x_1 \hat{\psi}_1(\cdot), \\ \dot{x}_3 = x_3\hat{\varphi}_2(\cdot) + \rho_2 x_2 \hat{\psi}_2(\cdot), \\ \vdots \\ \dot{x}_n = x_n\hat{\varphi}_{n-1}(\cdot) + \rho_{n-1} x_{n-1} \hat{\psi}_{n-1}(\cdot), \end{cases} \tag{1}$$

where x_i, $i = 0, 1, 2, \ldots, n$, are the system state variables, u_0 and u are the control variables,

$$\rho_i\left(x_0, u_0\right) \in \{x_0, u_0\}, \ i = 1, 2, \ldots, n-1, \tag{2}$$

and

$$\hat{\varphi}_i(\cdot) \triangleq \hat{\varphi}_i\left(x_0, x_{i+1\sim n}, u_0, t\right), \ \hat{\psi}_i(\cdot) \triangleq \hat{\psi}_i\left(x_0, x_{i+1\sim n}, u_0, t\right), \ i = 1, 2, \ldots, n-1 \tag{3}$$

are two sets of scalar functions with $\hat{\psi}_i(\cdot)$, $i = 1, 2, \ldots, n-1$, being required to satisfy the following assumption.

Assumption A. For all $x_0, u_0 \in \mathbb{R}$, $x_{i+1\sim n} \in \mathbb{R}^{n-i}$, and $t \geqslant 0$, there holds

$$\hat{\psi}_i\left(x_0, x_{i+1\sim n}, u_0, t\right) \neq 0, \ i = 1, 2, \ldots, n-1.$$

Clearly, system (1) is a time-invariant one if and only if $\hat{\varphi}_i(\cdot)$ and $\hat{\psi}_i(\cdot)$, $i = 1, 2, \ldots, n-1$, are all time-invariant.

In the special case of

$$\hat{\varphi}_i\left(x_0, x_{i+1\sim n}, u_0, t\right) = 0, \ \hat{\psi}_i\left(x_0, x_{i+1\sim n}, u_0, t\right) = 1, \ i = 1, 2, \ldots, n-1,$$

the above system (1) becomes

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = \rho_1 x_1, \\ \dot{x}_3 = \rho_2 x_2, \\ \vdots \\ \dot{x}_n = \rho_{n-1} x_{n-1}. \end{cases} \tag{4}$$

Further, choosing in (4) $\rho_i = x_0$, $i = 1, 2, \ldots, n-1$ and $\rho_i = u_0$, $i = 1, 2, \ldots, n-1$, gives, respectively, the following Brockett's first chained form:

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_0 x_1, \\ \vdots \\ \dot{x}_n = x_0 x_{n-1}, \end{cases} \tag{5}$$

and the Brockett's second chained form:

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_1 u_0, \\ \vdots \\ \dot{x}_n = x_{n-1} u_0. \end{cases} \tag{6}$$

These obviously contain the Brockett's first and second example systems proposed in [2] as special cases [2–4].

The following result is given in [1], which reveals the nonholonomic feature of the proposed system (1).

Proposition 1. The nonlinear system (1)–(3) does not have a smooth time-invariant exponentially stabilizing controller. Furthermore, for the special case that both $\hat{\varphi}_i(\cdot)$ and $\hat{\psi}_i(\cdot)$ are time-invariant, system (1)–(3) does not have a continuous time-invariant stabilizing controller if, for some $1 \leqslant i \leqslant n-1$, one of the following conditions holds:

(1) In the case of $\rho_i = x_0$,

$$\hat{\varphi}_i(\cdot)\,|_{u_0=0} = \hat{\psi}_i(\cdot)\,|_{u_0=0} = 0;$$

(2) In the case of $\rho_i = u_0$,

$$\hat{\varphi}_i(\cdot)\,|_{u_0=0} = 0.$$

The above fact states that, under certain circumstances, to realize stabilization of system (1)–(3) with a continuous controller, the controller must be a time-varying one. In [1], a discontinuous time-invariant stabilizing controller for system (1)–(3) is designed. While in this paper, a continuous time-varying stabilizing controller is sought.

Remark 1. In [1], several extensions of the above model (1) have also been given. These include the locally normal systems, sub-normal systems, the multivariable cases, and the time-delay cases. The time-delay cases also include the multiple time-delay case and the distributed time-delay case. Furthermore, adding three sets of system uncertainties, $\Delta\hat{\varphi}_i(\cdot)$, $\Delta\hat{\psi}_i(\cdot)$, $i = 1, 2, \ldots, n-1$, and $\Delta f_i(x_{0\sim n}, u_0, t)$, $i = 0, 1, \ldots, n-1$, to the model (1), gives the following uncertain system:

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u + \Delta f_0(x_{0\sim n}, u_0, t), \\ \dot{x}_2 = x_2[\hat{\varphi}_1(\cdot) + \Delta\hat{\varphi}_1(\cdot)] + \rho_1 x_1[\hat{\psi}_1(\cdot) + \Delta\hat{\psi}_1(\cdot)] + \Delta f_1(x_{0\sim n}, u_0, t), \\ \dot{x}_3 = x_3[\hat{\varphi}_2(\cdot) + \Delta\hat{\varphi}_2(\cdot)] + \rho_2 x_2[\hat{\psi}_2(\cdot) + \Delta\hat{\psi}_2(\cdot)] + \Delta f_2(x_{0\sim n}, u_0, t), \\ \vdots \\ \dot{x}_n = x_n[\hat{\varphi}_{n-1}(\cdot) + \Delta\hat{\varphi}_{n-1}(\cdot)] + \rho_{n-1} x_{n-1}[\hat{\psi}_{n-1}(\cdot) + \Delta\hat{\psi}_{n-1}(\cdot)] + \Delta f_{n-1}(x_{0\sim n}, u_0, t). \end{cases} \tag{7}$$

Robust stabilization of the above system can also be solved with the proposed FAS approach (under consideration).

3 Problem conversion

To make the treatment simpler, as done in [1], let us divide system (1) into the following two subsystems:

$$\dot{x}_0 = u_0, \tag{8}$$

and

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_2\hat{\varphi}_1(\cdot) + \rho_1 x_1 \hat{\psi}_1(\cdot), \\ \dot{x}_3 = x_3\hat{\varphi}_2(\cdot) + \rho_2 x_2 \hat{\psi}_2(\cdot), \\ \vdots \\ \dot{x}_n = x_n\hat{\varphi}_{n-1}(\cdot) + \rho_{n-1} x_{n-1} \hat{\psi}_{n-1}(\cdot). \end{cases} \tag{9}$$

3.1 Solution of u_0

Differentiating the first subsystem (8) gives

$$\ddot{x}_0 = \dot{u}_0. \tag{10}$$

Thus we can design the following simple controller based on (10):

$$\dot{u}_0 = -a_0 x_0 - a_1 \dot{x}_0, \tag{11}$$

or equivalently

$$u_0(t) = -a_0 \int_0^t x_0(s)\,\mathrm{d}s - a_1(x_0(t) - x_0(0)) + \dot{x}_0(0), \tag{12}$$

where a_i, $i = 0, 1$, are two positive scalars. The closed-loop system is

$$\ddot{x}_0 + a_1\dot{x}_0 + a_0 x_0 = 0. \tag{13}$$

Remark 2. The above treatment also has the ability to decouple any constant disturbances in the first subsystem (8). Specifically, when the subsystem (8) is replaced with

$$\dot{x}_0 = u_0 + d, \tag{14}$$

where d is a constant disturbance, after differentiating, the same system (10) is obtained. Hence, with the above controller (12), any constant disturbance d in the subsystem (14) is automatically decoupled.

Regarding the response of the above system (13), we have the following result [44].

Proposition 2. Let a_0 and a_1 be determined by

$$\alpha + \beta = a_1, \ \alpha\beta = a_0, \tag{15}$$

where $\alpha > \beta > 0$. Further, let $\zeta_0 = x_0(0)$ and $\eta_0 = \dot{x}_0(0)$ denote the initial values. Then the response of system (13) is given by

$$x_0(t) = -c_1 \mathrm{e}^{-\alpha t} + c_2 \mathrm{e}^{-\beta t}, \tag{16}$$

$$\dot{x}_0(t) = \alpha c_1 \mathrm{e}^{-\alpha t} - \beta c_2 \mathrm{e}^{-\beta t}, \tag{17}$$

where

$$c_1 = \frac{1}{\alpha - \beta}(\eta_0 + \beta\zeta_0), \ c_2 = \frac{1}{\alpha - \beta}(\eta_0 + \alpha\zeta_0). \tag{18}$$

Impose the following assumption on the initial values of the first closed-loop subsystem (13).

Assumption B. $\eta_0 = \dot{x}_0(0) \neq -\alpha x_0(0) = -\alpha\zeta_0$.

Then, under the above assumption, we clearly have

$$c_2 = \frac{1}{\alpha - \beta}(\eta_0 + \alpha\zeta_0) \neq 0.$$

With the help of the above Proposition 2, the following result can be immediately obtained.

Proposition 3. Let Assumption B be met, and $\alpha > \beta > 0$ be scalars satisfying (15). Further, let

$$\begin{cases} \sigma_0(t) = c_2 \mathrm{e}^{-\beta t} \neq 0, \ \forall t \geqslant 0, \\ \omega(t) = \frac{c_1}{c_2} \mathrm{e}^{-(\alpha-\beta)t}. \end{cases} \tag{19}$$

Then there hold

$$x_0(t) = \sigma_0(t)[1 - \omega(t)], \tag{20}$$

$$u_0(t) = \dot{x}_0(t) = \sigma_0(t)[-\beta + \alpha\omega(t)], \tag{21}$$

and

$$\rho_i(\gamma_i, \vartheta_i, t) \triangleq \rho_i(x_0, \dot{x}_0) = \sigma_0(t)[\gamma_i - \vartheta_i \omega(t)], \tag{22}$$

where

$$(\gamma_i, \vartheta_i) = \begin{cases} (1, 1), & \text{if } \rho_i = x_0, \\ (-\beta, -\alpha), & \text{if } \rho_i = u_0. \end{cases} \tag{23}$$

Owing to (16) and (17), we can now write

$$\begin{aligned} \varphi_i'(\cdot) &\triangleq \varphi_i'(\zeta_0, \eta_0, x_{i+1\sim n}, t) = \hat{\varphi}_i(x_0, x_{i+1\sim n}, u_0, t), \\ \psi_i'(\cdot) &\triangleq \psi_i'(\zeta_0, \eta_0, x_{i+1\sim n}, t) = \hat{\psi}_i(x_0, x_{i+1\sim n}, u_0, t), \ i = 1, 2, \ldots, n-1, \end{aligned} \tag{24}$$

and Assumption A correspondingly becomes the following.

Assumption A′. For all $\zeta_0, \eta_0 \in \mathbb{R}$, $x_{i+1\sim n} \in \mathbb{R}^{n-i}$, and $t \geqslant 0$, there holds

$$\psi_i'(\zeta_0, \eta_0, x_{i+1\sim n}, t) \neq 0, \ i = 1, 2, \ldots, n-1.$$

Furthermore, due to (22), the subsystem (9) can be written as

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_2 \varphi_1'(\cdot) + \rho_1(\gamma_1, \vartheta_1, t) x_1 \psi_1'(\cdot), \\ \dot{x}_3 = x_3 \varphi_2'(\cdot) + \rho_2(\gamma_2, \vartheta_2, t) x_2 \psi_2'(\cdot), \\ \vdots \\ \dot{x}_n = x_n \varphi_{n-1}'(\cdot) + \rho_{n-1}(\gamma_{n-1}, \vartheta_{n-1}, t) x_{n-1} \psi_{n-1}'(\cdot). \end{cases} \tag{25}$$

It is seen from the above that, when u_0 is designed as in (11), $x_0(t)$ and $\dot{x}_0(t)$ are given by (20) and (21), respectively. Hence, in the above subsystem (25), $\rho_i(\gamma_i, \vartheta_i, t)$ is really a time-varying parameter.

Based on the above deduction, it can be clearly seen that, to realize the stabilization of system (1) with the help of the partial controller (11) or (12), it suffices to solve the problem of stabilizing system (25) under Assumption B.

3.2 σ-process

To stabilize system (25), let us transform the system into an almost SFS using an extended σ-process.

Theorem 1. Let Assumption B be met. Then, under the following transformation:

$$z_i = \frac{x_{n-i+1}}{\sigma_0^{n-i}}, \ i = 1, 2, \ldots, n, \tag{26}$$

system (25) is equivalently transformed into the following system:

$$\begin{cases} \dot{z}_1 = g_1'(\cdot) + h_1'(\cdot) z_2 + \Delta f_1(\cdot), \\ \dot{z}_2 = g_2'(\cdot) + h_2'(\cdot) z_3 + \Delta f_2(\cdot), \\ \vdots \\ \dot{z}_{n-1} = g_{n-1}'(\cdot) + h_{n-1}'(\cdot) z_n + \Delta f_{n-1}(\cdot), \\ \dot{z}_n = u, \end{cases} \tag{27}$$

where

$$g_i'(\cdot) \triangleq g_i'(\zeta_0, \eta_0, z_{1\sim i}, t) = \left[(n-i)\beta + \varphi_{n-i}'(\zeta_0, \eta_0, x_{n-i+1\sim n}, t)\right] z_i, \tag{28}$$

$$h_i'(\cdot) \triangleq h_i'(\zeta_0, \eta_0, z_{1\sim i}, t) = \gamma_{n-i}\psi_{n-i}'(\zeta_0, \eta_0, x_{n-i+1\sim n}, t), \tag{29}$$

$$\Delta f_i(\cdot) \triangleq \Delta f_i(\zeta_0, \eta_0, z_{1\sim i+1}, t) = -\vartheta_{n-i}\omega\psi_{n-i}'(\zeta_0, \eta_0, x_{n-i+1\sim n}, t) z_{i+1}, \tag{30}$$

$i = 1, 2, \ldots, n-1$.

Proof. Recalling the definition of $\sigma_0(t)$ in (19), we have

$$\dot{\sigma}_0(t) = -\beta\sigma_0(t). \tag{31}$$

As in [15], let us introduce the following transformation:

$$y_i = \frac{x_i}{\sigma_0^{i-1}}, \; i = 1, 2, \ldots, n. \tag{32}$$

Then, noting (22), (31), and

$$\dot{x}_i = x_i\varphi_{i-1}'(\cdot) + \rho_{i-1}(\gamma_{i-1}, \vartheta_{i-1}, t) x_{i-1}\psi_{i-1}'(\cdot), \; i = 2, 3, \ldots, n,$$

we have

$$\dot{y}_1 = \dot{x}_1 = u, \tag{33}$$

and

$$\begin{aligned} \dot{y}_i &= \frac{\dot{x}_i\sigma_0^{i-1} - x_i(i-1)\sigma_0^{i-2}\dot{\sigma}_0}{\sigma_0^{2(i-1)}} \\ &= \frac{[x_i\varphi_{i-1}'(\cdot) + \rho_{i-1}(\gamma_{i-1}, \vartheta_{i-1}, t) x_{i-1}\psi_{i-1}'(\cdot)]\sigma_0^{i-1}}{\sigma_0^{2(i-1)}} + \frac{(i-1)\beta x_i\sigma_0^{i-1}}{\sigma_0^{2(i-1)}} \\ &= (\gamma_{i-1} - \vartheta_{i-1}\omega)\psi_{i-1}'(\cdot)\frac{x_{i-1}}{\sigma_0^{i-2}} + \left[(i-1)\beta + \varphi_{i-1}'(\cdot)\right]\frac{x_i}{\sigma_0^{i-1}} \\ &= (\gamma_{i-1} - \vartheta_{i-1}\omega)\psi_{i-1}'(\cdot) y_{i-1} + \left[(i-1)\beta + \varphi_{i-1}'(\cdot)\right] y_i, \\ & \qquad i = 2, 3, \ldots, n. \end{aligned} \tag{34}$$

Combining (33) with (34), gives the following equivalent system of the original system (25):

$$\begin{cases} \dot{y}_1 = u, \\ \dot{y}_2 = [\beta + \varphi_1'(\cdot)] y_2 + (\gamma_1 - \vartheta_1\omega)\psi_1'(\cdot) y_1, \\ \dot{y}_3 = [2\beta + \varphi_2'(\cdot)] y_3 + (\gamma_2 - \vartheta_2\omega)\psi_2'(\cdot) y_2, \\ \vdots \\ \dot{y}_n = \left[(n-1)\beta + \varphi_{n-1}'(\cdot)\right] y_n + (\gamma_{n-1} - \vartheta_{n-1}\omega)\psi_{n-1}'(\cdot) y_{n-1}. \end{cases} \tag{35}$$

Secondly, applying the following state transformation:

$$z_i = y_{n-i+1}, \; i = 1, 2, \ldots, n, \tag{36}$$

the above system (35) is equivalently transformed into

$$\begin{cases} \dot{z}_1 = \left[(n-1)\beta + \varphi_{n-1}'(\cdot)\right] z_1 + (\gamma_{n-1} - \vartheta_{n-1}\omega)\psi_{n-1}'(\cdot) z_2, \\ \dot{z}_2 = \left[(n-2)\beta + \varphi_{n-2}'(\cdot)\right] z_2 + (\gamma_{n-2} - \vartheta_{n-2}\omega)\psi_{n-2}'(\cdot) z_3, \\ \vdots \\ \dot{z}_{n-1} = [\beta + \varphi_1'(\cdot)] z_{n-1} + (\gamma_1 - \vartheta_1\omega)\psi_1'(\cdot) z_n, \\ \dot{z}_n = u, \end{cases} \tag{37}$$

which can be further written in the form of (27) with $g_i'(\zeta_0, \eta_0, z_{1\sim i}, t)$, $h_i'(\zeta_0, \eta_0, z_{1\sim i}, t)$, and $\Delta f_i(\zeta_0, \eta_0, z_{1\sim i}, t)$, $i = 1, 2, \ldots, n-1$, defined as in (28)–(30).

Finally, combining the transformations (32) and (36) gives the transformation (26).

Corresponding to Assumptions A and A′ on the series of functions $\psi_i'(\zeta_0, \eta_0, x_{i+1\sim n}, t)$, $i = 1, 2, \ldots, n-1$, we have the following assumption on the functions $h_i'(\zeta_0, \eta_0, z_{1\sim i}, t)$, $i = 1, 2, \ldots, n-1$.

Assumption A″. For all $\zeta_0, \eta_0 \in \mathbb{R}$, $z_{1\sim i} \in \mathbb{R}^i$, and $t \geqslant 0$, there holds

$$h_i'(\zeta_0, \eta_0, z_{1\sim i}, t) \neq 0, \ i = 1, 2, \ldots, n-1.$$

Clearly, when $\varphi_i'(\cdot)$ and $\psi_i'(\cdot)$, $i = 1, 2, \ldots, n-1$, are state-dependent, unlike the simple case treated in [15], the transformed system (27) is no longer a linear one.

With the above understanding, to realize the stabilization of system (25), it suffices to solve the following problem.

Problem 1. Find a UGE stabilizing controller for system (27) under Assumption A″.

4 Solvability

To find out the solvability of Problem 1, we first need some preliminary results.

4.1 A technical lemma

The following result performs an important function in the design of the continuous stabilizing controller for system (27).

Lemma 1. Consider the following perturbed nonlinear system:

$$\dot{x} = f(x,t) + \Delta f(x,t), \tag{38}$$

where (1) $f(x,t) \in \mathbb{R}^n$ is a continuous function, such that there exists a continuous function $V(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}$ that satisfies the inequalities:

$$\mu_1 \|x\|^{\mu_0} \leqslant V(x,t) \leqslant \mu_2 \|x\|^{\mu_0}, \tag{39}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leqslant -\mu_3 \|x\|^{\mu_0}, \tag{40}$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leqslant \rho_v(t) \|x\|^{(1-\vartheta)\mu_0}, \tag{41}$$

with μ_i, $i = 0, 1, 2, 3$, being a set of positive scalars, $0 < \vartheta < 1$, and $\rho_v(t)$ being a nonnegative scalar function; (2) $\Delta f(x,t) \in \mathbb{R}^n$ satisfies

$$\|\Delta f(x,t)\| \leqslant \rho_\Delta(t) \|x\|^{\vartheta \mu_0}, \tag{42}$$

with $\rho_\Delta(t)$ being a nonnegative function satisfying

$$\int_0^\infty \rho_v(s) \rho_\Delta(s) \mathrm{d}s = M < \infty. \tag{43}$$

Then system (38) is UGE stable.

Proof. Take the continuous function $V(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}$ as a Lyapunov function for the perturbed system (38). Using (39)–(42), we have

$$\begin{aligned}
\dot{V}(x,t) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f(x,t) + \Delta f(x,t)) \\
&\leqslant -\mu_3 \|x\|^{\mu_0} + \left\| \frac{\partial V}{\partial x} \right\| \|\Delta f(x,t)\| \\
&\leqslant -\mu_3 \|x\|^{\mu_0} + \rho_v(t) \rho_\Delta(t) \|x\|^{\mu_0} \\
&\leqslant -\frac{\mu_3}{\mu_2} V(x,t) + \frac{1}{\mu_1} \rho_v(t) \rho_\Delta(t) V(x,t) \\
&= \left(-\frac{\mu_3}{\mu_2} + \frac{1}{\mu_1} \rho_v(t) \rho_\Delta(t) \right) V(x,t).
\end{aligned} \tag{44}$$

By the Comparison Lemma (see the Lemma 3.4 of [45]), $V(x,t)$ is bounded above by the solution of the following first-order linear differential equation:

$$\dot{y}(t) = \left(-\frac{\mu_3}{\mu_2} + \frac{1}{\mu_1}\rho_v\left(t\right)\rho_\Delta(t)\right) y(t), \tag{45}$$

with the initial value

$$y(0) = V(x(0),0).$$

Since

$$\begin{aligned} y\left(t\right) &= y\left(0\right)\exp\left(\int_0^t \left(-\frac{\mu_3}{\mu_2} + \frac{1}{\mu_1}\rho_v\left(s\right)\rho_\Delta(s)\right)\mathrm{d}s\right) \\ &= y\left(0\right)\mathrm{e}^{-\frac{\mu_3}{\mu_2}t}\exp\left(\frac{1}{\mu_1}\int_0^t \rho_v\left(s\right)\rho_\Delta(s)\mathrm{d}s\right) \\ &\leqslant y\left(0\right)\mathrm{e}^{-\frac{\mu_3}{\mu_2}t}\exp\left(\frac{1}{\mu_1}\int_0^\infty \rho_v\left(s\right)\rho_\Delta(s)\mathrm{d}s\right) \\ &\leqslant y\left(0\right)\mathrm{e}^{\frac{1}{\mu_1}M}\mathrm{e}^{-\frac{\mu_3}{\mu_2}t}, \end{aligned}$$

we have

$$V\left(x,t\right) \leqslant y\left(t\right) \leqslant y\left(0\right)\mathrm{e}^{\frac{1}{\mu_1}M}\mathrm{e}^{-\frac{\mu_3}{\mu_2}t}.$$

This gives, in view of (39),

$$\|x\|^{\mu_0} \leqslant \frac{1}{\mu_1}V(x,t) \leqslant c\mathrm{e}^{-\frac{\mu_3}{\mu_2}t},$$

where

$$c = \frac{1}{\mu_1}y\left(0\right)\mathrm{e}^{\frac{1}{\mu_1}M}.$$

Therefore, system (38) is UGE stable.

Taking $\mu_0 = 2$ and $\vartheta = \frac{1}{2}$ in the above lemma, immediately gives the following corollary.

Corollary 1. The perturbed nonlinear system (38) is UGE stable if the following conditions are met:

(1) There exists a continuous function $V\left(x,t\right) : \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}$ that satisfies the inequalities

$$\mu_1\|x\|^2 \leqslant V(x,t) \leqslant \mu_2\|x\|^2, \tag{46}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(x,t) \leqslant -\mu_3\|x\|^2, \tag{47}$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leqslant \rho_v\left(t\right)\|x\|, \tag{48}$$

with μ_i, $i = 1,2,3$, being a set of positive scalars, and $\rho_v\left(t\right)$ being a nonnegative scalar function;

(2) $\Delta f(x,t) \in \mathbb{R}^n$ satisfies

$$\|\Delta f(x,t)\| \leqslant \rho_\Delta(t)\|x\|, \tag{49}$$

with $\rho_\Delta(t)$ being a nonnegative function satisfying (43).

When the assumption is further strengthened, the following corollaries can be given.

Corollary 2. The perturbed nonlinear system (38) is UGE stable if the following conditions are met:

(1) There exists a continuous function $V\left(x,t\right) : \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}$ that satisfies inequalities (46)–(48) with $\rho_v\left(t\right) = \rho_v$ being a nonnegative scalar;

(2) $\Delta f(x,t) \in \mathbb{R}^n$ satisfies (49) and

$$\int_0^\infty \rho_\Delta(s)\mathrm{d}s = M < \infty, \tag{50}$$

with $\rho_\Delta(t)$ being a nonnegative function.

Corollary 3. Suppose that $f(x,t) \in \mathbb{R}^n$ is continuously differentiable with respect to t, and satisfies one of the following conditions:

(1) The Jacobian matrix $[\partial f/\partial x]$ is globally bounded, uniformly in t;

(2) $f(x,t)$ is globally Lipschitz with respect to x, uniformly in t.

Furthermore, the system

$$\dot{x} = f(x,t), \tag{51}$$

is UGE stable, and $\Delta f(x,t) \in \mathbb{R}^n$ satisfies (49), with $\rho_\Delta(t)$ being a nonnegative function satisfying (50); then system (38) is UGE stable.

Proof. Firstly, note that the uniform global exponential stability of system (51) and the first condition in the corollary implies (see the Theorem 4.14 of [45]) the existence of a continuous function $V(x,t)$: $\mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}$ that satisfies the inequalities (46)–(48), with $\rho_v(t)$ being a constant. Therefore, when Eq. (50) is valid, the conclusion immediately follows from Corollary 1.

Secondly, in view of the fact that the second condition on the global Lipschitz property of $f(x,t)$ is equivalent to the first condition, the whole proof is completed.

4.2 Stabilizability

In this subsection, let us further give a stabilizability condition for system (27) based on the stability results in Subsection 4.1.

4.2.1 *Case of* $\eta_0 = -\beta\zeta_0$

In this case, we have

$$c_1 = \eta_0 + \beta\zeta_0 = 0. \tag{52}$$

Thus $\omega = 0$, and it further follows from (30) that

$$\Delta f_i(\cdot) = 0,\ i = 1,2,\ldots,n-1. \tag{53}$$

Consequently, system (27) reduces to

$$\begin{cases} \dot{z}_1 = g_1'(\zeta_0,\eta_0,z_1,t) + h_1'(\zeta_0,\eta_0,z_1,t)\,z_2, \\ \dot{z}_2 = g_2'(\zeta_0,\eta_0,z_{1\sim 2},t) + h_2'(\zeta_0,\eta_0,z_{1\sim 2},t)\,z_3, \\ \vdots \\ \dot{z}_{n-1} = g_{n-1}'(\zeta_0,\eta_0,z_{1\sim n-1},t) + h_{n-1}'(\zeta_0,\eta_0,z_{1\sim n-1},t)\,z_n, \\ \dot{z}_n = u, \end{cases} \tag{54}$$

which is clearly an SFS when Assumption A3$'$ is met. Therefore, in this case, in order to solve Problem 1, it is equivalent to solving the following one.

Problem 2. Find a UGE stabilizing controller for system (54) under Assumption A$''$.

There are two important things to be noted.

Firstly, under the condition of $\eta_0 = -\beta\zeta_0$, following Proposition 2 we can get $c_2 = \zeta_0$, and

$$x_0(t) = \zeta_0 \mathrm{e}^{-\beta t},\ \dot{x}_0(t) = -\beta\zeta_0\mathrm{e}^{-\beta t}. \tag{55}$$

Thus, by further using the relations in (15), the controller (12) is reduced to

$$\begin{aligned} u_0(t) &= -a_0\zeta_0\int_0^t \mathrm{e}^{-\beta s}\mathrm{d}s - a_1\zeta_0\mathrm{e}^{-\beta t} + (a_1-\beta)\,\zeta_0 \\ &= a_0\zeta_0\frac{1}{\beta}\left(\mathrm{e}^{-\beta t}-1\right) - a_1\zeta_0\mathrm{e}^{-\beta t} + \alpha\zeta_0 \\ &= \left(a_0\frac{1}{\beta} - a_1\right)\zeta_0\mathrm{e}^{-\beta t} \\ &= -\beta\zeta_0\mathrm{e}^{-\beta t}, \end{aligned} \tag{56}$$

that is,

$$u_0(t) = -\beta x_0(t). \tag{57}$$

Secondly, it is Assumption B, which guarantees $c_2 \neq 0$, and hence $\sigma_0(t) \neq 0$, $t \geqslant 0$, remembering that this is needed in the σ-process of deriving system (54). Considering the condition of $\eta_0 = -\beta\zeta_0$ and $\alpha > \beta$, we can obviously observe that Assumption B holds if and only if $\zeta_0 \neq 0$. Therefore, in this particular case, the initial value of x_0 must not be chosen to be zero.

Combining the above two aspects, it can be easily recognized that this particular case exactly coincides with the case treated in [1]. Eventually, the controller derived in [1] for this case should be no longer a continuous one due to the restriction of $x_0(0) \neq 0$.

Remark 3. As pointed out in [1], the above Problem 2 can be indeed solved by applying the well-known backstepping technique. However, as it is well-known, the method of backstepping suffers from the serious problem of "complexity explosion", which renders the application of the method of backstepping extremely difficult or even impossible when the dimension of the system is large. Furthermore, compared with the FAS approach given in Section 5, the method of backstepping may have two drawbacks:

(1) It is not guaranteed that a UGE stable closed-loop system can be always obtained as the UGE stabilizability of system (27) requires UGE stabilization of SFS (54);

(2) It does not always provide a linear closed-loop system like the FAS approach does.

Remark 4. Please note that, although η_0 is an initial value of system (10), relative to the original Problem 1, it is only an external design parameter. Hence it can be allowed to be set to any desired value. From this point of view, choosing $\eta_0 = -\beta\zeta_0$ may just sufficiently and perfectly solve the problem in the above sense of converting the problem into Problem 2. However, as an initial value of system (10), after all, η_0 may affect the response of system (10), that is, $x_0(t)$ and $u_0(t)$, and hence that of the whole system.

Owing to the above Remark 4, let us now also give a treatment for the case of $\eta_0 \neq -\beta\zeta_0$, which will produce a continuous controller.

4.2.2 *Case of* $\eta_0 \neq -\beta\zeta_0$

Please note that, in this case, we have $c_1 \neq 0$. Hence all the terms $\Delta f_i(\cdot)$, $i = 1, 2, \ldots, n-1$, may present in system (27). To cope with this case, let us introduce the following assumption.

Assumption C. There exists a series of positive functions $\rho_{\psi i}(t)$, $i = 1, 2, \ldots, n-1$, satisfying

$$\left|\psi'_{n-i}(\zeta_0, \eta_0, z_{1\sim i}, t)\, z_{i+1}\right| \leqslant \frac{1}{|\vartheta_{n-i}|}\rho_{\psi i}(t)\, \|z_{1\sim n}\|,\ t \geqslant 0,\ i = 1, 2, \ldots, n-1.$$

When a feedback stabilizing controller $u = u(z_{1\sim n}, t)$ for SFS (54) is found, the closed-loop system can be represented in the form of $\dot{x} = f(x, t)$ with

$$f(x,t) = \begin{bmatrix} g'_1(\cdot) + h'_1(\cdot)\, z_2 \\ g'_2(\cdot) + h'_2(\cdot)\, z_3 \\ \vdots \\ g'_{n-1}(\cdot) + h'_{n-1}(\cdot)\, z_n \\ u(z_{1\sim n}, t) \end{bmatrix},\ x = z_{1\sim n}. \tag{58}$$

When the same feedback stabilizing controller $u = u(z_{1\sim n}, t)$ is applied to system (27), the closed-loop system can be represented in the form of (38) with $f(x, t)$ being given by (58) and

$$\Delta f(x,t) = \begin{bmatrix} \Delta f_1^{\mathrm{T}}(\cdot) & \Delta f_2^{\mathrm{T}}(\cdot) & \cdots & \Delta f_{n-1}^{\mathrm{T}}(\cdot) & 0 \end{bmatrix}^{\mathrm{T}}. \tag{59}$$

Under Assumption C, we have, using the expression of $\omega(t)$ in (19), the following relations:

$$\begin{aligned} |\Delta f_i(\cdot)| &\leqslant |\vartheta_{n-i}\omega| \left|\psi'_{n-i}(\cdot)\, z_{i+1}\right| \\ &\leqslant \left|\frac{c_1}{c_2}\right| \rho_{\psi i}(t)\, \mathrm{e}^{-(\alpha-\beta)t} \|z_{1\sim n}\|,\ i = 1, 2, \ldots, n-1. \end{aligned}$$

Therefore,

$$\|\Delta f(\cdot)\| \leqslant \left|\frac{c_1}{c_2}\right| \|\rho_\psi(t)\| \mathrm{e}^{-(\alpha-\beta)t} \|z_{1\sim n}\|, \tag{60}$$

where

$$\rho_\psi(t) = \left[\begin{array}{cccc} \rho_{\psi 1}(t) & \rho_{\psi 2}(t) & \cdots & \rho_{\psi, n-1}(t) \end{array}\right]^{\mathrm{T}}. \tag{61}$$

This implies that, in view of $c_1 \neq 0$, inequality (42) is satisfied with

$$\rho_\Delta(t) = \|\rho_\psi(t)\| \mathrm{e}^{-(\alpha-\beta)t}. \tag{62}$$

With the above analysis, the following result can be immediately obtained by using Corollary 1.

Theorem 2. Let Assumption C be satisfied, $u = u(z_{1\sim n}, t)$ be a UGE stabilizing controller for system (54) such that, with $f(x,t)$ and $\Delta f(x,t)$ being given by (58) and (59), respectively, there exist a continuous function $V(x,t) : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ and a positive function $\rho_v(t)$ satisfying the inequalities (46)–(48). Further, if

$$\int_0^\infty \rho_v(s) \|\rho_\psi(s)\| \mathrm{e}^{-(\alpha-\beta)s} \mathrm{d}s = M < \infty, \tag{63}$$

then $u = u(z_{1\sim n}, t)$ is also a controller that UGE stabilizes system (27).

Similarly, using Corollary 3, the following result can also be immediately obtained.

Theorem 3. Let Assumption C be satisfied, $g_i'(\cdot)$ and $h_i'(\cdot)$, $i = 1, 2, \ldots, n-1$, be continuously differentiable with respect to t, and satisfy one of the following conditions:

(1) The Jacobian matrices $\frac{\partial g_i'(\cdot)}{\partial z_{1\sim i}}$ and $\frac{\partial h_i'(\cdot)}{\partial z_{1\sim i}}$, $i = 1, 2, \ldots, n-1$, are globally bounded, uniformly in t;

(2) $g_i'(\cdot)$ and $h_i'(\cdot) z_{i+1}$, $i = 1, 2, \ldots, n-1$, are globally Lipschitz with respect to $z_{1\sim i+1}$, uniformly in t.

If system (54) is UGE stabilizable with a controller $u = u(z_{1\sim n}, t)$, and

$$\int_0^\infty \|\rho_\psi(s)\| \mathrm{e}^{-(\alpha-\beta)s} \mathrm{d}s = M < \infty, \tag{64}$$

then system (27) is also UGE stabilizable with the same controller $u = u(z_{1\sim n}, t)$.

Conditions (63) and (64) are not strict at all. They allow $\rho_v(t)\|\rho_\psi(t)\|$ or $\|\rho_\psi(t)\|$ to diverge with exponential rates. Specifically, Eq. (63) allows

$$\rho_v(t)\|\rho_\psi(t)\| < \mathrm{e}^{(\alpha-\beta)t},$$

and Eq. (64) allows

$$\|\rho_\psi(t)\| < \mathrm{e}^{(\alpha-\beta)t}.$$

For given functions $\psi_i'(\cdot)$, $i = 1, 2, \ldots, n-1$, these conditions are often satisfied or can be satisfied by adjusting the design parameters α and β. Therefore, roughly speaking, in most cases the stabilization of system (27) can be realized by finding a UGE stabilizing controller of system (54).

With the above understanding, again, to solve Problem 1, under Assumption C and the mild condition (63) or (64), it often suffices to solve Problem 2.

5 FAS approach

In this section, let us give the solution to the original stabilization problem stated in Section 2. Firstly, let us present the FAS model of SFS (54).

5.1 FAS models

Note that the SFS (54) is in the exact same form of the SFS obtained in [1]. Simply modifying a result in [1] (see also [28]), gives the following theorem.

Theorem 4. Let Assumption A″ be satisfied, and $g'_k(\zeta_0, \eta_0, z_{1\sim k}, t)$ and $h'_k(\zeta_0, \eta_0, z_{1\sim k}, t)$, $k = 1, 2, \ldots, n-1$, be differentiable with respect to all variables. With the convention of $g'_n(\zeta_0, \eta_0, z_{1\sim n}, t) = 0$, let

$$\begin{aligned} B_k(\zeta_0, \eta_0, z_{1\sim k}, t) &= \prod_{i=1}^{k} h'_i(\zeta_0, \eta_0, z_{1\sim i}, t), \; k = 1, 2, \ldots, n-1, &(65)\\ B_n(\zeta_0, \eta_0, z_{1\sim n}, t) &= B_{n-1}(\zeta_0, \eta_0, z_{1\sim n-1}, t), &(66)\end{aligned}$$

and

$$\begin{aligned} f_k(\zeta_0, \eta_0, z_{1\sim k}, t) = \; & \dot{f}_{k-1}(\zeta_0, \eta_0, z_{1\sim k-1}, t) + \dot{B}_{k-1}(\zeta_0, \eta_0, z_{1\sim k-1}, t) z_k \\ & + B_{k-1}(\zeta_0, \eta_0, z_{1\sim k-1}, t) g'_k(\zeta_0, \eta_0, z_{1\sim k}, t), \; k = 2, 3, \ldots, n, \end{aligned} \tag{67}$$

with the initial value

$$f_1(\zeta_0, \eta_0, z_1, t) = g'_1(\zeta_0, \eta_0, z_1, t). \tag{68}$$

Then, under the following transformation:

$$z^{(0\sim n-1)} = \begin{bmatrix} z_1 \\ f_1(\zeta_0, \eta_0, z_1, t) + B_1(\zeta_0, \eta_0, z_1, t) z_2 \\ f_2(\zeta_0, \eta_0, z_{1\sim 2}, t) + B_2(\zeta_0, \eta_0, z_{1\sim 2}, t) z_3 \\ \vdots \\ f_{n-1}(\zeta_0, \eta_0, z_{1\sim n-1}, t) + B_{n-1}(\zeta_0, \eta_0, z_{1\sim n-1}, t) z_n \end{bmatrix}, \tag{69}$$

the SFS (54) is equivalently transformed into the following FAS:

$$z^{(n)} = f_n(\zeta_0, \eta_0, z_{1\sim n}, t) + B_n(\zeta_0, \eta_0, z_{1\sim n}, t) u, \tag{70}$$

with

$$B_n(\zeta_0, \eta_0, z_{1\sim n}, t) \neq 0, \; \forall \zeta_0 \in \mathbb{R}, \; z_{1\sim n} \in \mathbb{R}^n, \eta_0 \neq -\alpha\zeta_0, \text{ and } t \geqslant 0.$$

To complete this subsection, let us now consider the case of $n = 2$. For convenience, in the rest of this subsection let us omit the subscript in the variables $\rho_1(\gamma_1)$, $\hat{\varphi}_1(\cdot)$, $\hat{\psi}_1(\cdot)$, $\varphi'_1(\cdot)$, $\psi'_1(\cdot)$, $g'_1(\cdot)$, and $h'_1(\cdot)$. Then the original system (1) reduces to

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_2 \hat{\varphi}(\zeta_0, \eta_0, x_2, u_0, t) + \rho x_1 \hat{\psi}(\zeta_0, \eta_0, x_2, u_0, t), \end{cases} \tag{71}$$

which is a generalized form of the Brockett's first example system [2, 3]

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_0 x_1, \end{cases} \tag{72}$$

and also the well-known Brockett's integrator [2, 4]

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_1 u_0. \end{cases} \tag{73}$$

Design for the first scalar subsystem of system (71) the controller $u_0 = -\beta x_0$. Then the second subsystem of system (71), corresponding to (25), is

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_2 \varphi'(\zeta_0, \eta_0, x_2, t) + \rho x_1 \psi'(\zeta_0, \eta_0, x_2, t). \end{cases} \tag{74}$$

Applying the above Theorem 4, immediately gives the following result.

Corollary 4. Suppose

$$\psi'(\zeta_0, \eta_0, x_2, t) \neq 0,\ \forall \zeta_0, \eta_0, x_2 \in \mathbb{R}, \text{and } t \geqslant 0, \tag{75}$$

and define

$$g'(\zeta_0, \eta_0, z_1, t) = [\beta + \varphi'(\cdot)]\, z_1, \tag{76}$$
$$h'(\zeta_0, \eta_0, z_1, t) = \gamma\psi'(\cdot). \tag{77}$$

Then, under the following transformation:

$$\begin{cases} z = z_1, \\ \dot{z} = g'(\zeta_0, \eta_0, z_1, t) + h'(\zeta_0, \eta_0, z_1, t)\, z_2, \end{cases} \tag{78}$$

system (74) can be transformed into the following FAS:

$$\ddot{z} = f_2(\zeta_0, \eta_0, z_{1\sim 2}, t) + h'(\zeta_0, \eta_0, z_1, t)\, u, \tag{79}$$

where

$$f_2(\zeta_0, \eta_0, z_{1\sim 2}, t) = \dot{g}'(\zeta_0, \eta_0, z_1, t) + \dot{h}'(\zeta_0, \eta_0, z_1, t)\, z_2. \tag{80}$$

5.2 Stabilizing controller design

Once the FAS model (70) of SFS (54) is obtained, a stabilizing controller for system (1) can then be immediately designed by a standard procedure [24,28]. Combining the solution to Problem 2 given in [1] and controller (12), yields the following result.

Theorem 5. Let Assumptions A$''$, B, and C be met, a_0 and a_1 be determined by (15), with $\alpha > \beta > 0$, and $b_{0\sim n-1}$ be an arbitrary vector making $\Phi(b_{0\sim n-1})$ Hurwitz. Then (1) a stabilizing controller for system (1) is given by

$$\begin{cases} u_0 = -a_0 \int_0^t x_0(s)\, \mathrm{d}s - a_1(x_0(t) - \zeta_0) + \eta_0, \\ u = -\frac{1}{B_n(\zeta_0, \eta_0, z_{1\sim n}, t)}\left[f_n(\zeta_0, \eta_0, z_{1\sim n}, t) + u^*\right], \\ u^* = b_{0\sim n-1} z^{(0\sim n-1)}, \end{cases} \tag{81}$$

where $z_{1\sim n}$ is given by (69), $\zeta_0 = x_0(0)$, and $\eta_0 \neq -\alpha\zeta_0$ is a design parameter; and (2) the corresponding closed-loop system is given by

$$\begin{cases} \ddot{x}_0 + a_1\dot{x}_0 + a_0 x_0 = 0,\ \zeta_0 = x_0(0),\ \dot{x}_0(0) = \eta_0, \\ z^{(n)} + b_{0\sim n-1} z^{(0\sim n-1)} = 0, \end{cases} \tag{82}$$

or equivalently, by

$$\begin{cases} \ddot{x}_0 + a_1\dot{x}_0 + a_0 x_0 = 0,\ \zeta_0 = x_0(0),\ \dot{x}_0(0) = \eta_0, \\ \dot{z}^{(0\sim n-1)} = \Phi(b_{0\sim n-1}) z^{(0\sim n-1)}. \end{cases} \tag{83}$$

It should be noted that the states of the original system (1) include only $x_{0\sim n}$, but not $\dot{x}_0$. It can be recognized that the designed controller (81) is, in nature, a feedback of the system states $x_{0\sim n}$. Relative to the original system (1), η_0 is an external design parameter. It is only required to meet the condition $\eta_0 \neq -\alpha\zeta_0 = -\alpha x_0(0)$, and can be almost arbitrarily chosen.

Obviously, the well-known technique of pole assignment can be applied to solve for the vector $b_{0\sim n-1}$. Particularly, the complete parametric approach proposed in [24] (see also the Proposition 2 in [27]) can be readily applied. Please note that the controller (81) is smooth under Assumption B. Different from the discontinuous controller designed in [1], even when the open-loop system (1) is time-invariant, that is, when all the functions $g_i(\cdot)$ and $h_i(\cdot)$, $i = 1, 2, \ldots, n-1$, are time-invariant, the above controller (81) is still time-varying.

In the case of $n = 2$, we immediately have the following result.

Corollary 5. Let b_i, $i=1,2$, be two positive scalars, $f_2\left(\zeta_0,\eta_0,z_{1\sim 2},t\right)$ be given by (80), and

$$h'\left(\zeta_0,\eta_0,z_1,t\right)\neq 0,\ \forall\zeta_0,z_1\in\mathbb{R},\ \text{and}\ \eta_0\neq-\alpha\zeta_0,\ t\geqslant 0.$$

Then a stabilizing controller for system (71) is given by

$$\begin{cases} u_0=-a_0\int_0^t x_0\left(s\right)\mathrm{d}s-a_1\left(x_0\left(t\right)-\zeta_0\right)+\eta_0, \\ u=-\frac{1}{h'\left(\zeta_0,\eta_0,z_1,t\right)}\left(f_2\left(\zeta_0,\eta_0,z_{1\sim 2},t\right)+u^*\right), \\ u^*=b_0z_1+b_1\left[g'\left(\zeta_0,\eta_0,z_1,t\right)+h'\left(\zeta_0,\eta_0,z_1,t\right)z_2\right], \end{cases} \tag{84}$$

which results in the following constant linear closed-loop system:

$$\begin{cases} \ddot{x}_0+a_1\dot{x}_0+a_0x_0=0, \\ \ddot{z}+b_1\dot{z}+b_0z=0, \end{cases} \tag{85}$$

with the following initial value conditions:

$$x_0\left(0\right)=\zeta_0,\ \dot{x}_0\left(0\right)=\eta_0\neq-\alpha\zeta_0.$$

5.3 Response and stability analysis

It follows from the above results that, with the controller u given by (81) applied to FAS (70), the closed-loop system is given by

$$\dot{z}^{(0\sim n-1)}=\Phi(b_{0\sim n-1})z^{(0\sim n-1)}, \tag{86}$$

whose solution can be immediately given as

$$z^{(0\sim n-1)}\left(t\right)=Z_0\mathrm{e}^{-\Phi(b_{0\sim n-1})t},\ Z_0=z^{(0\sim n-1)}\left(0\right). \tag{87}$$

On the other side, recall that the open-loop system, that is, FAS (70) is equivalent to SFS (54) under the homeomorphism (69). Meanwhile, the controller u in (81) can be expressed as

$$u=u\left(\zeta_0,\eta_0,z^{(0\sim n-1)},t\right)=u\left(\zeta_0,\eta_0,z_{1\sim n},t\right).$$

Applying this controller to SFS (54), gives the closed-loop system

$$\begin{cases} \dot{z}_1=g_1'\left(\cdot\right)+h_1'\left(\cdot\right)z_2, \\ \dot{z}_2=g_2'\left(\cdot\right)+h_2'\left(\cdot\right)z_3, \\ \vdots \\ \dot{z}_{n-1}=g_{n-1}'\left(\cdot\right)+h_{n-1}'\left(\cdot\right)z_n, \\ \dot{z}_n=u\left(\zeta_0,\eta_0,z_{1\sim n},t\right). \end{cases} \tag{88}$$

Consequently, the above system is equivalent to the above system (86) under homeomorphism (69).

By Theorem 3 in [1], when the assumptions in Theorem 4 are met, the transformation (69) is one-to-one, keeps the origin unmoved, and simultaneously guarantees

$$\lim_{t\to\infty}z^{(0\sim n-1)}\left(t\right)=0\Longrightarrow\lim_{t\to\infty}z_{1\sim n}\left(t\right)=0. \tag{89}$$

That is, the response of system (88) converges to zero. Since $\Phi(b_{0\sim n-1})$ is Hurwitz, $z^{(0\sim n-1)}\left(t\right)$ UGE converges to zero. Now the question is, when $z_{1\sim n}\left(t\right)$ is also UGE approaching zero. To give an answer to this question, it suffices to give directly the response of system (88). This can be achieved with Algorithm 1.

Once the response $z_{1\sim n}\left(t\right)=z_{1\sim n}(Z_0,\zeta_0,\eta_0,t)$ is explicitly solved, the UGE stability of system (88) can be directly checked by verifying the UGE convergence of $z_{1\sim n}(Z_0,\zeta_0,\eta_0,t)$ with respect to $Z_0\in\mathbb{R}^n$, $\zeta_0\in\mathbb{R}$, and $\eta_0\neq-\beta\zeta_0$. Therefore, by Theorem 2, the following result is immediately derived.

Algorithm 1 Getting response of system (88)

1: The homeomorphism (69) gives an explicit transformation from $z_{1\sim n}$ to $z^{(0\sim n-1)}$. Therefore, we can define

$$\tilde{f}_i(\zeta_0, \eta_0, z^{(0\sim i-1)}, t) \triangleq f_i(\zeta_0, \eta_0, z_{1\sim i}, t), \tag{90}$$

$$\tilde{B}_i(\zeta_0, \eta_0, z^{(0\sim i-1)}, t) \triangleq B_i(\zeta_0, \eta_0, z_{1\sim i}, t), \ i = 1, 2, \ldots, n-1, \tag{91}$$

and get the expressions of these new functions.

2: Define

$$B_e(x_0, z^{(0\sim n-1)}, t) = \mathrm{diag}(1, \tilde{B}_1\,(x_0, z, t)\,, \tilde{B}_1(x_0, z^{(0\sim 1)}, t), \ldots, \tilde{B}_{n-1}(x_0, z^{(0\sim n-1)}, t)). \tag{92}$$

Then, in view of (90)–(92), transformation (69) can be rewritten as

$$z^{(0\sim n-1)} = B_e(x_0, z^{(0\sim n-1)}, t) z_{1\sim n} + \begin{bmatrix} 0 \\ \tilde{f}_1\,(x_0, z, t) \\ \tilde{f}_2(x_0, z^{(0\sim 1)}, t) \\ \vdots \\ \tilde{f}_{n-1}(x_0, z^{(0\sim n-1)}, t) \end{bmatrix}, \tag{93}$$

or, equivalently,

$$z_{1\sim n} = B_e^{-1}(x_0, z^{(0\sim n-1)}, t) \left(z^{(0\sim n-1)} - \begin{bmatrix} 0 \\ \tilde{f}_1\,(x_0, z, t) \\ \tilde{f}_2(x_0, z^{(0\sim 1)}, t) \\ \vdots \\ \tilde{f}_{n-1}(x_0, z^{(0\sim n-1)}, t) \end{bmatrix} \right). \tag{94}$$

3: Through substituting (87) into (90)–(92), we can define

$$\breve{f}_i(Z_0, \zeta_0, \eta_0, t) \triangleq \tilde{f}_i(\zeta_0, \eta_0, z^{(0\sim i-1)}, t), \tag{95}$$

$$\breve{B}_i(Z_0, \zeta_0, \eta_0, t) \triangleq \tilde{B}_i(\zeta_0, \eta_0, z^{(0\sim i-1)}, t), \ i = 1, 2, \ldots, n-1, \tag{96}$$

$$\breve{B}_e(Z_0, \zeta_0, \eta_0, t) \triangleq \mathrm{diag}(1, \breve{B}_1(Z_0, \zeta_0, \eta_0, t), \breve{B}_1(Z_0, \zeta_0, \eta_0, t), \ldots, \breve{B}_{n-1}(Z_0, \zeta_0, \eta_0, t)). \tag{97}$$

Therefore, by using (94)–(97), the response of system (88) can be directly given as

$$z_{1\sim n}(Z_0, \zeta_0, \eta_0, t) = \breve{B}_e^{-1}(\zeta_0, \eta_0, t) \left(Z_0 \mathrm{e}^{-\Phi(b_{0\sim n-1})t} - \begin{bmatrix} 0 \\ \breve{f}_1(\zeta_0, \eta_0, t) \\ \breve{f}_2(\zeta_0, \eta_0, t) \\ \vdots \\ \breve{f}_{n-1}(\zeta_0, \eta_0, t) \end{bmatrix} \right). \tag{98}$$

Theorem 6. Let Assumption C be satisfied, and $z_{1\sim n}(Z_0, \zeta_0, \eta_0, t)$ given by Algorithm 1 UGE converge to zero for all $Z_0 \in \mathbb{R}^n$, $\zeta_0 \in \mathbb{R}$, and $\eta_0 \neq -\beta\zeta_0$. If, further,

$$\int_0^\infty \|\rho_\psi\,(s)\|\, \mathrm{e}^{-(\alpha-\beta)s} \mathrm{d}s = M < \infty, \tag{99}$$

then $u = u\,(\zeta_0, \eta_0, z_{1\sim n}, t)$ given by (81) is a controller that UGE stabilizes the system

$$\begin{cases} \dot{z}_1 = g_1'\,(\cdot) + h_1'\,(\cdot)\, z_2 + \Delta f_1\,(\cdot)\,, \\ \dot{z}_2 = g_2'\,(\cdot) + h_2'\,(\cdot)\, z_3 + \Delta f_2\,(\cdot)\,, \\ \vdots \\ \dot{z}_{n-1} = g_{n-1}'\,(\cdot) + h_{n-1}'\,(\cdot)\, z_n + \Delta f_{n-1}\,(\cdot)\,. \\ \dot{z}_n = u\,(\zeta_0, \eta_0, z_{1\sim n}, t)\,. \end{cases} \tag{100}$$

Through examining (98), we can conclude that in many cases $z_{1\sim n}(Z_0, \zeta_0, \eta_0, t)$ exponentially converges to zero. The following corollary reveals a particular case.

Corollary 6. Let Assumption C be satisfied, and $\breve{f}_i(Z_0, \zeta_0, \eta_0, t)$, $i = 1, 2, \ldots, n-1$, and $\breve{B}_e(Z_0, \zeta_0, \eta_0, t)$ be given by Algorithm 1. If $\breve{B}_e(Z_0, \zeta_0, \eta_0, t)$ is uniformly lower bounded, $\breve{f}_i(Z_0, \zeta_0, \eta_0, t)$, $i = 1, 2, \ldots, n-1$, uniformly and exponentially converge to zero for all $Z_0 \in \mathbb{R}^n$, $\zeta_0 \in \mathbb{R}$, and $\eta_0 \neq -\beta\zeta_0$, and further

$$\int_0^{\infty} \|\rho_\psi(s)\| \, \mathrm{e}^{-(\alpha-\beta)s} \mathrm{d}s = M < \infty, \tag{101}$$

then $u = u(\zeta_0, \eta_0, z_{1\sim n}, t)$ given by (81) is a controller that UGE stabilizes the system (100).

Please note that the condition, $\breve{f}_i(Z_0, \zeta_0, \eta_0, t)$, $i = 1, 2, \ldots, n-1$, uniformly and exponentially converge to zero, is not very strict, because these functions have already been shown to converge to zero asymptotically.

6 Application to ship control

In this section, we will illustrate the validity of the proposed continuous stabilizing controller with a surface ship control.

6.1 System model

The dynamic equations of a considered surface ship in surge, yaw, and sway are given as follows [46]:

$$\begin{cases} m_x \dot{v}_x = m_y v_y \omega_o - d_{01} v_x - d_{02} v_x^2 \tanh(v_x/\varepsilon) - d_{03} v_x^3 + \tau_x, \\ m_\omega \dot{\omega}_o = (m_x - m_y) v_x v_y - d_{11} \omega_o - d_{12} \omega_o^2 \tanh(\omega_o/\varepsilon) - d_{13} \omega_o^3 + \tau_\omega, \\ m_y \dot{v}_y = -m_x v_x \omega_o - d_{21} v_y - d_{22} v_y^2 \tanh(v_y/\varepsilon) - d_{23} v_y^3, \end{cases} \tag{102}$$

where v_x, ω_o, and v_y represent the surge, yaw, and sway velocities, respectively; $m_x > 0$, $m_\omega > 0$, and $m_y > 0$ stand for the system inertia constants; constants $d_{ij} > 0$ $(i = 0, 1, 2; j = 1, 2, 3)$ denote the hydrodynamic damping terms; τ_x and τ_ω are the surge force and the yaw moment, respectively; ε is a proper small positive number.

Under the state transformation

$$\begin{cases} x_0 = m_x v_x, \\ x_1 = m_\omega \omega_o, \\ x_2 = m_y v_y, \end{cases} \tag{103}$$

and the input transformation

$$\begin{cases} u_0 = m_y v_y \omega_o - d_{01} v_x - d_{02} v_x^2 \tanh(v_x/\varepsilon) - d_{03} v_x^3 + \tau_x, \\ u = (m_x - m_y) v_x v_y - d_{11} \omega_o - d_{12} \omega_o^2 \tanh(\omega_o/\varepsilon) - d_{13} \omega_o^3 + \tau_\omega, \end{cases} \tag{104}$$

system (102) can be written into the following standard form:

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_2 \varphi'(x_2) + x_0 x_1 \psi'(x_2), \end{cases} \tag{105}$$

where

$$\psi'(x_2) = -\frac{1}{m_\omega}, \ \varphi'(x_2) = d_1 + d_2 x_2 \tanh \frac{x_2}{m_y \varepsilon} + d_3 x_2^2, \tag{106}$$

with

$$d_i = -\frac{d_{2i}}{m_y^i}, \ i = 1, 2, 3.$$

Obviously, the form of system (105) corresponds to the case of

$$\rho(x_0, u_0) = x_0 \text{ and } \gamma_1 = \vartheta_1 = 1.$$

Following the proposed approach, the almost SFS corresponds to (27) is obtained as

$$\begin{cases} \dot{z}_1 = g'(\zeta_0, \eta_0, z_1, t) + h'(\zeta_0, \eta_0, z_1, t)z_2 + \Delta f(\zeta_0, \eta_0, z_{1\sim 2}, t), \\ \dot{z}_2 = u, \end{cases} \tag{107}$$

where

$$h'(\zeta_0, \eta_0, z_1, t) = -\frac{1}{m_\omega},\ g'(\zeta_0, \eta_0, z_1, t) = [\beta + \varphi'(x_2)]z_1,$$

$$\Delta f(\zeta_0, \eta_0, z_{1\sim 2}, t) = \frac{1}{m_\omega}\omega z_2,\ \text{with } \omega(t) = \frac{c_1}{c_2}\mathrm{e}^{-(\alpha-\beta)t}.$$

6.2 Controller design

Applying the proposed controller development, the continuous stabilizing controller of (105) can be formulated as

$$\begin{cases} u_0 = -a_0 \int_0^t x_0(s)\mathrm{d}s - a_1(x_0(t) - \zeta_0) + \eta_0, \\ u = m_\omega f_2(\zeta_0, \eta_0, z_{1\sim 2}, t) + u^*, \\ u^* = m_\omega \left[b_0 + b_1 \left(\beta + \varphi'(x_2)\right)\right] z_1 - b_1 z_2, \end{cases} \tag{108}$$

where

$$\begin{aligned} f_2(\zeta_0, \eta_0, z_{1\sim 2}, t) &= \dot{g}'(\zeta_0, \eta_0, z_1, t) \\ &= [\beta + \varphi'(\zeta_0, \eta_0, z_1, t)]\dot{z}_1 + \dot{\varphi}'(\zeta_0, \eta_0, z_{1\sim 2}, t)z_1, \end{aligned}$$

with

$$\begin{cases} \dot{\varphi}'(\zeta_0, \eta_0, z_{1\sim 2}, t) = (\dot{z}_1\sigma_0 + z_1\dot{\sigma}_0)[d_2 \tanh \frac{z_1\sigma_0}{m_y\varepsilon} + z_1\sigma_0(2d_3 + \frac{d_2}{m_y\varepsilon}\mathrm{sech}^2 \frac{z_1\sigma_0}{m_y\varepsilon})], \\ \sigma_0\left(t\right) = c_2\mathrm{e}^{-\beta t}. \end{cases}$$

Furthermore, the variables $z_{1\sim 2}$ are given by the following transformation:

$$z_1 = \frac{x_2}{\sigma_0},\ z_2 = x_1.$$

When the initial values ζ_0 and η_0 satisfy the relationship $\eta_0 = -\beta\zeta_0$, we have $c_1 = 0$, which results in $\omega(t) \equiv 0$, and further $\Delta f(\zeta_0, \eta_0, z_{1\sim 2}, t) \equiv 0$. The UGE convergence of all the trajectories of the designed closed-loop system, starting from arbitrary initial values with $\zeta_0 = x_0\left(0\right) \neq 0$, is guaranteed. For the more general case of $\eta_0 \neq -\beta\zeta_0$ and $\eta_0 \neq -\alpha\zeta_0$, the designed system is UGE stable as shown below.

Recall that

$$f_1(\zeta_0, \eta_0, z_1, t) = g'(\zeta_0, \eta_0, z_1, t) = [\beta + \varphi'(x_2)]z_1 = [\beta + \varphi'(\sigma_0 z_1)]z_1, \tag{109}$$

with

$$\varphi'(\sigma_0 z_1) = d_1 + d_2 c_2 z_1 \mathrm{e}^{-\beta t} \tanh \frac{c_2 z_1 \mathrm{e}^{-\beta t}}{m_y\varepsilon} + d_3 c_2^2 z_1^2 \mathrm{e}^{-2\beta t}. \tag{110}$$

It can be easily observed that $f_1(\zeta_0, \eta_0, z_1, t)$ uniformly converges to zero when z_1 converges uniformly to zero.

Further, in view of (106), Assumption C can be obtained as

$$\left|\psi'(x_2)\right| \leqslant \frac{1}{m_\omega}\left|z_2\right| \leqslant \frac{1}{m_\omega}\left\|z_{1\sim 2}\right\|,$$

which gives

$$\rho_\psi = \frac{1}{m_\omega}.$$

Since

$$\int_0^\infty \left\|\rho_\psi\left(s\right)\right\| \mathrm{e}^{-(\alpha-\beta)s}\mathrm{d}s = \frac{1}{m_\omega\left(\alpha - \beta\right)} = M < \infty, \tag{111}$$

it follows from Corollary 6 that the designed closed-loop system is UGE stable.

Table 1 Hydrodynamic damping parameters d_{ij}

	$j=1$	$j=2$	$j=3$
$i=0$	0.0358	0.5×0.0358	0.25×0.0358
$i=1$	0.0308	0.5×0.0308	0.25×0.0308
$i=2$	0.1183	0.5×0.1183	0.25×0.1183

Table 2 Different cases of initial states

Case	$v_x(0)$	$v_y(0)$	$\omega(0)$	ζ_0	η_0
I	5	1	0.5	1.1274×5	-1.1274×0.5
II	5	1	-0.5	1.1274×5	-1.1274×0.5
III	6	2	0.5	1.1274×6	-1.1274×0.9
IV	6	2	-0.5	1.1274×6	-1.1274×0.9

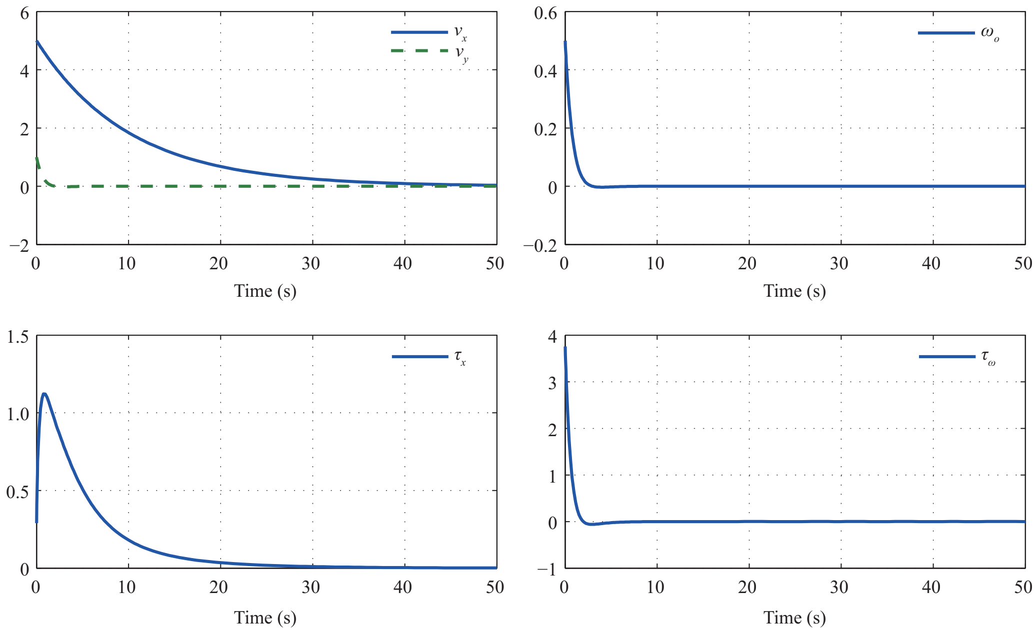

Figure 1 (Color online) Simulation results for Case I.

6.3 Simulation results

For simulation use, we select the same Bis-scale parameter values as in [46, 47], that is,

$$m_x = 1.1274,\ m_y = 1.8902,\ m_\omega = 0.1278,$$

and the values of hydrodynamic damping parameters d_{ij} are given in Table 1. The auxiliary physical parameter ε is set as $\varepsilon = 0.1$, and the related design parameters are chosen as

$$\alpha = 3,\ \beta = 0.1,\ \text{and}\ b_0 = 1,\ b_1 = 2.$$

To demonstrate the validity of the continuous stabilizing controller (108), we consider four different cases of initial states, as given in Table 2. In Cases I and II, the initial values ζ_0 and η_0 satisfy the relationship $\eta_0 = -\beta\zeta_0$. While, in Cases III and IV, the initial values ζ_0 and η_0 comply with the relationship $\eta_0 \neq -\beta\zeta_0$ and $-\alpha\zeta_0$.

Corresponding to the four cases, simulation results are respectively provided in Figures 1–4, from which we can observe that (1) under the four groups of initial values given in Table 2, all the state variables, that is, the surge velocity v_x, the sway velocity v_y, and the yaw velocity ω_o, exponentially converge to zero, with the sway velocity v_y and the yaw velocity ω_o converging much faster; (2) the system inputs τ_x and τ_ω maintain in a reasonable range and also die out exponentially.

All in all, the designed continuous stabilizing control method has produced a satisfactory performance for this surface ship control.

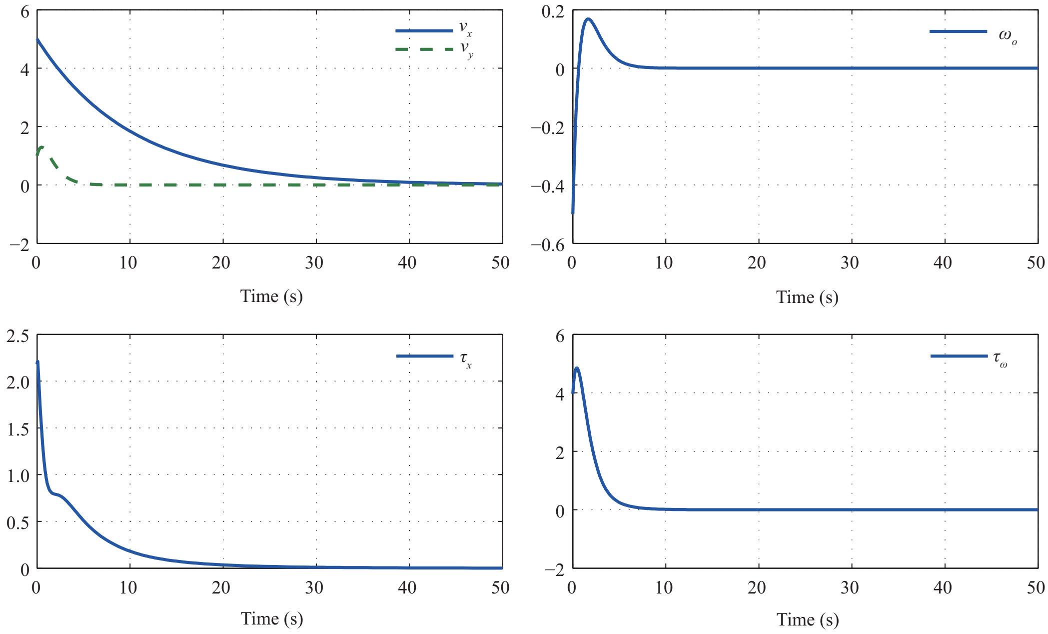

Figure 2 (Color online) Simulation results for Case II.

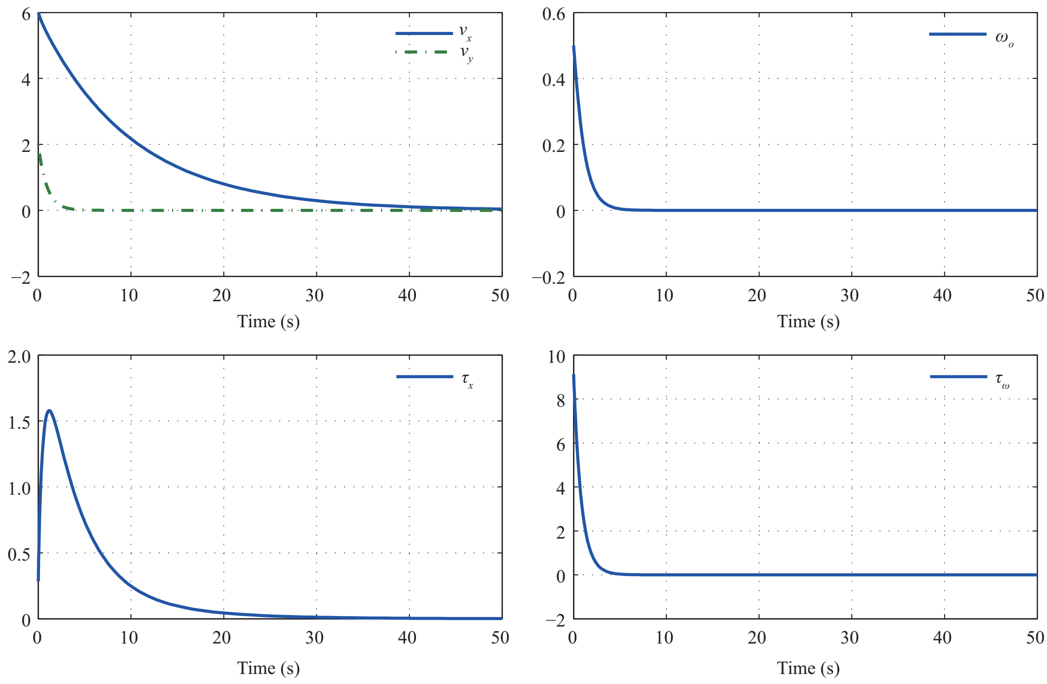

Figure 3 (Color online) Simulation results for Case III.

7 Conclusion

This paper reconsiders the general type of nonholonomic systems proposed and treated in [1], which involve two sets of nonlinear time-varying terms and a set of selective variables. It is shown that the FAS approach also renders the design of a continuous time-varying stabilizing controller for the system. Comparatively, the drawback of the discontinuous stabilizing controller proposed in [1] is that it may result in nonsmooth system responses, but the derived closed-loop system obeys a linear time-invariant one when the initial value of the first scalar subsystem is restricted to be nonzero, while the continuous stabilizing controller designed in this paper can no longer give a linear time-invariant closed-loop system, but guarantees the UGE stability of the closed-loop systems and provides smooth system responses under

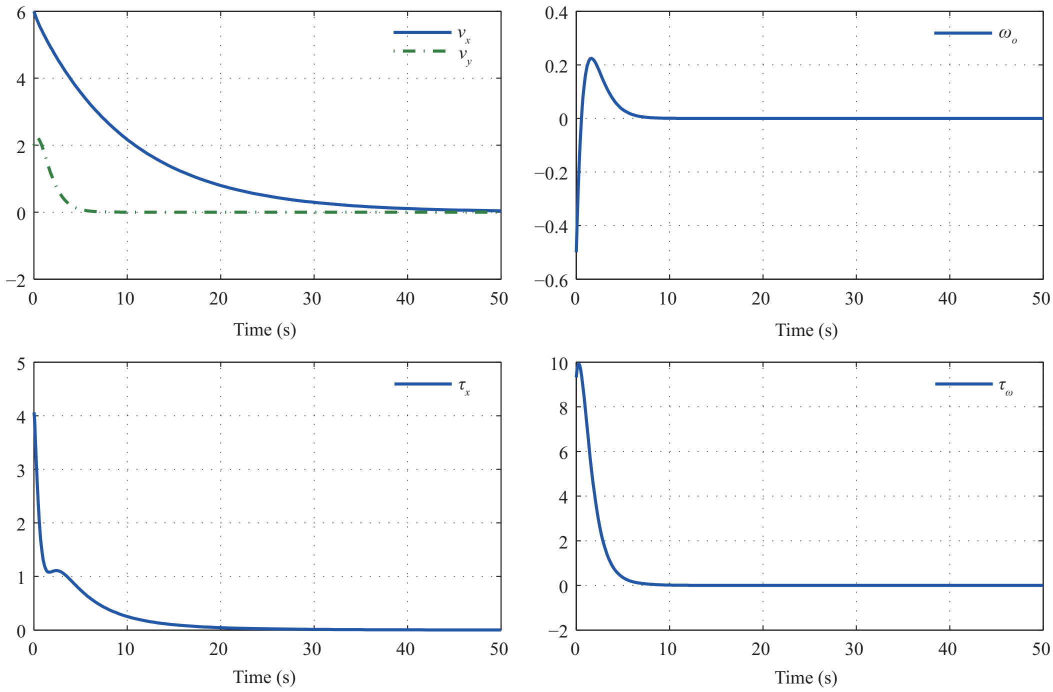

Figure 4 (Color online) Simulation results for Case IV.

very mild conditions.

Technically, an auxiliary variable is introduced by differentiating the first scalar subsystem, and the first control variable is then designed in a proportional plus integral form. With the solution of the extended first subsystem, the second subsystem formed by the rest equations is turned into a time-varying one with two external parameters, which can be further transformed into an almost SFS with the help of an extended σ-process. A stability result is derived to guarantee that a UGE stabilizing controller for the almost SFS is given by a UGE stabilizing controller for an SFS which is actually the main part of the obtained almost SFS. Finally, converting the SFS into a FAS readily gives the feedback form of the second control variable. The overall controller is then obtained by combining the feedback forms of both control variables.

The FAS approach proposed for this type of nonholonomic systems can be further generalized in several directions. As in [1], the proposed results can be extended to locally normal systems and time-delay systems. Particularly, robust stabilization of the type of uncertain nonholonomic systems in the form of (7) can be considered, and the type of systems can be further generalized into compact multivariable forms.

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