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• **Supplementary File** •

Event-Triggered Impulsive Synchronization of Heterogeneous Neural Networks

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Appendix A Definition and lemmas

Definition 1. The neural networks are said to achieve globally exponentially quasi-synchronization, if there exist $\eta > 0, T > 0, \theta > 0$ and $\theta_0 > 0$ such that for any initial values and $t > T$

$$||e_i(t)|| \leqslant \theta e^{-\eta t} + \theta_0, \quad i = 2, 3, \ldots, N.$$

Lemma 1 ([1]). If the directed network $\mathcal{G}$ contains a directed spanning tree, then one can select a positive diagonal matrix $\Theta = \mathrm{diag}\{\mu_2, \mu_3, \ldots, \mu_N\} \succ 0$ such that $\Theta\widetilde{L} + \widetilde{L}^T\Theta \succ 0$, where $\mu = [\mu_2, \mu_3, \ldots, \mu_N]^T = (\widetilde{L}^T)^{-1}1_{N-1}$.

Lemma 2 ([2]). Let $PC(l) = \{\varphi : [-\widetilde{\tau}, \infty) \to \mathbb{R}^l$, $\varphi(t)$ is continuous everywhere except for the finite points t_r at which $\varphi(t_r^+) = \varphi(t_r)$ and $\varphi(t_r^-)$ exist$\}$, and $0 \leqslant \tau(t) \leqslant \bar{\tau}$. For $u(t), v(t) \in PC(l)$, if there exist positive constants ϑ, $\widetilde{\vartheta}$, $\bar{\omega} > 0$ such that

$$\begin{cases} D^+u(t) \leqslant \vartheta u(t) + \widetilde{\vartheta} u(t-\tau(t)), & t \neq t_r, \\ u(t_r) \leqslant \overline{\omega} u(t_r^-), & r \in \mathbb{N}, \end{cases}$$

and

$$\begin{cases} D^+v(t) > \vartheta v(t) + \widetilde{\vartheta} v(t-\tau(t)), & t \neq t_r, \\ v(t_r) = \overline{\omega} v(t_r^-), & r \in \mathbb{N}, \end{cases}$$

then $u(t) \leqslant v(t)$ for $-\bar{\tau} \leqslant t \leqslant 0$ implies $u(t) \leqslant v(t)$ for $t > 0$.

Appendix B Proof of Theorem 1

Proof. It follows from (9) that the derivative of event-trigger function satisfies

$$\begin{aligned}
\dot{\psi}(t) =& \frac{d\left[e^T(t)e(t) - v(t_r)^T Q v(t_r)\right]}{dt} \\
=& 2e^T(t)\dot{e}(t) = -2e^T(t)\dot{v}(t) \\
=& -2e^T(t)\left[Dv(t) + BF(v(t)) - c(\widetilde{L}\otimes\Gamma)v(t_r) + \hat{I}\right] \\
\leqslant& 4e^T(t)e(t) + v^T(t)D^TDv(t) + F^T(v(t))B^TBF(v(t)) + c^2v^T(t_r)(\widetilde{L}\otimes\Gamma)^T(\widetilde{L}\otimes\Gamma)v(t_r) + \hat{I}^T\hat{I} \\
\leqslant& 4e^T(t)e(t) + \alpha_1 v^T(t)v(t) + \alpha_2 v^T(t_r)v(t_r) + \hat{I}^T\hat{I} \\
\leqslant& (4+2\alpha_1)e^T(t)e(t) + (2\alpha_1+\alpha_2)v^T(t_r)v(t_r) + \hat{I}^T\hat{I} \\
\leqslant& (4+2\alpha_1)\psi(t) + \hat{I}^T\hat{I} + \frac{2\alpha_1+\alpha_2+(4+2\alpha_1)\lambda_{\max}\{Q\}}{\lambda_{\min}\{Q\}}v^T(t_r)Qv(t_r),
\end{aligned}$$

where $\alpha_1 = \lambda_{\max}\left[D^TD + 2(\Xi\otimes E_n)^TB^TB(\Xi\otimes E_n)\right]$ and $\alpha_2 = c^2\lambda_{\max}\left[(\widetilde{L}\otimes\Gamma)^T(\widetilde{L}\otimes\Gamma)\right]$.

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Due to $\psi(t_r) = -v(t_r)^T Q v(t_r)$, when $t \in [t_r, t_{r+1}^-]$, the following inequality can be obtained

$$\begin{aligned}\psi(t) \leqslant & e^{(4+2\alpha_1)(t-t_r)} \int_{t_r}^{t} \Big[\frac{2\alpha_1 + \alpha_2 + (4+2\alpha_1)\lambda_{\max}\{Q\}}{\lambda_{\min}\{Q\}} v^T(t_r) Q v(t_r) + \hat{I}^T \hat{I}\Big] e^{-(4+2\alpha_1)(s-t_r)} ds - v(t_r)^T Q v(t_r) \\ = & \frac{\frac{2\alpha_1+\alpha_2+(4+2\alpha_1)\lambda_{\max}\{Q\}}{\lambda_{\min}\{Q\}} v^T(t_r) Q v(t_r) + \hat{I}^T \hat{I}}{4+2\alpha_1} (e^{(4+2\alpha_1)(t-t_r)} - 1) - v(t_r)^T Q v(t_r).\end{aligned}$$

Take $\varrho = \frac{\lambda_{\min}\{Q\} \hat{I}^T \hat{I}}{2\alpha_1+\alpha_2+(4+2\alpha_1)\lambda_{\max}\{Q\}}$. Combining the design of CETIC and $\psi(t_{r+1}^-) > \varrho$, the following relation of time interval can be deduced

$$\begin{aligned}& t_{r+1} - t_r \\ > & \frac{\ln \frac{(4+2\alpha_1)(\varrho + v(t_r)^T Q v(t_r))}{\frac{2\alpha_1+\alpha_2+(4+2\alpha_1)\lambda_{\max}\{Q\}}{\lambda_{\min}\{Q\}} v^T(t_r) Q v(t_r) + \hat{I}^T \hat{I}} + 1}{4+2\alpha_1} \\ = & \frac{1}{4+2\alpha_1} \ln \Big(\frac{(4+2\alpha_1)\lambda_{\min}\{Q\}}{2\alpha_1+\alpha_2+(4+2\alpha_1)\lambda_{\max}\{Q\}} + 1 + \frac{(4+2\alpha_1)\varrho - \frac{(4+2\alpha_1)\lambda_{\min}\{Q\}\hat{I}^T\hat{I}}{2\alpha_1+\alpha_2+(4+2\alpha_1)\lambda_{\max}\{Q\}}}{\frac{2\alpha_1+\alpha_2+(4+2\alpha_1)\lambda_{\max}\{Q\}}{\lambda_{\min}\{Q\}} v^T(t_r) Q v(t_r) + \hat{I}^T \hat{I}} \Big) \\ = & \frac{1}{4+2\alpha_1} \ln \Big(\frac{(4+2\alpha_1)\lambda_{\min}\{Q\}}{2\alpha_1+\alpha_2+(4+2\alpha_1)\lambda_{\max}\{Q\}} + 1 \Big) \triangleq T_{\min}.\end{aligned}$$

Therefore, the Zeno behavior is eliminated.

Appendix C Proof of Theorem 2

Proof. A Lyapunov function candidate is chosen as

$$V(t) = \frac{1}{2} \epsilon^T(t) (\Theta \otimes E_n) \epsilon(t). \tag{C1}$$

When $t \neq t_r$, calculating the derivative of $V(t)$ along the trajectories of error system (7) derives

$$\begin{aligned}\dot{V}(t) = & \epsilon^T(t)(\Theta \otimes E_n)\dot{\epsilon}(t) \\ = & \epsilon^T(t)(\Theta \otimes E_n)[D\epsilon(t) + BH(\epsilon(t)) - c(\widetilde{L} \otimes \Gamma)\epsilon(t_r) + I + G(v_1(t))] \\ = & \epsilon^T(t)(\Theta \otimes E_n)D\epsilon(t) + \epsilon^T(t)(\Theta \otimes E_n)BH(\epsilon(t) - c\epsilon^T(t)(\Theta \otimes E_n)(\widetilde{L} \otimes \Gamma)\epsilon(t_r) + \epsilon^T(t)(\Theta \otimes E_n)I \\ & + \epsilon^T(t)(\Theta \otimes E_n)G(v_1(t)) \\ \leqslant & \epsilon^T(t)\tilde{D}_s\epsilon(t) + \frac{1}{2}\epsilon^T(t)(\Theta \otimes E_n)BB^T(\Theta \otimes E_n)^T\epsilon(t) + \frac{1}{2}H^T(\epsilon(t))H(\epsilon(t)) - c\epsilon^T(t)(\Theta\widetilde{L} \otimes \Gamma)\epsilon(t_r) \\ & + \epsilon^T(t)\epsilon(t) + \frac{1}{2}G^T(v_1(t))(\Theta^T\Theta \otimes E_n)G(v_1(t)) + \frac{1}{2}I^T(\Theta^T\Theta \otimes E_n)I.\end{aligned} \tag{C2}$$

According to Assumption 1, one can get the following inequality,

$$H^T(\epsilon(t))H(\epsilon(t)) = \sum_{i=2}^{N} h_i^T(\epsilon_i(t))h_i(\epsilon_i(t)) \leqslant \sum_{i=2}^{N} \xi_i \epsilon_i^T(t)\epsilon_i(t) = \epsilon^T(t)(\Xi \otimes E_n)\epsilon(t). \tag{C3}$$

It follows from the synchronization error (3) that

$$\epsilon(t) = v(t_r) - 1_{N-1} \otimes v_1 - e(t). \tag{C4}$$

By utilizing Lemma 1 and the definition of $e(t)$, the following equation is available

$$\begin{aligned}& - c\epsilon^T(t)(\Theta\widetilde{L} \otimes \Gamma)\epsilon(t_r) \\ = & - c\epsilon^T(t)(\Theta\widetilde{L} \otimes \Gamma)(v(t_r) - 1_{N-1} \otimes v_1 - e(t_r)) \\ = & - c\epsilon^T(t)(\Theta\widetilde{L} \otimes \Gamma)v(t_r).\end{aligned} \tag{C5}$$

According to the Lemma 1 and the CETIC, when t is event-triggered moment, substituting (C4) into (C5) yields

$$\begin{aligned}& - c\epsilon^T(t)(\Theta\widetilde{L} \otimes \Gamma)v(t_r) \\ = & - c(v(t_r) - 1_{N-1} \otimes v_1 - e(t))^T(\Theta\widetilde{L} \otimes \Gamma)v(t_r) \\ = & - cv(t_r)^T(\Theta\widetilde{L} \otimes \Gamma)v(t_r) + ce^T(t)(\Theta\widetilde{L} \otimes \Gamma)v(t_r) + c(1_{N-1} \otimes v_1)^T(\Theta\widetilde{L} \otimes \Gamma)v(t_r) \\ \leqslant & \frac{c}{a_1}e^T(t)e(t) + a_1cv(t_r)^T(\Theta\widetilde{L} \otimes \Gamma)^T(\Theta\widetilde{L} \otimes \Gamma)v(t_r) - cv(t_r)^T(\Theta\widetilde{L} \otimes \Gamma)_s v(t_r) \\ \leqslant & \frac{c\varrho}{a_1}.\end{aligned} \tag{C6}$$

By combining (C2), (C3) and (C6), one obtains

$$\dot{V}(t) \leqslant \epsilon^T(t)\left[\tilde{D}_s + \frac{1}{2}(\Theta \otimes E_n)BB^T(\Theta \otimes E_n)^T + (\frac{1}{2}\Xi + E_{N-1}) \otimes E_n\right]\epsilon(t) + \frac{1}{2}I^T(\Theta^T\Theta \otimes E_n)I$$

$$
\begin{aligned}
&+\frac{c\varrho}{a_1}+\frac{1}{2}G^T(v_1(t))(\Theta^T\Theta\otimes E_n)G(v_1(t))\\
&\leqslant \sigma_1 V(t)+g_1,
\end{aligned}\tag{C7}
$$

where $g_1=\sup||\frac{1}{2}G^T(v_1(t))(\Theta^T\Theta\otimes E_n)G(v_1(t))+\frac{1}{2}I^T(\Theta^T\Theta\otimes E_n)I+\frac{c\varrho}{a_1}||$.

When $t=t_r$, one has

$$
\begin{aligned}
V(t_r)=&\ \frac{1}{2}\epsilon^T(t_r)(\Theta\otimes E_n)\epsilon(t_r)\\
\leqslant&\ \frac{1}{2}\epsilon^T(t_r^-)\big((E_{N-1}-\bar{c}\hat{D})\otimes E_n\big)^T(\Theta\otimes E_n)\big((E_{N-1}-\bar{c}\hat{D})\otimes E_n\big)\epsilon(t_r^-)\\
\leqslant&\ \frac{\lambda_{\max}\{(E_{N-1}-\bar{c}\hat{D})^T\Theta(E_{N-1}-\bar{c}\hat{D})\}}{\lambda_{\min}\{\Theta\}}V(t_r^-)=\rho_1 V(t_r^-).
\end{aligned}\tag{C8}
$$

For arbitrary $\varepsilon>0$, define the following impulsive comparison system,

$$
\begin{cases}
\dot{\omega}(t)=\sigma_1\omega(t)+g_1+\varepsilon, \quad t\neq t_r,\\
\omega(t_r)=\rho_1\omega(t_r^-),\\
\omega(0)=\dfrac{1}{2}\epsilon^T(0)(\Theta\otimes E_n)\epsilon(0),
\end{cases}
$$

its unique solution $\omega(t)$ can be represented as

$$
\omega(t)=Y(t,0)\omega(0)+\int_0^t Y(t,s)(g_1+\varepsilon)ds,\tag{C9}
$$

where

$$
Y(t,s)=e^{\sigma_1(t-s)}\prod_{s\leqslant t_r<t}\rho_1\leqslant\ e^{\sigma_1(t-s)}\rho_1^{N_0(t,s)}\leqslant e^{\sigma_1(t-s)}\rho_1^{\frac{t-s}{T_{\min}}}=e^{(\sigma_1+\frac{\ln\rho_1}{T_{\min}})(t-s)}.\tag{C10}
$$

By defining $\nu_1=\sigma_1+\frac{\ln\rho_1}{T_{\min}}$ and combining (C9) and (C10), one has

$$
\begin{aligned}
\omega(t)&\leqslant e^{\nu_1 t}\omega(0)+\int_0^t e^{\nu_1(t-s)}(g_1+\varepsilon)ds\\
&=e^{\nu_1 t}\omega(0)+\frac{g_1+\varepsilon}{-\nu_1}(1-e^{\nu_1 t})\\
&=\left(\omega(0)+\frac{g_1+\varepsilon}{\nu_1}\right)e^{\nu_1 t}+\frac{g_1+\varepsilon}{-\nu_1}.
\end{aligned}
$$

According to Lemma 2, one has $V(t)\leqslant\omega(t)$. When $\varepsilon\to 0$,

$$
\frac{\min\limits_{2\leqslant i\leqslant N}\mu_i}{2}||\epsilon(t)||^2\leqslant\ \frac{1}{2}\epsilon^T(t)(\Theta\otimes E_n)\epsilon(t)\leqslant\ V(t)\leqslant\ \omega(t)\leqslant\ \left(\omega(0)+\frac{g_1+\varepsilon}{\nu_1}\right)e^{\nu_1 t}+\frac{g_1+\varepsilon}{-\nu_1}.
$$

Furthermore, the synchronization error can satisfy

$$
||\epsilon(t)||\leqslant\sqrt{\frac{2\omega(0)}{\min\limits_{2\leqslant i\leqslant N}\mu_i}+\frac{2(g_1+\varepsilon)}{\nu_1\min\limits_{2\leqslant i\leqslant N}\mu_i}}\,e^{\frac{\nu_1}{2}t}+\sqrt{\frac{2(g_1+\varepsilon)}{-\nu_1\min\limits_{2\leqslant i\leqslant N}\mu_i}}.
$$

Therefore, the synchronization errors are bounded eventually and global exponential quasi-synchronization is reached for the heterogeneous neural networks.

Remark 1. According to Theorem 2, the synchronization errors eventually converge to the bounded set

$$
\left\{\epsilon(t)\ \middle|\ ||\epsilon(t)||\leqslant\sqrt{\frac{2(g_1+\varepsilon)}{-\nu_1\min\limits_{2\leqslant i\leqslant N}\mu_i}}\right\},
$$

where $\mu=\{\mu_2,\mu_3,\ldots,\mu_N\}^T$ can be obtained from Lemma 1. Therefore, regulating the value of $\min\limits_{2\leqslant i\leqslant N}\mu_i$ can reduce the upper bound of quasi-synchronization errors. One can also increase the value of $\bar{c}$ to obtain a smaller ν_1 and a lower upper bound of the synchronization errors.

Corollary 1. When Assumption 1 holds and $\mathcal{G}$ is undirected, global exponential quasi-synchronization of the heterogeneous neural networks (1) can be achieved and the Zeno behavior can also be eliminated for the event-triggered scheme CETIC, if

$$
(\Theta\widetilde{L}\otimes\Gamma)_s-a_1(\Theta\widetilde{L}\otimes\Gamma)^T(\Theta\widetilde{L}\otimes\Gamma)\succ 0,
$$

and

$$
\sigma_1+\frac{\ln\rho_1}{T_{\min}}<0,
$$

where $\sigma_1=\lambda_{\max}\left\{\tilde{D}+\frac{1}{2}(\Theta\otimes E_n)BB^T(\Theta\otimes E_n)^T+(\frac{1}{2}\Xi+E_{N-1})\otimes E_n\right\}$, $\rho_1=\lambda_{\max}\{(E_{N-1}-\bar{c}\hat{D})^T\Theta\ (E_{N-1}-\bar{c}\hat{D})\}/\lambda_{\min}\{\Theta\}$, and $\tilde{D}$ and $T_{\min}$ are the same as the definition of Theorem 2.

Corollary 2. Assume that graph $\mathcal{G}$ is undirected and the neural networks are homogeneous, i.e., $D_i = D_1, B_i = B_1$ and $f_i(t) = f_1(t)$, $i = 2, 3, \cdots, N$. Under Assumption 1 and selection $\xi_i = \xi$, global exponential quasi-synchronization of the heterogeneous neural networks (1) can be achieved and the Zeno behavior can further be eliminated via the event-triggered scheme CETIC, if

$$(\Theta\widetilde{L} \otimes \Gamma)_s - a_1(\Theta\widetilde{L} \otimes \Gamma)^T(\Theta\widetilde{L} \otimes \Gamma) \succ 0,$$

and

$$\sigma_1 + \frac{2\ln(1-\bar{c})}{T_{\min}} < 0,$$

where $\sigma_1 = \lambda_{\max}\left\{\Theta \otimes D_1 + \frac{1}{2}\Theta^2 \otimes (B_1B_1^T)\right\} + \frac{\xi+2}{2}$ and $T_{\min}$ is length of the minimal impulsive interval.

Remark 2. When the neural networks (1) are homogeneous, g_1 in the upper bound of quasi-synchronization errors becomes 0. The neural networks reach synchronization rather than quasi-synchronization. It also shows that the conclusion of Theorem 2 is applicable to undirected graphs and homogeneous neural networks.

Appendix D Proof of Theorem 3

Proof. To simplify the calculation, we define

$$\mathfrak{z}_i(t_r^i) = v_i^T(t_r^i)\sum_{j=2}^N l_{ij}\Gamma v_j(t_r^i) - \frac{\chi^2}{2} - (\frac{1}{2} + a_2^i)\left(\sum_{j=2}^N l_{ij}\Gamma v_j(t_r^i)\right)^T \sum_{j=2}^N l_{ij}\Gamma v_j(t_r^i).$$

Furthermore, let $\hat{v}_i(t) = \left[v_i^T(t), \left(\sum_{j=2,j\neq i}^N l_{ij}\Gamma v_j(t)\right)^T\right]^T$. Then, $\mathfrak{z}_i(t_r^i)$ can be rewritten as

$$\begin{aligned}\mathfrak{z}_i(t_r^i) =& v_i^T(t_r^i)l_{ii}\Gamma v_i(t_r^i) + v_i^T(t_r^i)\sum_{j=2,j\neq i}^N l_{ij}\Gamma v_j(t_r^i) - \frac{\chi^2}{2}\\ &- (\frac{1}{2} + a_2^i)\left(l_{ii}\Gamma v_i(t_r^i) + \sum_{j=2,j\neq i}^N l_{ij}\Gamma v_j(t_r^i)\right)^T\left(l_{ii}\Gamma v_i(t_r^i) + \sum_{j=2,j\neq i}^N l_{ij}\Gamma v_j(t_r^i)\right)\\ =& \hat{v}_i^T(t_r^i)\begin{bmatrix} l_{ii}\Gamma - (\frac{1}{2}+a_2^i)l_{ii}^2\Gamma^2 & \frac{1}{2}E_n - (\frac{1}{4}+\frac{a_2^i}{2})l_{ii}\Gamma \\ \frac{1}{2}E_n - (\frac{1}{4}+\frac{a_2^i}{2})l_{ii}\Gamma & -(\frac{1}{2}+a_2^i)E_n\end{bmatrix}\hat{v}_i(t_r^i) - \frac{\chi^2}{2} \triangleq \hat{v}_i^T(t_r^i)Q_{1,i}\hat{v}_i(t_r^i) - \frac{\chi^2}{2}.\end{aligned}$$

Taking the derivative of the event-triggering function $\psi_i(t)$ derives

$$\begin{aligned}\dot{\psi}_i(t) =& -2e_i^T\left[D_iv_i(t) + B_if_i(v_i(t)) - c\sum_{j=2}^N l_{ij}\Gamma v_j(t_r^i) + I_i\right]\\ \leqslant& 4e_i^T(t)e_i(t) + \lambda_{\max}\{D_i^TD_i + 2\xi_i^2B_i^TB_i\}v_i^T(t)v_i(t) + c^2\left(\sum_{j=2}^N l_{ij}\Gamma v_j(t_r^i)\right)^T\sum_{j=2}^N l_{ij}\Gamma v_j(t_r^i) + I_i^TI_i.\end{aligned}$$

Define $\alpha_1^i = \lambda_{\max}\{D_i^TD_i + 2\xi_i^2B_i^TB_i\}$. One obtains

$$\begin{aligned}\dot{\psi}_i(t) \leqslant& (4+2\alpha_1^i)e_i^T(t)e_i(t) + 2\alpha_1^iv_i^T(t_r^i)v_i(t_r^i) + c^2\left(\sum_{j=2}^N l_{ij}\Gamma v_j(t_r^i)\right)^T\sum_{j=2}^N l_{ij}\Gamma v_j(t_r^i) + I_i^TI_i\\ \leqslant& (4+2\alpha_1^i)\psi_i(t) + ((2-a_2^i)\alpha_1^i - 2a_2^i)v_i^T(t_r^i)v_i(t_r^i) + \left(c^2 - (4+2\alpha_1^i)(a_2^i + (a_2^i)^2)\right)\\ &\times\left(\sum_{j=2}^N l_{ij}\Gamma v_j(t_r^i)\right)^T\left(\sum_{j=2}^N l_{ij}\Gamma v_j(t_r^i)\right) - \frac{a_2^i\chi^2(4+2\alpha_1^i)}{2} + I_i^TI_i\\ =& (4+2\alpha_1^i)\psi_i(t) + \hat{v}_i^T(t_r^i)\begin{bmatrix} Q_{2,i}^{11} & Q_{2,i}^{12}\\ Q_{2,i}^{12} & Q_{2,i}^{22}\end{bmatrix}\hat{v}_i^T(t_r^i) + \hat{\beta}^i\\ \triangleq& (4+2\alpha_1^i)\psi_i(t) + \hat{v}_i^T(t_r^i)Q_{2,i}\hat{v}_i^T(t_r^i) + \hat{\beta}^i,\end{aligned}$$

where $Q_{2,i}^{11} = ((2-a_2^i)\alpha_1^i - 2a_2^i)E_n + \left(c^2 - (4+2\alpha_1^i)(a_2^i + (a_2^i)^2)\right)l_{ii}^2\Gamma^2$, $Q_{2,i}^{12} = \left(c^2 - (4+2\alpha_1^i)(a_2^i + (a_2^i)^2)\right)l_{ii}\Gamma$, $Q_{2,i}^{22} = \left(c^2 - (4+2\alpha_1^i)(a_2^i + (a_2^i)^2)\right)E_n$, $\hat{\beta}^i = -\frac{a_2^i\chi^2(4+2\alpha_1^i)}{2} + I_i^TI_i$.

A lower bound of time $T_{\min}^i$ can also be found by combining the method of Theorem 1 and will not repeat it here. Select $\varrho_2^i = \frac{\lambda_{\min}\{Q_{1,i}\}\hat{\beta}^i}{\lambda_{\max}\{Q_{2,i}\}}$. Therefore,

$$\begin{aligned}t_{k+1}^i - t_k^i >& \frac{\ln[\frac{(4+2\alpha_1^i)(\varrho_2^i + \hat{v}_i^T(t_r^i)Q_{1,i}\hat{v}_i^T(t_r^i))}{\hat{v}_i^T(t_r^i)Q_{2,i}\hat{v}_i^T(t_r^i) + \hat{\beta}^i} + 1]}{4+2\alpha_1^i}\\ >& \frac{\ln[\frac{(4+2\alpha_1^i)\lambda_{\min}\{Q_{1,i}\}}{\lambda_{\max}\{Q_{2,i}\}} + \frac{(4+2\alpha_1^i)(\varrho_2^i - \frac{\lambda_{\min}\{Q_{1,i}\}\hat{\beta}^i}{\lambda_{\max}\{Q_{2,i}\}})}{\hat{v}_i^T(t_r^i)Q_{2,i}\hat{v}_i^T(t_r^i) + \hat{\beta}^i} + 1]}{4+2\alpha_1^i}\\ =& \frac{\ln[\frac{(4+2\alpha_1^i)\lambda_{\min}\{Q_{1,i}\}}{\lambda_{\max}\{Q_{2,i}\}} + 1]}{4+2\alpha_1^i} \triangleq T_{\min}^i.\end{aligned}$$

Thus, Zeno behavior is eliminated.

Appendix E Proof of Theorem 4

Proof. A Lyapunov function candidate is chosen as

$$V_i(t) = \frac{\mu_i}{2}\epsilon_i^T(t)\epsilon_i(t). \tag{E1}$$

When $t \neq t_r$, taking the derivative of $V_i(t)$ gives

$$\begin{aligned}\dot{V}_i(t) &= \mu_i\epsilon_i^T(t)\dot{\epsilon}_i(t)\\ &= \mu_i\epsilon_i^T(t)\left[D_i\epsilon_i(t) + B_ih_i(\epsilon_i(t)) - c\sum_{j=2}^{N} l_{ij}\Gamma\epsilon_j(t_r^i) + I_i - I_1 + G_i(v_1(t))\right]\\ &\leqslant \mu_i\epsilon_i^T(t)\left[D_i + \frac{1}{2}B_iB_i^T + \frac{\xi_i+2}{2}E_n\right]\epsilon_i(t) + \frac{\mu_i}{2}((I_i - I_1)^T(I_i - I_1)\\ &\quad + G_i^T(v_1(t))G_i(v_1(t))) - c\mu_i\epsilon_i^T(t)\sum_{j=2}^{N} l_{ij}\Gamma\epsilon_j(t_r^i).\end{aligned} \tag{E2}$$

Furthermore, one gets

$$\begin{aligned}&-c\mu_i\epsilon_i^T(t)\sum_{j=2}^{N} l_{ij}\Gamma\epsilon_j(t_r^i)\\ =&-c\mu_i[v_i(t_r^i) - v_1(t) - e_i(t)]^T\sum_{j=2}^{N} l_{ij}\Gamma(v_j(t_r^i) - v_1(t_r^i))\\ \leqslant&-c\mu_i\left[v_i^T(t_r^i)\sum_{j=2}^{N} l_{ij}\Gamma v_j(t_r^i) - v_1^T(t)\sum_{j=2}^{N} l_{ij}\Gamma v_j(t_r^i) - \frac{1}{a_2^i}e_i(t)e_i^T(t) - a_2^i\sum_{j=2}^{N}\sum_{k=2}^{N} l_{ij}l_{ik}v_j^T(t_r^i)\Gamma^T\Gamma v_k(t_r^i)\right]\\ \leqslant&-c\mu_i\left[v_i^T(t_r^i)\sum_{j=2}^{N} l_{ij}\Gamma v_j(t_r^i) - \frac{1}{a_2^i}e_i^T(t)e_i(t) - \frac{\chi^2}{2} - (\frac{1}{2} + a_2^i)\left(\sum_{j=2}^{N} l_{ij}\Gamma v_j(t_r^i)\right)^T\sum_{j=2}^{N} l_{ij}\Gamma v_j(t_r^i)\right]\\ \leqslant&\frac{c\mu_i\varrho_2^i}{a_2^i}.\end{aligned} \tag{E3}$$

Under Assumption 1, substituting (E3) into (E2) yields

$$\dot{V}_i(t) \leqslant \sigma_2^iV_i(t) + g_2^i, \tag{E4}$$

where $g_2^i = \max||\frac{\mu_i}{2}\left((I_i - I_1)^T(I_i - I_1) + G_i^T(v_1)G_i(v_1)\right) + \frac{c\mu_i\varrho_2^i}{a_2^i}||$.

When $t = t_r^i$, one gets

$$V_i(t_r^i) \leqslant (1 - \bar{c}a_{i1})^2V_i(t_r^{i\,-}). \tag{E5}$$

For arbitrary constant $\varepsilon_2^i > 0$, define the following impulsive comparison system

$$\begin{cases}\dot{\omega}_2^i(t) = \sigma_2^i\omega_2^i(t) + g_2^i + \varepsilon_2^i, \quad t \neq t_r,\\ \omega_2^i(t_r^i) = (1 - \bar{c}a_{i1})^2\omega(t_2^{i\,-}),\\ \omega_2^i(0) = \frac{\mu_i}{2}||\epsilon_2^i(0)||^2.\end{cases}$$

In addition, let $\omega_2^i(t)$ be the unique solution of the above system. Similar to Theorem 2, defining $\nu_2^i = \sigma_2^i + \frac{2\ln(1-\bar{c}a_{i1})}{T_{\min}^i}$ yields

$$||\epsilon_i(t)|| \leqslant \sqrt{\frac{2\omega_2^i(0)}{\mu_i} + \frac{2(g_2^i + \varepsilon_2^i)}{\mu_i\nu_2^i}}e^{\frac{\nu_2^i}{2}t} + \sqrt{\frac{2(g_2^i + \varepsilon_2^i)}{-\mu_i\nu_2^i}}.$$

Therefore, the synchronization error of the ith neural network can be globally exponentially stable when t tends to infinity for $i = 2, 3, \ldots, N$. Therefore, the heterogeneous neural networks can reach globally exponentially quasi-synchronization under the DETIC.

Remark 3. From Theorem 4, one can find that each error system (13) eventually converges into the bounded set

$$\left\{\epsilon_i(t) \;\middle|\; ||\epsilon_i(t)|| \leqslant \sqrt{\frac{2(g_2^i + \varepsilon_2^i)}{-\mu_i\nu_2^i}}\right\},$$

$i = 2, 3, \cdots, N$. Therefore, regulating the value of μ_i and $\bar{c}$ can reduce the upper bound of quasi-synchronization errors.

Remark 4. Under the distributed impulsive strategy, since different neural networks have different impulsive sequences, different event-triggered functions should be designed separately for different neural networks. Meanwhile, according to the derived impulsive sequences, the sampling couplings (4) and (10) adopt synchronous sampling information and asynchronous sampling information, respectively.

Remark 5. When $t \neq t_r^i$, $\psi_i(t) < 0$, $r = 0, 1, 2, \cdots$. When $t = t_r^i$, $\psi_i(t)$ is negative again. Therefore, similar to $\psi(t)$, when the event-triggered function $\psi_i(t) \geqslant 0$ for any particular moment, the impulsive control will be triggered immediately. Then, $\psi_i(t) < 0$ will continue to be satisfied. Therefore, CEITC and DEITC are instantaneous and discrete.

Appendix F Numerical simulations

Six isolated neural networks are considered and five follower neural networks are defined as

$$\dot{v}(t) = Dv(t) + Bf(v(t)) + c(\widetilde{L} \otimes \Gamma)v(t_r) + I,$$

where $v(t) = [v_2(t), v_3(t), v_4(t), v_5(t), v_6(t)]^T$, $v_i(t) \in \mathbb{R}$, $\Gamma = 1.5$, $c = 1$, $f(v(t)) = [\tanh(v_2(t)), \arctan(v_3(t)), 0.7\tanh(v_4(t)), 0.6\arctan(v_5(t)), 1.2\tanh(v_6(t))]^T$, $I = [0.1, 0.1, 0.09, 0.08, 0.1]^T$,

$$D = \begin{bmatrix} -0.1 & 0 & 0 & 0 & 0 \\ 0 & -0.15 & 0 & 0 & 0 \\ 0 & 0 & -0.13 & 0 & 0 \\ 0 & 0 & 0 & -0.05 & 0 \\ 0 & 0 & 0 & 0 & -0.1 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 1.3 & 0 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 2.3 \end{bmatrix}.$$

The leader neural network is defined as

$$\dot{v}_1(t) = D_1 v_1(t) + B_1 f_1(v_1(t)) + I_1,$$

where $v_1(t) \in \mathbb{R}$, $D_1 = -2$, $B_1 = 5$, $f(v_1(t)) = \tanh(v_1(t))$ and $I_1 = 0$.

The topology is sketched for five followers in the Fig. F1 with the Laplacian matrix

$$\widetilde{L} = \begin{bmatrix} 2 & 0 & -1 & 0 & -1 \\ -1 & 2 & 0 & 0 & -1 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix},$$

with $\Theta = \mathrm{diag}\{\frac{50}{3}, \frac{71}{15}, \frac{377}{15}, \frac{172}{15}, \frac{112}{15}\}$.

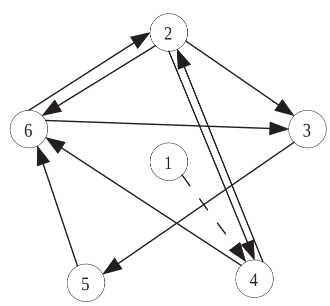

Figure F1 The topology of neural networks.

Example 1. Select $v_1(0) = 0.35$ and $v(0) = [1.7, 0.88, 0.5, 2.1, 1.57]^T$ as the initial values of the leader and five followers, respectively. For the CETIC, take $\bar{c} = 0.6$, $a_1 = 0.06$, $a_{41} = 1$, $a_{21} = a_{31} = a_{51} = a_{61} = 0$, and $\varrho = 1$. It follows from Fig. F2 that quasi-synchronization can be reached under the CETIC. From Fig. F3, quasi-synchronization errors can eventually converge to a bounded set for each impulsive interval. Fig. F4 shows the value of the event-triggered function at each moment. As shown in Fig. F4 that once the value exceeds 0, it immediately returns below 0, which is exactly consistent with the description of Remark 1 of the letter.

Example 2. Suppose that the initial values of the followers and the leader are the same as those in Example 1. In the DETIC, set $\bar{c} = 0.6$, $a_2^i = 0.06$, and $\varrho_2^i = 1, i = 2, 3, \cdots, 6$. Besides, $a_{41} = 1$, and $a_{21} = a_{31} = a_{51} = a_{61} = 0$. It follows from Fig. F5 that quasi-synchronization of the heterogeneous neural networks can be reached faster under the DETIC. In addition, Fig. F6 shows that quasi-synchronization errors can be kept within a bounded range. Compared with CETIC, DETIC has a better control efficiency. In fact, the maximal absolute value of synchronization errors under DETIC is less than the maximal absolute value of synchronization errors under CETIC by Figs. F3 and F6. Fig. F7 shows the value of the event-triggered function of each neural network at each moment, which also validates the description in Remark 1 of the letter.

Remark 6. In the simulations, the description of the state of the neural networks are recursive in discrete time, therefore the value of the event-triggered function will be greater than 0. To avoid this situation, we can reduce the criterion threshold of the event-triggered function to a negative number.

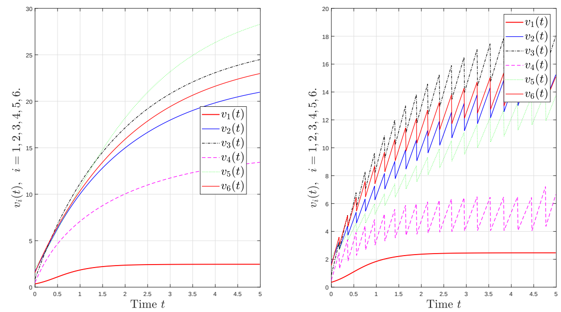

Figure F2 The state trajectories of neural networks without control and with CETIC.

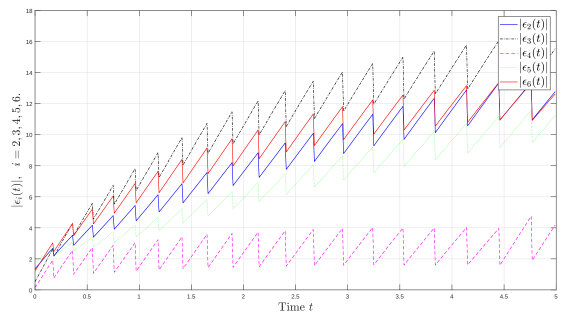

Figure F3 The absolute value of quasi-synchronization errors under CETIC.

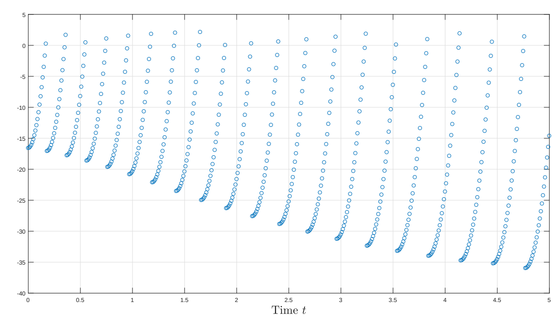

Figure F4 The value of event-triggered function under CETIC.

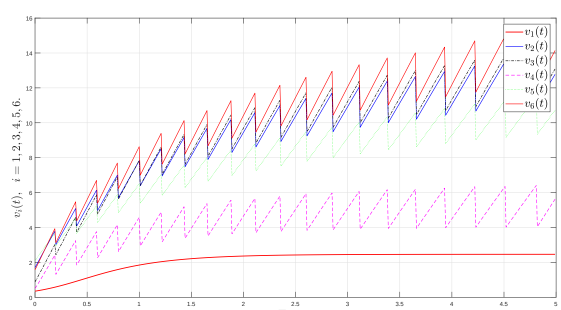

Figure F5 The state trajectories of neural networks under DETIC.

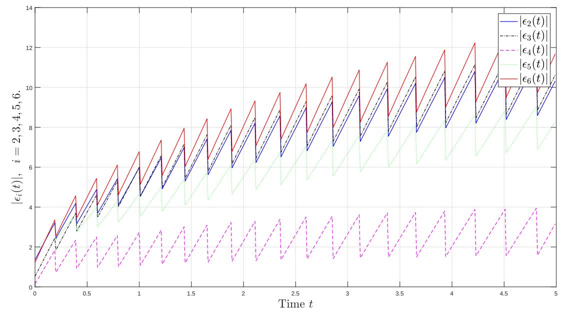

Figure F6 The synchronization errors of neural networks under DETIC.

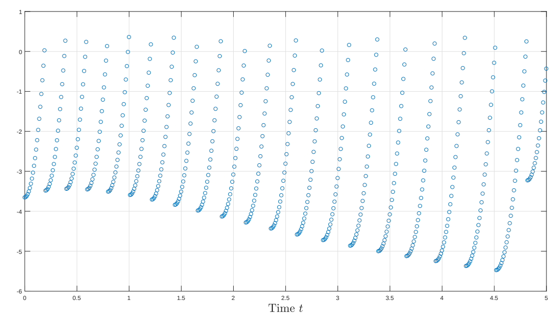

Figure F7 The value of event-triggered function under DETIC.

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