SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

February 2024, Vol. 67, Iss. 2, 122201:1–122201:26
https://doi.org/10.1007/s11432-022-3730-8

A FAS approach for stabilization of generalized chained forms: part 1. Discontinuous control laws

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Received 1 September 2022/Revised 1 January 2023/Accepted 10 February 2023/Published online 11 January 2024

Abstract In this paper, a type of general nonholonomic systems is proposed, which contains both the Brockett's two example systems, and their extended n-dimensional chained forms, as special cases. For the stabilization of such systems, a stabilizing controller is proposed based on the fully actuated system (FAS) approach, which is discontinuous at the origin but time-invariant when the open-loop system is time-invariant, and drives the feasible trajectories of the system to the origin exponentially. Furthermore, the proposed FAS approach is also extended to the sub-normal system case and the time-delay system case.

Keywords nonholonomic systems, feedback stabilization, exponential convergence, sub-fully actuated systems, time-delay systems

1 Introduction

In the celebrated work [1] of Brockett in 1983, the following two example systems:

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_0 x_1, \end{cases} \tag{1}$$

and

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_1 u_0 - x_0 u, \end{cases} \tag{2}$$

are proposed, where x_i, $i = 0, 1, 2$, are the scalar state variables, u_0 and u are two scalar control variables. The first example system (1), although has a continuous time-invariant feedback stabilizing controller, does not have a smooth time-invariant stabilizing controller, while the second one (2) even does not have a continuous time-invariant feedback stabilizing controller although it is open-loop controllable in a certain nonlinear system sense [1].

The two example systems (1) and (2) are respectively referred to as the Brockett's first and second example systems in [2, 3]. Particularly, the above Brockett's second example system (2), as well as its following equivalent form:

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_1 u_0, \end{cases} \tag{3}$$

is commonly called the Brockett integrator or the nonholonomic integrator in [4, 5]. This "deceivingly simple looking nonlinear system" [6] turns out to be very tough and thorny, and has stirred considerable research attention in the nonlinear control field (e.g., [7–10]).

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Obviously, the Brockett's first and second example systems (1) and (3) have the following extended chained forms:

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_0 x_1, \\ \vdots \\ \dot{x}_n = x_0 x_{n-1}, \end{cases} \tag{4}$$

and

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_1 u_0, \\ \vdots \\ \dot{x}_n = x_{n-1} u_0, \end{cases} \tag{5}$$

respectively. For convenience, in this paper, systems (4) and (5) are respectively called the Brockett's first and second chained forms. These typical nonholonomic systems arise from many applications such as car-type vehicles, mobile robots, surface vessels, and underwater vehicles [3]. As a matter of fact, many nonholonomic systems can be represented by kinematic models in these chained forms or are feedback equivalent to these chained forms [10, 11]. Particularly, it has been shown that the Brockett's second chained form (5) is in fact a kind of canonical form for a wide class of nonholonomic systems, and therefore has attracted the attention of numerous researchers. Considerable efforts have been made in seeking the related feedback controllers, e.g., discontinuous control [12–19], sliding mode control [20, 21], time-varying stabilizers [22–27], and switching control [28].

The absence of continuous time-invariant state-feedback asymptotic stabilizers has oriented research towards two main stabilization techniques: by time-invariant but discontinuous laws, and by continuous but time-varying laws. Particularly, for discontinuous control of the Brockett's second chained form, the basic σ-process introduced in [12] is an important invention, which transforms the system into a controllable constant linear one. Very recently, robust control of the Brockett's second chained system with uncertainties is investigated in [13], and a robust finite-time stabilizing algorithm is presented for the arbitrary-order nonholonomic system in chained form with the locally bounded, homogeneous, and discontinuous state-feedback law. For time-varying controller designs, global asymptotical stability with exponential convergence is achieved in [29] about any desired configuration by using a nonsmooth, time-varying feedback control law. In [30], a recursive technique is proposed which appears to be an extension of the popular integrator backstepping idea to the tracking of nonholonomic control systems. For continuous time-varying controllers for the Brockett's second chained form, the approach proposed in [22] is wise and convenient, which employs an extended σ-process and converts the problem into the stabilization of controllable time-varying linear systems with the time-varying terms decaying exponentially and therefore being neglected in the design.

Most of the reported results for control of the Brockett's chained forms are based on the general state-space approach. In essence, these chained systems, although nonholonomic, are relatively simple because they are "close to" linear, since they can be transformed into a linear, sometimes time-varying, state-space system by the standard σ-process. In cases where complicated generalized forms of these chained systems are considered, the transformed systems by the σ-process (if applicable) may not be linear anymore. Since the stabilization of some general time-varying linear systems even may not be solvable, such generalizations eventually put us into the awkward situation where there is no way to apply but the Lyapunov function methods.

Parallel to the state-space approach, recently, a new approach called the fully actuated system (FAS) approach is introduced for control system designs (see [31–43]), which owns an advantage that the closed-loop systems are often constant linear ones or constant linear ones with a certain initial value restriction. The FAS approach has also been generalized to the discrete-time system case [44] and the time-varying and time-delay system cases [45–48]. More relevantly, the FAS approach has recently been successfully applied to solve the stabilization of some nonholonomic systems [2,3,49]. Specifically, in [2], the Brockett's first example system (1) is treated, and in [3], the Brockett's second example system (2) or (3) is further treated, both with the FAS approach.

The contributions of this paper are the following.

Firstly, by adding two series of nonlinear time-varying functions into the Brockett's chained forms, a type of general nonholonomic systems is proposed. It is obvious that the original Brockett's chained forms are time-invariant, but the generalized ones may be time-varying. It is well-known that the original Brockett's chained forms can be equivalently converted into linear systems by the well-known σ-process, but the generalized ones, when transformed by the well-known σ-process, are still nonlinear. Furthermore, the generalized ones may contain the locally normal and sub-normal cases (see definitions in Section 2), and may also contain the forms with time-varying delays.

Secondly, under the condition that the initial value of the first scalar subsystem is restricted to be nonzero, as in [12], the well-known σ-process is employed and a strict-feedback system (SFS) model for the second subsystem is obtained, which is then converted into a global FAS. Based on such a treatment, a discontinuous stabilizing controller is obtained, which turns out to be time-invariant if the two series of nonlinear functions introduced are time-invariant. As a consequence, the designed controller drives all the states of the designed system to zero exponentially provided that the initial value of the first scalar subsystem is restricted to be nonzero. At this stage, the well-known method of backstepping can still be applied to the derived SFS model. However, the FAS approach possesses two advantages, one is the simplicity of the approach, and the other is that it always guarantees a constant linear closed-loop system.

Thirdly, the proposed FAS approach is also extended to the sub-normal system case and the time-delay system case. For the sub-normal system case, eventually a sub-FAS model for the second subsystem is derived after the application of the "σ-process"-like transformation. Therefore, unlike the globally or locally normal case, now the transformed second subsystem even does not have a smooth or even continuous controller. Eventually, the well-known method of backstepping is no longer applicable. Yet, applying the typical sub-FAS approach, the largest region in the initial value space of the system, which is termed the region of exponential attraction (ROEA), is characterized, and a controller is designed which drives all the trajectories starting from this ROEA to the origin exponentially. For control of time-delay systems, it is well-recognized that the problem is very difficult, especially for those systems with time-varying delays. Particularly, in the literature of nonholonomic system control, most of the reported results are concentrated on systems without time-delays. Yet it is shown in this paper that the FAS approach allows the investigation to be carried over to the generalized type of systems with time-varying delays.

In the subsequent sections, the n-dimensional vector space and the space of dimension $m \times n$ matrices are defined as $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$, respectively. I_n denotes the identity matrix of order n, $\emptyset$ denotes the null set, and $\Omega\backslash\Theta$ represents the complement of the set Θ in the set Ω. For a square matrix $A \in \mathbb{R}^{n\times n}$, the notations $\det A$ and A^{-1} represent its determinant and inverse, respectively. Moreover, for vector functions $x, x_i \in \mathbb{R}^m$ of time t, and $A_i \in \mathbb{R}^{m\times m}$, $i = 1, 2, \ldots, n$, the following standard symbols for the FAS approach are used in the paper:

$$x^{(0\sim n)} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(n)} \end{bmatrix}, \; x_{i\sim j} = \begin{bmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_j \end{bmatrix}, \; i \leqslant j,$$

$$x_{i\sim j}^{(0\sim n)} = \begin{bmatrix} x_i^{(0\sim n)} \\ x_{i+1}^{(0\sim n)} \\ \vdots \\ x_j^{(0\sim n)} \end{bmatrix}, \; x_k^{(n_k)}|_{k=i\sim j} = \begin{bmatrix} x_i^{(n_i)} \\ x_{i+1}^{(n_{i+1})} \\ \vdots \\ x_j^{(n_j)} \end{bmatrix}, \; i \leqslant j,$$

$$A_{0\sim n} = \begin{bmatrix} A_0 & A_1 & \cdots & A_n \end{bmatrix}, \; \Phi\left(A_{0\sim n}\right) = \begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -A_0 & -A_1 & \cdots & -A_n \end{bmatrix}.$$

The paper consists of 8 sections. Section 2 proposes the type of general nonholonomic systems, and the problem of stabilizing controller design is formulated in Section 3. Section 4 provides the FAS approach, and Section 5 particularly treats the case of $n = 2$, which corresponds to the Brockett's first and second example systems. Extensions of the FAS approach to the sub-normal and time-delay cases are presented in Sections 6 and 7, respectively, followed by some concluding remarks in Section 8.

2 System models

2.1 The basic model

In this paper, we introduce several more general types of nonholonomic systems, with the two basic ones described by

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_2\hat{\varphi}_1(\cdot) + \rho_1 x_1\hat{\psi}_1(\cdot), \\ \dot{x}_3 = x_3\hat{\varphi}_2(\cdot) + \rho_2 x_2\hat{\psi}_2(\cdot), \\ \quad\vdots \\ \dot{x}_n = x_n\hat{\varphi}_{n-1}(\cdot) + \rho_{n-1} x_{n-1}\hat{\psi}_{n-1}(\cdot), \end{cases} \tag{6}$$

where x_i, $i = 0, 1, 2, \ldots, n$, are the system state variables, u_0 and u are the control variables,

$$\rho_i \in \{x_0, u_0\},\ i = 1, 2, \ldots, n-1, \tag{7}$$

and

$$\hat{\varphi}_i(\cdot) \triangleq \hat{\varphi}_i(x_0, x_{i+1\sim n}, u_0, t),\ \hat{\psi}_i(\cdot) \triangleq \hat{\psi}_i(x_0, x_{i+1\sim n}, u_0, t),\ i = 1, 2, \ldots, n-1, \tag{8}$$

are two sets of scalar functions, with $\hat{\psi}_i(\cdot)$, $i = 1, 2, \ldots, n-1$, satisfying the following assumption.

Assumption A1. For all $x_0, u_0 \in \mathbb{R}, x_{i+1\sim n} \in \mathbb{R}^{n-i}$, and $t \geqslant 0$, there holds

$$\hat{\psi}_i(x_0, x_{i+1\sim n}, u_0, t) \neq 0,\ i = 1, 2, \ldots, n-1.$$

Clearly, system (6) is a time-invariant one if and only if $\hat{\varphi}_i(\cdot) \triangleq \hat{\varphi}_i(x_0, x_{i+1\sim n}, u_0, t)$ and $\hat{\psi}_i(\cdot) \triangleq \hat{\psi}_i(x_0, x_{i+1\sim n}, u_0, t)$, $i = 1, 2, \ldots, n-1$, are all time-invariant (dependent on x_i, $i = 0, 1, \ldots, n$, and u_0 only). Furthermore, because of (7), the above system (6) is in fact a unified representation of 2^{n-1} number of systems.

In the special case of

$$\hat{\varphi}_i(x_0, x_{i+1\sim n}, u_0, t) = 0,\ \hat{\psi}_i(x_0, x_{i+1\sim n}, u_0, t) = 1,\ i = 1, 2, \ldots, n-1,$$

the above system (6), with $\rho_i = x_0$, $i = 1, 2, \ldots, n-1$, becomes the Brockett's first chained form (4) which clearly contains the Brockett's first example system (1) as a special case [1,2]; while the above (6), with $\rho_i = u_0$, $i = 1, 2, \ldots, n-1$, becomes the Brockett's second chained form (5) which obviously contains the famous Brockett integrator (3) as a special case [1,2].

The following result reveals some special features of the proposed system (6).

Proposition 1. The nonlinear system (6)–(8) does not have a smooth exponentially stabilizing controller. Furthermore, for the special case that both $\hat{\varphi}_i(\cdot)$ and $\hat{\psi}_i(\cdot)$ are time-invariant, system (6)–(8) does not have a time-invariant continuous stabilizing controller if, for $i = 1, 2, \ldots, n-1$, one of the following conditions holds:

(1) In the case of $\rho_i = x_0$,

$$\hat{\varphi}_i(\cdot)\,|_{u_0=0} = \hat{\psi}_i(\cdot)\,|_{u_0=0} = 0;$$

(2) In the case of $\rho_i = u_0$,

$$\hat{\varphi}_i(\cdot)\,|_{u_0=0} = 0.$$

Proof. For simplicity, let

$$F_i\left(x_0, x_{i+1\sim n}, u_0, \rho_i, t\right) = x_{i+1}\hat{\varphi}_i\left(\cdot\right) + \rho_i x_i \hat{\psi}_i\left(\cdot\right).$$

Since

$$\begin{aligned}
&\frac{\partial F_i}{\partial u} = 0,\\
&\frac{\partial F_i}{\partial x_p} = 0,\ 1 \leqslant p \leqslant i-1,\\
&\frac{\partial F_i}{\partial x_i} = \rho_i \hat{\psi}_i\left(\cdot\right),\\
&\frac{\partial F_i}{\partial x_{i+1}} = \hat{\varphi}_i\left(\cdot\right) + x_{i+1}\frac{\partial}{\partial x_{i+1}}\hat{\varphi}_i\left(\cdot\right) + \rho_i x_i \frac{\partial}{\partial x_{i+1}}\hat{\psi}_i\left(\cdot\right),\\
&\frac{\partial F_i}{\partial x_q} = x_{i+1}\frac{\partial}{\partial x_q}\hat{\varphi}_i\left(\cdot\right) + \rho_i x_i \frac{\partial}{\partial x_q}\hat{\psi}_i\left(\cdot\right),\ i+1 < q \leqslant n,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial F_i|_{\rho_i=u_0}}{\partial x_0} &= x_{i+1}\frac{\partial}{\partial x_0}\hat{\varphi}_i\left(\cdot\right) + u_0 x_i \frac{\partial}{\partial x_0}\hat{\psi}_i\left(\cdot\right),\\
\frac{\partial F_i|_{\rho_i=x_0}}{\partial x_0} &= x_{i+1}\frac{\partial}{\partial x_0}\hat{\varphi}_i\left(\cdot\right) + x_i\hat{\psi}_i\left(\cdot\right) + x_0 x_i \frac{\partial}{\partial x_0}\hat{\psi}_i\left(\cdot\right),\\
\frac{\partial F_i|_{\rho_i=x_0}}{\partial u_0} &= x_{i+1}\frac{\partial}{\partial u_0}\hat{\varphi}_i\left(\cdot\right) + x_0 x_i \frac{\partial}{\partial u_0}\hat{\psi}_i\left(\cdot\right),\\
\frac{\partial F_i|_{\rho_i=u_0}}{\partial u_0} &= x_{i+1}\frac{\partial}{\partial u_0}\hat{\varphi}_i\left(\cdot\right) + x_i\hat{\psi}_i\left(\cdot\right) + u_0 x_i \frac{\partial}{\partial u_0}\hat{\psi}_i\left(\cdot\right),
\end{aligned}$$

the linearized model of (6) at $x_i = 0$, $i = 0, 1, 2, \ldots, n$, and $u_0 = 0$, $u = 0$, is obviously given by

$$\begin{cases}
\dot{x}_0 = u_0,\\
\dot{x}_1 = u,\\
\dot{x}_{i+1} = \hat{\varphi}_i^{\circ}\left(t\right) x_{i+1},\ i = 1, 2, \ldots, n-1,
\end{cases} \tag{9}$$

where

$$\hat{\varphi}_i^{\circ}\left(t\right) \triangleq \hat{\varphi}_i\left(x_0, x_{i+1\sim n}, u_0, t\right)|_{x_{0\sim n}=0,\, u_0=0},\ i = 1, 2, \ldots, n-1.$$

Since the above linear system (9) is not controllable, the nonlinear system (6)–(8) does not have a smooth exponentially stabilizing controller.

To proceed, let

$$f\left(x_{0\sim n}, u_0, u\right) = \begin{bmatrix} u_0 \\ u \\ x_2\hat{\varphi}_1\left(\cdot\right) + \rho_1 x_1 \hat{\psi}_1\left(\cdot\right) \\ \vdots \\ x_n\hat{\varphi}_{n-1}\left(\cdot\right) + \rho_{n-1} x_{n-1}\hat{\psi}_{n-1}\left(\cdot\right) \end{bmatrix}.$$

Then, for an arbitrarily small positive scalar ε, under either condition of the proposition, setting

$$f\left(x_{0\sim n}, u_0, u\right) = \begin{bmatrix} 0 & 0 & \varepsilon & \cdots & \varepsilon \end{bmatrix}^{\mathrm{T}},$$

gives

$$u_0 = u = 0,$$

and

$$0 = \varepsilon. \tag{10}$$

Recall that ε is a positive scalar. The above equation (10) gives a contradiction which states that the mapping $f\left(x_{0\sim n},u_0,u\right):\mathbb{R}^{n+1}\times\mathbb{R}\times\mathbb{R}\longmapsto\mathbb{R}^{n+1}$ is not a surjection onto an open set containing 0. Therefore, by the well-known Brockett Theorem [1], under either condition of the proposition, the system does not have a time-invariant continuous stabilizing controller.

Remark 1. An important development in the generalization of the chained form systems was made in 2000 by Jiang [50], in which the robust stabilization of the following closely related model was considered with the method of backstepping:

$$\begin{cases}\dot{x}_0=\psi_0\left(t\right)u_0+\Delta f_0\left(t,x_0\right),\\ \dot{x}_1=\psi_1\left(t\right)x_2u_0+\Delta f_1\left(t,x_{0\sim n},u_0\right),\\ \quad\vdots\\ \dot{x}_{n-1}=\psi_{n-1}\left(t\right)x_nu_0+\Delta f_{n-1}\left(t,x_{0\sim n},u_0\right),\\ \dot{x}_n=\psi_n\left(t\right)u+\Delta f_n\left(t,x_{0\sim n},u_0\right),\end{cases}\tag{11}$$

where $\psi_0\left(t\right)$, $\Delta f_0\left(t,x\right)$, $\psi_i\left(t\right)$, and $\Delta f_i\left(t,x_{0\sim n},u_0\right)$, $i=1,2,\ldots,n$ represent the system uncertainties satisfying certain conditions. We point out that similar uncertainties can also be added to the model (6), and the corresponding problem of robust stabilization can be also solved with the proposed FAS approach (see [37,39,51]).

2.2 Further extended forms

In this subsection, let us further point out that the proposed system (6) has generalizations in several directions.

2.2.1 *Multivariable systems*

Firstly, we mention that the above system (6) can be further naturally generalized into the following multivariable one:

$$\begin{cases}\dot{x}_0=u_0,\\ \dot{x}_{i1}=u_i,\\ \dot{x}_{i2}=x_{i2}\hat{\varphi}_{i1}\left(\cdot\right)+\rho_{i1}x_{i1}\hat{\psi}_{i1}\left(\cdot\right),\\ \dot{x}_{i3}=x_{i3}\hat{\varphi}_{i2}\left(\cdot\right)+\rho_{i2}x_{i2}\hat{\psi}_{i2}\left(\cdot\right),\\ \quad\vdots\\ \dot{x}_{in_i}=x_{i,n_i}\hat{\varphi}_{i,n_i-1}\left(\cdot\right)+\rho_{i,n_i-1}x_{i,n_i-1}\hat{\psi}_{i,n_i-1}\left(\cdot\right),\end{cases}\quad i=1,2,\ldots,p,\tag{12}$$

where x_0 and x_{ij}, $j=1,2,\ldots,n_i$, $i=1,2,\ldots,p$, are the state variables, u_0 and u_i, $i=1,2,\ldots,p$, are control variables, and

$$\hat{\varphi}_{ij}\left(\cdot\right)\triangleq\hat{\varphi}_{ij}\left(x_0,x_{i,j+1\sim n_i},u_0,t\right),\ \hat{\psi}_{ij}\left(\cdot\right)\triangleq\hat{\psi}_{ij}\left(x_0,x_{i,j+1\sim n_i},u_0,t\right),\quad j=1,2,\ldots,n_i-1,\ i=1,2,\ldots,p,\tag{13}$$

are sets of scalar functions. Remember that the variables ρ_{ij} take the value of either x_0 or u_0.

2.2.2 *Locally normal systems*

For convenience, let us introduce the following definition.

Definition 1. A nonlinear system in the form of (6) is called globally normal if Assumption A1 is satisfied.

To introduce the locally normal system, let us introduce the following assumption for system (6).

Assumption B1. There exists a ball (neighborhood) $\hat{\mathbb{B}}_0$ centered at $(x_{0\sim n},u_0)=(0,0)$, such that

$$\hat{\psi}_i\left(x_0,x_{i+1\sim n},u_0,t\right)\neq0,\ \forall\left(x_{0\sim n},u_0\right)\in\hat{\mathbb{B}}_0,\ t\geqslant0,i=1,2,\ldots,n-1.$$

Corresponding to the above Definition 1, we also have the following one.

Definition 2. The nonlinear system (6) is called locally normal if Assumption B1 is satisfied. Otherwise, it is called abnormal.

2.2.3 *Sub-normal systems*

Let us first introduce the concepts of singular and feasible sets.

Definition 3. Consider the nonlinear system in the form of (6). Let

$$\hat{\mathbb{S}}_x = \left\{ (x_{0\sim n}, u_0) \,\middle|\, \begin{array}{l} \hat{\psi}_i\left(x_0, x_{i+1\sim n}, u_0, t\right) = 0, \ \ (x_{0\sim n}, u_0) \in \mathbb{R}^{n+2}, \ t \geqslant 0, \\ i = 1, 2, \ldots, n-1. \end{array} \right\}. \tag{14}$$

Then $\hat{\mathbb{S}}_x$ is called the singular set of system (6), and $\hat{\mathbb{F}}_x = \mathbb{R}^{n+2} \backslash \hat{\mathbb{S}}_x$ is called the feasible set of system (6). Furthermore, system (6) is called sub-normal if $\hat{\mathbb{S}}_x \neq \emptyset$.

The following definition further classifies a very important category.

Definition 4. The sub-normal system (6) is said to be rational if there exists a ball $\hat{\mathbb{B}}_0$ centered at $(x_{0\sim n}, u_0) = (0, 0)$, such that $\hat{\mathbb{B}}_0 \subset \hat{\mathbb{F}}_x$.

Obviously, a rational sub-normal system is locally normal.

2.2.4 *Time-delay systems*

The system (6)–(8) can be further extended to the following time-delay case:

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_2 \hat{\varphi}_1^{\mathrm{d}}\left(\cdot\right) + \rho_1 x_1 \hat{\psi}_1^{\mathrm{d}}\left(\cdot\right), \\ \dot{x}_3 = x_3 \hat{\varphi}_2^{\mathrm{d}}\left(\cdot\right) + \rho_2 x_2 \hat{\psi}_2^{\mathrm{d}}\left(\cdot\right), \\ \quad \vdots \\ \dot{x}_n = x_n \hat{\varphi}_{n-1}^{\mathrm{d}}\left(\cdot\right) + \rho_{n-1} x_{n-1} \hat{\psi}_{n-1}^{\mathrm{d}}\left(\cdot\right), \end{cases} \tag{15}$$

with

$$\hat{\varphi}_i^{\mathrm{d}}\left(\cdot\right) \triangleq \hat{\varphi}_i^{\mathrm{d}}\left(x_j\left(t - \tau_{i,j}\right)|_{j=0,i+1\sim n}, u_0\left(t - \tau_{i0}\right), t - \tau_i\right), \tag{16}$$

$$\hat{\psi}_i^{\mathrm{d}}\left(\cdot\right) \triangleq \hat{\psi}_i^{\mathrm{d}}\left(x_j\left(t - \sigma_{i,j}\right)|_{j=0,i+1\sim n}, u_0\left(t - \sigma_{i0}\right), t - \sigma_i\right), \ i = 1, 2, \ldots, n-1, \tag{17}$$

where $\tau_{i,j} = \tau_{i,j}\left(t\right), \sigma_{i,j} = \sigma_{i,j}\left(t\right), \tau_i = \tau_i\left(t\right), \sigma_i = \sigma_i\left(t\right), \ j = 0, i+1, i+2, \ldots, n$, and $i = 1, 2, \ldots, n-1$, are known time-varying time delays in the system, which are all bounded by a positive constant h, while the other variables are as stated before.

Corresponding to Assumption A1, we have Assumption C1.

Assumption C1. For all $X_i \in \mathbb{R}^{n-i+1}$, $i = 1, 2, \ldots, n-1$, $y \in \mathbb{R}$, and $t \geqslant -h$, there holds

$$\hat{\psi}_i^{\mathrm{d}}\left(X_i, y, t\right) \neq 0, \ i = 1, 2, \ldots, n-1.$$

The above description for time-delay systems gives only the globally normal case as guaranteed by the above Assumption C1. We point out that the proposed FAS approach can also be extended to treat the time-delay locally normal case and the sub-normal case.

For simplicity, in this paper, we will only treat the above case where the functions $\hat{\varphi}_i^{\mathrm{d}}\left(\cdot\right)$ and $\hat{\psi}_i^{\mathrm{d}}\left(\cdot\right)$, $i = 1, 2, \ldots, n-1$ possess the forms in (16) and (17), or, in other words, u_0 and each x_j have only simple delays. As a matter of fact, the proposed approach is also applicable to the cases where these two sets of functions are in much more complicated forms as shown in the following two cases.

(1) Multiple delays. When u_0 and each x_j in $\hat{\varphi}_i^{\mathrm{d}}\left(\cdot\right)$ have the multiple delays $\tau_{i,j,k}, \ k = 1, 2, \ldots, p_{ij}$, and u_0 and each x_j in $\hat{\psi}_i^{\mathrm{d}}\left(\cdot\right)$ have known multiple delays $\sigma_{i,j,k}$, $k = 1, 2, \ldots, q_{ij}$, the functions $\hat{\varphi}_i^{\mathrm{d}}\left(\cdot\right)$ and $\hat{\psi}_i^{\mathrm{d}}\left(\cdot\right)$, $i = 1, 2, \ldots, n-1$ may appear as follows:

$$\hat{\varphi}_i^{\mathrm{d}}\left(\cdot\right) \triangleq \hat{\varphi}_i^{\mathrm{d}}\left(\xi_i, u_0\left(t - \tau_{i,0,1\sim p_{i0}}\right), t - \tau_i\right), \ \hat{\psi}_i^{\mathrm{d}}\left(\cdot\right) \triangleq \hat{\psi}_i^{\mathrm{d}}\left(\zeta_i, u_0\left(t - \sigma_{i,0,1\sim q_{i0}}\right), t - \sigma_i\right), \tag{18}$$

where

$$\xi_i = x_j\left(t - \tau_{i,j,1\sim p_{ij}}\right)|_{j=0,i+1\sim n},\ \zeta_i = x_j\left(t - \sigma_{i,j,1\sim q_{ij}}\right)|_{j=0,i+1\sim n},\ i = 1,2,\ldots,n-1.$$

(2) Distributed delays. When u_0 and each x_j in $\hat{\varphi}_i^{\mathrm{d}}(\cdot)$ and $\hat{\psi}_i^{\mathrm{d}}(\cdot)$ have distributed delays, the functions $\hat{\varphi}_i^{\mathrm{d}}(\cdot)$ and $\hat{\psi}_i^{\mathrm{d}}(\cdot)$, $i = 1,2,\ldots,n-1$ may appear as follows:

$$\hat{\varphi}_i^{\mathrm{d}}(\cdot) \triangleq \hat{\varphi}_i^{\mathrm{d}}\left(\xi_i, \int_0^{\tau_{i0}} p_{i0}(\theta)u_0(t-\theta)\mathrm{d}\theta, t-\tau_i\right),\ \hat{\psi}_i^{\mathrm{d}}(\cdot) \triangleq \hat{\psi}_i^{\mathrm{d}}\left(\zeta_i, \int_0^{\sigma_{i0}} q_{i0}(\theta)u_0(t-\theta)\mathrm{d}\theta, t-\sigma_i\right), \tag{19}$$

where

$$\xi_i = \int_0^{\tau_{ij}} k_{ij}^{\varphi}(\theta)x_j(t-\theta)\mathrm{d}\theta\bigg|_{j=0,i+1\sim n},\ \zeta_i = \int_0^{\sigma_{ij}} k_{ij}^{\psi}(\theta)x_j(t-\theta)\mathrm{d}\theta\bigg|_{j=0,i+1\sim n},\ i = 1,2,\ldots,n-1, \tag{20}$$

with $k_{ij}^{\varphi} : [0,\tau_{ij}] \to \mathbb{R}$ and $k_{ij}^{\psi} : [0,\sigma_{ij}] \to \mathbb{R}$ denoting the delay kernels.

3 Problem formulation

3.1 The basic problem

The purpose of the paper is to present a stabilizing controller for the generalized chained form nonlinear system (6). To make the treatment simpler, let us divide system (6) into the following two subsystems:

$$\dot{x}_0 = u_0, \tag{21}$$

and

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_2\hat{\varphi}_1(\cdot) + \rho_1 x_1 \hat{\psi}_1(\cdot), \\ \dot{x}_3 = x_3\hat{\varphi}_2(\cdot) + \rho_2 x_2 \hat{\psi}_2(\cdot), \\ \quad \vdots \\ \dot{x}_n = x_n\hat{\varphi}_{n-1}(\cdot) + \rho_{n-1} x_{n-1} \hat{\psi}_{n-1}(\cdot). \end{cases} \tag{22}$$

For the first subsystem (21), we can design the following simple controller:

$$u_0 = -\beta x_0, \tag{23}$$

where β is a positive scalar. The closed-loop system is

$$\dot{x}_0 = -\beta x_0, \tag{24}$$

whose response is clearly given by

$$x_0(t) = x_0(0)\,\mathrm{e}^{-\beta t}. \tag{25}$$

Therefore, we have

$$u_0(t) = \dot{x}_0(t) = -\beta x_0(t) = -\beta x_0(0)\,\mathrm{e}^{-\beta t}.$$

Furthermore, it is easy to note that

$$x_0(t) \neq 0,\ t \geqslant 0 \Longleftrightarrow x_0(0) \neq 0. \tag{26}$$

Due to (23), we can now simply write

$$\begin{aligned} \varphi_i(\cdot) &\triangleq \varphi_i(x_0, x_{i+1\sim n}, t) = \hat{\varphi}_i(x_0, x_{i+1\sim n}, u_0, t), \\ \psi_i(\cdot) &\triangleq \psi_i(x_0, x_{i+1\sim n}, t) = \hat{\psi}_i(x_0, x_{i+1\sim n}, u_0, t),\ i = 1,2,\ldots,n-1, \end{aligned} \tag{27}$$

and Assumption A1 correspondingly becomes the following one.

Assumption A2. For all $x_0 \in \mathbb{R}$, $x_{i+1\sim n} \in \mathbb{R}^{n-i}$, and $t \geqslant 0$, there holds

$$\psi_i\left(x_0, x_{i+1\sim n}, t\right) \neq 0,\ i = 1, 2, \ldots, n-1.$$

Furthermore, with (23), the subsystem (22) can be rewritten as

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_2\varphi_1\left(\cdot\right) + \gamma_1 x_0 x_1 \psi_1\left(\cdot\right), \\ \dot{x}_3 = x_3\varphi_2\left(\cdot\right) + \gamma_2 x_0 x_2 \psi_2\left(\cdot\right), \\ \quad\vdots \\ \dot{x}_n = x_n\varphi_{n-1}\left(\cdot\right) + \gamma_{n-1} x_0 x_{n-1} \psi_{n-1}\left(\cdot\right), \end{cases} \tag{28}$$

where

$$\gamma_i = \begin{cases} 1, & \text{if } \rho_i = x_0, \\ -\beta, & \text{if } \rho_i = u_0. \end{cases} \tag{29}$$

In the case that system (28) is looked upon as an individual problem, we can replace (29) with

$$\gamma_i \in \{1, -\beta\},\ i = 1, \ldots, n-1. \tag{30}$$

In this case, system (28) is a unified representation of 2^{n-1} number of systems.

Parallel to Definition 1, a nonlinear system in the form of (28) is also called globally normal if Assumption A2 is satisfied. For system (28), we can also introduce the local assumption.

Assumption B2. There exists a ball (neighborhood) $\mathbb{B}_0$ centered at $x_{0\sim n} = 0$, such that

$$\psi_i\left(x_0, x_{i+1\sim n}, t\right) \neq 0,\ \forall x_{0\sim n} \in \mathbb{B}_0,\ t \geqslant 0, i = 1, 2, \ldots, n-1.$$

Corresponding to Definition 1, the nonlinear system (28) is also called locally normal if Assumption B2 is satisfied. Otherwise, it is called abnormal.

When u_0 is designed as in (23), $x_0\left(t\right)$ is given by (25). Hence, in the above subsystem (28), $x_0\left(t\right) = x_0\left(0\right)\mathrm{e}^{-\beta t}$ is a time-varying parameter rather than a state variable. Since $x_0\left(t\right) = 0$, $t \geqslant 0$ makes the subsystem (28) uncontrollable, let us impose the following assumption.

Assumption X. $x_0\left(t\right) = x_0\left(0\right)\mathrm{e}^{-\beta t}$, $x_0\left(0\right) \neq 0$.

Based on the above deduction, it can be clearly seen that, to stabilize system (6), it suffices to solve the following basic problem.

Problem 1. Find a uniformly globally exponentially (UGE) stabilizing feedback controller for system (28) under Assumption X.

Remark 2. Once a state feedback controller $u = u\left(x_0, x_{1\sim n}, t\right)$ to the above problem is found, a state feedback controller for system (6) is obtained, under $x_0\left(0\right) \neq 0$, as

$$\begin{cases} u_0 = -\beta x_0, \\ u = u\left(x_{0\sim n}, t\right). \end{cases} \tag{31}$$

As we will soon find that, when the functions $\varphi_i\left(\cdot\right)$ and $\psi_i\left(\cdot\right)$, $i = 1, 2, \ldots, n-1$ are not time-varying, the control $u\left(x_{0\sim n}, t\right)$ is also not time-varying. In this case, the above controller (31) turns out to be a time-invariant state feedback controller for system (6).

To conclude this subsection, let us finally point out that the requirement of $x_0\left(0\right) \neq 0$ is not a crucial restriction. Anyhow, to get rid of such a restriction, the idea of [22] can be adopted, and eventually a time-varying continuous state feedback controller can be designed.

3.2 The converted problem

In order to solve Problem 1, let us firstly transform system (28) into an SFS via the σ-process.

Theorem 1. Let Assumption X be met. Then, under the following transformation:

$$z_i = \frac{x_{n-i+1}}{x_0^{n-i}},\ i = 1, 2, \ldots, n, \tag{32}$$

system (28) is equivalently transformed into the following system:

$$\begin{cases} \dot{z}_1 = g_1(x_0, z_1, t) + h_1(x_0, z_1, t)\, z_2, \\ \dot{z}_2 = g_2(x_0, z_{1\sim 2}, t) + h_2(x_0, z_{1\sim 2}, t)\, z_3, \\ \ \vdots \\ \dot{z}_{n-1} = g_{n-1}(x_0, z_{1\sim n-1}, t) + h_{n-1}(x_0, z_{1\sim n-1}, t)\, z_n, \\ \dot{z}_n = u, \end{cases} \tag{33}$$

where

$$g_i(x_0, z_{1\sim i}, t) = [(n-i)\beta + \varphi_{n-i}(x_0, x_{n-i+1\sim n}, t)]\, z_i, \tag{34}$$

$$h_i(x_0, z_{1\sim i}, t) = \gamma_{n-i}\psi_{n-i}(x_0, x_{n-i+1\sim n}, t),\ i = 1, 2, \ldots, n-1. \tag{35}$$

Proof. Firstly, as in [12], let us introduce the following transformation:

$$y_i = \frac{x_i}{x_0^{i-1}},\ i = 1, 2, \ldots, n. \tag{36}$$

Then, noting (24) and

$$\dot{x}_i = x_i\varphi_{i-1}(\cdot) + \gamma_{i-1}x_0x_{i-1}\psi_{i-1}(\cdot),\ i = 2, 3, \ldots, n,$$

we have

$$\dot{y}_1 = \dot{x}_1 = u, \tag{37}$$

and

$$\begin{aligned} \dot{y}_i &= \frac{\dot{x}_i x_0^{i-1} - x_i(i-1)x_0^{i-2}\dot{x}_0}{x_0^{2(i-1)}} \\ &= \frac{[x_i\varphi_{i-1}(\cdot) + \gamma_{i-1}x_0x_{i-1}\psi_{i-1}(\cdot)]\, x_0^{i-1} + (i-1)\beta x_i x_0^{i-1}}{x_0^{2(i-1)}} \\ &= \gamma_{i-1}\frac{x_{i-1}\psi_{i-1}(\cdot)}{x_0^{i-2}} + \frac{(i-1)\beta x_i + x_i\varphi_{i-1}(\cdot)}{x_0^{i-1}} \\ &= \gamma_{i-1}\psi_{i-1}(\cdot)\, y_{i-1} + [(i-1)\beta + \varphi_{i-1}(\cdot)]\, y_i,\ i = 2, 3, \ldots, n. \end{aligned} \tag{38}$$

Combining (37) and (38) gives the following equivalent system of the original system (28):

$$\begin{cases} \dot{y}_1 = u, \\ \dot{y}_2 = [\beta + \varphi_1(\cdot)]\, y_2 + \gamma_1\psi_1(\cdot)\, y_1, \\ \dot{y}_3 = [2\beta + \varphi_2(\cdot)]\, y_3 + \gamma_2\psi_2(\cdot)\, y_2, \\ \ \vdots \\ \dot{y}_n = [(n-1)\beta + \varphi_{n-1}(\cdot)]\, y_n + \gamma_{n-1}\psi_{n-1}(\cdot)\, y_{n-1}. \end{cases} \tag{39}$$

Secondly, applying the following state transformation:

$$z_i = y_{n-i+1},\ i = 1, 2, \ldots, n, \tag{40}$$

the above system (39) is equivalently transformed into

$$\begin{cases} \dot{z}_1 = [(n-1)\beta + \varphi_{n-1}(\cdot)]\, z_1 + \gamma_{n-1}\psi_{n-1}(\cdot)\, z_2, \\ \dot{z}_2 = [(n-2)\beta + \varphi_{n-2}(\cdot)]\, z_2 + \gamma_{n-2}\psi_{n-2}(\cdot)\, z_3, \\ \ \vdots \\ \dot{z}_{n-1} = [\beta + \varphi_1(\cdot)]\, z_{n-1} + \gamma_1\psi_1(\cdot)\, z_n, \\ \dot{z}_n = u, \end{cases} \tag{41}$$

which can be further written in the form of (33) with $g_i(x_0, z_{1\sim i}, t)$ and $h_i(x_0, z_{1\sim i}, t)$, $i = 1, 2, \ldots, n-1$ defined as in (34) and (35).

Finally, combining the transformations (36) and (40) gives the transformation (32).

Remark 3. Due to (25), the transformation (32) can be rewritten as

$$z_i = \frac{x_{n-i+1}}{x_0^{n-i}} = \frac{\mathrm{e}^{\beta(n-i)t}}{x_0^{n-i}(0)} x_{n-i+1},\ i = 1, 2, \ldots, n, \tag{42}$$

which is clearly a differentiable homeomorphism between $z_{1\sim n}$ and $x_{1\sim n}$ when $x_0(0)$ is taken differently from zero. Therefore, a stabilizing controller for system (28) can be immediately obtained via (42) when a stabilizing controller for system (33) is designed.

Corresponding to Assumption A2 on the series of functions $\psi_i(x_0, x_{i+1\sim n}, t)$, $i = 1, 2, \ldots, n-1$, we have the following assumption on the functions $h_i(x_0, z_{1\sim i}, t)$, $i = 1, 2, \ldots, n-1$.

Assumption A3. For all $x_0 \neq 0$, $z_{1\sim i} \in \mathbb{R}^i$ and $t \geqslant 0$, there holds

$$h_i(x_0, z_{1\sim i}, t) \neq 0,\ i = 1, 2, \ldots, n-1.$$

Clearly, system (33) is an SFS when the above Assumption A3 is met.

According to the above Theorem 1, Problem 1 is clearly equivalently converted into the following one.

Problem 2. Find a UGE stabilizing controller for system (33)–(35) under Assumption A3.

Remark 4. When $\varphi_i(\cdot)$ and $\psi_i(\cdot)$, $i = 1, 2, \ldots, n-1$ are absent, the transformed system (39) is a time-invariant linear controllable system, based on which the control u can then be easily designed as done in [12]. However, when these functions are indeed present, the linear system design approach proposed in [12] is no longer valid since the corresponding system (39) is no longer a constant linear one.

4 The FAS approach

Due to Theorem 1, Problem 1 can indeed be solved by applying the well-known backstepping technique to the above SFS (33) under Assumptions X and A3. However, compared with the FAS approach to be applied, the method of backstepping has the following two drawbacks:

(1) It suffers from the serious problem of "differential explosion", which renders the application of the method of backstepping extremely difficult or even impossible when the dimension of the system is large;

(2) It does not always provide, like the FAS approach does, a linear closed-loop system which guarantees the uniform exponential stability of the designed system.

4.1 The FAS model

In this subsection, the FAS approach recently proposed in [35,39,41–44] is applied for stabilizing controller design of the proposed system (6). To get the FAS model of the above SFS (33), let us make use of a result proposed in [36].

In [36], the following SFS is considered:

$$\begin{cases} \dot{x}_1 = F_1(x_1, \zeta, t) + G_1(x_1, \zeta, t)x_2, \\ \dot{x}_2 = F_2(x_{1\sim 2}, \zeta, t) + G_2(x_{1\sim 2}, \zeta, t)x_3, \\ \vdots \\ \dot{x}_{n-1} = F_{n-1}(x_{1\sim n-1}, \zeta, t) + G_{n-1}(x_{1\sim n-1}, \zeta, t)x_n, \\ \dot{x}_n = u, \end{cases} \tag{43}$$

where $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$ are the state vectors, $u \in \mathbb{R}^r$ is the control input vector, and $\zeta \in \mathbb{R}^m$ is a parameter vector. Furthermore, $F_i(x_{1\sim i}, \zeta, t) \in \mathbb{R}^r$ and $G_i(x_{1\sim i}, \zeta, t) \in \mathbb{R}^{r\times r}$, $i = 1, 2, \ldots, n-1$ are sufficiently smooth vector functions and matrix functions, respectively, and for arbitrary $x_i \in \mathbb{R}^r$, $i = 1, 2, \ldots, n$, and $\zeta \in \mathbb{R}^m$, there holds

$$\det G_i(x_{1\sim i}, \zeta, t) \neq 0,\ \forall t \geqslant 0,\ i = 1, 2, \ldots, n-1. \tag{44}$$

In the Theorem 2.1 of [36], a method is provided to convert equivalently the SFS (43) into a FAS model in the following form:

$$z^{(n)} = f_n(z^{(0\sim n-1)}, \zeta, t) + B_n(z^{(0\sim n-1)}, \zeta, t)u, \tag{45}$$

where the matrix function $B_n(z^{(0\sim n-1)}, \zeta, t)$ and the vector function $f_n(z^{(0\sim n-1)}, \zeta, t)$ are given recursively. Based on this result, we can now obtain the FAS model of SFS (33).

Corresponding to SFS (43), we now have

$$\zeta = x_0,\ x_i = z_i,\ i = 1, 2, \ldots, n,$$

and

$$F_i\left(x_{1\sim i}, \zeta, t\right) = g_i\left(x_0, z_{1\sim i}, t\right),\ G_i\left(x_{1\sim i}, \zeta, t\right) = h_i\left(x_0, z_{1\sim i}, t\right),\ i = 1, 2, \ldots, n-1.$$

Applying the Theorem 2.1 in [36] to SFS (33) gives the following result.

Theorem 2. Let Assumptions X and A3 be satisfied, and $g_k\left(x_0, z_{1\sim k}, t\right)$ and $h_k\left(x_0, z_{1\sim k}, t\right), k = 1, 2, \ldots, n-1$, be differentiable with respect to all variables. With the convention of $g_n\left(x_0, z_{1\sim n}, t\right) = 0$, let

$$\begin{aligned} B_k\left(x_0, z_{1\sim k}, t\right) &= \prod_{i=1}^{k} h_i\left(x_0, z_{1\sim i}, t\right),\ k = 1, 2, \ldots, n-1, & (46)\\ B_n\left(x_0, z_{1\sim n}, t\right) &= B_{n-1}\left(x_0, z_{1\sim n-1}, t\right), & (47) \end{aligned}$$

and

$$\begin{aligned} f_k(x_0, z_{1\sim k}, t) = {} & \dot{f}_{k-1}(x_0, z_{1\sim k-1}, t) + \dot{B}_{k-1}(x_0, z_{1\sim k-1}, t) z_k \\ & + B_{k-1}(x_0, z_{1\sim k-1}, t) g_k\left(x_0, z_{1\sim k}, t\right),\ k = 2, 3, \ldots, n, \end{aligned} \tag{48}$$

with the initial value

$$f_1\left(x_0, z_1, t\right) = g_1\left(x_0, z_1, t\right). \tag{49}$$

Then, under the following transformation:

$$z^{(0\sim n-1)} = \begin{bmatrix} z_1 \\ f_1\left(x_0, z_1, t\right) + B_1\left(x_0, z_1, t\right) z_2 \\ f_2(x_0, z_{1\sim 2}, t) + B_2(x_0, z_{1\sim 2}, t) z_3 \\ \vdots \\ f_{n-1}(x_0, z_{1\sim n-1}, t) + B_{n-1}(x_0, z_{1\sim n-1}, t) z_n \end{bmatrix}, \tag{50}$$

the SFS (33)–(35) is equivalently transformed into the following FAS:

$$z^{(n)} = f_n(x_0, z_{1\sim n}, t) + B_n(x_0, z_{1\sim n}, t)u. \tag{51}$$

The following result confirms some properties of the above transformation (50).

Theorem 3. Let the assumptions in Theorem 2 be met. Then the transformation (50)

(1) is one-to-one;

(2) keeps the origin unmoved; and

(3) guarantees

$$\lim_{t\to\infty} z^{(0\sim n-1)} = 0 \Longrightarrow \lim_{t\to\infty} z_{1\sim n} = 0.$$

Proof. For each $z_{1\sim n}$, Eq. (50) clearly gives a unique $z^{(0\sim n-1)}$. Conversely, let $z^{(0\sim n-1)}$ be given; then we have

$$\begin{aligned} z_1 &= z, \\ z_2 &= B_1^{-1}\left(x_0, z_1, t\right)\left[\dot{z} - f_1\left(x_0, z_1, t\right)\right] \\ &= B_1^{-1}\left(x_0, z, t\right)\left[\dot{z} - f_1\left(x_0, z, t\right)\right], \\ z_3 &= B_2^{-1}(x_0, z_{1\sim 2}, t)\left[\ddot{z} - f_2(x_0, z_{1\sim 2}, t)\right] \end{aligned}$$

$$
\begin{aligned}
&= B_2^{-1}(x_0, z^{(0\sim 1)}, t)[\ddot{z} - f_2(x_0, z^{(0\sim 1)}, t)],\\
&\cdots.
\end{aligned}
$$

Therefore, Eq. (50) also gives a unique $z_{1\sim n}$ in the way described in Figure 1. Thus the transformation (50) is one-to-one.

Next, from (34) we have

$$
g_i\left(x_0, z_{1\sim i}, t\right)|_{z_{1\sim n}=0} = 0,\ i = 1, 2, \ldots, n-1, \tag{52}
$$

and

$$
\begin{aligned}
\dot{g}_i\left(x_0, z_{1\sim i}, t\right) = {}& \dot{\varphi}_{n-i}\left(x_0, x_{n-i+1\sim n}, t\right) z_i\\
&+ \left[(n-i)\,\beta + \varphi_{n-i}\left(x_0, x_{n-i+1\sim n}, t\right)\right] \dot{z}_i,\quad i = 1, 2, \ldots, n-1.
\end{aligned} \tag{53}
$$

Combining (52) with (33) gives

$$
\dot{z}_i|_{z_{1\sim n}=0} = 0,\ i = 1, 2, \ldots, n-1. \tag{54}
$$

Substituting (52) and (54) into (53) further produces

$$
\dot{g}_i\left(x_0, z_{1\sim i}, t\right)|_{z_{1\sim n}=0} = 0,\ i = 1, 2, \ldots, n-1. \tag{55}
$$

These relations in (55), together with (48), finally yield the relations

$$
f_i\left(x_0, z_{1\sim i}, t\right)|_{z_{1\sim n}=0} = 0,\ i = 1, 2, \ldots, n-1. \tag{56}
$$

Now rewrite the transformation (50) as

$$
z^{(0\sim n-1)} = B_e(x_0, z_{1\sim n-1}, t) z_{1\sim n} + \begin{bmatrix} 0 \\ f_1\left(x_0, z_1, t\right) \\ f_2(x_0, z_{1\sim 2}, t) \\ \vdots \\ f_{n-1}(x_0, z_{1\sim n-1}, t) \end{bmatrix}, \tag{57}
$$

where

$$
B_e(x_0, z_{1\sim n-1}, t) = \mathrm{diag}\left(1, B_1\left(x_0, z_1, t\right), \ldots, B_{n-1}(x_0, z_{1\sim n-1}, t)\right).
$$

With (56), it is clearly seen that the transformation (57), or equivalently (50), keeps the origin unmoved.

Finally, let us prove the conclusion (3). Since $g_k\left(x_0, z_{1\sim k}, t\right)$ and $h_k\left(x_0, z_{1\sim k}, t\right), k = 1, 2, \ldots, n$ are differentiable with respect to all variables, from the relations in (56) we can get

$$
\lim_{z_{1\sim n}\to 0} f_i\left(x_0, z_{1\sim i}, t\right) = 0,\ i = 1, 2, \ldots, n-1. \tag{58}
$$

Using (58) and Assumption A3, we have

$$
\begin{aligned}
z_1 &= z \to 0,\ \text{as } z^{(0\sim n-1)} \to 0,\\
z_2 &= B_1^{-1}\left(x_0, z_1, t\right)\left[\dot{z} - f_1\left(x_0, z_1, t\right)\right]\\
&= B_1^{-1}\left(x_0, z, t\right)\left[\dot{z} - f_1\left(x_0, z, t\right)\right]\\
&\to 0,\ \text{as } z^{(0\sim n-1)} \to 0,\\
z_3 &= B_2^{-1}(x_0, z_{1\sim 2}, t)\left[\ddot{z} - f_2(x_0, z_{1\sim 2}, t)\right]\\
&= B_2^{-1}(x_0, z^{(0\sim 1)}, t)[\ddot{z} - f_2(x_0, z^{(0\sim 1)}, t)]\\
&\to 0,\ \text{as } z^{(0\sim n-1)} \to 0,\\
&\cdots.
\end{aligned}
$$

Therefore, following the chart given in Figure 1, we can also get $\lim_{t\to\infty} z_{1\sim n} = 0$. Thus the whole proof is completed.

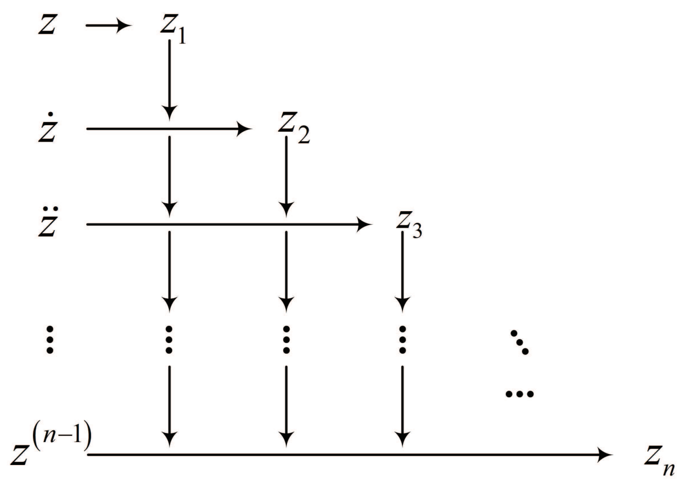

Figure 1 Solution flow chart.

4.2 Stabilizing controller design

Once the FAS model (51) of system (33) is obtained, a stabilizing controller for system (6) can then be immediately designed by a standard procedure [31, 36].

Theorem 4. Let Assumption A3 be met, $\beta > 0$, $x_0(0) \neq 0$, and $b_{0\sim n-1}$ be an arbitrary vector making $\Phi(b_{0\sim n-1})$ Hurwitz. Then,

(1) a stabilizing controller for system (6) is given by

$$\begin{cases} u_0 = -\beta x_0, \\ u = -\frac{1}{B_n(x_0, z_{1\sim n}, t)}\left[f_n(x_0, z_{1\sim n}, t) + b_{0\sim n-1} z_1^{(0\sim n-1)}\right], \end{cases} \tag{59}$$

where $z_1^{(0\sim n-1)}$ is given by (50), and $z_{1\sim n}$ is given by (32); and

(2) the corresponding closed-loop system is a constant linear one given by

$$\begin{cases} \dot{x}_0 = -\beta x_0, \\ z_1^{(n)} + b_{0\sim n-1} z_1^{(0\sim n-1)} = 0. \end{cases} \tag{60}$$

Please note that the closed-loop system (60) can also be equivalently written in the following state-space form:

$$\begin{cases} \dot{x}_0 = -\beta x_0, \\ \dot{z}_1^{(0\sim n-1)} = \Phi(b_{0\sim n-1}) z_1^{(0\sim n-1)}. \end{cases} \tag{61}$$

Since $\Phi(b_{0\sim n-1})$ is Hurwitz, $z_1^{(0\sim n-1)}$ converges to zero exponentially. Therefore, by Theorem 3, $z_{1\sim n}$ also converges to zero.

Obviously, the well-known technique of pole assignment can be applied to solve for the vector $b_{0\sim n-1}$. Particularly, the complete parametric approach proposed in [31] (see also the Proposition 2 in [35]) can be readily applied. Please note that the controller (59) is smooth under the condition $x_0(0) \neq 0$. Furthermore, when the open-loop system (6) is time-invariant, that is, when all the functions $g_i(\cdot)$ and $h_i(\cdot), i = 1, 2, \ldots, n-1$ are time-invariant, the above controller (59) is also time-invariant.

Remark 5. Since the stabilizing controller (59) is designed via the FAS model (51), it technically depends on the term $z_1^{(0\sim n-1)}$. This often gives the reader the wrong impression that the controller is relies on the high-order derivatives of the system variables and is thus difficult to apply due to problems of measuring and implementing the derivative signals. While as a matter of fact, this is not the case. When system (33) is transformed into FAS (51), as long as the state vector $z_{1\sim n}$ of the original system

(33) is known, the derivative term $z_1^{(0\sim n-1)} = z^{(0\sim n-1)}$ can be immediately expressed by $z_{1\sim n}$ via the transformation (50). We do not really need to measure these high-order derivatives and implement them practically, but instead, we use the measured state information $z_{1\sim n}$ of the original system (33).

In Section 5, let us examine an important special case.

5 Case of $n = 2$

In this case, let us omit the subscripts in ρ_1, $\hat{\varphi}_1(\cdot)$, and $\hat{\psi}_1(\cdot)$. Then the original system (6) reduces to

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_2\hat{\varphi}(x_0, x_2, u_0, t) + \rho x_1 \hat{\psi}(x_0, x_2, u_0, t), \end{cases} \tag{62}$$

which is a generalized form of the well-known Brockett integrator (3), and also the Brockett's first example system (1). Due to its particular importance, let us present the result for this case in some detail.

In this case, when the first scalar subsystem is controlled using the controller $u_0 = -\beta x_0$, the second subsystem becomes

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_2\varphi(x_0, x_2, t) + \rho x_1 \psi(x_0, x_2, t). \end{cases} \tag{63}$$

Further, by Theorem 1, under the following transformation:

$$z_1 = \frac{x_2}{x_0},\ z_2 = x_1, \tag{64}$$

system (63) is equivalent to the SFS:

$$\begin{cases} \dot{z}_1 = g(x_0, z_1, t) + h(x_0, z_1, t)\, z_2, \\ \dot{z}_2 = u, \end{cases} \tag{65}$$

where

$$g(x_0, z_1, t) = \beta z_1 + z_1\varphi(x_0, x_2, t)\,|_{x_2 = x_0 z_1},\ h(x_0, z_1, t) = \gamma\psi(x_0, x_2, t)\,|_{x_2 = x_0 z_1}, \tag{66}$$

with γ denoting γ_1 for simplicity.

5.1 Stabilizing controller

5.1.1 *Control of SFS (65)*

With the initial values

$$B_1(x_0, z, t) = h(x_0, z_1, t),\ f_1(x_0, z, t) = g(x_0, z_1, t), \tag{67}$$

the iterative formulae (46)–(48) give

$$B_2(x_0, z_1, t) = h(x_0, z_1, t),\ f_2(x_0, z_{1\sim 2}, t) = \dot{g}(x_0, z_1, t) + \dot{h}(x_0, z_1, t)\, z_2. \tag{68}$$

Therefore, the following result immediately follows from Theorem 2.

Corollary 1. Let Assumption X be satisfied, b_i, $i = 1, 2$, be two positive scalars, $g(x_0, z_1, t)$ and $h(x_0, z_1, t)$ be differentiable with respect to all variables, and

$$h(x_0, z_1, t) \neq 0,\ \forall x_0, z_1 \in \mathbb{R} \text{ and } t \geqslant 0.$$

Then,

(1) under the following transformation:

$$\begin{cases} z = z_1, \\ \dot{z} = g(x_0, z_1, t) + h(x_0, z_1, t)\, z_2, \end{cases} \tag{69}$$

the SFS (65) can be transformed into the following FAS:

$$\ddot{z} = f_2\left(x_0, z_{1\sim 2}, t\right) + h\left(x_0, z_1, t\right) u, \tag{70}$$

where $f_2\left(x_0, z_{1\sim 2}, t\right)$ is given by (68);

(2) a stabilizing controller for the SFS (65) is given by

$$\begin{cases} u = -\frac{1}{h(x_0,z_1,t)}\left(f_2\left(x_0, z_{1\sim 2}, t\right) + u^*\right), \\ u^* = b_0 z_1 + b_1\left[g\left(x_0, z_1, t\right) + h\left(x_0, z_1, t\right) z_2\right], \end{cases} \tag{71}$$

which results in the following constant linear closed-loop system:

$$\ddot{z} + b_1\dot{z} + b_0 z = 0. \tag{72}$$

The following remark gives, in the case of $n = 2$, a more intuitive explanation of the third conclusion of Theorem 3.

Remark 6. Once z and $\dot{z}$ are obtained, in view of (69), the variables z_1 and z_2 can be immediately given by

$$\begin{cases} z_1 = z, \\ z_2 = \frac{1}{h(x_0,z_1,t)}\left[\dot{z} - g\left(x_0, z_1, t\right)\right]. \end{cases} \tag{73}$$

Since b_i, $i = 0, 1$ are positive, both $z = z_1$ and $\dot{z}$ converge to zero exponentially. Furthermore, since

$$g\left(x_0, z_1, t\right) = \left[\beta + \varphi\left(x_0, x_2, t\right)\right] z_1 \to 0,$$

from (73) we clearly have $\lim_{t\to\infty} z_{1\sim 2} = 0$.

5.1.2 *Control of system (62)*

Coming back to the stabilization of the original system (62), we have the following result.

Theorem 5. Let b_i, $i = 0, 1$ be two positive scalars, $x_0(0) \neq 0$, and

$$\psi\left(x_0, x_2, t\right) \neq 0,\ \forall x_0, x_2 \in \mathbb{R} \ \text{ and } t \geqslant 0. \tag{74}$$

Further, let $g\left(x_0, z_1, t\right)$ and $h\left(x_0, z_1, t\right)$ be defined by (66). Then the following controller for the nonholonomic system (62) drives all the state variables x_i, $i = 0, 1, 2$ to zero:

$$\begin{cases} u_0 = -\beta x_0, \\ u = -\frac{1}{\gamma\psi(x_0,x_2,t)}\left[f_2(x_{0\sim 2}, t) + u^*\right], \\ u^* = b_1\gamma\psi\left(x_0, x_2, t\right) x_1 + \left(b_0 + b_1\left[\beta + \varphi\left(x_0, x_2, t\right)\right]\right)\frac{x_2}{x_0}, \end{cases} \tag{75}$$

where $f_2(x_{0\sim 2}, t)$ is given,

(1) when $g\left(x_0, z_1, t\right)$ and $h\left(x_0, z_1, t\right)$ are differentiable with respect to all variables, by

$$\begin{cases} f_2\left(x_0, z_{1\sim 2}, t\right) = \dot{g}\left(x_0, z_1, t\right) + \dot{h}\left(x_0, z_1, t\right) z_2, \\ z_1 = \frac{x_2}{x_0}, \\ z_2 = x_1; \end{cases} \tag{76}$$

(2) when $\varphi\left(x_0, x_2, t\right)$ and $\psi\left(x_0, x_2, t\right)$ are differentiable with respect to all variables, explicitly by

$$\begin{cases} f_2(x_{0\sim 2}, t) = \Delta_1 x_1 + \Delta_2 x_2, \\ \Delta_1 = \gamma\left[\beta + \varphi\left(x_0, x_2, t\right)\right]\psi\left(x_0, x_2, t\right) + \gamma\dot{\psi}\left(x_0, x_2, t\right), \\ \Delta_2 = \frac{1}{x_0}(\dot{\varphi}\left(x_0, x_2, t\right) + \left[\beta + \varphi\left(x_0, x_2, t\right)\right]^2). \end{cases} \tag{77}$$

Proof. Firstly, it follows from Corollary 1 that a stabilizing controller for the SFS (65) is given by (71), which only depends on x_0, z_1, and z_2.

Secondly, note that in this case the state transformation (32) is (64). Therefore, from (35), we have

$$h(x_0, z_1, t) = \gamma\psi(x_0, x_2, t), \tag{78}$$

$$g(x_0, z_1, t) = \beta z_1 + \varphi(x_0, x_2, t) z_1 = [\beta + \varphi(x_0, x_2, t)] \frac{x_2}{x_0}. \tag{79}$$

Substituting (64), (78) and (79) into (71) gives

$$\begin{aligned} u^* &= b_0 z_1 + b_1 \left([\beta + \varphi(x_0, x_2, t)] \frac{x_2}{x_0} + \gamma\psi(x_0, x_2, t) z_2 \right) \\ &= b_1 \gamma\psi(x_0, x_2, t) x_1 + (b_0 + b_1 [\beta + \varphi(x_0, x_2, t)]) \frac{x_2}{x_0}. \end{aligned} \tag{80}$$

Therefore, substituting (80) into (71), and incorporating with the controller $u_0 = -\beta x_0$, yield the controller (75) for the original system (62).

When $g(x_0, z_1, t)$ and $h(x_0, z_1, t)$ are differentiable with respect to all variables, Eq. (76) can be immediately derived by (64) and (68).

When $\varphi(x_0, x_2, t)$ and $\psi(x_0, x_2, t)$ are differentiable with respect to all variables, substituting (63), (64), (78), and (79) into (68) gives

$$\begin{aligned} f_2(x_0, z_{1\sim 2}, t) &= \frac{\mathrm{d}}{\mathrm{d}t} \left([\beta + \varphi(x_0, x_2, t)] \frac{x_2}{x_0} \right) + \gamma\dot{\psi}(x_0, x_2, t) x_1 \\ &= \dot{\varphi}(x_0, x_2, t) \frac{x_2}{x_0} + \gamma\dot{\psi}(x_0, x_2, t) x_1 + [\beta + \varphi(x_0, x_2, t)] \frac{\dot{x}_2 x_0 - x_2 \dot{x}_0}{x_0^2} \\ &= \dot{\varphi}(x_0, x_2, t) \frac{x_2}{x_0} + \gamma\dot{\psi}(x_0, x_2, t) x_1 \\ &\quad + \frac{1}{x_0} [\beta + \varphi(x_0, x_2, t)] (x_2 \varphi(x_0, x_2, t) + \gamma x_0 x_1 \psi(x_0, x_2, t) + \beta x_2) \\ &= \frac{1}{x_0} (\dot{\varphi}(x_0, x_2, t) + [\beta + \varphi(x_0, x_2, t)]^2) x_2 \\ &\quad + (\gamma [\beta + \varphi(x_0, x_2, t)] \psi(x_0, x_2, t) + \gamma\dot{\psi}(x_0, x_2, t)) x_1, \end{aligned}$$

which can be arranged into (77).

Finally, let us show the convergence of x_i, $i = 0, 1, 2$. Note that the exponential convergence of x_0 is guaranteed by (25). Since system (72) is stable, z and $\dot{z}$ converge to zero. Therefore, we can obtain from either Theorem 3 or Remark 6 that z_i, $i = 1, 2$ also converge to zero. In the end, the convergence of x_i, $i = 1, 2$ can then be deduced from that of z_i, $i = 1, 2$, and the relation (64). The proof is completed.

5.2 Application to ship control

This subsection is devoted to illustrating the validity of the proposed stabilizing controller. To this goal, the control of the speed system of a surface ship is considered.

The dynamic equations of the surface ship in surge, yaw, and sway are as follows [52, 53]:

$$\begin{cases} m_x \dot{v}_x = m_y v_y \omega - \sum_{j=1}^3 d_{0j} |v_x|^{j-1} v_x + \tau_x, \\ m_\omega \dot{\omega} = (m_x - m_y) v_x v_y - \sum_{j=1}^3 d_{1j} |\omega|^{j-1} \omega + \tau_\omega, \\ m_y \dot{v}_y = -m_x v_x \omega - \sum_{j=1}^3 d_{2j} |v_y|^{j-1} v_y, \end{cases} \tag{81}$$

where v_x, ω and v_y respectively represent the surge, yaw, and sway velocities; $m_x > 0$, $m_\omega > 0$, and $m_y > 0$ stand for the system inertia constants; constants $d_{ij} > 0$ ($i = 0, 1, 2$; $j = 1, 2, 3$) denote the hydrodynamic damping terms; and τ_x and τ_ω are the ship control inputs.

Define the state transformation

$$\begin{cases} x_0 = m_x v_x, \\ x_1 = m_\omega \omega, \\ x_2 = m_y v_y, \end{cases} \tag{82}$$

and the input transformation

$$\begin{cases} u_0 = m_y v_y \omega - \sum_{j=1}^{3} d_{0j} |v_x|^{j-1} v_x + \tau_x, \\ u = (m_x - m_y) v_x v_y - \sum_{j=1}^{3} d_{1j} |\omega|^{j-1} \omega + \tau_\omega. \end{cases} \tag{83}$$

Then, system (81) can be written into the following compact form:

$$\begin{cases} \dot{x}_0 = u_0, \\ \dot{x}_1 = u, \\ \dot{x}_2 = x_2 \hat{\varphi}(x_2) + x_0 x_1 \hat{\psi}(x_2), \end{cases} \tag{84}$$

where

$$\hat{\varphi}(x_2) = \sum_{j=1}^{3} d_j |x_2|^{j-1}, \ \hat{\psi}(x_2) = -\frac{1}{m_\omega},$$

with

$$d_j = -\frac{d_{2j}}{m_y^j}, \ j = 1, 2, 3.$$

Obviously, system (84) is in the form of (62) with $\rho = x_0$. Correspondingly, we have $\gamma = 1$.

Since $\hat{\varphi}(x_2)$ and $\hat{\psi}(x_2)$ are not dependent on u_0, we have

$$\varphi(x_2) = \hat{\varphi}(x_2) = \sum_{j=1}^{3} d_j |x_2|^{j-1},$$
$$\psi(x_2) = \hat{\psi}(x_2) = -\frac{1}{m_\omega},$$

where $\psi(x_2)$ clearly satisfies condition (74). However, as it can be easily seen that $\varphi(x_2)$ is not differentiable at $x_2 = 0$, by (66), we have

$$g(x_0, z_1) = (\beta + d_1) z_1 + (d_2 |x_0| z_1^2 + d_3 x_0^2 |z_1|^3) \mathrm{sign}(z_1), \ h = -\frac{1}{m_\omega}.$$

Recall that $x_0 = \mathrm{e}^{-\beta t} x_0(0)$ and $\dot{x}_0 = -\beta x_0$, and notice that the function $f(x) = \mathrm{sign}(x)|x|^\alpha$ $(\alpha \geqslant 1)$ is continuously differentiable. Unlike $\varphi(x_2)$, it can be easily checked that $g(x_0, z_1)$ is differentiable with respect to x_0 and z_1. Hence, it follows from (76) that

$$\begin{cases} f_2(x_0, z_{1\sim 2}) = (\beta + d_1 + 2d_2 |x_0||z_1| + 3d_3 x_0^2 z_1^2) \dot{z}_1 \\ \qquad\qquad - (\beta d_2 |x_0| z_1^2 + 2\beta d_3 x_0^2 |z_1|^3) \mathrm{sign}(z_1), \\ z_1 = \frac{x_2}{x_0}, \\ z_2 = x_1. \end{cases} \tag{85}$$

Finally, it follows from (75) that a controller for the ship velocity system is given by

$$\begin{cases} u_0 = -\beta x_0, \\ u = m_\omega (f_2(x_0, z_{1\sim 2}) + u^*), \\ u^* = -\frac{1}{m_\omega} b_1 x_1 + [b_0 + b_1(\beta + \sum_{j=1}^{3} d_j |x_2|^{j-1})] \frac{x_2}{x_0}. \end{cases} \tag{86}$$

For simulation use, we choose the same Bis-scaled parameter values as in [52], that is,

$$m_x = 1.1274, \ m_y = 1.8902, \ m_\omega = 0.1278,$$

and the values of hydrodynamic damping parameters d_{ij} are given in Table 1.

The design parameters are taken as

$$\beta = 0.5, \ b_0 = 1, \ b_1 = 2.$$

Table 1 Hydrodynamic damping parameters d_{ij}

	$j=1$	$j=2$	$j=3$
$i=0$	0.0358	0.5×0.0358	0.25×0.0358
$i=1$	0.0308	0.5×0.0308	0.25×0.0308
$i=2$	0.1183	0.5×0.1183	0.25×0.1183

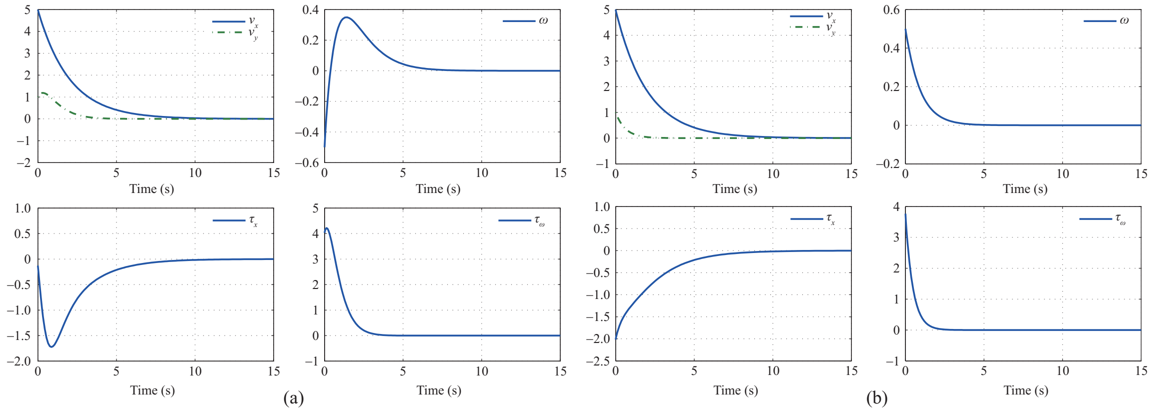

Figure 2 (Color online) Simulation results. (a) Case of $\omega(0)=-0.5$; (b) case of $\omega(0)=0.5$.

Two sets of system initial values are chosen as follows:

$$\begin{aligned}&\text{Case (a):}\quad v_x(0)=5,\ v_y(0)=1,\ \omega(0)=-0.5,\\&\text{Case (b):}\quad v_x(0)=5,\ v_y(0)=1,\ \omega(0)=0.5.\end{aligned}$$

The simulation results are provided in Figure 2. It can be seen from Figure 2 that, with both sets of initial values, the surge velocity v_x exponentially converges to zero, and the sway velocity v_y and the yaw velocity ω also tend to zero quickly. Meanwhile, the system inputs τ_x and τ_ω also converge to zero. These results show that the control method has produced satisfactory control performance.

5.3 Special cases

Consider the case of

$$\varphi(x_0,x_2,t)=\hat{\varphi}(x_0,x_2,u_0,t)=0. \tag{87}$$

Now the original system (6) reduces to

$$\begin{cases}\dot{x}_0=u_0,\\ \dot{x}_1=u,\\ \dot{x}_2=x_0x_1\hat{\psi}(x_0,x_2,u_0,t).\end{cases} \tag{88}$$

Since in this case, Eq. (66) gives

$$g(x_0,z_1,t)=\beta z_1, \tag{89}$$

and it then follows from Theorem 1 that the SFS of (88) is

$$\begin{cases}\dot{z}_1=\beta z_1+h(x_0,z_1,t)\,z_2,\\ \dot{z}_2=u.\end{cases} \tag{90}$$

Furthermore, with (89), it can be verified that Eq. (77) becomes

$$f_2(x_{0\sim 2},t)=\left[\beta\psi(x_0,x_2,t)+\dot{\psi}(x_0,x_2,t)\right]x_1+\beta^2\frac{x_2}{x_0}. \tag{91}$$

The above facts, together with Theorem 5, immediately suggest the following result.

Theorem 6. Let the assumptions in Theorem 5 be met. Then a controller for the nonholonomic system (88) is given by (75), with $f_2(x_{0\sim 2}, t)$ being defined as in (91), which drives all the state variables x_i, $i = 0, 1, 2$ to zero.

If, further, letting $\psi(x_0, x_2, t) \equiv 1$ in addition to (87), following the above Theorem 6, we obtain the following corollary.

Corollary 2. Let b_i, $i = 0, 1$ be two positive scalars, and $x_0(0) \neq 0$. Then a controller for the nonholonomic system (1) is given by

$$\begin{cases} u_0 = -\beta x_0, \\ u = -(\beta + b_1) x_1 - \left[\beta^2 + b_0 + \beta b_1\right] \frac{x_2}{x_0}, \end{cases} \tag{92}$$

which drives all the state variables x_i, $i = 0, 1, 2$ to zero.

Parallelly, letting

$$\hat{\psi}(x_0, x_2, u_0, t) = \frac{u_0}{x_0} = \frac{-\beta x_0}{x_0} = -\beta,$$

in the above Theorem 6 gives the following corollary.

Corollary 3. Let b_i, $i = 0, 1$ be two positive scalars, and $x_0(0) \neq 0$. Then a controller for the Brockett integrator (3) is given by

$$\begin{cases} u_0 = -\beta x_0, \\ u = -(\beta + b_1) x_1 + \frac{1}{\beta} \left[\beta^2 + b_0 + \beta b_1\right] \frac{x_2}{x_0}, \end{cases} \tag{93}$$

which drives all the state variables x_i, $i = 0, 1, 2$ to zero exponentially.

Before ending this section, let us make the following remark.

Remark 7. In this section, we have given, for the case of $n = 2$, an explicit solution to the stabilizing controller for the original system (6). However, to derive an explicit solution for a larger n is, although theoretically feasible, practically very difficult. The main point is, for the case of $n \geqslant 3$, one may simply use the general solution, that is, the implicit solution given by (59), with variables $z_1^{(0\sim n-1)}$ and $z_{1\sim n}$ being produced by the transformations (50) and (32), respectively.

6 Extension to sub-normal systems

Before getting onto the stabilization of sub-normal systems, let us comment on the stabilization of the multivariable system (12). In fact, the proposed FAS approach can be easily applied to the multivariable system (12). The idea is as follows.

Firstly, design the controller for the first subsystem as

$$u_0 = -\beta x_0, \tag{94}$$

with β being a proper positive scalar. Secondly, for each i, $i = 1, 2, \ldots, p$, design a controller u_i using the above proposed FAS approach for the following subsystem:

$$\begin{cases} \dot{x}_{i1} = u_i, \\ \dot{x}_{i2} = x_{i2}\hat{\varphi}_{i1}(\cdot) + \gamma_{i1} x_0 x_{i1} \hat{\psi}_{i1}(\cdot), \\ \quad \vdots \\ \dot{x}_{in_i} = x_{i,n_i}\hat{\varphi}_{i,n_i-1}(\cdot) + \gamma_{i,n_i-1} x_0 x_{i,n_i-1} \hat{\psi}_{i,n_i-1}(\cdot). \end{cases} \tag{95}$$

Since a locally normal system is really a special sub-normal system, in Subsections 6.1 and 6.2, let us directly treat the case of sub-normal systems.

6.1 Singular and feasible sets

The aim of this subsection is to design a stabilizing controller for the sub-normal system (6)–(8) which does not satisfy Assumption A1, or even does not necessarily satisfy Assumption B1. In the following development, some basic concepts in the FAS theory, such as sub-FAS, sub-stabilization, and singular and feasible sets (see [2, 49]), will be encountered.

Recall that, when the control u_0 is given in the form of (94), we have

$$x_0(t) = x_0(0)\,\mathrm{e}^{-\beta t}.$$

In this case, the sub-normal system (6) can be also turned into the form of (28), but bear in mind that Assumption A2, or even Assumption B2, is not necessarily satisfied. To deal with this complicated case, parallel to Definitions 3 and 4, we also introduce the following ones for system (28).

Definition 5. Consider the nonlinear system in the form of (28). Let

$$\mathbb{S}_x(x_0) = \left\{ x_{1\sim n} \,\middle|\, \begin{array}{l} \psi_i(x_0, x_{i+1\sim n}, t) = 0,\ x_{0\sim n} \in \mathbb{R}^{n+1},\ t \geqslant 0, \\ i = 1, 2, \ldots, n-1. \end{array} \right\}. \tag{96}$$

Then $\mathbb{S}_x(x_0)$ is called the singular set of system (28), and $\mathbb{F}_x(x_0) = \mathbb{R}^n \backslash \mathbb{S}_x(x_0)$ is called the feasible set of system (28). Furthermore, system (28) is called sub-normal if $\mathbb{S}_x(x_0) \neq \emptyset$ or $\mathbb{R}^n$.

Definition 6. The sub-normal system (28) is said to be rational if there exists a ball $\mathbb{B}_0$ centered at $x_{1\sim n} = 0$, such that $\mathbb{B}_0 \subset \mathbb{F}_x(x_0)$.

Based on the above analysis, the problem becomes the one of finding the control u to sub-stabilize system (28) under the condition $x_{1\sim n} \in \mathbb{F}_x(x_0)$. To start with, let us first give an extension of Theorem 1, which can be proven similarly as Theorem 1.

Theorem 7. Let Assumption X be met. Then, under transformation (32), the sub-normal system (28) with singular set $\mathbb{S}_x(x_0)$ is equivalently transformed into system (33)–(35) with the following singular set:

$$\mathbb{S}_z(x_0) = \left\{ z_{1\sim n} \,|\, h_i(x_0, z_{1\sim i}, t) = 0,\ z_{1\sim n} \in \mathbb{R}^n,\ t \geqslant 0,\ i = 1, 2, \ldots, n-1 \right\}. \tag{97}$$

Similarly, define the feasible set for the transformed system (33)–(35) as

$$\mathbb{F}_z(x_0) = \mathbb{R}^n \backslash \mathbb{S}_z(x_0).$$

Then, there holds

$$h_i(x_0, z_{1\sim i}, t) \neq 0,\ \forall z_{1\sim n} \in \mathbb{F}_z(x_0),\ i = 1, 2, \ldots, n-1. \tag{98}$$

With this understanding, following the proof of Theorem 2, we can also prove the following result.

Theorem 8. Let Assumption X be satisfied, $g_k(x_0, z_{1\sim k}, t)$ and $h_k(x_0, z_{1\sim k}, t), k = 1, 2, \ldots, n$ be differentiable with respect to all variables, and $B_k(x_0, z^{(0\sim k-1)}, t)$ and $f_k(x_0, z^{(0\sim k-1)}, t)$, $k = 1, 2, \ldots, n$ be given by (46)–(49). Then, for $z_{1\sim n} \in \mathbb{F}_z(x_0)$, the transformation (50) is one-to-one, guarantees $\lim_{t\to\infty} z_{1\sim n} = 0$ if $\lim_{t\to\infty} z^{(0\sim n-1)} = 0$, and transforms the SFS (33)–(35) into the sub-FAS (51) with the singular set $\mathbb{S}_z(x_0)$ defined in (97).

Proof. Following the same line of the proof of Theorem 3, we can show the fact that the transformation (50) is one-to-one in view of (98), and $\lim_{t\to\infty} z_{1\sim n} = 0$ if $\lim_{t\to\infty} z^{(0\sim n-1)} = 0$. Again, similar to the proof of the Theorem 2.1 in [36], it can be shown that the transformation (50) turns the SFS (33)–(35) into the sub-FAS (51) for $z_{1\sim n} \in \mathbb{F}_z(x_0)$.

By the FAS theory, the singular set of the sub-FAS (51) is defined to be [41, 47, 48]

$$\mathbb{S} = \left\{ z_{1\sim n} \,|\, \det B_n(x_0, z_{1\sim n}, t) = 0,\ z_{1\sim n} \in \mathbb{R}^n,\ t \geqslant 0 \right\},$$

which, in view of the expression of $B_n(x_0, z_{1\sim n}, t)$ in (46) and (47), is obviously coincident with $\mathbb{S}_z(x_0)$.

6.2 Sub-stabilization

Based on the above theorem, we can now give the following result for the original sub-normal system described by (6) with the singular set $\hat{\mathbb{S}}_x$.

Theorem 9. Let $g_k(x_0, z_{1\sim k}, t)$ and $h_k(x_0, z_{1\sim k}, t), k = 1, 2, \ldots, n$ be differentiable with respect to all variables. Further, let $\beta > 0$, $x_0(0) \neq 0$ and $b_{0\sim n-1}$ be an arbitrary vector making $\Phi(b_{0\sim n-1})$ Hurwitz. Then, under the condition of $z_{1\sim n} \in \mathbb{F}_z(x_0)$,

(1) a sub-stabilizing controller for system (6) is given by

$$\begin{cases} u_0 = -\beta x_0, \\ u = -\frac{1}{B_n(x_0, z_{1\sim n}, t)} \left[f_n(x_0, z_{1\sim n}, t) + b_{0\sim n-1} z^{(0\sim n-1)} \right], \end{cases} \tag{99}$$

where $z^{(0\sim n-1)}$ is given by (50), and all z_i, $i = 1, 2, \ldots, n$ are given by (32); and

(2) the resultant closed-loop system is a constrained constant linear one given by

$$\begin{cases} \dot{x}_0 = -\beta x_0, \\ z^{(n)} + b_{0\sim n-1} z^{(0\sim n-1)} = 0, \\ z_{1\sim n} \in \mathbb{F}_z(x_0). \end{cases} \tag{100}$$

By standard notations for the FAS approach, the constrained linear system can also be represented in the following state-space form:

$$\begin{cases} \dot{x}_0 = -\beta x_0, \\ \dot{z}^{(0\sim n-1)} = \Phi(b_{0\sim n-1}) z^{(0\sim n-1)}, \\ z_{1\sim n} \in \mathbb{F}_z(x_0). \end{cases} \tag{101}$$

Recall that, in the globally normal case treated in Sections 4 and 5, a UGE stabilizing controller for the SFS (33), or equivalently, the second subsystem (28), has been derived. For the sub-normal case, one may wonder what the function of the designed sub-stabilizing controller (99) is. When $\mathbb{F}_z(x_0)$ is not empty, the designed sub-stabilizing controller (99) is no longer a UGE stabilizing controller in the Lyapunov sense, it may even not be a locally asymptotically stabilizing controller. To better understand the above result, let us introduce the following definition.

Definition 7. Let $\Phi(b_{0\sim n-1})$ be Hurwitz, and Assumption X be satisfied. Then the set $\mathbb{Q} \subset \mathbb{R}^n$ is called the ROEA of the system

$$\begin{cases} z^{(n)} + b_{0\sim n-1} z^{(0\sim n-1)} = 0, \\ z_{1\sim n} \in \mathbb{F}_z(x_0), \end{cases} \tag{102}$$

if it is the largest set contained in $\mathbb{F}_z(x_0)$, such that any trajectory of the system

$$\dot{z}^{(0\sim n-1)} = \Phi(b_{0\sim n-1}) z^{(0\sim n-1)},$$

starting from $\mathbb{Q}$ converges exponentially to zero within $\mathbb{F}_z(x_0)$.

For a particularly given system (33)–(35), $\mathbb{F}_z(x_0)$ can be exactly determined, and so can the ROEA $\mathbb{Q}$ (see the work in [2,3,49]). When the system initial values are chosen within the ROEA $\mathbb{Q}$, the above controller (99) can be directly applied, which results in the exponential convergence of the state variables to the origin. However, when the system initial values are not chosen within this ROEA $\mathbb{Q}$, the above controller (99) does not make sense. Such a fact immediately implies the following result.

Corollary 4. There exists an exponentially stabilizing controller for the sub-FAS (51) if and only if one of the following conditions is met:

(1) the feasible set $\mathbb{F}_z(x_0)$, or equivalently, the corresponding ROEA $\mathbb{Q}$, contains the origin as an inner point;

(2) the sub-FAS (51) is a local FAS; and

(3) the SFS (33) is locally normal.

For the case that $\mathbb{F}_z(x_0)$ or the corresponding ROEA $\mathbb{Q}$ does not contain the origin as an inner point, there does not exist even a locally exponentially stabilizing controller for the second subsystem. Yet we do have a so-called sub-stabilizing controller which drives, as mentioned above, all trajectories of the system starting from the ROEA $\mathbb{Q}$ exponentially to the origin. In many practical applications, the feasible set $\mathbb{F}_z(x_0)$ or the corresponding ROEA $\mathbb{Q}$ is sufficiently large such that the singular case seldom or even never happens. In such a case, although the designed controller is not even a locally stabilizing one in the Lyapunov sense, it does work practically. This clearly suggests a kind of stability which is termed sub-stability in [2,49], and the designed controller is called a sub-stabilizing controller.

When the initial values are inevitably chosen out of the ROEA $\mathbb{Q}$, a pre-controller can be easily designed (see also [2, 3, 49]), to drive the trajectory of the closed-loop system onto the ROEA $\mathbb{Q}$ first, and then let the above controller (99) take over. Clearly, the design of such a pre-controller is much easier than solving a stabilizing controller because the ROEA is generally a very large region, while stabilization in the Lyapunov sense requires convergence of the states to a single point, which is usually the origin.

Finally, let us point out that the general theory of sub-FAS provides a general framework for dealing with nonholonomic systems. As a matter of fact, sub-FAS describes a larger category covering nonholonomic systems. Eventually, the nonholonomic constraints generally produce a singular set containing the origin, often as a boundary point. The FAS approach provides a general approach for dealing with systems possessing constraints on the state variables and their derivatives.

7 Extension to time-delay systems

In this subsection, let us consider the stabilization of the time-delay system (15) with $\hat{\varphi}_i^{\mathrm{d}}(\cdot)$ and $\hat{\psi}_i^{\mathrm{d}}(\cdot)$, $i = 1, 2, \ldots, n-1$, given by (16) and (17) and satisfying Assumption C1.

7.1 The SFS

Similarly, the controller u_0 for the first scalar subsystem of (15)–(17) is again given by

$$u_0 = -\beta x_0,$$

and the rest of the time-delay system described by (15)–(17) is turned into the form of

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_2\varphi_1^{\mathrm{d}}(\cdot) + \gamma_1 x_0 x_1 \psi_1^{\mathrm{d}}(\cdot), \\ \dot{x}_3 = x_3\varphi_2^{\mathrm{d}}(\cdot) + \gamma_2 x_0 x_2 \psi_2^{\mathrm{d}}(\cdot), \\ \quad\vdots \\ \dot{x}_n = x_n\varphi_{n-1}^{\mathrm{d}}(\cdot) + \gamma_{n-1} x_0 x_{n-1} \psi_{n-1}^{\mathrm{d}}(\cdot), \end{cases} \tag{103}$$

with

$$\varphi_i^{\mathrm{d}}(\cdot) \triangleq \varphi_i^{\mathrm{d}}\left(x_j\left(t-\tau_{i,j}\right)|_{j=0,i+1\sim n}, t-\tau_i\right), \tag{104}$$

$$\psi_i^{\mathrm{d}}(\cdot) \triangleq \psi_i^{\mathrm{d}}\left(x_j\left(t-\sigma_{i,j}\right)|_{j=0,i+1\sim n}, t-\sigma_i\right),\ i = 1, 2, \ldots, n-1. \tag{105}$$

Furthermore, Assumption C1 becomes Assumption C2.

Assumption C2. For all $X_i \in \mathbb{R}^{n-i+1}$, $i = 1, 2, \ldots, n-1$, and $t \geqslant -h$, there holds

$$\psi_i^{\mathrm{d}}(X_i, t) \neq 0,\ i = 1, 2, \ldots, n-1.$$

Therefore, in this case the problem becomes the one of finding, under Assumption C2, a control u to stabilize system (103)–(105).

Once again, let us start with giving an extension of Theorem 1.

Theorem 10. Let Assumption X be met. Then, under transformation (32), the time-delay system described by (103)–(105) is equivalently transformed into the following system:

$$\begin{cases} \dot{z}_1 = g_1^{\mathrm{d}}(\cdot) + h_1^{\mathrm{d}}(\cdot) z_2, \\ \dot{z}_2 = g_2^{\mathrm{d}}(\cdot) + h_2^{\mathrm{d}}(\cdot) z_3, \\ \quad\vdots \\ \dot{z}_{n-1} = g_{n-1}^{\mathrm{d}}(\cdot) + h_{n-1}^{\mathrm{d}}(\cdot) z_n, \\ \dot{z}_n = u, \end{cases} \tag{106}$$

where

$$g_i^{\mathrm{d}}(\cdot) = \left[(n-i)\beta + \varphi_{n-i}^{\mathrm{d}}(\cdot)\right] z_i,\ h_i^{\mathrm{d}}(\cdot) = \gamma_{n-i}\psi_{n-i}^{\mathrm{d}}(\cdot),\ i = 1, 2, \ldots, n-1,$$

with $\varphi_i^{\mathrm{d}}(\cdot)$ and $\psi_i^{\mathrm{d}}(\cdot)$, $i = 1, 2, \ldots, n-1$, given by (104) and (105).

The above result can be proven similarly as Theorem 1.

It can be easily figured out that $g_i^{\rm d}(\cdot)$ is a function of the arguments $x_0(t-\tau_{n-i,0})$, $z_{1\sim i}(t-\tau_{n-i,j})|_{j=n-i+1\sim n}$, and $t-\tau_{n-i}$, while $h_i^{\rm d}(\cdot)$ is a function of the arguments $x_0(t-\sigma_{n-i,0})$, $z_{1\sim i}(t-\sigma_{n-i,j})|_{j=n-i+1\sim n}$, and $t-\sigma_{n-i}$, for $i=1,2,\ldots,n-1$. Furthermore, corresponding to Assumptions C1 and C2 on the series of functions $\hat{\psi}_i^{\rm d}(\cdot)$ and $\psi_i^{\rm d}(\cdot), i=1,2,\ldots,n-1$, we have the following assumption on the functions $h_i^{\rm d}(\cdot)$, $i=1,2,\ldots,n-1$.

Assumption C3. For all $x\in\mathbb{R}$, $Z_i\in\mathbb{R}^{i^2}, i=1,2,\ldots,n-1$, and $t\geqslant -h$, there holds

$$h_i^{\rm d}(x,Z_i,t)\neq 0,\ i=1,2,\ldots,n-1.$$

Under the above Assumption C3, it is clear that the above system (106) is an SFS with time delays.

7.2 The FAS approach

In order to give a generalized version of Theorem 2, for a continuous function $x:[-r,+\infty)\to\mathbb{R}^n$, let us first define the following functional operator $[x]_t$, which is commonly used in time-delay system theory (see [54]):

$$[x]_t(\theta)=x(t+\theta),\ \theta\in[-r,0],$$

where r is a positive scalar representing the upper bound of all the possible time delays in x.

Again, following the same line as in the proof of Theorem 2, we can obtain the result below.

Theorem 11. Let Assumptions X and C3 be satisfied, and with the convention of $g_n^{\rm d}(\cdot)=0$, define

$$\tilde{B}_k([x_0]_t,[z_{1\sim k}]_t,t)=\prod_{i=1}^{k}h_i^{\rm d}(\cdot),\ k=1,2,\ldots,n-1, \tag{107}$$

$$\tilde{B}_n([x_0]_t,[z_{1\sim n}]_t,t)=\tilde{B}_{n-1}([x_0]_t,[z_{1\sim n-1}]_t,t), \tag{108}$$

and

$$\begin{aligned}\tilde{f}_k([x_0]_t,[z_{1\sim k}]_t,t)=&\ \dot{\tilde{f}}_{k-1}([x_0]_t,[z_{1\sim k-1}]_t,t)+\dot{\tilde{B}}_{k-1}([x_0]_t,[z_{1\sim k-1}]_t,t)z_k\\&+\tilde{B}_{k-1}([x_0]_t,[z_{1\sim k-1}]_t,t)g_k^{\rm d}(\cdot),\ k=2,3,\ldots,n,\end{aligned} \tag{109}$$

with the initial value

$$\tilde{f}_1([x_0]_t,[z_1]_t,t)=g_1^{\rm d}(\cdot). \tag{110}$$

Then, under the following transformation:

$$\begin{cases}z=z_1,\\ \dot{z}=\tilde{f}_1(\cdot)+\tilde{B}_1(\cdot)z_2,\\ \ \vdots\\ z^{(n-1)}=\tilde{f}_{n-1}(\cdot)+\tilde{B}_{n-1}(\cdot)z_n,\end{cases} \tag{111}$$

the time-delay SFS (33)–(35) can be transformed into the following FAS with time delays:

$$z^{(n)}=\tilde{f}_n([x_0]_t,[z_{1\sim n}]_t,t)+\tilde{B}_n([x_0]_t,[z_{1\sim n}]_t,t)u. \tag{112}$$

Clearly, it follows from (107) that Assumption C3 is equivalent to

$$\tilde{B}_n([x_0]_t,[z_{1\sim n}]_t,t)\neq 0,$$

for all involved variables. Therefore, under Assumption C3, the above system (112) is a global FAS.

Theorem 12. Let Assumption C3 be met, $\beta>0$, $x_0(0)\neq 0$ and $b_{0\sim n-1}$ be an arbitrary vector making $\Phi(b_{0\sim n-1})$ Hurwitz. Then,

(1) a stabilizing controller for system (15)–(17) is given by

$$\begin{cases}u_0=-\beta x_0,\\ u=-\frac{1}{\tilde{B}_n([x_0]_t,[z_{1\sim n}]_t,t)}\left[\tilde{f}_n([x_0]_t,[z_{1\sim n}]_t,t)+b_{0\sim n-1}z^{(0\sim n-1)}\right],\end{cases} \tag{113}$$

where $z^{(0\sim n-1)}$ is given by (111), and $z_{1\sim n}$ is given by (32); and

(2) the resultant closed-loop system is a constant linear one given by (60) or (61).

To end this subsection, let us point out that the above FAS approach for stabilization of the globally normal system (15)–(17) with time delays can also be generalized to the time-delay locally normal and sub-normal cases.

8 Concluding remarks

A more general type of nonholonomic systems is proposed, which contains both the two well-known Brockett's chained forms as special cases. It is shown that, for this type of systems, the well-known σ-process can still be applied, but the existing linear (time-varying) systems approaches do not work since the transformed system is still nonlinear if the introduced function terms in the open-loop system are nonlinear. Furthermore, even if these functions are not ones of the system state variables, but only pure functions of time, in which case the transformed system does fall into a linear form, the transformed system is still a general time-varying one and may be still too difficult to be dealt with by linear state-space system theories.

Fortunately, it is shown in this paper that this type of systems can be equivalently transformed into a global FAS, and hence its stabilizing controller can be well designed by the FAS approach. Consequently, a stabilizing controller is derived, which drives the trajectories of the designed system to the origin exponentially when the initial value of the first scalar subsystem is restricted nonzero. It is also shown that, whilst some existing methods proposed for the chained systems fail to stabilize the proposed type of generalized nonholonomic systems, the proposed FAS approach can also be further extended to solve the stabilization of the corresponding types of sub-normal systems and time-delay systems. These facts once again demonstrate the power of the FAS approach.

The FAS approach proposed for this type of nonholonomic systems can be further generalized in several directions. A natural extension is to design a time-varying continuous stabilizing controller for the proposed system. Furthermore, a type of uncertain nonholonomic systems can be proposed by adding uncertainties into the presented type of nonholonomic systems, and the corresponding robust stabilization problem can be considered. Parallelly, the type of nonholonomic systems subject to disturbances can be also proposed, and the corresponding disturbance rejection problem can be investigated.

Acknowledgements This work has been partially supported by Shenzhen Key Laboratory of Control Theory and Intelligent Systems (Grant No. ZDSYS20220330161800001), Major Program of National Natural Science Foundation of China (Grant Nos. 61690210, 61690212), National Natural Science Foundation of China (Grant No. 61333003), and Science Center Program of the National Natural Science Foundation of China (Grant No. 62188101). The author is grateful to his Ph.D. students, Weizhen LIU, Guangtai TIAN, Qin ZHAO, etc., for helping him with reference selection and proofreading. His thanks also go to Prof. Zhongping JIANG, Drs. Wei SUN, Xiang XU, and Tao LIU for helpful discussions and comments. He particularly thanks Dr. Zhongcai ZHANG for helping work out the simulation results. Finally, the author thanks the anonymous reviewers for the helpful suggestions.

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