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Prescribed-time leader-following consensus of linear multi-agent systems by bounded linear time-varying protocols

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Abstract This paper considers the prescribed-time leader-following consensus problem of input-constrained linear multi-agent systems under generally directed communication topology in two cases: the Laplacian matrix related to the entire communication topology between agents is either known or unknown. In particular, the consensus problem for the former case is solved by a novel bounded linear time-varying (LTV) protocol, where the feedback gain is formulated by the parametric Lyapunov equation and the knowledge of the Laplacian matrix. Moreover, by utilizing a distributed observer, a fully bounded LTV protocol is proposed for the latter case. It should be noted that, compared with the existing results, the system under consideration is more general, the designed protocols are linear, and the consensus problem is accomplished even in a fully distributed manner. Finally, the effectiveness of the proposed approach is verified by a numerical example.

Keywords leader-following consensus, time-vary feedback, prescribed-time control, fully distributed consensus

1 Introduction

The coordinative control problem has attracted a lot of research due to its many application scenarios, such as sensor networks [1], military surveillance [2], and spacecraft formation flying [3]. As a coordinative control task, the consensus problem aims to design a distributed protocol such that the states of all agents reach an agreement. Various distributed protocols were developed for this problem [4–9]. However, these distributed protocols can only achieve asymptotic consensus, that is the state achieves leader-follower consensus with infinite settling time. Engineering requirements usually require a faster convergence rate than the exponential rate, called finite-time consensus.

In [10], signed gradient descent flows-based discontinuous protocols were developed for finite-time consensus (FTC) of first-order systems for the first time. Since then, many efforts have been made to extend discontinuous control protocols [11–13]. Since there are several practical second-order systems, it is natural to solve the FTC problem using these systems. A series of classical protocols using homogeneity theory were developed in [14, 15] to achieve FTC. However, these protocols cannot explicitly estimate convergence time. The authors in [16] solved this problem by "adding a power integrator" based on FTC protocols. Such an approach can estimate the convergence time as $T \leqslant V^{(1-\alpha)}(x_0)/c\,(1-\alpha)$ by constructing a nonnegative function $V(t)$ whose derivative is $\dot{V}(t) + cV^{\alpha}(t) \leqslant 0$. Recently, the authors in [17–19] also addressed the FTC problem for high-order systems. It should be emphasized that although the convergence time for the FTC protocols [16, 17] can be estimated, it heavily relies on initial state conditions. The fixed-time stability theorem shows that the upper bound of the convergence time of the fixed-time control-based closed-loop system is independent of the initial state conditions [20]. Therefore, the fixed-time cooperative control problem has also gained attention (see [21–26] and the references therein).

In practice, there are two limitations of finite- and fixed-time consensus protocols mentioned above [27]. The former is that these protocols involve non-smooth control signals, and the latter is that the user

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cannot define the required time to achieve consensus. To overcome these limitations, a new approach, prescribed-time stability, was first studied in [28]. Because the convergence time of a prescribed-time stable system is independent of initial state conditions and can be precisely designated by the user, the prescribed-time consensus problem has been a hot topic in recent years [29–31]. For example, the bipartite consensus problem with signed and directed graphs was studied in [30], the leader-follower prescribed-time consensus for linear high-order systems with simple input was studied in [32], and the leaderless specified-time consensus problem for general linear multi-agent systems was studied in [31].

Nonetheless, issues may arise when implementing these protocols in [29–33] for input-constrained systems, as actuator saturation nonlinearity can significantly degrade the control performance and even lead to unstable behavior if it is not considered in the distributed protocol design, especially in the time-varying high-gain distributed protocol design. To address such a problem, further research should be conducted on bounded protocols. Many studies on the design of bounded protocols have been reported [34–39]. However, these results have only considered the asymptotic consensus case. Consequently, research on prescribed-time consensus by the bounded protocol is one of the primary motivations for conducting this study. On the other hand, the protocols in [30,31] are not fully distributed. Roughly speaking, these protocols are designed by a common control gain that must be greater than a Laplacian matrix-determined threshold. To calculate such a matrix, however, each agent must obtain the complete communication topology, which is the global information. Adaptive distributed protocols were proposed in [40] to address this issue. Such a protocol is designed for a general linear system with a directed communication graph and utilizes only local information, and can be considered a fundamental solution to the fully distributed consensus problem. This protocol has not yet been effectively applied to the prescribed-time consensus problem of input-constrained systems. Consequently, designing a fully distributed bounded protocol for the prescribed-time consensus problem is another reason for conducting this study.

To the best of our knowledge, the prescribed-time consensus problem for general linear multi-agent systems by bounded protocols, particularly for the case where each agent cannot collect a priori knowledge of the entire communication topology, has not been satisfactorily addressed in the literature. From a mathematical standpoint, there are at least two obstacles to solving this issue. On the one hand, the estimation of the upper bound of controllers plays an important role in the analysis of input-constrained control systems. however, prescribed-time controllers typically involve time-varying terms [32,41], making the estimation of the upper bound of controllers difficult and the prescribed-time control problem of input-constrained control systems theoretically difficult. On the other hand, the communication topology among agents determines how the local signal propagates throughout the entire group, which has a significant impact on the group behaviors of multi-agent systems and may lead to unstable group behavior if ignored in the design of protocols. Consequently, it is difficult to design a fully distributed protocol without using the a priori global information of the communication topology (namely, the spectrum of the associated Laplacian matrix) and to demonstrate its stability.

Motivated by the aforementioned observations and partially inspired by [42], in this work, we investigate the leader-follower prescribed-time consensus problem of input-constrained linear multi-agent systems over directed graphs in two cases (i.e., the Laplacian matrix is either known or unknown). The three principal contributions of this paper are the following.

- This paper generalizes at least two existing results on the prescribed-time consensus problem. On the one hand, compared with the existing result [32] where single input systems were studied, a general input-constrained system with multiple inputs is considered here, which can be used to model most engineering control systems. On the other hand, one of the proposed protocols does not require the Laplacian matrix information and can achieve consensus in a fully distributed manner.
- This paper proposes a novel, fully distributed, observer-based, bounded linear time-varying protocol that enables the estimation of the protocol's amplitude and the achievement of prescribed-time consensus for input-constrained systems. Moreover, the authors in [43] proposed an additional fully distributed protocol for input-constrained systems. However, a protocol designed for a system with a specific number of agents cannot be applied to other systems with different agent counts. This disadvantage is avoided in this paper because the proposed protocol can achieve consensus with any number of agents and any communication topology.
- The relationship among the size of the domain of attraction, the regulation time, and the saturation level is studied quantitatively in either the known or unknown Laplacian matrix case.

Notation: For an n-dimensional matrix A, $\|A\|$, A^{T}, $\mathrm{tr}(A)$, $\lambda(A)$, $\lambda_i(A)$, $\mathrm{Re}\{\lambda_i(A)\}$, $\lambda_{\min}(A)$, $\lambda_{\max}(A)$

are its 2-norm, transpose, trace, eigenvalue set, i-th eigenvalue, real part of the i-th eigenvalue, the minimal eigenvalue, and maximal eigenvalue, respectively. Let $\phi(A) = \min_{i=1,2,\ldots,n} \{\mathrm{Re}(\lambda_i(A))\}$. $A > 0$ denotes that A is positive definite. Let $u = [u_1, u_2, \ldots, u_m]$ and $a > 0$ be the saturation level. Then let $\sigma_a(u) = [\sigma_a(u_1), \sigma_a(u_2), \ldots, \sigma_a(u_m)]^{\mathrm{T}}$ in which $\sigma_a(u_i) = \mathrm{sign}(u_i)\min\{a, |u_i|\}$, $i = 1, 2, \ldots, m$. For brevity, let $\sigma(u)$ denote $\sigma_1(u)$.

2 Motivation and problem statement

We consider a group of agents composed of one leader and N followers. The dynamics of the ith follower agent is

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t),\ i = 1, 2, \ldots, N, \tag{1}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ are the system state vector (with initial state $x_{i0} = x_i(0)$) and the control input, respectively. The dynamics of the leader is

$$\dot{x}_{N+1}(t) = Ax_{N+1}(t). \tag{2}$$

Suppose that the communication topology among agents is described by the graph $\mathcal{G}(\mathcal{N}, \varepsilon, \mathcal{R})$, where $\mathcal{N} = \{1, 2, \ldots, N+1\}$, $\varepsilon \subseteq \mathcal{N} \times \mathcal{N}$ is the edge set, and $\mathcal{R} = [r_{ij}] \in \mathbb{R}^{(N+1)\times(N+1)}$ is the weighted adjacency matrix. Let $(i, j) \in \varepsilon$ denote that node j can obtain information from node i. If $(j, i) \in \varepsilon$, $r_{ij} > 0$, otherwise, $r_{ij} = 0$. In addition, $r_{ii} = 0$. With the denotation that $\mathcal{D} = \mathrm{diag}(\mathcal{D}_1, \ldots, \mathcal{D}_{N+1}) \in \mathbb{R}^{(N+1)\times(N+1)}$ with $\mathcal{D}_i = \sum_{j\in\mathcal{F}_i} r_{ij}$, the corresponding Laplacian can be described by $L = [l_{ij}] = \mathcal{D} - \mathcal{R}$. We consider here the multi-agent systems with one leader that has no in-neighbors. In such a case, L can also be represented as $L = \left[\begin{smallmatrix} L_1 & L_2 \\ 0_{1\times N} & 0 \end{smallmatrix}\right]$ with $L_1 \in \mathbb{R}^{N\times N}$ and $L_2 \in \mathbb{R}^{N\times 1}$.

It is shown in [40] that the following assumptions are always required for studying the asymptotic (or finite-time) consensus problem.

Assumption 1. The matrix pair (A, B) is controllable.

Assumption 2. The graph $\mathcal{G}(\mathcal{N}, \varepsilon, \mathcal{R})$ contains a directed spanning tree rooted at the leader.

For ensuring a faster convergence rate than exponential, we are devoted to the design of linear time-varying (LTV) state feedback protocols such that the prescribed-time leader-following consensus is achieved in two cases: L_1 is known and L_1 is unknown. To this end, we consider here the parametric Lyapunov equation (PLE),

$$A^{\mathrm{T}}P + PA - PBB^{\mathrm{T}}P = -\gamma P, \tag{3}$$

and its some interesting properties.

Lemma 1 ([42])**.** Let $\gamma_0 > \max\{0, -2\phi(A)\}$ and the matrix pair (A, B) be controllable. Then for any $\gamma \geqslant \gamma_0$, the PLE (3) has a unique positive definite solution $P(\gamma)$ that has the following properties:

$$\mathrm{tr}\left(B^{\mathrm{T}}P(\gamma)B\right) = \pi(\gamma) \triangleq 2\mathrm{tr}(A) + n\gamma > 0, \tag{4}$$

$$\frac{\mathrm{d}P(\gamma)}{\mathrm{d}\gamma} > 0, \tag{5}$$

$$\frac{\mathrm{d}P(\gamma)}{\mathrm{d}\gamma} \leqslant \frac{\delta_{\mathrm{c}}P(\gamma)}{\pi(\gamma)},\quad \forall\gamma \geqslant \gamma_0, \tag{6}$$

where $\delta_{\mathrm{c}} \geqslant 1$ is a constant independent of γ.

We finally recall the following lemma.

Lemma 2 ([40])**.** Let Assumption 2 be satisfied. Then L_1 is nonsingular. Furthermore, let the matrix S be described by $S = \mathrm{diag}(s_1, s_2, \ldots, s_N)$, in which $[s_1, s_2, \ldots, s_N]^{\mathrm{T}} = \left(L_1^{\mathrm{T}}\right)^{-1} 1_N$; then there holds $SL_1 + L_1^{\mathrm{T}}S > 0$.

3 Prescribed-time consensus with known L_1

Let $e = [e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \ldots, e_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{nN}$, in which

$$e_i(t) = x_i(t) - x_{N+1}(t),\quad i = 1, 2, \ldots, N, \tag{7}$$

is the state error between the ith follower and the leader. Let $\alpha_1 = \frac{2\mathrm{tr}(A)}{n}$ and $\alpha_2 = \frac{n}{n+\delta_c}$. Let γ_* and α_3 be two constants to be designed,

$$T_1 = \frac{1}{\alpha_1(\alpha_2-\alpha_3)} \ln\left(1+\frac{\alpha_1}{\gamma_*}\right), \tag{8}$$

and

$$\gamma_1(t) = \frac{\mathrm{e}^{\alpha_1(\alpha_2-\alpha_3)T_1}-1}{\mathrm{e}^{\alpha_1(\alpha_2-\alpha_3)(T_1-t)}-1}\gamma_*, \quad \forall t\in[0,T_1). \tag{9}$$

Let

$$\mathcal{E}_1(\gamma_1) = \{e : g\pi(\gamma_1)e^{\mathrm{T}}(S\otimes P(\gamma_1(t)))e \leqslant 1\}, \tag{10}$$

where $P(\gamma_1(t))$ is the solution to (3) with $\gamma = \gamma_1(t)$ and $\pi(\gamma_1) \triangleq 2\mathrm{tr}(A) + n\gamma_1$, and $g = g_1 g_2^2$ with $g_1 = \lambda_{\max}\left(L_1^{\mathrm{T}}L_1\right)\lambda_{\max}\left(S^{-1}\right)$ and $g_2 = 1/\left(\lambda_{\min}\left(SL_1 + L_1^{\mathrm{T}}S\right)\lambda_{\min}\left(S^{-1}\right)\right)$. Then we can give the following theorem.

Theorem 1. Let Assumptions 1 and 2 be satisfied. Let $\gamma_* > \max\{0, -2\phi(A)\}$ and α_3 be a constant satisfying $0 < \alpha_3 < \alpha_2$. Consider the regulation time T_1 shown in (8) that can be arbitrarily specified in the time interval $(0,\infty)$. Consider the protocol

$$u_i(t) = \sigma\left(-g_2 B^{\mathrm{T}} P(\gamma_1(t)) \sum_{j=1}^{N+1} r_{ij}\left(x_i(t) - x_j(t)\right)\right), \quad i = 1,2,\ldots,N. \tag{11}$$

Then for any $e(0) \in \mathcal{E}_1(\gamma_*)$, the system consisting of (1), (2) and (11) can achieve consensus within the prescribed finite-time T_1; that is $\lim_{t\uparrow T_1}\|x_i(t) - x_{N+1}(t)\| = 0$, $i = 1,2,\ldots,N$.

Proof. Whether T_1 is well defined should be checked firstly. We consider here two cases: $\alpha_1 \neq 0$ (namely, $\mathrm{tr}(A) \neq 0$) and $\alpha_1 = 0$ (namely, $\mathrm{tr}(A) = 0$). For the former case $\alpha_1 \neq 0$, by the definition of $\phi(A)$ we have that

$$\mathrm{tr}(A) \geqslant n\phi(A), \tag{12}$$

which, together with the condition $\gamma_* > \max\{0, -2\phi(A)\}$ in Theorem 1, implies that

$$1 + \frac{\alpha_1}{\gamma_*} = \frac{2\mathrm{tr}(A) + n\gamma_*}{n\gamma_*} \geqslant \frac{2\phi(A) + \gamma_*}{\gamma_*} > 0.$$

In addition, we have $\alpha_2 - \alpha_3 > 0$. Thus, T_1 is well defined for $\alpha_1 \neq 0$. Notice that

$$\lim_{\alpha_1\to 0} T_1 = \lim_{\alpha_1\to 0}\frac{1}{\alpha_1(\alpha_2-\alpha_3)}\ln\left(1+\frac{\alpha_1}{\gamma_*}\right) = \frac{1}{(\alpha_2-\alpha_3)\gamma_*},$$

$$\lim_{\alpha_1\to 0}\gamma_1(t) = \lim_{\alpha_1\to 0}\frac{\mathrm{e}^{\alpha_1(\alpha_2-\alpha_3)T_1}-1}{\mathrm{e}^{\alpha_1(\alpha_2-\alpha_3)(T_1-t)}-1}\gamma_* = \frac{T_1\gamma_*}{T_1-t},$$

from which we know that both T_1 and $\gamma_1(t)$ are also well defined for $\alpha_1 = 0$. Thus, one can respectively rewrite both T_1 and $\gamma_1(t)$ as

$$T_1 = \begin{cases} \frac{1}{\alpha_1(\alpha_2-\alpha_3)}\ln\left(1+\frac{\alpha_1}{\gamma_*}\right), & \alpha_1 \neq 0, \\ \frac{1}{(\alpha_2-\alpha_3)\gamma_*}, & \alpha_1 = 0, \end{cases} \tag{13}$$

and

$$\gamma_1(t) = \begin{cases} \frac{\mathrm{e}^{\alpha_1(\alpha_2-\alpha_3)T_1}-1}{\mathrm{e}^{\alpha_1(\alpha_2-\alpha_3)(T_1-t)}-1}\gamma_*, & \alpha_1 \neq 0, \\ \frac{T_1\gamma_*}{T_1-t}, & \alpha_1 = 0, \end{cases} \tag{14}$$

from which we have $\dot{\gamma}_1(t) > 0, \forall t \in [0,T_1)$ (thus, $\gamma_1(t) \geqslant \gamma_*, \forall t \in [0,T_1)$).

It follows from the definition in (7) that

$$e(t) = \left[x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)\right]^{\mathrm{T}} - 1_N \otimes x_{N+1}(t), \tag{15}$$

and

$$\dot{e}(t) = (I_N \otimes A)\, e(t) + (I_N \otimes B)\, u(t), \tag{16}$$

where $u(t) = [u_1^{\mathrm{T}}(t), u_2^{\mathrm{T}}(t), \ldots, u_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, from which it is clear to see that $\|e(t)\| = 0$ implies $\|x_i(t) - x_{N+1}(t)\| = 0$, $i = 1, 2, \ldots, N$, namely, the consensus is achieved. Therefore, we next turn our attention to system (16) and aim to show that $\lim_{t\uparrow T_1} \|e(t)\| = 0$. In the remaining of the proof, let $\gamma_1 = \gamma_1(t)$, $e = e(t)$, and $P_1 = P(\gamma_1) = P(\gamma_1(t))$ for brevity.

Consider the Lyapunov-like function

$$V(t, e) = g\pi(\gamma_1)e^{\mathrm{T}}\,(S \otimes P_1)\, e. \tag{17}$$

Then we will prove that

$$V(t, e) \leqslant 1 \Rightarrow \dot{V}(t, e) \leqslant 0, \quad \forall t \in [0, T_1). \tag{18}$$

Let $b_k, k = 1, 2, \ldots, m$ be the k-th column of matrix B. Then for any $b_k, k = 1, 2, \ldots, m$, we have

$$\begin{aligned}
\left|-g_2 b_k^{\mathrm{T}} P_1 \sum_{j=1}^{N+1} r_{ij}\,(x_i - x_j)\right|^2 &= \left|-g_2 b_k^{\mathrm{T}} P_1 \sum_{j=1}^{N+1} r_{ij}\,(e_i - e_j)\right|^2 \\
&\leqslant \left\|-g_2\left(L_1 \otimes b_k^{\mathrm{T}} P_1\right) e\right\|_\infty^2 \\
&\leqslant g_2^2 e^{\mathrm{T}}\left(L_1^{\mathrm{T}} L_1 \otimes P_1 b_k b_k^{\mathrm{T}} P_1\right) e \\
&\leqslant g_2^2 e^{\mathrm{T}}\left(L_1^{\mathrm{T}} L_1 \otimes P_1^{\frac{1}{2}} \operatorname{tr}\left(P_1^{\frac{1}{2}} b_k b_k^{\mathrm{T}} P_1^{\frac{1}{2}}\right) P_1^{\frac{1}{2}}\right) e \\
&\leqslant g_2^2 e^{\mathrm{T}}\left(L_1^{\mathrm{T}} L_1 \otimes P_1^{\frac{1}{2}} \operatorname{tr}\left(\sum_{k=1}^{m} b_k^{\mathrm{T}} P_1 b_k\right) P_1^{\frac{1}{2}}\right) e \\
&= g_2^2 \pi(\gamma_1) e^{\mathrm{T}}\left(L_1^{\mathrm{T}} L_1 \otimes P_1\right) e \\
&\leqslant g_1 g_2^2 \pi(\gamma_1) e^{\mathrm{T}}\,(S \otimes P_1)\, e \\
&= V(t, e) \leqslant 1, \quad \forall t \in [0, T_1),
\end{aligned} \tag{19}$$

by which we have that, for any $i = 1, 2, \ldots, N$,

$$u_i = \sigma\left(-g_2 B^{\mathrm{T}} P_1 \sum_{j=1}^{N+1} r_{ij}\,(x_i - x_j)\right) = -g_2 B^{\mathrm{T}} P_1 \sum_{j=1}^{N+1} r_{ij}\,(x_i - x_j) = -g_2 B^{\mathrm{T}} P_1 \sum_{j=1}^{N+1} r_{ij}\,(e_i - e_j)\,. \tag{20}$$

Thus, with the definitions of $u\,(t)$ and $e\,(t)$ in hand, we have

$$u = -g_2\left(L_1 \otimes B^{\mathrm{T}} P_1\right) e. \tag{21}$$

As a result, the system consisting of (16) and (21) can be written as

$$\dot{e} = \left((I_N \otimes A) - g_2\left(L_1 \otimes BB^{\mathrm{T}} P_1\right)\right) e, \tag{22}$$

along the trajectories of which the time derivative of $V(t, e)$ is

$$\begin{aligned}
\dot{V}(t, e) =& g\left(\dot{\pi}(\gamma_1) e^{\mathrm{T}}\,(S \otimes P_1)\, e + \pi(\gamma_1) e^{\mathrm{T}}\left(S \otimes \dot{P}_1\right) e + \pi(\gamma_1)\dot{e}^{\mathrm{T}}\,(S \otimes P_1)\, e + \pi(\gamma_1) e^{\mathrm{T}}\,(S \otimes P_1)\,\dot{e}\right) \\
\leqslant& g\left(n\dot{\gamma}_1 e^{\mathrm{T}}\,(S \otimes P_1)\, e + \delta_{\mathrm{c}}\dot{\gamma}_1 e^{\mathrm{T}}\,(S \otimes P_1)\, e + \pi(\gamma_1) e^{\mathrm{T}}\left(S \otimes \left(A^{\mathrm{T}} P_1 + P_1 A\right)\right) e\right. \\
&\left.- g_2 \pi(\gamma_1) e^{\mathrm{T}}\left(\left(S L_1 + L_1^{\mathrm{T}} S\right) \otimes P_1 BB^{\mathrm{T}} P_1\right) e\right) \\
\leqslant& g\left(n\dot{\gamma}_1 e^{\mathrm{T}}\,(S \otimes P_1)\, e + \delta_{\mathrm{c}}\dot{\gamma}_1 e^{\mathrm{T}}\,(S \otimes P_1)\, e + \pi(\gamma_1) e^{\mathrm{T}}\left(S \otimes \left(A^{\mathrm{T}} P_1 + P_1 A\right)\right) e\right. \\
&\left.- \pi(\gamma_1) e^{\mathrm{T}}\left(S \otimes P_1 BB^{\mathrm{T}} P_1\right) e\right) \\
=& g\left(n\dot{\gamma}_1 e^{\mathrm{T}}\,(S \otimes P_1)\, e + \delta_{\mathrm{c}}\dot{\gamma}_1 e^{\mathrm{T}}\,(S \otimes P_1)\, e - \gamma_1 \pi(\gamma_1) e^{\mathrm{T}}\,(S \otimes P_1)\, e\right) \\
=& g\left(\frac{(n\dot{\gamma}_1 + \delta_{\mathrm{c}}\dot{\gamma}_1)}{\pi(\gamma_1)} - \gamma_1\right)\pi(\gamma_1) e^{\mathrm{T}}\,(S \otimes P_1)\, e
\end{aligned}$$

$$=\frac{(n+\delta_{\rm c})\left(\dot{\gamma}_1-\alpha_1\alpha_2\gamma_1-\alpha_2\gamma_1^2\right)}{n\left(\alpha_1+\gamma_1\right)}V(t,e), \tag{23}$$

where we have used (3), (6) and $g_2 = 1/\ (\lambda_{\min}(SL_1 + L_1^{\rm T}S)\lambda_{\min}(S^{-1}))$. By noting (9), the term $\dot{\gamma}_1 - \alpha_1\alpha_2\gamma_1 - \alpha_2^2\gamma_1$ in (23) can be further written as

$$\dot{\gamma}_1-\alpha_1\alpha_2\gamma_1-\alpha_2\gamma_1^2=-\alpha_3\gamma_1\left(\alpha_1+\gamma_1\right),\quad t\in[0,T_1), \tag{24}$$

by which Eq. (23) can be continued as

$$\dot{V}(t,e)\leqslant-\frac{\alpha_3}{\alpha_2}\gamma_1V(t,e), \tag{25}$$

which exactly proves (18). Therefore, $V(t,e)\leqslant 1, t\in[0,T_1)$ holds if $V(0,e(0))\leqslant 1$, namely, $e\,(0)\in\mathcal{E}_1(\gamma_*)$.

By the comparison lemma [44], we have from (25) that

$$V(t,e)\leqslant\psi\left(t,0\right)V\left(0,e(0)\right), \tag{26}$$

where

$$\psi\left(t,0\right)=\exp\left(-\frac{\alpha_3}{\alpha_2}\int_0^t\gamma_1\left(\tau\right){\rm d}\tau\right)=\left(\frac{{\rm e}^{\alpha_1(\alpha_2-\alpha_3)T_1}-{\rm e}^{\alpha_1(\alpha_2-\alpha_3)t}}{{\rm e}^{\alpha_1(\alpha_2-\alpha_3)T_1}-1}\right)^{\rho_1}, \tag{27}$$

with $\rho_1=\frac{\alpha_3}{\alpha_2(\alpha_2-\alpha_3)}$. By using (27), we rewrite (26) as

$$V(t,e)\leqslant g\pi(\gamma_*)\lambda_{\max}(S_1)\lambda_{\max}(P(\gamma_*))\left(\frac{{\rm e}^{\alpha_1(\alpha_2-\alpha_3)T_1}-{\rm e}^{\alpha_1(\alpha_2-\alpha_3)t}}{{\rm e}^{\alpha_1(\alpha_2-\alpha_3)T_1}-1}\right)^{\rho_1}\|e(0)\|^2. \tag{28}$$

On the other hand,

$$\begin{aligned}V(t,e)&=g\pi(\gamma_1)e^{\rm T}\left(S\otimes P(\gamma_1)\right)e\\&\geqslant g\pi(\gamma_*)e^{\rm T}\left(S\otimes P(\gamma_*)\right)e\\&\geqslant g\pi(\gamma_*)\lambda_{\min}(S)\lambda_{\min}(P(\gamma_*))\left\|e\right\|^2,\end{aligned} \tag{29}$$

where we have used $\lambda_{\min}(P(\gamma_*))I_n\leqslant P(\gamma_*)\leqslant P(\gamma_1)$, $\forall\gamma_1\geqslant\gamma_*$. It follows from (28) and (29) that

$$\|e\|^2\leqslant\frac{\lambda_{\max}(S)\lambda_{\max}(P(\gamma_*))}{\lambda_{\min}(S)\lambda_{\min}(P(\gamma_*))}\left(\frac{{\rm e}^{\alpha_1(\alpha_2-\alpha_3)T_1}-{\rm e}^{\alpha_1(\alpha_2-\alpha_3)t}}{{\rm e}^{\alpha_1(\alpha_2-\alpha_3)T_1}-1}\right)^{\rho_1}\|e(0)\|^2, \tag{30}$$

which implies that $\lim_{t\uparrow T_1}\|e(t)\|=0$.

Finally, we will show that T_1 can be specified arbitrarily, that is, for a given constant $T_*\in(0,+\infty)$, we can always find two well defined constants γ_* and α_3 such that

$$T_1=\frac{1}{\alpha_1(\alpha_2-\alpha_3)}\ln\left(1+\frac{\alpha_1}{\gamma_*}\right)=T_*. \tag{31}$$

It follows from (8) that,

$$\frac{{\rm d}T_1}{{\rm d}\alpha_3}=\frac{\ln(1+\frac{\alpha_1}{\gamma_*})}{\alpha_1(\alpha_2-\alpha_3)^2}>0,\quad\forall\alpha_3\in(0,\alpha_2), \tag{32}$$

$$\frac{{\rm d}T_1}{{\rm d}\gamma_*}=\frac{{\rm d}\delta_{\rm c}\left(\gamma_*\right)}{{\rm d}\gamma_*}\frac{\ln(1+\frac{\alpha_1}{\gamma_*})\alpha_2^2}{n\alpha_1^2(\alpha_2-\alpha_3)^2}-\frac{1}{\gamma_*(\alpha_2-\alpha_3)\left(\gamma_*+\alpha_1\right)}<0, \tag{33}$$

where we have used $\frac{{\rm d}\delta_{\rm c}(\gamma_*)}{{\rm d}\gamma_*}\leqslant 0$ [42]. It can be seen from (32) that the smaller the constant α_3, the smaller the regulation time T_1. Moreover, $T_1\to T_{\min}=\frac{1}{\alpha_1\alpha_2}\ln(1+\frac{\alpha_1}{\gamma_*})$ ($T_{\min}=\frac{1}{\alpha_2\gamma_*}$ when $\alpha_1=0$) as $\alpha_3\to 0$, and $T_1\to T_{\max}=+\infty$ as $\alpha_3\to\alpha_2$. On the other hand, it follows from (33) that the larger the parameter γ_*, the smaller the finite-time T_1. Let us consider $T_{\min}$ again with the condition $\gamma_*\to+\infty$; it is clear that $T_{\min}\to 0$. According to the above analysis, we know that T_1 can be selected arbitrarily in the interval $(T_{\min},T_{\max})$ (namely, $(0,+\infty)$), which implies that Eq. (31) holds. This completes the proof.

Noting that the control gain in (11) tends to infinity as $t \to T_1$. To facilitate physical implementation, the time-varying function (9) can be replaced by

$$\begin{cases} \gamma_1(t) = \frac{e^{\alpha_1(\alpha_2-\alpha_3)T_1}-1}{e^{\alpha_1(\alpha_2-\alpha_3)(T_1+\kappa-t)}-1}\gamma_*, \ t \in [0, T_1), \\ \gamma_1(t) = \frac{e^{\alpha_1(\alpha_2-\alpha_3)T_1}-1}{e^{\alpha_1(\alpha_2-\alpha_3)\kappa}-1}\gamma_* \triangleq \bar{\gamma}, \ t \in [T_1, +\infty), \end{cases} \tag{34}$$

with κ being an arbitrarily small positive number to be designed.

Remark 1. We have considered the case $\|u_i(t)\|_\infty \leqslant 1, i = 1, 2, \ldots, N$ in Theorem 1. Notice that such a result can be extended to the general case $\|u_i(t)\|_\infty \leqslant u_{\max}, i = 1, 2, \ldots, N$ with $u_{\max} > 0$ being a constant by considering the transformation $u_i(t) \mapsto u_i(t)/u_{\max}$. In this case, the set in (10) can be rewritten as

$$\mathcal{E}_1(\gamma_1) = \{e \in \mathbb{R}^{nN} : g\pi(\gamma_1)e^{\mathrm{T}}(S \otimes P(\gamma_1))\, e \leqslant u_{\max}^2\}. \tag{35}$$

Remark 2. As shown in (8) and (35) that a short convergence time T_1 corresponds to a large constant parameter γ_* which will lead to a small domain of attraction $\mathcal{E}_1(\gamma_*)$. Hence, the chosen of the parameter γ_* should weigh the regulation time T_1 and the set $\mathcal{E}_1(\gamma_*)$ at the same time. In addition, we consider the case that $\alpha_1 = 0$ (namely, $\mathrm{tr}(A) = 0$). In such a case, the function $\gamma(t)$ is the form of the second equation in (14) where the parameter γ_* only needs to satisfy $\gamma_* > 0$ by noting the condition $\gamma_* > \max\{0, -2\phi(A)\}$ in Theorem 1 and (12). Notice that the domain of attraction $\mathcal{E}_1(\gamma_*)$ satisfies $\lim_{\gamma_* \downarrow 0} \mathcal{E}_1(\gamma_*) = \infty$, which implies that the semi-global consensus problem is solved, while the regulation time T_1 satisfies $\lim_{\gamma_* \downarrow 0} T_1 = \lim_{\gamma_* \downarrow 0} \frac{1}{(\alpha_2-\alpha_3)\gamma_*} = \infty$.

Remark 3. In Theorem 1, there are two parameters that need to be selected, namely, γ_* and α_3. In practice, it is easy to choose these two parameters by the known information of the system. In what follows, we will show how to select γ_* and α_3 for a given system. To begin with, it is easy to know the information of the saturation level $u_{\max}$ and the initial conditions $e(0)$. To ensure the stability of the closed-loop system, γ_* can be directly obtained by solving the inequality $g\pi(\gamma_*)e^{\mathrm{T}}(0)\,(S \otimes P(\gamma_*))\, e(0) \leqslant u_{\max}^2$. Then by using γ_*, the minimum convergence time $T_{\min}$ for the closed-loop system can be directly obtained by solving $T_{\min} = \frac{1}{\alpha_1\alpha_2}\ln(1 + \frac{\alpha_1}{\gamma_*})$ (if $\alpha_1 = 0$, $T_{\min} = \frac{1}{\alpha_2\gamma_*}$). The convergence time T in the interval $[T_{\min}, \infty)$ can be finally determined by choosing appropriate parameter α_3.

4 Prescribed-time consensus with unknown L_1

In Section 3, the prescribed-time consensus problem was analyzed, and a bounded linear time-varying protocol was proposed. It is worth noting that such a protocol requires knowledge of the Laplacian matrix and, therefore, cannot be fully distributed. Adaptive nonlinear distributed protocols have been proposed in [6, 40] to address this issue. Even though these protocols only utilize local information, the upper bound of their amplitudes may still depend on the Laplacian matrix and is, therefore, difficult to estimate. Consequently, these design methods may not apply to the input-constrained system. To solve the prescribed-time consensus problem, we intend to design a novel fully distributed consensus protocol in this section.

To begin with, partially inspired by [16], we consider here the distributed observer,

$$\dot{\varepsilon}_i(t) = \frac{1}{\sum_{j=1}^{N+1} r_{ij}} \sum_{j=1}^{N+1} r_{ij}\dot{\varepsilon}_j(t) - \frac{\gamma_2(t)}{2} \frac{1}{\sum_{j=1}^{N+1} r_{ij}} \sum_{j=1}^{N+1} r_{ij}\left(\varepsilon_i(t) - \varepsilon_j(t)\right), \ i = 1, 2, \ldots, N, \tag{36}$$

with $\varepsilon_{N+1}(t) = x_{N+1}(t)$ and $\gamma_2(t)$ being a time-varying function to be designed.

By using the estimated values, we design the protocol

$$u_i(t) = \sigma\left(-B^{\mathrm{T}}P(\gamma_2(t))\,(x_i(t) - \varepsilon_i(t))\right), \ i = 1, 2, \ldots, N, \tag{37}$$

where $P(\gamma_2)$ is the solution to (3) with $\gamma = \gamma_2(t)$. Define

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdots \\ \beta_N \end{bmatrix} = \begin{bmatrix} \varepsilon_1 - \varepsilon_{N+1} \\ \varepsilon_2 - \varepsilon_{N+1} \\ \cdots \\ \varepsilon_N - \varepsilon_{N+1} \end{bmatrix} = \begin{bmatrix} \varepsilon_1 - x_{N+1} \\ \varepsilon_2 - x_{N+1} \\ \cdots \\ \varepsilon_N - x_{N+1} \end{bmatrix} = \varepsilon - 1_N \otimes x_{N+1},$$

where $\varepsilon = \left[\varepsilon_1^{\mathrm{T}}, \varepsilon_2^{\mathrm{T}}, \ldots, \varepsilon_N^{\mathrm{T}}\right]^{\mathrm{T}}$. Define the following set

$$\mathcal{E}_2(\gamma_2) = \left\{\left[e^{\mathrm{T}}, \beta^{\mathrm{T}}\right]^{\mathrm{T}} : 2\pi(\gamma_2) e^{\mathrm{T}} \left(I_N \otimes P(\gamma_2)\right) e + 4n\pi(\gamma_2) \mathrm{tr}\left(P(\gamma_2)\right) \beta^{\mathrm{T}} \beta \leqslant 1\right\}. \tag{38}$$

Let

$$T_2 = \frac{1}{\alpha_6} \ln\left(\frac{1 + \frac{\alpha_1}{\gamma_{**}}}{1 - \frac{\alpha_1}{(1-2\alpha_4)\gamma_{**}}}\right), \tag{39}$$

and

$$\gamma_2 = \gamma_2\left(t\right) = \frac{\left(\mathrm{e}^{\alpha_6 T_2} - 1\right)\left(\frac{\mathrm{e}^{\alpha_6 (T_2 - t)}}{1-2\alpha_4} + 1\right)}{\left(\mathrm{e}^{\alpha_6 (T_2 - t)} - 1\right)\left(\frac{\mathrm{e}^{\alpha_6 T_2}}{1-2\alpha_4} + 1\right)} \gamma_{**}, \tag{40}$$

where γ_{**} and α_4 are two constants to be designed, $\alpha_5 = \frac{n}{2(n+\delta_{\mathrm{c}})}$, and $\alpha_6 = 2\alpha_1\alpha_5\left(1 - \alpha_4\right)$. Then we can give the following theorem.

Theorem 2. Consider the input-constrained multi-agent system consisting of (1) and (2) that satisfies Assumptions 1 and 2. Let $\gamma_{**} > \max\{0, -2\phi(A)\}$ and $\alpha_4 \in (0, \frac{\gamma_{**} - \alpha_1}{2\gamma_{**}})$. Consider the regulation time T_2 shown in (39) that can be arbitrarily specified by selecting the appropriate parameters γ_{**} and α_4. Then for any $[e^{\mathrm{T}}\left(0\right), \beta^{\mathrm{T}}\left(0\right)]^{\mathrm{T}} \in \mathcal{E}_2(\gamma_{**})$, the consensus of the system consisting of (1), (2) and (37) is achieved within the regulation time T_2, namely, for any $i = 1, 2, \ldots, N$, $\lim_{t \uparrow T_2} \left\|x_i\left(t\right) - x_{N+1}\left(t\right)\right\| = 0$ and $\lim_{t \uparrow T_2} \left\|\varepsilon_i\left(t\right) - x_{N+1}\left(t\right)\right\| = 0$.

Proof. Similar to Theorem 1, whether T_2 is well defined should be checked. We also consider here two cases: $\alpha_1 \neq 0$ and $\alpha_1 = 0$. For the former case $\alpha_1 \neq 0$, T_2 is well defined. Notice that

$$\lim_{\alpha_1 \to 0} T_2 = \lim_{\alpha_1 \to 0} \frac{1}{\alpha_6} \ln\left(\frac{1 + \frac{\alpha_1}{\gamma_{**}}}{1 - \frac{\alpha_1}{(1-2\alpha_4)\gamma_{**}}}\right) = \frac{1}{\alpha_5 (1-2\alpha_4)\gamma_{**}},$$

$$\lim_{\alpha_1 \to 0} \gamma_2(t) = \lim_{\alpha_1 \to 0} \frac{(\mathrm{e}^{\alpha_6 T_2} - 1)(\frac{\mathrm{e}^{\alpha_6 (T_2 - t)}}{1-2\alpha_4} + 1)\gamma_{**}}{(\mathrm{e}^{\alpha_6 (T_2 - t)} - 1)(\frac{\mathrm{e}^{\alpha_6 T_2}}{1-2\alpha_4} + 1)} = \frac{T_2 \gamma_{**}}{T_2 - t},$$

from which we have that both T_2 and $\gamma_2(t)$ are also well defined for the latter case $\alpha_1 = 0$. Therefore, we can respectively rewrite both T_2 and $\gamma_2(t)$ as

$$T_2 = \begin{cases} \frac{1}{\alpha_6} \ln\left(\frac{1 + \frac{\alpha_1}{\gamma_{**}}}{1 - \frac{\alpha_1}{(1-2\alpha_4)\gamma_{**}}}\right), & \alpha_1 \neq 0, \\ \frac{1}{\alpha_5 (1-2\alpha_4)\gamma_{**}}, & \alpha_1 = 0, \end{cases}$$

and

$$\gamma_2(t) = \begin{cases} \frac{(\mathrm{e}^{\alpha_6 T_2} - 1)(\frac{\mathrm{e}^{\alpha_6 (T_2 - t)}}{1-2\alpha_4} + 1)}{(\mathrm{e}^{\alpha_6 (T_2 - t)} - 1)(\frac{\mathrm{e}^{\alpha_6 T_2}}{1-2\alpha_4} + 1)} \gamma_{**}, & \alpha_1 \neq 0, \\ \frac{T_2 \gamma_{**}}{T_2 - t}, & \alpha_1 = 0. \end{cases}$$

The second step is to check whether the distributed observer in (36) is well defined. By Lemma 2, it is clear that $\sum_{j=1}^{N+1} r_{ij} \neq 0$, which implies that such an observer is well defined. Thus Eq. (36) can be continued as

$$\sum_{j=1}^{N+1} r_{ij}\left(\dot{\varepsilon}_i\left(t\right) - \dot{\varepsilon}_j\left(t\right)\right) = -\frac{\gamma_2\left(t\right)}{2} \sum_{j=1}^{N+1} r_{ij}\left(\varepsilon_i\left(t\right) - \varepsilon_j\left(t\right)\right). \tag{41}$$

Let $\upsilon_i\left(t\right) = \sum_{j=1}^{N+1} r_{ij}\left(\varepsilon_i\left(t\right) - \varepsilon_j\left(t\right)\right)$. Then it follows from (41) that

$$\dot{\upsilon}_i\left(t\right) = -\frac{\gamma_2\left(t\right)}{2} \upsilon_i\left(t\right). \tag{42}$$

Let $\upsilon = \left[\upsilon_1^{\mathrm{T}}, \upsilon_2^{\mathrm{T}}, \ldots, \upsilon_N^{\mathrm{T}}\right]^{\mathrm{T}}$. Then Eq. (42) becomes

$$\dot{\upsilon}\left(t\right) = -\frac{\gamma_2\left(t\right)}{2} \upsilon\left(t\right). \tag{43}$$

In addition, it is clear that

$$\upsilon\left(t\right)=\left(L_{1}\otimes I_{n}\right)\beta\left(t\right). \tag{44}$$

As is known that L_1 is nonsingular as long as Assumption 2 is satisfied, by which Eq. (44) becomes

$$\beta\left(t\right)=\left(L_{1}^{-1}\otimes I_{n}\right)\upsilon\left(t\right),$$

whose time derivative is $\dot{\beta}\left(t\right)=-\frac{\gamma_2(t)}{2}\beta\left(t\right)$ by noting (43). With the definition of $e(t)$, $u(t)$, $\varepsilon(t)$, and $\beta\left(t\right)$, the system consisting of (1), (2), (36), and (37) is rewritten as

$$\begin{cases}\dot{\beta}\left(t\right)=-\frac{\gamma_2(t)}{2}\beta\left(t\right),\\ \dot{e}(t)=\left(I_N\otimes A\right)e(t)+\left(I_N\otimes B\right)u(t),\\ u(t)=\sigma\left(-\left(I_N\otimes B^{\mathrm{T}}P\left(\gamma_2\left(t\right)\right)\right)\left(e(t)-\beta\left(t\right)\right)\right).\end{cases} \tag{45}$$

We next aim to show that $\lim_{t\uparrow T_2}\|e(t)\|=0$ and $\lim_{t\uparrow T_2}\|\beta(t)\|=0$. For brevity, let $\gamma_2=\gamma_2(t)$, $e=e(t)$, $\beta=\beta\left(t\right)$, and $P_2=P(\gamma_2)=P(\gamma_2(t))$ in the remaining of the proof.

Consider the Lyapunov-like function,

$$V(t,e,\beta)=V_1(t,e)+V_2(t,\beta), \tag{46}$$

where $V_1(t,e)=2\pi(\gamma_2)e^{\mathrm{T}}\left(I_N\otimes P_2\right)e$ and $V_2(t,\beta)=4n\pi(\gamma_2)\mathrm{tr}\left(P_2\right)\beta^{\mathrm{T}}\beta$. Then we will show that

$$V(t,e,\beta)\leqslant 1\Rightarrow \dot{V}(t,e,\beta)\leqslant 0,\quad \forall t\in[0,T_2). \tag{47}$$

With the definition of $b_k, k=1,2,\ldots,m$ shown in (19), we have that for any x_i and ε_i, $i=1,2\ldots,N$,

$$\begin{aligned}\left|-b_k^{\mathrm{T}}P_2\left(x_i-\varepsilon_i\right)\right|^2=&\left|-b_k^{\mathrm{T}}P_2\left(e_i-\beta_i\right)\right|^2\\ \leqslant&2\left|-b_k^{\mathrm{T}}P_2e_i\right|^2+2\left|b_k^{\mathrm{T}}P_2\beta_i\right|^2\\ =&2e_i^{\mathrm{T}}P_2b_kb_k^{\mathrm{T}}P_2e_i+2\beta_i^{\mathrm{T}}P_2b_kb_k^{\mathrm{T}}P_2\beta_i\\ \leqslant&2e^{\mathrm{T}}\left(I_N\otimes P_2b_kb_k^{\mathrm{T}}P_2\right)e+2\beta^{\mathrm{T}}\left(I_N\otimes P_2b_kb_k^{\mathrm{T}}P_2\right)\beta\\ \leqslant&2e^{\mathrm{T}}\left(I_N\otimes P_2^{\frac{1}{2}}\mathrm{tr}\left(P_2^{\frac{1}{2}}b_kb_k^{\mathrm{T}}P_2^{\frac{1}{2}}\right)P_2^{\frac{1}{2}}\right)e+2\beta^{\mathrm{T}}\left(I_N\otimes P_2^{\frac{1}{2}}\mathrm{tr}\left(P_2^{\frac{1}{2}}b_kb_k^{\mathrm{T}}P_2^{\frac{1}{2}}\right)P_2^{\frac{1}{2}}\right)\beta\\ \leqslant&2e^{\mathrm{T}}\left(I_N\otimes P_2^{\frac{1}{2}}\mathrm{tr}\left(b_k^{\mathrm{T}}P_2b_k\right)P_2^{\frac{1}{2}}\right)e+2\beta^{\mathrm{T}}\left(I_N\otimes P_2^{\frac{1}{2}}\mathrm{tr}\left(\sum_{k=1}^{m}b_k^{\mathrm{T}}P_2b_k\right)P_2^{\frac{1}{2}}\right)\beta\\ =&2\pi(\gamma_2)e^{\mathrm{T}}\left(I_N\otimes P_2\right)e+2\pi(\gamma_2)\beta^{\mathrm{T}}\left(I_N\otimes P_2\right)\beta\\ \leqslant&2\pi(\gamma_2)e^{\mathrm{T}}\left(I_N\otimes P_2\right)e+2\pi(\gamma_2)\mathrm{tr}\left(P_2\right)\beta^{\mathrm{T}}\beta\\ \leqslant&2\pi(\gamma_2)e^{\mathrm{T}}\left(I_N\otimes P_2\right)e+4n\pi(\gamma_2)\mathrm{tr}\left(P_2\right)\beta^{\mathrm{T}}\beta\\ =&V(t,e,\beta)\leqslant 1,\ \forall t\in[0,T_2),\end{aligned} \tag{48}$$

which implies that

$$u_i\left(t\right)=\sigma\left(-B^{\mathrm{T}}P_2\left(x_i\left(t\right)-\varepsilon_i\left(t\right)\right)\right)=-B^{\mathrm{T}}P_2\left(x_i\left(t\right)-\varepsilon_i\left(t\right)\right)=-B^{\mathrm{T}}P_2\left(e_i\left(t\right)-\beta_i\left(t\right)\right).$$

In such a case, we rewrite system (45) as

$$\begin{cases}\dot{\beta}\left(t\right)=-\frac{\gamma_2(t)}{2}\beta\left(t\right),\\ \dot{e}(t)=\left(I_N\otimes A-I_N\otimes BB^{\mathrm{T}}P_2\right)e(t)+\left(I_N\otimes BB^{\mathrm{T}}P_2\right)\beta\left(t\right),\end{cases} \tag{49}$$

along the trajectories of which, the time derivative of $V(t,e,\beta)$ is

$$\dot{V}(t,e,\beta)=\dot{V}_1\left(t,e\right)+\dot{V}_2\left(t,\beta\right), \tag{50}$$

where

$$\dot{V}_1\left(t,e\right)=2\dot{\pi}(\gamma_2)e^{\mathrm{T}}\left(I_N\otimes P_2\right)e+2\pi(\gamma_2)e^{\mathrm{T}}\left(I_N\otimes\dot{P}_2\right)e+2\pi(\gamma_2)\dot{e}^{\mathrm{T}}\left(I_N\otimes P_2\right)e+2\pi(\gamma_2)e^{\mathrm{T}}\left(I_N\otimes P_2\right)\dot{e}$$

$$
\begin{aligned}
&=\frac{\dot{\pi}(\gamma_2)}{\pi(\gamma_2)}V_1(t,e)+\frac{\delta_c\dot{\gamma}_2}{\pi(\gamma_2)}V_1(t,e)+2\pi(\gamma_2)e^{\mathrm{T}}\left(I_N\otimes\left(P_2A+A^{\mathrm{T}}P_2-P_2BB^{\mathrm{T}}P_2\right)\right)e\\
&\quad-2\pi(\gamma_2)e^{\mathrm{T}}\left(I_N\otimes P_2BB^{\mathrm{T}}P_2\right)e+4\pi(\gamma_2)e^{\mathrm{T}}\left(I_N\otimes P_2BB^{\mathrm{T}}P_2\right)\beta\\
&\leqslant\frac{\dot{\pi}(\gamma_2)}{\pi(\gamma_2)}V_1(t,e)+\frac{\delta_c\dot{\gamma}_2}{\pi(\gamma_2)}V_1(t,e)-\gamma_2V_1(t,e)+2\pi(\gamma_2)\beta^{\mathrm{T}}\left(I_N\otimes P_2BB^{\mathrm{T}}P_2\right)\beta\\
&\leqslant\left(\frac{(n+\delta_c)\dot{\gamma}_2}{\pi(\gamma_2)}-\gamma_2\right)V_1(t,e)+2\pi^2(\gamma_2)\mathrm{tr}\left(P_2\right)\beta^{\mathrm{T}}\beta\\
&=\left(\frac{(n+\delta_c)\dot{\gamma}_2}{\pi(\gamma_2)}-\gamma_2\right)V_1(t,e)+\frac{\pi(\gamma_2)}{2n}V_2(t,\beta),
\end{aligned}
\tag{51}
$$

in which the first inequality has used (3) and the Young's inequality, and

$$
\begin{aligned}
\dot{V}_2(t,\beta)&=4n\dot{\pi}(\gamma_2)\mathrm{tr}\left(P_2\right)\beta^{\mathrm{T}}\beta+4n\pi(\gamma_2)\mathrm{tr}\left(\dot{P}_2\right)\beta^{\mathrm{T}}\beta+4n\pi(\gamma_2)\mathrm{tr}\left(P_2\right)\dot{\beta}^{\mathrm{T}}\beta+4n\pi(\gamma_2)\mathrm{tr}\left(P_2\right)\beta^{\mathrm{T}}\dot{\beta}\\
&\leqslant\frac{\dot{\pi}(\gamma_2)}{\pi(\gamma_2)}V_2(t,\beta)+\frac{\delta_c\dot{\gamma}_2}{\pi(\gamma_2)}V_2(t,\beta)-\gamma_2V_2(t,\beta)\\
&=\left(\frac{(n+\delta_c)\dot{\gamma}_2}{\pi(\gamma_2)}-\gamma_2\right)V_2(t,\beta),
\end{aligned}
\tag{52}
$$

in which the first inequality holds by noting (6). Substituting (51) and (52) into (50) gives

$$
\dot{V}(t,e,\beta)\leqslant\varphi_1(\gamma_2)V_1(t,e)+\varphi_2(\gamma_2)V_2(t,\beta),
\tag{53}
$$

where $\varphi_1(\gamma_2)=\frac{(n+\delta_c)\dot{\gamma}_2}{\pi(\gamma_2)}-\gamma_2$ and $\varphi_2(\gamma_2)=\frac{(n+\delta_c)\dot{\gamma}_2}{\pi(\gamma_2)}-\gamma_2+\frac{\pi(\gamma_2)}{2n}$. It is easy to know that $\varphi_1(\gamma_2)<\varphi_2(\gamma_2)$ due to $\frac{\pi(\gamma_2)}{2n}>0$, by which Eq. (53) becomes

$$
\dot{V}(t,e,\beta)\leqslant\varphi_2(\gamma_2)V(t,e,\beta).
\tag{54}
$$

Noting (40), it follows that

$$
\varphi_2(\gamma_2)=-\alpha_4\gamma_2,
\tag{55}
$$

by which we rewrite (54) as

$$
\dot{V}(t,e,\beta)\leqslant-\alpha_4\gamma_2V(t,e,\beta),
\tag{56}
$$

which exactly proves (47). Therefore, $V(t,e,\beta)\leqslant 1, t\in[0,T_2)$ holds if $V(0,e(0),\beta(0))\leqslant 1$, that is, $[e^{\mathrm{T}}(0),\beta^{\mathrm{T}}(0)]^{\mathrm{T}}\in\mathcal{E}_2(\gamma_{**})$. By reviewing (25) and (56), it is clear to see that both of them have the same structure. Therefore, we can use the same argument as in the proof of Theorem 1 to show that $\lim_{t\uparrow T_2}\|e(t)\|=0$ and $\lim_{t\uparrow T_2}\|\beta(t)\|=0$.

Finally, similar to the proof of Theorem 1, we can also determine a prescribed finite-time $T_2=T_{**}$ in the interval $(0,\infty)$ by appropriately choosing γ_{**} and α_4. This completes the proof.

Remark 4. For the follower with $r_{iN+1}>0$, the observer (36) does not require to be designed since, in such a case, the follower can collect information $x_{N+1}(t)$ from the leader directly. In addition, although the distributed observer (36) is partly inspired by [16], the distributed observer (36)-based protocol (37) is still theoretically attractive. On the one hand, Ref. [16] proposed a switching algorithm to ensure the boundedness of the control input, while a smooth time-varying protocol is developed here. On the other hand, we study the relationship among the size of the attraction domain, the regulation time, and the saturation level quantitatively without using a priori knowledge of the Laplacian matrix by employing a new Lyapunov function.

Remark 5. Compared with the protocol in (11), the protocol (37) has two advantages. Firstly, the protocol (37) is designed in a fully distributed manner since it does not collect the a priori information of the Laplacian matrix L_1. Secondly, suppose that the saturation level and initial conditions of these two different cases are the same; then the setting time to achieve consensus under the protocol (37) is shorter than that under the protocol (11) by selecting appropriate initial conditions of the distributed observer (36). Recently, using adaptive control techniques, other interesting fully distributed protocols have been proposed [45–47]. These protocols can only achieve asymptotic consensus instead of prescribed time consensus. Moreover, these protocols cannot be directly applied to input-constrained systems since they

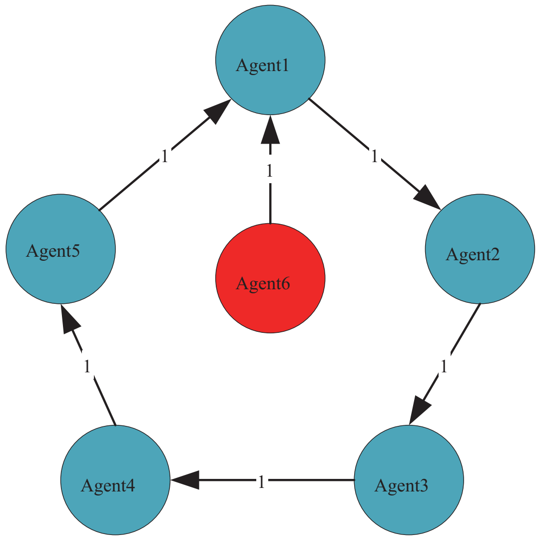

Figure 1 (Color online) Directed communication graph among agents.

do not consider the effect of actuator saturation on the closed-loop system. These two limitations were eliminated in the proposed protocol (37). However, the protocol (37) needs to collect its edge weights and those of its neighbors (namely, $\sum_{j=1}^{N+1} r_{ij}$).

Finally, Remarks 1 and 2 are also applicable to Theorem 2 and are omitted due to limited space.

5 A numerical example

In this section, the effectiveness of the proposed protocols is demonstrated on a group of two-mass-spring systems. Such a system has five followers and one leader, in which the ith identical follower agent can be described by (1) with

$$A=\begin{bmatrix}0 & 1 & 0 & 0\\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0\end{bmatrix},\quad B=\begin{bmatrix}0\\ \frac{1}{m_1}\\ 0\\ 0\end{bmatrix},$$

where $m_1=m_2=0.75$ kg and $k_1=k_2=2$ N/m, which implies that $\mathrm{Re}\{\lambda(A)\}=0$, namely $\alpha_1=0$. The dynamics of the leader agent can be described by (2). The communication topology among agents is shown in Figure 1. In such a case, the Laplacian matrix is constructed as

$$L=\begin{bmatrix}L_1 & L_2\\ 0_{1\times 5} & 0\end{bmatrix}=\left[\begin{array}{ccccc|c}2 & 0 & 0 & 0 & -1 & -1\\ -1 & 1 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 0 & -1 & 1 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 0\end{array}\right],$$

which satisfies Assumption 2. With the Laplacian L, simple calculations show that $S=\mathrm{diag}(5,9,8,7,6)$. In such a case, $g_1=1.33$ and $g_2=5.2$.

Let the initial states for the agents be $x_1(0)=[1,0,0.1,0.1]^{\mathrm{T}}$, $x_2(0)=[-1,2,-0.1,0.3]^{\mathrm{T}}$, $x_3(0)=[1,-1,0.2,0.1]^{\mathrm{T}}$, $x_4(0)=[-1,1,0.1,-0.1]^{\mathrm{T}}$, $x_5(0)=[2,1,0.2,-0.1]^{\mathrm{T}}$, $x_6(0)=[1,1,0.2,0.1]^{\mathrm{T}}$, by which, it is clear that $e(0)=[0,-1,-0.1,0,-2,1,-0.3,0.2,0,-2,0,0,\ -2,0,-0.1,-0.2,1,0,0,-0.2]^{\mathrm{T}}$. In addition, without losing generality, we assume that $\|u_i(t)\|_\infty\leqslant 30, i=1,2,\ldots,5$. By choosing $\delta_{\mathrm{c}}=16$, $\alpha_2=0.2$ can be obtained according to $\alpha_2=\frac{n}{n+\delta_{\mathrm{c}}}$ and $\alpha_5=0.1$ can be obtained according to $\alpha_5=\frac{n}{2(n+\delta_{\mathrm{c}})}$.

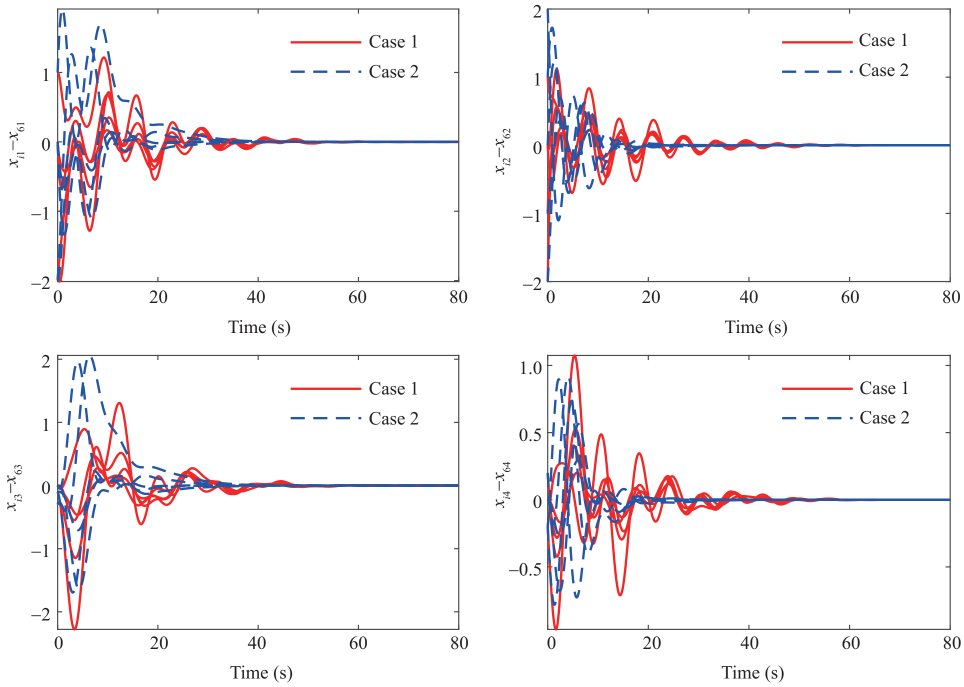

Figure 2 (Color online) State errors between followers and their leader for Cases 1 and 2.

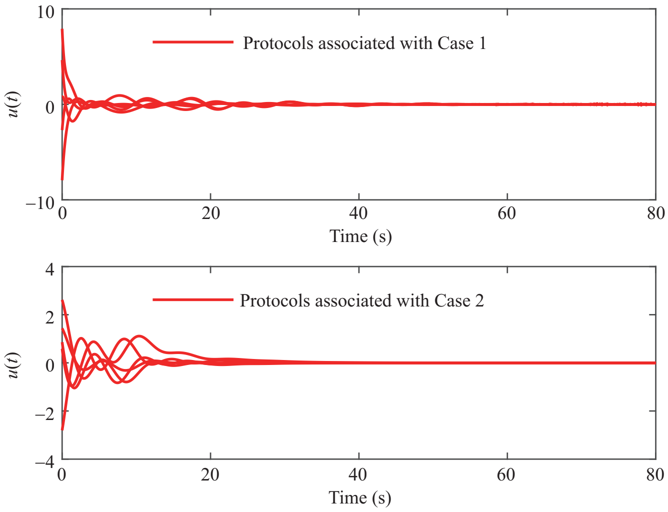

Figure 3 (Color online) The protocol signals for Cases 1 and 2.

Let us consider two different cases. Case 1: The protocol is designed by (11) with $\kappa = 5$ shown in (34). Then, solving $g\pi(\gamma_*)e^{\mathrm{T}}(0)(S \otimes P(\gamma_*))e(0) \leqslant 900$ immediately yields $\gamma_* = 0.07$. Thus, $T_{\min} = \frac{1}{\alpha_2\gamma_*} = 71.43$, by which, it is clear that T_1 can only be selected in the time interval $[71.43, \infty)$. We finally chose $T_1 = 71.43$ by selecting $\alpha_3 = 0$. Case 2: The protocol is designed by (37) with $\kappa = 5$. According to Remark 4, we only design the distributed observer for Agents 2–5, and the initial states for the distributed observer are randomly chosen as $\varepsilon_2(0) = [1, 2, 0.1, 0.1]^{\mathrm{T}}$, $\varepsilon_3(0) = [0, 1, 0.1, 0.2]^{\mathrm{T}}$, $\varepsilon_4(0) = [1, 1, 0.2, 0]^{\mathrm{T}}$, and $\varepsilon_5(0) = [2, 1, 0.1, 0]^{\mathrm{T}}$. Thus, $\beta(0) = [0, 0, 0, 0, 0, 1, -0.1, 0, -1, 0 - 0.1, 0.1,\ 0, 0, 0, -0.1, 1, 0, -0.1, -0.1]^{\mathrm{T}}$. Then, by the inequality $2\pi(\gamma_{**})e^{\mathrm{T}}(0)(I_N \otimes P(\gamma_{**}))e(0) + 4n\pi(\gamma_{**})\mathrm{tr}(P(\gamma_{**}))\beta^{\mathrm{T}}(0)\beta(0) \leqslant 900$, we have $\gamma_{**} = 0.2$. In such a case, $T_{\min} = \frac{1}{\alpha_5\gamma_{**}} = 50$ for the second case, by which, T_2 can be selected in the time interval $[50, \infty)$. We finally chose $T_2 = 50$ by selecting $\alpha_4 = 0$. The state errors between followers and their leader for these two cases are plotted in Figure 2. The protocol signals associated with Cases 1 and 2 are given in Figure 3. As shown in Figure 2, the prescribed-time consensus problem is achieved for Cases 1 and 2. Moreover, the regulation time associated with Case 2 is shorter than that associated with

Case 1. Figure 3 shows that the control signals associated with Cases 1 and 2 can eliminate saturation (namely, $\|u_i(t)\|_\infty \leqslant 30$, $i = 1, 2, \ldots, 5$) during the convergence.

6 Conclusion

In this paper, the prescribed-time consensus problem for general linear multi-agent systems has been investigated. Two distinct time-varying linear bounded protocols have been proposed. Specifically, the first protocol was designed in a non-fully distributed manner (i.e., using the Laplacian matrix knowledge), whereas the second protocol was designed in a fully distributed manner. By investigating the properties of the parametric Lyapunov equation, it has been established that these two protocols can solve the prescribed-time consensus problem under consideration. Additionally, the quantitative relationship among the size of the domain of attraction, the regulation time, and the saturation level has been studied. Simulation results were employed to validate the efficacy of the proposed methods. In this paper, the system and input matrices of the leader are identical to those of the followers. Thus, a future research direction is to extend the results of this paper to a general case in which the followers have different systems and input matrices. Another interesting topic for future study is to extend the results of this paper to triangular nonlinear systems that satisfy a linear growth condition.

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