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Distributed optimal consensus of multiagent systems with Markovian switching topologies: synchronous and asynchronous communications

Juan ZHANG[1*], Huaguang ZHANG[1,2], Bowen ZHOU[1] & Xiangpeng XIE[3]

[1]*College of Information Science and Engineering, Northeastern University, Shenyang* 110004*, China;*
[2]*State Key Laboratory of Synthetical Automation for Process Industries, Shenyang* 110004*, China;*
[3]*School of Internet of Things, Nanjing University of Posts and Telecommunications, Nanjing* 210023*, China*

Appendix A Proof of Theorem 1

(1) Stability proof. Construct the following Lyapunov funcation $V = V_1 + V_2 + V_3$, where $V_1 = \mathbb{E}[\sum_{i\in I[1,N]} \xi_i^{\mathrm{T}} P\xi_i]$, $V_2 = \mathbb{E}[\sum_{i\in I[1,N]}\sum_{j\in I[1,N]} \frac{(\mu_{ij}-\mu)^2}{4v_{ij}} + \frac{(\iota_{ij}-\iota)^2}{2o_{ij}}]$ with $\mu > \{\frac{4}{\lambda_2}, \frac{2}{\delta}\}$ and ι being positive constant to be specified, and $V_3 = \mathbb{E}[\mu\sum_{i\in I[1,N]} \pi_i + \iota_1 \sum_{i\in I[1,N]} \pi_i^1]$ with ι_1 being positive constant to be specified.

Let $V_1^m = \mathbb{E}[\sum_{i\in I[1,N]} \xi_i^{\mathrm{T}} P\xi_i \mathbf{1}_{\{\sigma(t)=m\}}]$, in which $\mathbf{1}_{\{\sigma(t)=m\}}$ represents the Dirac measure satisfying $\mathbf{1}_{\{\sigma(t)=m\}} = 1$ if $\sigma(t) = m$, and $\mathbf{1}_{\{\sigma(t)=m\}} = 0$ if $\sigma(t) \neq m$. In combination with (9) and Lemma 2, one has

$$\begin{aligned}
\mathrm{d}V_1^m =& \mathbb{E}\left[\sum_{i\in I[1,N]} \xi_i^{\mathrm{T}} P\xi_i \mathrm{d}\mathbf{1}_{\{\sigma(t)=m\}}\right] + \mathbb{E}\left[\sum_{i\in I[1,N]} 2\xi_i^{\mathrm{T}} P\dot{\xi}_i \mathbf{1}_{\{\sigma(t)=m\}}\right]\mathrm{d}t \\
=& \sum_{q\in I[1,s]} \lambda_{qm} V_1^q \mathrm{d}t + \mathbb{E}\Bigg[\Bigg(\sum_{i\in I[1,N]} \xi_i^{\mathrm{T}}(PA + A^{\mathrm{T}}P)\xi_i + 2\sum_{i\in I[1,N]}\sum_{j\in I[1,N]} a_{ij}^{\sigma(t)}\mu_{ij}\xi_i^{\mathrm{T}} PBF(\tilde{x}_i - \tilde{x}_j) \\
&+ 2\sum_{i\in I[1,N]}\sum_{j\in I[1,N]} a_{ij}^{\sigma(t)}\iota_{ij}\xi_i^{\mathrm{T}} PB\mathrm{sgn}[F(\tilde{x}_i - \tilde{x}_j)] \\
&+ 2\sum_{i\in I[1,N]} \xi_i^{\mathrm{T}} PB\left[\varphi_i(\tilde{x}_i) - \frac{1}{N}\sum_{j\in I[1,N]} \varphi_j(\tilde{x}_j)\right]\Bigg)\mathbf{1}_{\{\sigma(t)=m\}}\Bigg]\mathrm{d}t. \qquad \text{(A1)}
\end{aligned}$$

Based on $\lambda_{mm} + \sum_{q\in I[1,s],q\neq m} \lambda_{mq} = 0$, one can easily obtain

$$\sum_{m\in I[1,s]}\sum_{q\in I[1,s]} \lambda_{qm} V_1^q = 0. \qquad \text{(A2)}$$

By Young's inequality and $\varsigma^{\mathrm{T}}\mathrm{sgn}(\varsigma) = \|\varsigma\|_1$, one obtains

$$\begin{aligned}
&2\sum_{i\in I[1,N]}\sum_{j\in I[1,N]} a_{ij}^{\sigma(t)}\mu_{ij}\xi_i^{\mathrm{T}} PBF(\tilde{x}_i - \tilde{x}_j) \\
&\leqslant -\frac{1}{2}\sum_{i\in I[1,N]}\sum_{j\in I[1,N]} a_{ij}^{\sigma(t)}\mu_{ij}(\tilde{x}_i - \tilde{x}_j)^{\mathrm{T}} PBB^{\mathrm{T}}P(\tilde{x}_i - \tilde{x}_j) + 2\sum_{i\in I[1,N]}\sum_{j\in I[1,N]} a_{ij}^{\sigma(t)}\mu_{ij} e_i^{\mathrm{T}} PBB^{\mathrm{T}} Pe_i, \qquad \text{(A3a)}
\end{aligned}$$

and

$$\begin{aligned}
&2\sum_{i\in I[1,N]}\sum_{j\in I[1,N]} a_{ij}^{\sigma(t)}\iota_{ij}\xi_i^{\mathrm{T}} PB\mathrm{sgn}[F(\tilde{x}_i - \tilde{x}_j)] \\
&= -\sum_{i\in I[1,N]}\sum_{j\in I[1,N]} a_{ij}^{\sigma(t)}\iota_{ij}(\xi_i - \xi_j)^{\mathrm{T}} F^{\mathrm{T}}\mathrm{sgn}[F(\tilde{x}_i - \tilde{x}_j)] \\
&\leqslant -\sum_{i\in I[1,N]}\sum_{j\in I[1,N]} a_{ij}^{\sigma(t)}\iota_{ij}\|F(\tilde{x}_i - \tilde{x}_j)\|_1 + 2N\sum_{i\in I[1,N]}\sum_{j\in I[1,N]} a_{ij}^{\sigma(t)}\iota_{ij}\|Fe_i\|_1. \qquad \text{(A3b)}
\end{aligned}$$

Set $\varphi(\tilde{x}) = (\varphi_1^{\mathrm{T}}(\tilde{x}_1), \ldots, \varphi_N^{\mathrm{T}}(\tilde{x}_N))^{\mathrm{T}}$. Based on Assumption 4, one has

$$2\sum_{i\in I[1,N]} \xi_i^{\mathrm{T}} PB\left[\varphi_i(\tilde{x}_i) - \frac{1}{N}\sum_{j\in I[1,N]} \varphi_j(\tilde{x}_j)\right]$$

* Corresponding author (email: zjneu11@163.com)

$$= 2\xi^{\rm T}(M \otimes PB)\varphi(\tilde{x}) \leqslant 2\|(M \otimes B^{\rm T}P)\xi\|_1 \|(M \otimes I_n)\varphi(\tilde{x})\|_\infty \\ \leqslant \varpi(N-1) \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} \|F(\tilde{x}_i - \tilde{x}_j)\|_1 + 2\varpi(N-1) \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} \|Fe_i\|_1, \tag{A4}$$

where ϖ represents the supremum of $\|(M \otimes I_n)\varphi(\tilde{x})\|_\infty$.

Substituting (A2), (A3), and (A4) into (A1) and combining with $V_1 = \sum_{m\in I[1,s]} V_1^m$ yield

$$\begin{aligned} \dot{V}_1 \leqslant \mathbb{E}\Bigg[& \sum_{i\in I[1,N]} \xi_i^{\rm T}(PA + A^{\rm T}P)\xi_i - \frac{1}{2} \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} \mu_{ij} a_{ij}^{\sigma(t)} (\tilde{x}_i - \tilde{x}_j)^{\rm T} G(\tilde{x}_i - \tilde{x}_j) \\ & + 2 \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} \mu_{ij} a_{ij}^{\sigma(t)} e_i^{\rm T} G e_i - \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} \iota_{ij} \|F(\tilde{x}_i - \tilde{x}_j)\|_1 \\ & + 2N \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} \iota_{ij} \|Fe_i\|_1 + \varpi(N-1) \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} \|F(\tilde{x}_i - \tilde{x}_j)\|_1 \\ & + 2\varpi(N-1) \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} \|Fe_i\|_1 \Bigg]. \end{aligned} \tag{A5}$$

Let $V_2^m = \mathbb{E}[(\sum_{i\in I[1,N]} \sum_{j\in I[1,N]} \frac{(\mu_{ij}-\mu)^2}{4v_{ij}} + \frac{(\iota_{ij}-\iota)^2}{2o_{ij}}) \mathbf{1}_{\{\sigma(t)=m\}}]$. In combination with (3b) and (3c), one has

$$\begin{aligned} {\rm d}V_2^m = & \mathbb{E}\Bigg[\sum_{i\in I[1,N]} \sum_{j\in I[1,N]} \left(\frac{(\mu_{ij}-\mu)^2}{4v_{ij}} + \frac{(\iota_{ij}-\iota)^2}{2o_{ij}} \right) {\rm d}\mathbf{1}_{\sigma(t)=m} \Bigg] \\ & + \mathbb{E}\Bigg[\Bigg(\sum_{i\in I[1,N]} \sum_{j\in I[1,N]} \frac{\mu_{ij}-\mu}{2} a_{ij}^{\sigma(t)} \|F(\tilde{x}_i - \tilde{x}_j)\|_2^2 \\ & + \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} (\iota_{ij} - \iota) a_{ij}^{\sigma(t)} \|F(\tilde{x}_i - \tilde{x}_j)\|_1 \Bigg) \mathbf{1}_{\{\sigma(t)=m\}} \Bigg] {\rm d}t. \end{aligned} \tag{A6}$$

By $x_i - x_j = \xi_i - \xi_j + e_i - e_j$ and Young's inequality, one obtains

$$\begin{aligned} & -\frac{\mu}{4} \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} (\tilde{x}_i - \tilde{x}_j)^{\rm T} G(\tilde{x}_i - \tilde{x}_j) \\ & \leqslant -\frac{\mu}{8} \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} (\xi_i - \xi_j)^{\rm T} G(\xi_i - \xi_j) + \mu \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} e_i^{\rm T} G e_i. \end{aligned} \tag{A7}$$

Substituting (A7) into (A6) and combining with $V_2 = \sum_{m\in I[1,s]} V_2^m$ yield

$$\begin{aligned} \dot{V}_2 \leqslant \mathbb{E}\Bigg[& \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} \frac{2\mu_{ij}-\mu}{4} a_{ij}^{\sigma(t)} \|F(\tilde{x}_i - \tilde{x}_j)\|_2^2 + \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} (\iota_{ij} - \iota) a_{ij}^{\sigma(t)} \|F(\tilde{x}_i - \tilde{x}_j)\|_1 \\ & - \frac{\mu}{8} \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} (\xi_i - \xi_j)^{\rm T} G(\xi_i - \xi_j) + \mu \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} e_i^{\rm T} G e_i \Bigg]. \end{aligned} \tag{A8}$$

Similar to (A5) and (A8), and combining with (7b) and (7d), one has

$$\begin{aligned} \dot{V}_3 = \mathbb{E}\Bigg[& \frac{\mu\gamma_2}{4} \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} (\tilde{x}_i - \tilde{x}_j)^{\rm T} G(\tilde{x}_i - \tilde{x}_j) \\ & - \mu\gamma_2 \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} (1 + \delta\mu_{ij}) a_{ij}^{\sigma(t)} e_i^{\rm T} G e_i - \mu\gamma_3 \sum_{i\in I[1,N]} \pi_i \\ & + \iota_1 \sum_{i\in I[1,N]} \left(\gamma_2^1 \left\{ \delta_2 \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} \|F(\tilde{x}_i - \tilde{x}_j)\|_1 - \sum_{j\in I[1,N]} (1 + \delta^1 \iota_{ij}) a_{ij}^{\sigma(t)} \|Fe_i\|_1 \right\} - \gamma_3^1 \pi_i^1 \right) \Bigg]. \end{aligned} \tag{A9}$$

Then, combining $\dot{V}_1$ in (A5), $\dot{V}_2$ in (A8), $\dot{V}_3$ in (A9) and triggering mechanism in (6), we obtain

$$\begin{aligned} \dot{V} \leqslant \mathbb{E}\Bigg[& \sum_{i\in I[1,N]} \xi_i^{\rm T}(PA + A^{\rm T}P)\xi_i - \frac{\mu}{4} \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} \xi_i^{\rm T} G(\xi_i - \xi_j) \\ & + \mu\big[(1-\gamma_2)\gamma_1 - \gamma_3\big] \sum_{i\in I[1,N]} \pi_i + \big[2\varpi N\delta_2 + \varpi(N-1) - \iota\big] \sum_{i\in I[1,N]} \sum_{j\in I[1,N]} a_{ij}^{\sigma(t)} \|F(\tilde{x}_i - \tilde{x}_j)\|_1 \\ & + \big[(2\varpi N - \iota_1\gamma_2^1)\gamma_1^1 - \iota_1\gamma_3\big] \sum_{i\in I[1,N]} \pi_i^1 \Bigg], \end{aligned} \tag{A10}$$

where $\iota > 2\varpi N\delta_2 + \varpi(N-1)$ and $(2\varpi N - \iota_1\gamma_2^1)\gamma_1^1 - \iota_1\gamma_3 < 0$ can be ensured by selecting ι_1.

When $t \in [t_k^i, t_{k+1}^i)$, it follows from triggering mechanism in (15) that $\dot{\pi}_i \geqslant -(\gamma_1\gamma_2 + \gamma_3)\pi_i$, which implies that $\pi_i \geqslant \pi_i(t_k^i){\rm e}^{-(\gamma_1\gamma_2+\gamma_3)(t-t_k^i)} \geqslant \cdots \geqslant \pi_i(0){\rm e}^{-(\gamma_1\gamma_2+\gamma_3)t} > 0$. Similarly, $\pi_i^1 > \pi_i^1(0){\rm e}^{-(\gamma_1^1\gamma_2^1+\gamma_3^1)t} > 0$ can be ensured.

Since $\gamma_3 > (1-\gamma_2)\gamma_1$, Eq. (A10) can be written as

$$\dot{V} \leqslant \xi^{\mathrm{T}}\Big[I_N \otimes (AP + PA^{\mathrm{T}}) - \frac{\mu}{4}\mathbb{E}\big[\mathcal{L}^{\sigma(t)}\big] \otimes G\Big]\xi. \tag{A11}$$

Based on Assumptions 1 and 3 and Lemma 1, there must exist an orthogonal matrix $\mathcal{O}$ such that $\mathcal{O}^{\mathrm{T}}\mathbb{E}[\mathcal{L}^{\sigma(t)}]\mathcal{O} = \mathrm{diag}\{0, \lambda_2, \ldots, \lambda_N\}$ ascending. Set $\bar{\xi} = [\bar{\xi}_1^{\mathrm{T}}, \ldots, \bar{\xi}_N^{\mathrm{T}}]^{\mathrm{T}} = (\mathcal{O}^{\mathrm{T}} \otimes I_n)\xi$, from which we can obtain $\bar{\xi}_1 = \mathbf{0}$. Then, Eq. (A11) can be written as

$$\dot{V} \leqslant \sum_{i\in I[2,N]} \bar{\xi}_i^{\mathrm{T}}\left(AP + PA^{\mathrm{T}} - \frac{\mu\lambda_i}{4}PBB^{\mathrm{T}}P\right)\bar{\xi}_i \leqslant -\sum_{i\in I[2,N]} \bar{\xi}_i^{\mathrm{T}}\bar{\xi}_i \tag{A12}$$

which implies that ξ_i is stable. Therefore, $x_i \to \frac{1}{N}\sum_{i\in I[1,N]} x_i \triangleq \bar{x}$ can be obtained as $t \to \infty$.

(2) Optimization proof. Based on Assumption 5 and extreme value theorem, we can obtain that $\sum_{i\in I[1,N]} f_i(\bar{x})$ has a lower bound. Therefore, one obtains that $\frac{\mathrm{d}\sum_{i\in I[1,N]} f_i(\bar{x})}{\mathrm{d}t} = \mathbf{0}$. Then, consider the following formula:

$$\begin{aligned}&\lim_{t\to\infty}\left\{\frac{\mathrm{d}\sum_{i\in I[1,N]} f_i(\bar{x})}{\mathrm{d}t} - \frac{1}{N}\left(\frac{\partial\sum_{i\in I[1,N]} f_i(\bar{x})}{\partial\bar{x}}\right)^{\mathrm{T}}\mathcal{R}^{-1}\left(\frac{\partial\sum_{i\in I[1,N]} f_i(\bar{x})}{\partial\bar{x}}\right)\right\}\\ &= \lim_{t\to\infty}\left\{\left(\frac{\partial\sum_{i\in I[1,N]} f_i(\bar{x})}{\partial\bar{x}}\right)^{\mathrm{T}}\left[\frac{\mathrm{d}\bar{x}}{\mathrm{d}t} - \frac{1}{N}\mathcal{R}^{-1}\left(\frac{\partial\sum_{i\in I[1,N]} f_i(\bar{x})}{\partial\bar{x}}\right)\right]\right\}.\end{aligned} \tag{A13}$$

Since the following formula is satisfied:

$$\begin{aligned}&\lim_{t\to\infty}\left\{\frac{\mathrm{d}\bar{x}}{\mathrm{d}t} - \frac{1}{N}\mathcal{R}^{-1}\left(\frac{\partial\sum_{i\in I[1,N]} f_i(\bar{x})}{\partial\bar{x}}\right)\right\}\\ &= \lim_{t\to\infty}\left\{\frac{1}{N}\sum_{i\in I[1,N]}[Ax_i + B\varphi(x_i)] - \frac{1}{N}\mathcal{R}^{-1}\sum_{i\in I[1,N]}\mathcal{R}A[\bar{x} + B\varphi(\bar{x})]\right\} = \mathbf{0}.\end{aligned} \tag{A14}$$

Combining (A13) and (A14), and $\lim_{t\to\infty}\frac{\mathrm{d}\sum_{i\in I[1,N]} f_i(\bar{x})}{\mathrm{d}t} = \mathbf{0}$ one obtains

$$\lim_{t\to\infty}\left(\frac{\partial\sum_{i\in I[1,N]} f_i(\bar{x})}{\partial\bar{x}}\right)^{\mathrm{T}}\mathcal{R}^{-1}\left(\frac{\partial\sum_{i\in I[1,N]} f_i(\bar{x})}{\partial\bar{x}}\right) = \mathbf{0}.$$

Since matrix $\mathcal{R}$ is nonsingular, one can further obtain $\lim_{t\to\infty}\frac{\partial\sum_{i\in I[1,N]} f_i(\bar{x})}{\partial\bar{x}} = \mathbf{0}$, which implies $\bar{x}$ is the optimal state by Lemma 3.

(3) Feasibility proof. Here, the proof by contradiction method is adopted. Suppose agent i exhibits Zeno behavior, that is, $\lim_{t\to\infty} T_k^i = 0$ with $T_k^i \triangleq t_{k+1}^i - t_k^i$, $\lim_{k\to\infty} t_k^i = t_\infty^i < \infty$. At triggering instant t_{k+1}^i, at least one of the following inequations holds:

$$\sum_{j\in I[1,N]}(1+\delta\mu_{ij})a_{ij}^{\sigma(t)}\|Fe_i(t_{k+1}^i)\|_2^2 \geqslant \gamma_1\pi_1(t_{k+1}^i) \geqslant \gamma_1\pi_i(0)e^{-(\gamma_1\gamma_2+\gamma_3)t_{k+1}^i}$$

and

$$\sum_{j\in I[1,N]}(1+\delta'\iota_{ij})a_{ij}^{\sigma(t)}\|Fe_i\|_1 \geqslant \gamma_1^1\pi_1^1(t_{k+1}^i) \geqslant \gamma_1^1\pi_i^1(0)e^{-(\gamma_1^1\gamma_2^1+\gamma_3^1)t_{k+1}^i}.$$

Therefore, $0 = \lim_{t\to\infty}\sum_{j\in I[1,N]}(1+\delta\mu_{ij})a_{ij}^{\sigma(t)}\|Fe_i(t_{k+1}^i)\|_2^2 \geqslant \gamma_1\pi_i(0)\mathrm{e}^{-(\gamma_1\gamma_2+\gamma_3)t_{k+1}^i} > 0$ or $0 = \lim_{t\to\infty}\sum_{j\in I[1,N]}(1+\delta'\iota_{ij})a_{ij}^{\sigma(t)}\|Fe_i(t_{k+1}^i)\|_1 \geqslant \gamma_1^1\pi_i^1(0)\mathrm{e}^{-(\gamma_1^1\gamma_2^1+\gamma_3^1)t_{k+1}^i} > 0$, which is obviously contradictory. Thus, no Zeno behavior is exhibited under the proposed triggering mechanism in (6). The proof is completed.

Appendix B Proof of Theorem 2

Stability proof. Construct the following Lyapunov function $V = V_1 + V_2 + V_3$, where $V_1 = \mathbb{E}[\sum_{i\in I[1,N]}\xi_i^{\mathrm{T}}P\xi_i]$, $V_2 = \mathbb{E}[\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}(\frac{(d_{ij}-d)^2}{4m_{ij}} + \frac{(c_{ij}-c)^2}{2\tau_{ij}})]$ with $d > \{\frac{4}{\lambda_2}, \frac{2}{\sigma}\}$ and c being positive constant to be specified, and $V_3 = \mathbb{E}[d\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}\pi_{ij} + c_1\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}\pi_{ij}^1]$ with c_1 being positive constant to be specified.

Similar to (A5), calculate the derivative of V_1 along (16) as

$$\begin{aligned}\dot{V}_1 \leqslant \mathbb{E}\Bigg[&\sum_{i\in I[1,N]}\xi_i^{\mathrm{T}}(PA + A^TP)\xi_i - \frac{1}{2}\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}d_{ij}a_{ij}^{\sigma(t)}(\tilde{x}_{ij}-\tilde{x}_{ji})^{\mathrm{T}}G(\tilde{x}_{ij}-\tilde{x}_{ji})\\ &+ 2\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}d_{ij}a_{ij}^{\sigma(t)}e_{ij}^{\mathrm{T}}Ge_{ij} - \sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}c_{ij}\|F(\tilde{x}_{ij}-\tilde{x}_{ji})\|_1\\ &+ 2N\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}c_{ij}\|Fe_{ij}\|_1 + \varpi(N-1)\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}\|F(\tilde{x}_{ij}-\tilde{x}_{ji})\|_1\\ &+ 2\varpi(N-1)\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}\|Fe_{ij}\|_1\Bigg].\end{aligned} \tag{B1}$$

Similar to (A8), one can obtain

$$\begin{aligned}\dot{V}_2 \leqslant \mathbb{E}\Bigg[&\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}\frac{d_{ij}}{2}a_{ij}^{\sigma(t)}(\tilde{x}_{ij}-\tilde{x}_{ji})^{\mathrm{T}}G(\tilde{x}_{ij}-\tilde{x}_{ji})-\frac{d}{4}\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}(\tilde{x}_{ij}-\tilde{x}_{ji})^{\mathrm{T}}G(\tilde{x}_{ij}-\tilde{x}_{ji})\\ &-\frac{d}{8}\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}(\xi_i-\xi_j)^{\mathrm{T}}G(\xi_i-\xi_j)+d\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}e_{ij}^{\mathrm{T}}Ge_{ij}\\ &+\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}(c_{ij}-c)a_{ij}^{\sigma(t)}\|F(\tilde{x}_{ij}-\tilde{x}_{ji})\|_1\Bigg].\end{aligned} \tag{B2}$$

Similar to (A9), calculate the derivative of V_3 along (14b) and (14d)

$$\begin{aligned}\dot{V}_3 = \mathbb{E}\Bigg[&\frac{\mathrm{d}\bar{\gamma}_2}{4}\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}\|F(\tilde{x}_{ij}-\tilde{x}_{ji})\|_2^2-\mathrm{d}\bar{\gamma}_2\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}(1+\sigma d_{ij})a_{ij}^{\sigma(t)}\|Fe_{ij}\|_2^2\\ &-\mathrm{d}\bar{\gamma}_3\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}\pi_{ij}+c_1\bar{\gamma}_2^1\sigma_2\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}\|F(\tilde{x}_{ij}-\tilde{x}_{ji})\|_1\\ &-c_1\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}(1+\sigma^1 c_{ij})\|Fe_{ij}\|_1-c_1\bar{\gamma}_3^1\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}\pi_{ij}^1\Bigg].\end{aligned} \tag{B3}$$

Combining (B1)–(B3) and triggering mechanism in (13) obtain

$$\begin{aligned}\dot{V} \leqslant \mathbb{E}\Bigg[&\sum_{i\in I[1,N]}\xi_i^{\mathrm{T}}(PA+A^TP)\xi_i-\frac{d}{4}\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}\xi_i^{\mathrm{T}}G(\xi_i-\xi_j)+d\big[(1-\bar{\gamma}_2)\bar{\gamma}_1-\bar{\gamma}_3\big]\\ &\otimes\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}\pi_{ij}+\big[2\varpi N\sigma_2+\varpi(N-1)-c\big]\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}\|F(\tilde{x}_{ij}-\tilde{x}_{ji})\|_1\\ &+\big[(2\varpi N-c_1\bar{\gamma}_2^1)\bar{\gamma}_1^1-c_1\bar{\gamma}_3\big]\sum_{i\in I[1,N]}\sum_{j\in I[1,N]}a_{ij}^{\sigma(t)}\pi_{ij}^1\Bigg],\end{aligned} \tag{B4}$$

where $c>2\varpi N\sigma_2+\varpi(N-1)$ and $(2\varpi N-c_1\bar{\gamma}_2^1)\bar{\gamma}_1^1-c_1\bar{\gamma}_3^1<0$ can be ensured by selecting c_1.
Similar to (A11), Eq. (B4) can be written as

$$\dot{V} \leqslant \mathbb{E}\Big[\xi^{\mathrm{T}}\big[I_N\otimes(AP+PA^{\mathrm{T}})-\frac{\mu}{4}\mathcal{L}^{\sigma(t)}\otimes G\big]\xi\Big]. \tag{B5}$$

Then, similar to Theorem 1, one can obtain $\lim_{t\to\infty}\xi=\mathbf{0}$.
The proof of optimization and feasibility is the same as Theorem 1, which is omitted here.