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Multicase finite-time stabilization of stochastic memristor neural network with adaptive PI control

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Abstract In this paper, a unified control framework is proposed to investigate the multicase finite-time stabilization of stochastic delayed memristor neural networks (SDMNNs). With this framework, stochastic finite-time stabilization (SFTS), stochastic fixed-time stabilization (SFXTS), and stochastic prescribed-time stabilization (SPTS) of SDMNNs are achieved. Subsequently, unlike existing results, a bridge between the proportional-integral (PI) control protocol and the SDMNN is established, allowing the system to perform SFTS/SFXTS/SPTS without a separately designed controller. By appropriately adjusting the control factors of the unified framework, appropriate time of settlement estimates is derived. Furthermore, the controller is improved to the appropriate adaptive controller using the PI control protocol. The SFTS/SFXTS/SPTS of the system are obtained, and the corresponding upper bounds for the estimation of the settling time functions are acquired. Finally, the feasibility of the obtained theoretical results is demonstrated by two examples.

Keywords stochastic delayed memristor neural networks, multicase finite-time stabilization, PI control, adaptive PI control

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1 Introduction

The memristor is a fundamental component of the implemented circuit that describes the interrelationship between the circuit and the magnetic flux. The existence of memristors was predicted by Chua [1] in 1971, following the principle of completeness of combinations of circuit components. The existence of memristors was first verified in 2008 by Hewlett-Packard's labs when they appeared in nature, which prompted widespread interest in the scientific field [2]. With the advancement of research, memristors have the merits of nanoscale size, non-volatility, synapse-like properties, and low energy consumption. Combining memristors with neural networks (NNs) has the potential for a profound impact in areas such as image processing [3,4], pattern recognition [5,6], and associative memory [7,8]. As indicated by Guo et al. [9], n-neuron cell memristor neural networks (MNNs) with time-varying delays dramatically expand their balance number from 2^n to 2^{2n^2+n}, which significantly increases memory storage capacity.

Simulating human synapses with memristors makes it possible to transmit information in a neuronal system with stochastic probability, which is susceptible to external disturbances such as noise, which deteriorate the system's performance to the extent that it may render the system more unstable [10–14]. Accordingly, it is significant and challenging to study the dynamical behavior of stochastic memristor neural networks (SMNNs) in synchronization [15–17], dissipative [18–20], and stability [21–23]. Ren et al. [15] considered the problem of fixed-time synchronization of SMNNs using two different controllers. Shen et al. [18] addressed the problem of dissipative synchronization control of Markovian jump MNNs by factoring in time-varying delays and fragility when implementing gain-scheduling controllers. Chen et al. [21] investigated the analysis and control of the global generalized exponential stability and global finite-time stability of adaptive switching systems by establishing Lyapunov-based logical switching rules.

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In practice-related situations, regarding the dynamical behavior of SMNNs, one should focus on their occurrence in finite-time. However, finite-time control theory still has certain weaknesses; see [24, 25]. Existing finite-time control theories obtain settling time (ST) evaluations, which are always related to the initial value, when the initial value is enormous or insignificant, the finite-time estimated by us loses its practical value. To address this problem, Polyakov proposed a class of control algorithms that do not depend on the original conditions to render an uncertain system globally finite-time stable, referred to as fixed-time control [26]. Recently, researchers advanced prescribed-time control; see [27, 28]. Prescribed-time control is not associated with any control parameters; its ST is provided randomly, unlike fixed-time control, which is constrained by the control parameters.

Notably, Refs. [29, 30] investigated some similarities between the two types of controllers usually designed by studies addressed finite-time and fixed-time control. To better control the cost economy, some researchers have introduced a harmonized control framework to achieve finite- and fixed-time control implemented by a system with adjusted control parameters [31, 32]. Moreover, some studies gradually imported the prescribed-time control to discuss the multicase finite-time control within a unified framework; see [33, 34].

However, most of the currently available control protocols use only 'current' feedback. For improving the stability or consistency performance of NNs [35], utilizing 'current' feedback and integrating 'past' information and/or 'future' trends for stability and consistency protocols emerge as an influential theme of research. The classical control theory literature [36] describes proportional-integral-derivative (PID) control and its evolution as broadly available. In PID controllers, the proportional, integral, and differential terms represent the current feedback, past records, and prospective estimations of the system state, respectively. However, PID control and its use for stabilization or consistency changes in control systems have been neglected in the literature. Gu et al. [37] addressed the synchronization problem of nonlinear dynamic complex networks with stochastic coupling for adaptive proportional-integral (PI) control. Gu et al. [38] aimed to establish a connection between PID control protocols and the synchronization of complex dynamical networks with directed topologies. Zhao et al. [39] attempted to address output synchronization and H_∞ output synchronization problems for multiple output coupled complex networks with PD and PI controllers.

Despite the recent work on the finite-time stabilization of stochastic delayed memristor neural networks (SDMNNs), PID control and unified control frameworks implementing stochastic finite-time stabilization (SFTS), stochastic fixed-time stabilization (SFXTS), and stochastic prescribed-time stabilization (SPTS) control have not been considered. Consequently, designing PID control protocols to make SDMNNs stable by combining multicase finite-time control theory in a unified framework is challenging. This study aims to combine PI control protocols and multicase finite-time control theory in a unified framework by utilizing the tools of control theory [40], algebraic graph theory [41], and matrix theory [42] to enable the stabilization of SDMNNs. The main contributions of the study are as follows.

(1) First, the corresponding linear matrix inequalities (LMIs) are derived by designing controllers that combine PI control and multicase finite-time control in a unified framework to ensure that SDMNNs achieve SFTS, SFXTS, and SPTS.

(2) Subsequently, to reduce control costs, the controller is added as adaptive control under the PI control protocol, and the appropriate updated control rules are designed to obtain the suitable conditions expressed in the form of LMIs, which lead SDMNNs to achieve SFTS, SFXTS, and SPTS.

(3) Finally, the proposed theoretical results are tested with two examples, which confirm that the control gain of adaptive PI control is smaller than that without adaptive PI control, resulting in control cost savings.

The remainder of this paper is structured as follows: Section 2 presents the model and network preparation. Section 3 introduces the implementation of SFTS, SFXTS, and SPTS for SDMNNs using PI control and the adaptive PI control. Two simulation examples are presented in Section 4. Finally, the conclusion is presented in Section 5.

2 Preliminaries and network model

In this section, we propose a mathematical model for SDMNNs and introduce some assumptions, definitions, and lemmas.

According to Kirchhoff's current law, we considered a type of DMNNs:

$$\begin{aligned}\boldsymbol{C}_i\dot{\chi}_i(t) =& -\left[\sum_{j=1}^{n}\boldsymbol{W}_{1ij}+\boldsymbol{W}_{2ij}+\frac{1}{\boldsymbol{R}_i}\right]\chi_i(t)+\sum_{j=1}^{n}\mathrm{sign}_{ij}\,\boldsymbol{W}_{1ij}g_j\left(\chi_j(t)\right)\\ &+\sum_{j=1}^{n}\mathrm{sign}_{ij}\,\boldsymbol{W}_{2ij}g_j\left(\chi_j(t-\tilde{h}(t))\right),\end{aligned}\tag{1}$$

where $i,j=1,2,\ldots,n$, $\chi_i(t)$ is the capacitor voltage of $\boldsymbol{C}_i$, $\boldsymbol{R}_i$ is the resistance of the resistor,

$$\mathrm{sign}_{ij}=\begin{cases}-1, & i=j,\\ 1, & i\neq j,\end{cases}$$

$g_j(\cdot)$ is the activation function, and $\tilde{h}(t)$ is the time-varying delay. $\boldsymbol{W}_{1ij},\boldsymbol{W}_{2ij}$ are the memristance between $g_j(\chi_j(t))$ and $\chi_j(t)$, $g_j(\chi_j(t-\tilde{h}(t)))$ and $\chi_j(t-\tilde{h}(t))$, respectively.

Define $\chi(t)=(\chi_1(t),\chi_2(t),\ldots,\chi_n(t))^{\mathrm{T}}$, $g(\chi(t))=(g_1(\chi_1(t)),g_2(\chi_2(t)),\ldots,g_n(\chi_n(t)))^{\mathrm{T}}$, and $g(\chi(t-\tilde{h}(t)))=(g_1(\chi_1(t-\tilde{h}(t))),g_2(\chi_2(t-\tilde{h}(t))),\ldots,g_n(\chi_n(t-\tilde{h}(t))))^{\mathrm{T}}$. Stochastic noise is unavoidable in practical neural networks. The entire system (1) can then be reformulated by taking the shape of the matrices as follows:

$$\begin{aligned}\mathrm{d}\chi(t)=&\left[-A(\chi(t))\chi(t)+S(\chi(t))g(\chi(t))+D(\chi(t-\tilde{h}(t)))g(\chi(t-\tilde{h}(t)))\right]\mathrm{d}t\\ &+\delta(t,\chi(t))\mathrm{d}\omega(t),\end{aligned}\tag{2}$$

where $A(\chi(t))=\mathrm{diag}\{a_1(\chi_1(t)),a_2(\chi_2(t)),\ldots,a_n(\chi_n(t))\}$ with $a_i(\chi_i(t))>0$, $S(\chi(t))=s_{ij}(\chi_j(t))_{n\times n}$, $D(\chi(t))=d_{ij}(\chi_j(t))_{n\times n}$, $\tilde{h}(t)$ is the time-varying delay and satisfies $0\leqslant\tilde{h}(t)\leqslant\tilde{h},\dot{\tilde{h}}(t)\leqslant\tilde{h}^{\star}$, $\tilde{h},\tilde{h}^{\star}$ are positive constants. The parameters related to the memristor state should satisfy

$$a_i\left(\chi_i(t)\right)=\frac{1}{\boldsymbol{C}_i}\left[\sum_{j=1}^{n}(\boldsymbol{W}_{1ij}(t)+\boldsymbol{W}_{2ij}(t))+\frac{1}{\boldsymbol{R}_i}\right]=\begin{cases}\hat{a}_i, & |\chi_i(t)|\leqslant T_i,\\ \check{a}_i, & |\chi_i(t)|>T_i,\end{cases}$$

$$s_{ij}\left(\chi_j(t)\right)=\frac{\mathrm{sign}_{ij}}{\boldsymbol{C}_i}\boldsymbol{W}_{1ij}=\begin{cases}\hat{s}_{ij}, & |\chi_i(t)|\leqslant T_i,\\ \check{s}_{ij}, & |\chi_i(t)|>T_i,\end{cases}$$

$$d_{ij}\left(\chi_j(t-\tilde{h}(t))\right)=\frac{\mathrm{sign}_{ij}}{\boldsymbol{C}_i}\boldsymbol{W}_{2ij}=\begin{cases}\hat{d}_{ij}, & \left|\chi_i(t-\tilde{h}(t))\right|\leqslant T_i,\\ \check{d}_{ij}, & \left|\chi_i(t-\tilde{h}(t))\right|>T_i,\end{cases}$$

where $T_i>0$ are switching jumps, and $\hat{a}_i,\check{a}_i$, $\hat{s}_{ij},\check{s}_{ij},\hat{d}_{ij},\check{d}_{ij}$ are known constants. $\delta:\mathbb{R}^{+}\times\mathbb{R}^{n}\rightarrow\mathbb{R}^{n}$ denote the noise function matrix. $w(t)=(w_1(t),w_2(t),\ldots,w_n(t))^{\mathrm{T}}$ is a Brownian motion defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},\boldsymbol{P})$ satisfying $E\{\mathrm{d}w(t)\}=0,E\{\mathrm{d}w(t)^2\}=\mathrm{d}t$. The initial values of the system (2) are given by $\chi_i(t)=\phi_i(t)\in\boldsymbol{C}([-\tilde{h},0],\mathbb{R})$.

Assumption 1. The neuron activation function $g_j(\cdot)$ is bounded, and there exist positive number $l_j, j=1,2,\ldots,n$, such that

$$|g_j(\mathfrak{a}_1)-g_j(\mathfrak{a}_2)|\leqslant l_j|\mathfrak{a}_1-\mathfrak{a}_2|$$

holds for all $\mathfrak{a}_1,\mathfrak{a}_2\in\mathbb{R}^n,\mathfrak{a}_1\neq\mathfrak{a}_2$.

Assumption 2. The noise disturbance function $\delta(t,\chi(t))$ with $\delta(0,0)=0$ satisfies the uniform Lipschitz continuity conditions,

$$|\delta^{\mathrm{T}}(t,\chi(t))\delta(t,\chi(t))|\leqslant\beta\chi^{\mathrm{T}}(t)\chi(t),$$

where β is a positive constant.

Assumption 3. By allowing $0<\varrho<1$ and $\Phi>0$, a continuous function $g:[0,\infty)\rightarrow[0,\infty)$ with $g(0)>0$ exists for an arbitrary $0\leqslant\upsilon\leqslant\alpha$, such that

$$g(\alpha)-g(\upsilon)\leqslant-\Phi\int_{\upsilon}^{\alpha}(g(s))^{\varrho}\mathrm{d}s.$$

Let $\overline{a}_i = \max\{\hat{a}_i, \check{a}_i\}$, $\underline{a}_i = \min\{\hat{a}_i, \check{a}_i\}$, $\overline{s}_{ij} = \max\{\hat{s}_{ij}, \check{s}_{ij}\}$, $\underline{s}_{ij} = \min\{\hat{s}_{ij}, \check{s}_{ij}\}$, $\overline{d}_{ij} = \max\{\hat{d}_{ij}, \check{d}_{ij}\}$, $\underline{d}_{ij} = \min\{\hat{d}_{ij}, \check{d}_{ij}\}$, $\grave{a}_i = \frac{\underline{a}_i+\overline{a}_i}{2}$, $\acute{a}_i = \frac{\overline{a}_i-\underline{a}_i}{2}$, $\grave{s}_{ij} = \frac{\underline{s}_{ij}+\overline{s}_{ij}}{2}$, $\acute{s}_{ij} = \frac{\overline{s}_{ij}-\underline{s}_{ij}}{2}$, $\grave{d}_{ij} = \frac{\underline{d}_{ij}+\overline{d}_{ij}}{2}$, $\acute{d}_{ij} = \frac{\overline{d}_{ij}-\underline{d}_{ij}}{2}$.

From differential inclusion theories and some interval matrix transformations, the SDMNNs (2) can be equivalently rewritten as

$$\begin{aligned} \mathrm{d}\chi(t) =& \Big[-\left(\grave{A} + G_A\Theta_1(t)F_A\right)\chi(t) + \left(\grave{S} + G_S\Theta_2(t)F_S\right) g(\chi(t)) \\ &+ \left(\grave{D} + G_D\Theta_3(t)F_D\right) g(\chi(t-\tilde{h}(t)))\Big]\,\mathrm{d}t + \delta(t,\chi(t))\mathrm{d}\omega(t), \end{aligned} \tag{3}$$

where $\grave{A} = \mathrm{diag}\{\grave{a}_1, \grave{a}_2, \ldots, \grave{a}_n\}$, $S = (\grave{s}_{ij})_{n\times n}$, $D = (\grave{d}_{ij})_{n\times n}$, $\gamma_i \in \mathbb{R}^n$ denotes the column vector with the ith elements to be 1 and others to be 0; $G_A = \mathrm{diag}(\sqrt{\acute{a}_1}\gamma_1, \ldots, \sqrt{\acute{a}_n}\gamma_n)$, $F_A = \mathrm{diag}(\sqrt{\acute{a}_1}\gamma_1, \ldots, \sqrt{\acute{a}_n}\gamma_n)^{\mathrm{T}}$, $G_S = (\sqrt{\acute{s}_{11}}\gamma_1, \ldots, \sqrt{\acute{s}_{1n}}\gamma_1, \ldots, \sqrt{\acute{s}_{n1}}\gamma_n, \ldots, \sqrt{\acute{s}_{nn}}\gamma_n)$, $F_S = (\sqrt{\acute{s}_{11}}\gamma_1, \ldots, \sqrt{\acute{s}_{1n}}\gamma_1, \ldots, \sqrt{\acute{s}_{n1}}\gamma_n, \ldots, \sqrt{\acute{s}_{nn}}\gamma_n)^{\mathrm{T}}$, $G_D = (\sqrt{\acute{d}_{11}}\gamma_1, \ldots, \sqrt{\acute{d}_{1n}}\gamma_1, \ldots, \sqrt{\acute{d}_{n1}}\gamma_n, \ldots, \sqrt{\acute{d}_{nn}}\gamma_n)$, $F_D = (\sqrt{\acute{d}_{11}}\gamma_1, \ldots, \sqrt{\acute{d}_{1n}}\gamma_1, \ldots, \sqrt{\acute{d}_{n1}}\gamma_n, \ldots, \sqrt{\acute{d}_{nn}}\gamma_n)^{\mathrm{T}}$, $\Theta_\aleph(t) = \{\mathrm{diag}\{\theta^{\aleph}_{11}(t), \ldots, \theta^{\aleph}_{1n}(t), \ldots, \theta^{\aleph}_{n1}(t), \ldots, \theta^{\aleph}_{nn}(t)\} : |\theta_{ij}(t)| \leqslant 1, \aleph = 1, 2, 3\}$. Evidently, $\Theta^{\mathrm{T}}_\aleph(t)\Theta_\aleph(t) \leqslant I$.

Subsequently, the stochastic nonlinear system of system (3) can be expressed as

$$\mathrm{d}\chi(t) = g(t,\chi(t))\mathrm{d}t + \delta(t,\chi(t))\mathrm{d}\omega(t), \tag{4}$$

where $\omega(t)$ is the Brownian motion and it is clearly $E\{\mathrm{d}w(t)\} = 0$. $\mathscr{L}$ is an operator that is defined as

$$\begin{aligned} \mathscr{L}V(t,\chi(t)) =& \frac{\partial V(t,\chi(t))}{\partial t} + V_\chi(t,\chi(t))g(t,\chi(t)) \\ &+ \frac{1}{2}\,\mathrm{trace}\left[\delta^{\mathrm{T}}(t,\chi(t))\left(\frac{\partial^2 V(t,\chi(t))}{\partial\chi_i\partial\chi_j}\right)\delta(t,\chi(t))\right], \end{aligned} \tag{5}$$

where $V_\chi(t,\chi(t)) = (\frac{\partial V(t,\chi(t))}{\partial\chi_1}, \frac{\partial V(t,\chi(t))}{\partial\chi_2}, \ldots, \frac{\partial V(t,\chi(t))}{\partial\chi_n})^{\mathrm{T}}$.

Definition 1. The system achieves multicase stochastic finite-time stability (MCSFTS) if both of the following criteria are established:

(i) The initial value of the system (3) is probabilistically stochastic stable.

(ii) The initial value of the system (3) is stochastically finite-time (or fixed-time, or prescribed-time) convergent if ST is correlated with initial conditions and controller factors (or with controller factors only, or without respect to any initial conditions or controller factors other than ST) such that

$$\lim_{t\to T} E\left\|\chi(t)\right\|^2 = 0, \tag{6}$$

and $E\left\|\chi(t)\right\|^2 \equiv 0$, for $t > T$.

Lemma 1 ([43])**.** If $V(t,\chi(t)) : \mathbb{R}^n \to \mathbb{R}^+$ is a $\boldsymbol{C}$-regular regarding $\chi(t) : [0,+\infty) \to \mathbb{R}^n$ on arbitrary tight interval in $[0,+\infty)$ that is, it is also strictly continuous, such that

$$\zeta_1(\|\chi(t)\|) \leqslant V(\chi(t)) \leqslant \zeta_2(\|\chi(t)\|),$$

$$\mathscr{L}V(\chi(t)) \leqslant -\mathfrak{P}(V(\chi(t))),$$

where ζ_1 and ζ_2 are $\boldsymbol{K}_\infty$ class functions and $\mathfrak{P}$ is a positive real numbers with $\mathfrak{P}(\epsilon) > 0$, for all $\epsilon > 0$.

(i) If $E\{T_\dagger\} = E\{\int_0^{V(\chi(0))} \frac{1}{\mathfrak{P}(\epsilon)}\mathrm{d}\epsilon\}$ is finite, the system (3) stabilizes stochastically at finite time and the ST is $E\{T_\dagger\}$.

(ii) If $T_\ddagger = \int_0^{+\infty} \frac{1}{\mathfrak{P}(\epsilon)}\mathrm{d}\epsilon$ is finite, the system (3) stabilizes stochastically at fixed time and the ST is $T_\ddagger$.

Lemma 2 ([44])**.** If a Lyapunov function $V(t) : \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\}$ is a positive radially unbounded function that satisfies the following requirements:

$$\mathscr{L}V(t) \leqslant -pV^{\eta_1}(t) - qV^{\eta_2}(t),$$

where $p, q > 0$, and $0 \leqslant \eta_1 < 1, \eta_2 > 1$, hence, the initial value of the system (3) is probabilistically stable at a fixed time, and the ST is

$$T_{\ddagger} = \frac{1}{q}\left(\frac{q}{p}\right)^{\frac{1-\eta_1}{\eta_2-\eta_1}}\left(\frac{1}{1-\eta_1} + \frac{1}{\eta_2 - 1}\right).$$

Moreover, if

$$\mathscr{L}V(t) \leqslant -\frac{T_{\ddagger}}{T_{\S}}\left(pV^{\eta_1}(t) + qV^{\eta_2}(t)\right),$$

where $p, q > 0$, and $0 \leqslant \eta_1 < 1, \eta_2 > 1$, hence, the initial value of the system (3) is probabilistically stable at a prescribed time, and the ST is $T_{\S}$.

Lemma 3 ([45])**.** Given y_i, $i = 1, 2, \ldots, n$, $0 < \eta_1 \leqslant 1$ and $\eta_2 > 1$, the inequalities below are satisfied:

$$\sum_{i=1}^{n} y_i^{\eta_1} \geqslant \left(\sum_{i=1}^{n} y_i\right)^{\eta_1}, \quad \sum_{i=1}^{n} y_i^{\eta_2} \geqslant n^{1-\eta_2}\left(\sum_{i=1}^{n} y_i\right)^{\eta_2}.$$

Lemma 4 (Schur complement [46])**.** If a symmetric matrix $\mathcal{S} = \mathcal{S}^{\mathrm{T}} = \left[\begin{smallmatrix}\mathcal{S}_{11} & \mathcal{S}_{12} \\ * & \mathcal{S}_{22}\end{smallmatrix}\right]$ is available, where $\mathcal{S}_{11} \in \mathbb{R}^{r\times r}$, the below requirements are equivalent:

(1) $\mathcal{S} < 0$,

(2) $\mathcal{S}_{11} < 0$, $\mathcal{S}_{22} - \mathcal{S}_{12}^{\mathrm{T}}\mathcal{S}_{11}^{-1}\mathcal{S}_{12} < 0$, and

(3) $\mathcal{S}_{22} < 0$, $\mathcal{S}_{11} - \mathcal{S}_{12}\mathcal{S}_{22}^{-1}\mathcal{S}_{12}^{\mathrm{T}} < 0$.

Lemma 5 ([47])**.** For some matrices $Q = Q^{\mathrm{T}}, H$, and E that satisfy

$$Q + H\Theta(t)E + E^{\mathrm{T}}\Theta^{\mathrm{T}}(t)H^{\mathrm{T}} < 0,$$

if there exists $\xi > 0$ with $\Theta(t)^{\mathrm{T}}\Theta(t) \leqslant I$,

$$Q + \xi^{-1}HH^{\mathrm{T}} + \xi E^{\mathrm{T}}E < 0.$$

3 Main results

3.1 Multicase stochastic finite-time stability with PI control

In this subsection, a control PI protocol (7) is designed to enable SDMNNs to achieve MCSFTS. The control protocol is as follows:

$$\begin{aligned}\mathcal{U}(t) = &-\mathcal{K}_P\chi(t) - \mathcal{K}_{\mathrm{Fit}}\mathrm{sign}(\chi(t))|\chi(t)|^{\vartheta} - \mathcal{K}_{\mathrm{Fix}}\mathrm{sign}(\chi(t))|\chi(t)|^{\nu} \\ &- \mathcal{K}_I\left(\int_{t-\tilde{h}(t)}^{\mathrm{T}} \chi(s)^{\mathrm{T}}M\chi(s)\mathrm{d}s\right)^{\frac{1+\nu}{2}}\left(\frac{M^{-1}|\chi(t)|}{\|\chi(t)\|^2}\right),\end{aligned} \tag{7}$$

where $\mathcal{K}_P, \mathcal{K}_{\mathrm{Fit}}, \mathcal{K}_{\mathrm{Fix}}, \mathcal{K}_I$ are control strength matrices, M is a real matrix, the real number ϑ, ν satisfies $\vartheta, \nu \geqslant 0$, and the sign $(\cdot)$ is the signum function.

Remark 1. In the controller $\mathcal{U}(t)$, $\mathcal{K}_P$ is employed to ensure that the current system reaches stability in the Lyapunov sense, $\mathcal{K}_I$ eliminates the influence of the time-varying delay of the system, adjusting $\mathcal{K}_{\mathrm{Fix}}$ keeps the system (3) is SFTS, adjusting $\mathcal{K}_{\mathrm{Fit}}$ guarantees that the system is SFXTS, while adjusting $\mathcal{K}_{\mathrm{Fit}}$, $\mathcal{K}_{\mathrm{Fix}}$ to maintain the system is SPTS.

Remark 2. To further enhance the stability performance of SDMNNs, the state variables are dependent on 'current' feedback and correlated with 'past' information. Therefore, PI control strategies have attracted considerable attention and an impressive number of research results have been published [37–39]. However, compared with the existing studies, most of the authors consider the coupling structure in the network to implement the PI control strategy and have not considered constructing an integral inequality to implement the PI control strategy. Therefore, it is fascinating to investigate the MCSFTS with PI control protocols in this study.

Theorem 1. Given the conditions of Assumptions 1–3, if there exist some real matrices $Y_P, Y_{\rm Fit}$, a symmetric positive matrix Z, and some scalars $\xi_1 > 0, \xi_2 > 0, \xi_3 > 0, \xi_4 > 0, \xi_5 > 0$, the following LMI holds:

$$\begin{bmatrix} \Xi_1 & 0 & \Xi_2 \\ * & -\left(1-\tilde{h}^{\star}\right) Z^{\rm T} - 2Y_{\rm Fit} & \Xi_3 \\ * & * & \Xi_4 \end{bmatrix} < 0, \tag{8}$$

where

$$\begin{aligned} \Xi_1 =& -2Y_P - 2\grave{A}Z + \xi_1 G_A G_A^{\rm T} + \xi_2 \grave{S}\grave{S}^{\rm T} + \xi_3 G_S G_S^{\rm T} + \xi_4 \grave{D}\grave{D}^{\rm T} \\ &+ \xi_5 G_D G_D^{\rm T} + \beta Z^{\rm T} + Z^{\rm T} + 2Y_{\rm Fit}, \\ \Xi_2 =& (ZF_A^{\rm T}, ZL^{\rm T}, ZL^{\rm T}F_S^{\rm T}, 0, 0), \\ \Xi_3 =& (0, 0, 0, ZL^{\rm T}, ZL^{\rm T}F_D^{\rm T}), \\ \Xi_4 =& {\rm diag}(-\xi_1 I, -\xi_2 I, -\xi_3 I, -\xi_4 I, -\xi_5 I). \end{aligned}$$

Subsequently, the controller factors can be devised as $\mathcal{K}_P = Y_P Z^{-1}, \mathcal{K}_{\rm Fit} = Y_{\rm Fit} Z^{-1}, \mathcal{K}_{\rm Fix}^{\rm T}\mathcal{K}_{\rm Fix} > 0, \mathcal{K}_I^{\rm T}\mathcal{K}_I > 0$. Additionally, for the system (3) to achieve SFTS, SFXTS, or SPTS by the choice of the controller (7) control factors, ST is estimated as follows:

$$\begin{cases} {\rm SFTS}: E\{T_1\} \leqslant \frac{E\{V(\chi(0))\}^{\frac{1-\vartheta}{2}}}{\varpi(1-\vartheta)}, 0 \leqslant \vartheta < 1, \mathcal{K}_{\rm Fix} = {\rm diag}\underbrace{(0,\ldots,0)}_{n}, \mathcal{K}_I = {\rm diag}\underbrace{(0,\ldots,0)}_{n}, \\ {\rm SFTS}: E\{T_2\} \leqslant \frac{\ln(1+\frac{\varpi_1}{\varpi}E\{V(\chi(0))\}^{\frac{1-\vartheta}{2}})}{\varpi_1(1-\vartheta)}, 0 \leqslant \vartheta < 1, \nu = 1, \\ {\rm SFXTS}: T_3 \leqslant \frac{1}{(2n)^{\frac{1-\nu}{2}}\varpi_1}\left(\frac{(2n)^{\frac{1-\nu}{2}}\varpi_1}{\varpi}\right)^{\frac{1-\vartheta}{\nu-\vartheta}}\left(\frac{1}{1-\vartheta}+\frac{1}{\nu-1}\right), 0 \leqslant \vartheta < 1, \nu > 1, \\ {\rm SPTS}: T_4 \leqslant T_{\rm Pre}, 0 \leqslant \vartheta < 1, \nu > 1, \mathcal{K}_{\rm Fit} = \mathcal{K}_{\rm Fit}\frac{T_3}{T_{\rm Pre}}, \mathcal{K}_{\rm Fix} = \mathcal{K}_{\rm Fix}\frac{T_3}{T_{\rm Pre}}, \end{cases}$$

where $T_{\rm Pre} < T_3$ is available at an arbitrary given time.

Remark 3. One of the main innovations of this paper is that the complexity of the Lyapunov function is decreased by constructing the integral term corresponding to the controller gain $\mathcal{K}_{\rm Fit}$ with Assumption 3 in the process of proving Theorem 1. Therefore, the inequality (8) can be calculated for the control gain $\mathcal{K}_P$ and for the control gain $\mathcal{K}_{\rm Fit}$. In [31–34], the control gains of both $\mathcal{K}_{\rm Fit}$ and $\mathcal{K}_{\rm Fix}$ were tuned depending on the actual system to make the system achieve SFTS, SFXTS, and SPTS, respectively. Therefore, it follows from Theorem 1 that the control gains $\mathcal{K}_{\rm Fix}$ and $\mathcal{K}_I$ must be adjusted in a unified control framework such that the system achieves MCSFTS and reduces the cost of some controls.

Remark 4. Theorem 1 achieved the MCSFTS of SDMNN in a new unified framework. By suitably adjusting the controller parameters, the system (3) was implemented with SFTS (cf. $E\{T_1\}$ and $E\{T_2\}$), SFXTS (cf. T_3), and SPTS (cf. T_4). Evidently, $E\{T_1\}$ and $E\{T_2\}$ are linked to the initial conditions and the controller parameters, whereas T_3 relies solely on the controller parameters, and T_4 can be freely prescribed.

Proof. The candidate for constructing the below Lyapunov-Krasovskii functional is

$$V(t) = \chi^{\rm T}(t)M\chi(t) + \int_{t-\tilde{h}(t)}^{\rm T} \chi^{\rm T}(s)M\chi(s){\rm d}s. \tag{9}$$

Subsequently, by employing the Itô-formula, we obtain the stochastic derivative of $V(t)$ together with the trajectories of system (3) as follows:

$${\rm d}V(t) = \mathscr{L}V(t){\rm d}t + 2\chi^{\rm T}(t)M\delta(t,\chi(t)){\rm d}\omega(t), \tag{10}$$

where

$$\begin{aligned} \mathscr{L}V(t) =& 2\chi^{\rm T}(t)M\left[-\left(\grave{A} + G_A\Theta_1(t)F_A\right)\chi(t) + \left(\grave{S} + G_S\Theta_2(t)F_S\right)g(\chi(t)) \right. \\ &\left. + \left(\grave{D} + G_D\Theta_3(t)F_D\right)g(\chi(t-\tilde{h}(t))) + \mathcal{U}(t)\right]{\rm d}t + {\rm trace}\left[\delta^{\rm T}(t,\chi(t))M \right. \\ &\left. \times \delta(t,\chi(t))\right]{\rm d}t + \chi^{\rm T}(t)M\chi(t) - \left(1-\tilde{h}^{\star}\right)\chi^{\rm T}(t-\tilde{h}(t))M\chi(t-\tilde{h}(t)). \end{aligned} \tag{11}$$

From Assumption 1 and Lemma 5, we have

$$
\begin{aligned}
2\chi^{\mathrm{T}}(t)M\left(\left(\grave{S}+G_S\Theta_2(t)F_S\right)g(\chi(t))\right) \leqslant & \xi_2\chi^{\mathrm{T}}(t)M\grave{S}\grave{S}^{\mathrm{T}}M\chi(t)+\xi_2^{-1}\chi^{\mathrm{T}}(t)L^{\mathrm{T}}L\chi(t)\\
&+\xi_3\chi^{\mathrm{T}}(t)MG_SG_S^{\mathrm{T}}M\chi(t)+\xi_3^{-1}\chi^{\mathrm{T}}(t)L^{\mathrm{T}}F_S^{\mathrm{T}}F_SL\chi(t),
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
2\chi^{\mathrm{T}}(t)M\left(\left(\grave{D}+G_D\Theta_3(t)F_D\right)g(\chi(t-\tilde{h}(t)))\right) \leqslant & \xi_4\chi^{\mathrm{T}}(t)M\grave{D}\grave{D}^{\mathrm{T}}M\chi(t)\\
&+\xi_4^{-1}\chi^{\mathrm{T}}(t-\tilde{h}(t))L^{\mathrm{T}}L\chi(t-\tilde{h}(t))\\
&+\xi_5\chi^{\mathrm{T}}(t)MG_DG_D^{\mathrm{T}}M\chi(t)\\
&+\xi_5^{-1}\chi^{\mathrm{T}}(t-\tilde{h}(t))L^{\mathrm{T}}F_D^{\mathrm{T}}F_DL\chi(t-\tilde{h}(t)).
\end{aligned} \tag{13}
$$

From Assumption 2, we obtain

$$
\operatorname{trace}\left[\delta^{\mathrm{T}}\left(t,\chi(t)\right)M\delta\left(t,\chi(t)\right)\right] \leqslant \beta\chi^{\mathrm{T}}(t)M\chi(t). \tag{14}
$$

From Assumption 3, there exists a $\Phi>0$. We give

$$
2\mathcal{K}_{\mathrm{Fit}}\left(\chi^{\mathrm{T}}(t)M\chi(t)-\chi^{\mathrm{T}}(t-\tilde{h}(t))M\chi(t-\tilde{h}(t))\right) \leqslant -2\mathcal{K}_{\mathrm{Fit}}\Phi\int_{t-\tilde{h}(t)}^{\mathrm{T}}\left(\chi^{\mathrm{T}}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}\mathrm{d}s. \tag{15}
$$

Therefore,

$$
\begin{aligned}
\mathscr{L}V(t) \leqslant & \chi^{\mathrm{T}}(t)\Gamma_{11}\chi(t)+\chi^{\mathrm{T}}(t-\tilde{h}(t))\Gamma_{22}\chi(t-\tilde{h}(t))\\
&-2\Bigg[\left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fit}}\chi(t)\right)^{\frac{\vartheta+1}{2}}+\left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fix}}\chi(t)\right)^{\frac{\nu+1}{2}}\\
&+\mathcal{K}_I\left(\int_{t-\tilde{h}(t)}^{\mathrm{T}}\chi^{\mathrm{T}}(s)M\chi(s)\mathrm{d}s\right)^{\frac{1+\nu}{2}}\Bigg]-2\mathcal{K}_{\mathrm{Fit}}\Phi\int_{t-\tilde{h}(t)}^{\mathrm{T}}\left(\chi^{\mathrm{T}}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}\mathrm{d}s,
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
\Gamma_{11}=&-2M\mathcal{K}_P-2M\grave{A}+\xi_1MG_AG_A^{\mathrm{T}}M+\xi_1^{-1}F_A^{\mathrm{T}}F_A+\xi_2M\grave{S}\grave{S}^{\mathrm{T}}M+\xi_2^{-1}L^{\mathrm{T}}L+\xi_3MG_SG_S^{\mathrm{T}}M\\
&+\xi_3^{-1}L^{\mathrm{T}}F_S^{\mathrm{T}}F_SL+\xi_4M\grave{D}\grave{D}^{\mathrm{T}}M+\xi_5MG_DG_D^{\mathrm{T}}M+\beta M+M+2M\mathcal{K}_{\mathrm{Fit}},\\
\Gamma_{22}=&-(1-\tilde{h}^{\star})M+\xi_4^{-1}L^{\mathrm{T}}L+\xi_5^{-1}L^{\mathrm{T}}F_D^{\mathrm{T}}F_DL-2M\mathcal{K}_{\mathrm{Fit}}.
\end{aligned}
$$

From Schur's lemma, inequality (8) is equivalent to pre- and post-multiplying the aforementioned Γ_{11},Γ_{22} by $M=Z^{-1}$, and we have

$$
\begin{aligned}
\mathscr{L}V(t) \leqslant &-2\Bigg[\left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fit}}\chi(t)\right)^{\frac{\vartheta+1}{2}}+\left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fix}}\chi(t)\right)^{\frac{\nu+1}{2}}\\
&+\mathcal{K}_I\left(\int_{t-\tilde{h}(t)}^{\mathrm{T}}\chi^{\mathrm{T}}(s)M\chi(s)\mathrm{d}s\right)^{\frac{1+\nu}{2}}\Bigg]-2\mathcal{K}_{\mathrm{Fit}}\Phi\int_{t-\tilde{h}(t)}^{\mathrm{T}}\left(\chi^{\mathrm{T}}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}\mathrm{d}s.
\end{aligned} \tag{17}
$$

Case 1: $0\leqslant\vartheta<1$, $\mathcal{K}_{\mathrm{Fix}}=\operatorname{diag}\underbrace{(0,\ldots,0)}_{n}$, $\mathcal{K}_I=\operatorname{diag}\underbrace{(0,\ldots,0)}_{n}$. From Lemma 3, we obtain

$$
\int_{t-\tilde{h}(t)}^{\mathrm{T}}\left(\chi^{\mathrm{T}}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}\mathrm{d}s \geqslant \left(\int_{t-\tilde{h}(t)}^{\mathrm{T}}\chi^{\mathrm{T}}(s)M\chi(s)\mathrm{d}s\right)^{\frac{1+\vartheta}{2}}. \tag{18}
$$

Letting $\varpi=\min\{\rho_1,\rho_2\}$, $\rho_1=\sqrt{\lambda_{\min}(\mathcal{K}_{\mathrm{Fit}}^{\mathrm{T}}\mathcal{K}_{\mathrm{Fit}})}$, $\rho_2=\sqrt{\lambda_{\min}((\mathcal{K}_{\mathrm{Fit}}\Phi)^{\mathrm{T}}(\mathcal{K}_{\mathrm{Fit}}\Phi))}$, we get

$$
\begin{aligned}
E\{\mathscr{L}V(t)\} \leqslant &-2E\left\{\left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fit}}\chi(t)\right)^{\frac{\vartheta+1}{2}}+\mathcal{K}_{\mathrm{Fit}}\Phi\int_{t-\tilde{h}(t)}^{\mathrm{T}}\left(\chi^{\mathrm{T}}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}\mathrm{d}s\right\}\\
\leqslant &-2\varpi E\left\{V(t)\right\}^{\frac{1+\vartheta}{2}}.
\end{aligned} \tag{19}
$$

From Lemma 1, the system (3) is SFTS with a PI controller (7), and the ST is

$$E\{T_1\} \leqslant \frac{E\{V(\chi(0))\}^{\frac{1-\vartheta}{2}}}{\varpi(1-\vartheta)}.$$

Case 2: $0 \leqslant \vartheta < 1$, $\nu = 1$. It has

$$\begin{aligned} E\{\mathscr{L}V(t)\} \leqslant & -2E\Bigg\{ \left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fit}}\chi(t)\right)^{\frac{\vartheta+1}{2}} + \left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fix}}\chi(t)\right) \\ & + \mathcal{K}_I\left(\int_{t-\tilde{h}(t)}^{\mathrm{T}} \chi^{\mathrm{T}}(s)M\chi(s)\mathrm{d}s\right) + \mathcal{K}_{\mathrm{Fit}}\Phi\int_{t-\tilde{h}(t)}^{\mathrm{T}} \left(\chi^{\mathrm{T}}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}\mathrm{d}s\Bigg\} \\ \leqslant & -2\varpi E\left\{V(t)\right\}^{\frac{1+\vartheta}{2}} - 2\varpi_1 E\left\{V(t)\right\}, \end{aligned} \tag{20}$$

where $\varpi_1 = \min\{\rho_3, \rho_4\}$, $\rho_3 = \sqrt{\lambda_{\min}\left(\mathcal{K}_{\mathrm{Fix}}^{\mathrm{T}}\mathcal{K}_{\mathrm{Fix}}\right)}$, and $\rho_4 = \sqrt{\lambda_{\min}\left(\mathcal{K}_I^{\mathrm{T}}\mathcal{K}_I\right)}$.

From Lemma 1, the system (3) is SFTS with a PI controller (7), and the ST is

$$E\{T_2\} \leqslant \frac{\ln\left(1+\frac{\varpi_1}{\varpi}E\{V(\chi(0))\}^{\frac{1-\vartheta}{2}}\right)}{\varpi_1(1-\vartheta)}.$$

Case 3: $0 \leqslant \vartheta < 1$, $\nu > 1$. From Lemma 3, we deduce that

$$\left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fix}}\chi(t)\right)^{\frac{\nu+1}{2}} + \mathcal{K}_I\left(\int_{t-\tilde{h}(t)}^{\mathrm{T}} \chi^{\mathrm{T}}(s)M\chi(s)\mathrm{d}s\right)^{\frac{1+\nu}{2}} \geqslant (2n)^{\frac{1-\nu}{2}}\varpi_1 V(t)^{\frac{1+\nu}{2}}. \tag{21}$$

Consequently,

$$\begin{aligned} E\{\mathscr{L}V(t)\} \leqslant & -2E\Bigg\{ \left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fit}}\chi(t)\right)^{\frac{\vartheta+1}{2}} + \left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fix}}\chi(t)\right)^{\frac{\nu+1}{2}} \\ & + \mathcal{K}_I\left(\int_{t-\tilde{h}(t)}^{\mathrm{T}} \chi^{\mathrm{T}}(s)M\chi(s)\mathrm{d}s\right)^{\frac{1+\nu}{2}} + \mathcal{K}_{\mathrm{Fit}}\Phi\int_{t-\tilde{h}(t)}^{\mathrm{T}} \left(\chi^{\mathrm{T}}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}\mathrm{d}s\Bigg\} \\ \leqslant & -2\varpi E\left\{V(t)\right\}^{\frac{1+\vartheta}{2}} - 2(2n)^{\frac{1-\nu}{2}}\varpi_1 E\left\{V(t)\right\}^{\frac{1+\nu}{2}}. \end{aligned} \tag{22}$$

From Lemma 1, the system (3) reaches the SFXTS with the PI controller (7), and the ST is

$$T_3 \leqslant \frac{1}{(2n)^{\frac{1-\nu}{2}}\varpi_1}\left(\frac{(2n)^{\frac{1-\nu}{2}}\varpi_1}{\varpi}\right)^{\frac{1-\vartheta}{\nu-\vartheta}}\left(\frac{1}{1-\vartheta}+\frac{1}{\nu-1}\right).$$

Case 4: $0 \leqslant \vartheta < 1$, $\nu > 1$, $\mathcal{K}_{\mathrm{Fit}} = \mathcal{K}_{\mathrm{Fit}}\frac{T_3}{T_{\mathrm{Pre}}}$, $\mathcal{K}_{\mathrm{Fix}} = \mathcal{K}_{\mathrm{Fix}}\frac{T_3}{T_{\mathrm{Pre}}}$, where $0 < T_{\mathrm{Pre}} < T_3$ is a constant that can be prescribed randomly. Like Case 3, one has

$$E\{\mathscr{L}V(t)\} \leqslant -2\frac{T_3}{T_{\mathrm{Pre}}}\left(\varpi E\left\{V(t)\right\}^{\frac{1+\vartheta}{2}} + (2n)^{\frac{1-\nu}{2}}\varpi_1 E\left\{V(t)\right\}^{\frac{1+\nu}{2}}\right). \tag{23}$$

From Lemma 1, the system (3) reaches the SPTS with the PI controller (7), and the ST is

$$T_4 \leqslant T_{\mathrm{Pre}}.$$

3.2 Multicase stochastic finite-time stability with adaptive PI control

In this subsection, we describe the design of an adaptive state feedback controller for the SDMNNs (3) to enable the system to achieve MCSFTS. The aforementioned controller $\mathcal{U}(t)$ is designed as

$$\begin{aligned}\mathcal{U}(t) = &-\mathcal{K}_P(t)\chi(t) - \mathcal{K}_{\text{Fit}}\text{sign}(\chi(t))|\chi(t)|^{\vartheta} - \mathcal{K}_{\text{Fix}}\text{sign}(\chi(t))|\chi(t)|^{\nu} \\ &- \mathcal{K}_I\Big(\int_{t-\tilde{h}(t)}^{\text{T}} \chi^{\text{T}}(s)M\chi(s)\text{d}s\Big)^{\frac{1+\nu}{2}}\Big(\frac{M^{-1}|\chi(t)|}{\|\chi(t)\|^2}\Big),\end{aligned} \tag{24}$$

where $\mathcal{K}_P(t)$ is the adaptive adjustment of the feedback gain, and the other parameters are defined using the same controller (7). The updated law is presented,

$$\begin{aligned}\text{d}\mathcal{K}_P(t) = \big\{&\chi^{\text{T}}(t)\chi(t) - K_1\text{sign}(\mathcal{K}_P(t)-\mathcal{K}_\diamond)|\mathcal{K}_P(t)-\mathcal{K}_\diamond|^{\vartheta} \\ &- K_2\text{sign}(\mathcal{K}_P(t)-\mathcal{K}_\diamond)\ |\mathcal{K}_P(t)-\mathcal{K}_\diamond|^{\nu}\big\}\,\text{d}t.\end{aligned} \tag{25}$$

Remark 5. Typically, the calculated control gain of the feedback controller will be much larger than the actual required control gain because of the conservative nature of the theory, while the adaptive feedback controller can effectively avoid the inflated control gain and economize certain control costs. Compared with the existing studies [15, 30, 45], most scholars only consider the adaptive control strategy for proportional terms and have not considered the adaptive control strategy for PI terms. Therefore, the designed adaptive PI control strategy has better applicability and comprehensiveness for handling multicase stochastic finite-time stabilization of discontinuous SDMNNs.

Subsequently, the MCSFTS criterion is obtained by utilizing the adaptive controller.

Theorem 2. Given the conditions of Assumptions 1–3, if there exist some real matrices $Y_\diamond, Y_{\text{Fit}}$, a symmetric positive matrix Z, and some scalars $\xi_1 > 0, \xi_2 > 0, \xi_3 > 0, \xi_4 > 0, \xi_5 > 0$, the following LMI holds:

$$\begin{bmatrix} \Xi_1^\diamond & 0 & \Xi_2 \\ * & -\left(1-\tilde{h}^\star\right)Z^{\text{T}} - 2Y_{\text{Fit}} & \Xi_3 \\ * & * & \Xi_4 \end{bmatrix} < 0, \tag{26}$$

where

$$\begin{aligned}\Xi_1^\diamond =& 2Y_\diamond - 2\grave{A}Z + \xi_1 G_A G_A^{\text{T}} + \xi_2 \grave{S}\grave{S}^{\text{T}} + \xi_3 G_S G_S^{\text{T}} + \xi_4 \grave{D}\grave{D}^{\text{T}} \\ &+ \xi_5 G_D G_D^{\text{T}} + \beta Z^{\text{T}} + Z^{\text{T}} + 2Y_{\text{Fit}}, \\ \Xi_2 =& (ZF_A^{\text{T}}, ZL^{\text{T}}, ZL^{\text{T}}F_S^{\text{T}}, 0, 0), \\ \Xi_3 =& (0,0,0,ZL^{\text{T}}, ZL^{\text{T}}F_D^{\text{T}}), \\ \Xi_4 =& \text{diag}(-\xi_1 I, -\xi_2 I, -\xi_3 I, -\xi_4 I, -\xi_5 I).\end{aligned}$$

Subsequently, the controller factors can be devised as $\mathcal{K}_\diamond = Y_\diamond Z^{-1}, \mathcal{K}_{\text{Fit}} = Y_{\text{Fit}}Z^{-1}, \mathcal{K}_{\text{Fix}}^{\text{T}}\mathcal{K}_{\text{Fix}} > 0, \mathcal{K}_I^{\text{T}}\mathcal{K}_I > 0, K_1^{\text{T}}K_1 > 0, K_2^{\text{T}}K_2 > 0$. Additionally, for the system (3) to achieve SFTS, SFXTS, or SPTS by the choice of the controller (24) control parameters, ST is estimated as follows:

$$\begin{cases} \text{SFTS}: E\{T_1\} \leqslant \frac{E\{V(\chi(0))\}^{\frac{1-\vartheta}{2}}}{\hat{\varpi}(1-\vartheta)}, 0 \leqslant \vartheta < 1, \mathcal{K}_{\text{Fix}} = \text{diag}\,\underbrace{(0,\ldots,0)}_{n}, \\ \qquad \mathcal{K}_I = \text{diag}\,\underbrace{(0,\ldots,0)}_{n}, K_2 = \text{diag}\,\underbrace{(0,\ldots,0)}_{n}, \\ \text{SFTS}: E\{T_2\} \leqslant \frac{\ln(1+\frac{\hat{\varpi}_1}{\hat{\varpi}}E\{V(\chi(0))\}^{\frac{1-\vartheta}{2}})}{\hat{\varpi}_1(1-\vartheta)}, 0 \leqslant \vartheta < 1, \nu = 1, \\ \text{SFXTS}: T_3 \leqslant \frac{1}{(2n)^{\frac{1-\nu}{2}}\hat{\varpi}_1}\left(\frac{(2n)^{\frac{1-\nu}{2}}\hat{\varpi}_1}{\hat{\varpi}}\right)^{\frac{1-\vartheta}{\nu-\vartheta}}\left(\frac{1}{1-\vartheta}+\frac{1}{\nu-1}\right), 0 \leqslant \vartheta < 1, \nu > 1, \\ \text{SPTS}: T_4 \leqslant \hat{T}_{\text{Pre}}, 0 \leqslant \vartheta < 1, \nu > 1, \mathcal{K}_{\text{Fit}} = \mathcal{K}_{\text{Fit}}\frac{T_3}{T_{\text{Pre}}}, \mathcal{K}_{\text{Fix}} = \mathcal{K}_{\text{Fix}}\frac{T_3}{T_{\text{Pre}}}. \end{cases}$$

Remark 6. Theorem 2 differs from Theorem 1 in that the control gain $\mathcal{K}_\diamond$ is restrained in inequality (26), which is the control gain function $\mathcal{K}_P(t)$ approximation value while increasing the control gain K_1 and K_2 for fine-tuning.

Proof. Selecting Lyapunov functional candidates

$$V(t) =\chi^{\mathrm{T}}(t)M\chi(t) + \int_{t-\tilde{h}(t)}^{\mathrm{T}} \chi^{\mathrm{T}}(s)M\chi(s)\mathrm{d}s + (\mathcal{K}_P(t) - \mathcal{K}_\diamond)^{\mathrm{T}} M (\mathcal{K}_P(t) - \mathcal{K}_\diamond). \tag{27}$$

Following the identical line of Theorem 1 and computing the derivative of $V(t)$ along system (3), we obtain

$$\begin{aligned}
\mathscr{L}V(t) \leqslant & \chi(t)^{\mathrm{T}}\Gamma'_{11}\chi(t) + \chi^{\mathrm{T}}(t-\tilde{h}(t))\Gamma_{22}\chi(t-\tilde{h}(t)) \\
& - 2\Bigg[\left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fit}}\chi(t)\right)^{\frac{\vartheta+1}{2}} + \left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fix}}\chi(t)\right)^{\frac{\nu+1}{2}} \\
& + \mathcal{K}_I \left(\int_{t-\tilde{h}(t)}^{\mathrm{T}} \chi^{\mathrm{T}}(s)M\chi(s)\mathrm{d}s\right)^{\frac{1+\nu}{2}}\Bigg] - 2\mathcal{K}_{\mathrm{Fit}}\Phi \int_{t-\tilde{h}(t)}^{\mathrm{T}} \left(\chi^{\mathrm{T}}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}\mathrm{d}s \\
& + 2(\mathcal{K}_P(t) - \mathcal{K}_\diamond)^{\mathrm{T}} M\dot{\mathcal{K}}_P(t) \\
\leqslant & \chi^{\mathrm{T}}(t)\Gamma^{\diamond}_{11}\chi(t) + \chi^{\mathrm{T}}(t-\tilde{h}(t))\Gamma_{22}\chi(t-\tilde{h}(t)) \\
& - 2\chi^{\mathrm{T}}(t)M(\mathcal{K}_P(t) - \mathcal{K}_\diamond)\chi(t) - 2\Bigg[\left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fit}}\chi(t)\right)^{\frac{\vartheta+1}{2}} \\
& + \left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fix}}\chi(t)\right)^{\frac{\nu+1}{2}} + \mathcal{K}_I\left(\int_{t-\tilde{h}(t)}^{\mathrm{T}} \chi^{\mathrm{T}}(s)M\chi(s)\mathrm{d}s\right)^{\frac{1+\nu}{2}}\Bigg] \\
& - 2\mathcal{K}_{\mathrm{Fit}}\Phi\int_{t-\tilde{h}(t)}^{\mathrm{T}} \left(\chi^{\mathrm{T}}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}\mathrm{d}s + 2(\mathcal{K}_P(t) - \mathcal{K}_\diamond)^{\mathrm{T}} M\left\{\chi^{\mathrm{T}}(t)\chi(t)\right. \\
& \left. - K_1\mathrm{sign}(\mathcal{K}_P(t) - \mathcal{K}_\diamond)|\mathcal{K}_P(t) - \mathcal{K}_\diamond|^{\vartheta} - K_2\mathrm{sign}(\mathcal{K}_P(t) - \mathcal{K}_\diamond)|\mathcal{K}_P(t) - \mathcal{K}_\diamond|^{\nu}\right\} \\
\leqslant & \chi^{\mathrm{T}}(t)\Gamma^{\diamond}_{11}\chi(t) + \chi^{\mathrm{T}}(t-\tilde{h}(t))\Gamma_{22}\chi(t-\tilde{h}(t)) \\
& - 2\Bigg[\left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fit}}\chi(t)\right)^{\frac{\vartheta+1}{2}} \left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fix}}\chi(t)\right)^{\frac{\nu+1}{2}} \\
& + \mathcal{K}_I\left(\int_{t-\tilde{h}(t)}^{\mathrm{T}} \chi^{\mathrm{T}}(s)M\ \chi(s)\mathrm{d}s\right)^{\frac{1+\nu}{2}}\Bigg] - 2\mathcal{K}_{\mathrm{Fit}}\Phi\int_{t-\tilde{h}(t)}^{\mathrm{T}} \left(\chi^{\mathrm{T}}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}\mathrm{d}s \\
& + 2(\mathcal{K}_P(t) - \mathcal{K}_\diamond)^{\mathrm{T}} M\left\{K_1\mathrm{sign}(\mathcal{K}_P(t) - \mathcal{K}_\diamond)|\ \mathcal{K}_P(t) - \mathcal{K}_\diamond|^{\vartheta}\right. \\
& \left. - K_2\mathrm{sign}(\mathcal{K}_P(t) - \mathcal{K}_\diamond)|\mathcal{K}_P(t) - \mathcal{K}_\diamond|^{\nu}\right\},
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\Gamma'_{11} =& -2M\mathcal{K}_P(t) - 2M\grave{A} + \xi_1 MG_AG_A^{\mathrm{T}}M + \xi_1^{-1}F_A^{\mathrm{T}}F_A + \xi_2 M\grave{S}\grave{S}^{\mathrm{T}}M + \xi_2^{-1}L^{\mathrm{T}}L \\
& + \xi_3 MG_SG_S^{\mathrm{T}}M + \xi_3^{-1}L^{\mathrm{T}}F_S^{\mathrm{T}}F_SL + \xi_4 M\grave{D}\grave{D}^{\mathrm{T}}M + \xi_5 MG_DG_D^{\mathrm{T}}M + \beta M + M + 2M\mathcal{K}_{\mathrm{Fit}}, \\
\Gamma^{\diamond}_{11} =& -2M\mathcal{K}_\diamond - 2M\grave{A} + \xi_1 MG_AG_A^{\mathrm{T}}M + \xi_1^{-1}F_A^{\mathrm{T}}F_A + \xi_2 M\grave{S}\grave{S}^{\mathrm{T}}M + \xi_2^{-1}L^{\mathrm{T}}L \\
& + \xi_3 MG_SG_S^{\mathrm{T}}M + \xi_3^{-1}L^{\mathrm{T}}F_S^{\mathrm{T}}F_SL + \xi_4 M\grave{D}\grave{D}^{\mathrm{T}}M + \xi_5 MG_DG_D^{\mathrm{T}}M + \beta M + M + 2M\mathcal{K}_{\mathrm{Fit}}.
\end{aligned}$$

From inequality (26), we deduce that

$$\begin{aligned}
\mathscr{L}V(t) \leqslant & -2\Bigg[\left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fit}}\chi(t)\right)^{\frac{\vartheta+1}{2}} + \left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fix}}\chi(t)\right)^{\frac{\nu+1}{2}} \\
& + \mathcal{K}_I\left(\int_{t-\tilde{h}(t)}^{\mathrm{T}} \chi^{\mathrm{T}}(s)M\chi(s)\mathrm{d}s\right)^{\frac{1+\nu}{2}}\Bigg] - 2\mathcal{K}_{\mathrm{Fit}}\Phi\int_{t-\tilde{h}(t)}^{\mathrm{T}} \left(\chi^{\mathrm{T}}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}\mathrm{d}s \\
& + 2(\mathcal{K}_P(t) - \mathcal{K}_\diamond)^{\mathrm{T}} M\left\{K_1\mathrm{sign}(\mathcal{K}_P(t) - \mathcal{K}_\diamond)|\mathcal{K}_P(t) - \mathcal{K}_\diamond|^{\vartheta}\right. \\
& \left. - K_2\mathrm{sign}(\mathcal{K}_P(t) - \mathcal{K}_\diamond)|\mathcal{K}_P(t) - \mathcal{K}_\diamond|^{\nu}\right\}.
\end{aligned} \tag{29}$$

Case 1: $0 \leqslant \vartheta < 1$, $\mathcal{K}_{\rm Fix} = {\rm diag}\underbrace{(0,\ldots,0)}_{n}$, $\mathcal{K}_I = {\rm diag}\underbrace{(0,\ldots,0)}_{n}$, $K_2 = {\rm diag}\underbrace{(0,\ldots,0)}_{n}$. From Lemma 3, we obtain

$$\begin{aligned}&(\mathcal{K}_P(t)-\mathcal{K}_\diamond)^{\rm T} M\left(K_1{\rm sign}(\mathcal{K}_P(t)-\mathcal{K}_\diamond)\ |\mathcal{K}_P(t)-\mathcal{K}_\diamond|^{\vartheta}\right)\\&\geqslant K_1\left((\mathcal{K}_P(t)-\mathcal{K}_\diamond)^{\rm T} M|\mathcal{K}_P(t)-\mathcal{K}_\diamond|\right)^{\frac{1+\vartheta}{2}}.\end{aligned} \tag{30}$$

Letting $\hat{\varpi} = \min\{\rho_1, \rho_2, \hat{\rho}_1\}$, $\hat{\rho}_1 = \sqrt{\lambda_{\min}\left(K_1^{\rm T}K_1\right)}$, we get

$$\begin{aligned}E\left\{\mathscr{L}V(t)\right\} \leqslant &-2E\left\{\left(\chi^{\rm T}(t)M\mathcal{K}_{\rm Fit}\chi(t)\right)^{\frac{\vartheta+1}{2}} + \mathcal{K}_{\rm Fit}\Phi\int_{t-\tilde{h}(t)}^{\rm T}\left(\chi^{\rm T}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}{\rm d}s\right.\\&\left.+K_1\left((\mathcal{K}_P(t)-\mathcal{K}_\diamond)^{\rm T}M|\mathcal{K}_P(t)-\mathcal{K}_\diamond|\right)^{\frac{1+\vartheta}{2}}\right\}\\\leqslant &-2\hat{\varpi}E\left\{V(t)\right\}^{\frac{1+\vartheta}{2}}.\end{aligned} \tag{31}$$

From Lemma 1, the system (3) is SFTS with the adaptive PI controller (24), and the ST is

$$E\{T_1\} \leqslant \frac{E\{V(\chi(0))\}^{\frac{1-\vartheta}{2}}}{\hat{\varpi}(1-\vartheta)}.$$

Case 2: $0 \leqslant \vartheta < 1$, $\nu = 1$. It has

$$\begin{aligned}E\{\mathscr{L}V(t)\} \leqslant &-2E\left\{\left(\chi^{\rm T}(t)M\mathcal{K}_{\rm Fit}\chi(t)\right)^{\frac{\vartheta+1}{2}} + \left(\chi^{\rm T}(t)M\mathcal{K}_{\rm Fix}\chi(t)\right)\right.\\&+\mathcal{K}_I\left(\int_{t-\tilde{h}(t)}^{\rm T}\chi^{\rm T}(s)M\chi(s){\rm d}s\right)+\mathcal{K}_{\rm Fit}\Phi\int_{t-\tilde{h}(t)}^{\rm T}\left(\chi^{\rm T}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}{\rm d}s\\&\left.+K_1\left((\mathcal{K}_P(t)-\mathcal{K}_\diamond)^{\rm T}M|\mathcal{K}_P(t)-\mathcal{K}_\diamond|\right)^{\frac{1+\vartheta}{2}}+K_2\left((\mathcal{K}_P(t)-\mathcal{K}_\diamond)^{\rm T}M|\mathcal{K}_P(t)-\mathcal{K}_\diamond|\right)\right\}\\\leqslant &-2\hat{\varpi}E\left\{V(t)\right\}^{\frac{1+\vartheta}{2}}-2\hat{\varpi}_1E\left\{V(t)\right\},\end{aligned} \tag{32}$$

where $\hat{\varpi}_1 = \min\{\rho_3, \rho_4, \hat{\rho}_2\}$, $\hat{\rho}_2 = \sqrt{\lambda_{\min}\left(K_2^{\rm T}K_2\right)}$.

From Lemma 1, the system (3) is SFTS with the adaptive PI controller (24), and the ST is

$$E\{T_2\} \leqslant \frac{\ln(1+\frac{\hat{\varpi}_1}{\hat{\varpi}}E\{V(\chi(0))\}^{\frac{1-\vartheta}{2}})}{\hat{\varpi}_1(1-\vartheta)}.$$

Case 3: $0 \leqslant \vartheta < 1$, $\nu > 1$. From Lemma 3, we deduce the following:

$$\begin{aligned}&\left(\chi^{\rm T}(t)M\mathcal{K}_{\rm Fix}\chi(t)\right)^{\frac{\nu+1}{2}}+\mathcal{K}_I\left(\int_{t-\tilde{h}(t)}^{\rm T}\chi^{\rm T}(s)M\chi(s){\rm d}s\right)^{\frac{1+\nu}{2}}\\&+K_2\left((\mathcal{K}_P(t)-\mathcal{K}_\diamond)^{\rm T}M|\mathcal{K}_P(t)-\mathcal{K}_\diamond|\right)^{\frac{1+\nu}{2}}\\&\geqslant (2n)^{\frac{1-\nu}{2}}\hat{\varpi}_1V(t)^{\frac{1+\nu}{2}}.\end{aligned} \tag{33}$$

Hence,

$$\begin{aligned}E\{\mathscr{L}V(t)\} \leqslant & -2E\Bigg\{\left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fit}}\chi(t)\right)^{\frac{\vartheta+1}{2}}+\left(\chi^{\mathrm{T}}(t)M\mathcal{K}_{\mathrm{Fix}}\chi(t)\right)^{\frac{\nu+1}{2}} \\ & +\mathcal{K}_I\left(\int_{t-\tilde{h}(t)}^{\mathrm{T}}\chi^{\mathrm{T}}(s)M\chi(s)\mathrm{d}s\right)^{\frac{1+\nu}{2}}+\mathcal{K}_{\mathrm{Fit}}\Phi\int_{t-\tilde{h}(t)}^{\mathrm{T}}\left(\chi^{\mathrm{T}}(s)M\chi(s)\right)^{\frac{1+\vartheta}{2}}\mathrm{d}s \\ & +K_1\left(\left(\mathcal{K}_P(t)-\mathcal{K}_{\diamond}\right)^{\mathrm{T}}M|\mathcal{K}_P(t)-\mathcal{K}_{\diamond}|\right)^{\frac{1+\vartheta}{2}}+K_2\left(\left(\mathcal{K}_P(t)-\mathcal{K}_{\diamond}\right)^{\mathrm{T}}M|\mathcal{K}_P(t)-\mathcal{K}_{\diamond}|\right)^{\frac{1+\nu}{2}}\Bigg\} \\ \leqslant & -2\hat{\varpi}E\left\{V(t)\right\}^{\frac{1+\vartheta}{2}}-2(2n)^{\frac{1-\nu}{2}}\hat{\varpi}_1E\left\{V(t)\right\}^{\frac{1+\nu}{2}}.\end{aligned} \tag{34}$$

From Lemma 1, the system (3) reaches the SFXTS with the adaptive PI controller (24), and the ST is

$$T_3 \leqslant \frac{1}{(2n)^{\frac{1-\nu}{2}}\hat{\varpi}_1}\left(\frac{(2n)^{\frac{1-\nu}{2}}\hat{\varpi}_1}{\hat{\varpi}}\right)^{\frac{1-\vartheta}{\nu-\vartheta}}\left(\frac{1}{1-\vartheta}+\frac{1}{\nu-1}\right).$$

Case 4: $0 \leqslant \vartheta < 1$, $\nu > 1$, $\mathcal{K}_{\mathrm{Fit}} = \mathcal{K}_{\mathrm{Fit}}\frac{T_3}{T_{\mathrm{Pre}}}$, $\mathcal{K}_{\mathrm{Fix}} = \mathcal{K}_{\mathrm{Fix}}\frac{T_3}{T_{\mathrm{Pre}}}$, where $0 < T_{\mathrm{Pre}} < T_3$ is a constant that can be prescribed randomly. Like Case 3, one has

$$E\{\mathscr{L}V(t)\} \leqslant -2\frac{T_3}{T_{\mathrm{Pre}}}\left(\hat{\varpi}E\left\{V(t)\right\}^{\frac{1+\vartheta}{2}}+(2n)^{\frac{1-\nu}{2}}\hat{\varpi}_1E\left\{V(t)\right\}^{\frac{1+\nu}{2}}\right). \tag{35}$$

From Lemma 1, the system (3) reaches SPTS with the adaptive PI controller (24), and the ST is

$$T_4 \leqslant T_{\mathrm{Pre}}.$$

4 Numerical simulations

In this section, two examples are presented to demonstrate the multicase stochastic finite-time stabilization method for SDMNNs.

Example 1. Consider the SDMNNs (3) with the following parameters:

$$A(\chi(t))=\begin{bmatrix}a_{11}(\chi_1(t)) & 0\\ 0 & a_{22}(\chi_2(t))\end{bmatrix},\ S(\chi(t))=\begin{bmatrix}b_{11}(\chi_1(t)) & b_{12}(\chi_1(t))\\ b_{21}(\chi_2(t)) & b_{22}(\chi_2(t))\end{bmatrix},\ g(\chi(t))=\begin{bmatrix}\tanh(\chi_1(t))\\ \tanh(\chi_2(t))\end{bmatrix},$$

$$D(\chi(t-\tilde{h}(t)))=\begin{bmatrix}d_{11}\left(\chi_1(t-\tilde{h}(t))\right) & d_{12}\left(\chi_1(t-\tilde{h}(t))\right)\\ d_{21}\left(\chi_2(t-\tilde{h}(t))\right) & d_{22}\left(\chi_2(t-\tilde{h}(t))\right)\end{bmatrix},\ g(\chi(t-\tilde{h}(t)))=\begin{bmatrix}\tanh\left(\chi_1(t-\tilde{h}(t))\right)\\ \tanh\left(\chi_2(t-\tilde{h}(t))\right)\end{bmatrix},$$

where

$$a_{11}(\chi_1(t))=\begin{cases}1.0, & |\chi_1(t)|<1,\\ 1.1, & |\chi_1(t)|\geqslant 1,\end{cases}\quad a_{22}(\chi_2(t))=\begin{cases}1.1, & |\chi_2(t)|<1,\\ 1.0, & |\chi_2(t)|\geqslant 1,\end{cases}$$

$$b_{11}(\chi_1(t))=\begin{cases}1.9, & |\chi_1(t)|<1,\\ 2.0, & |\chi_1(t)|\geqslant 1,\end{cases}\quad b_{12}(\chi_2(t))=\begin{cases}-0.1, & |\chi_2(t)|<1,\\ -0.08, & |\chi_2(t)|\geqslant 1,\end{cases}$$

$$b_{21}(\chi_1(t))=\begin{cases}-4.8, & |\chi_1(t)|<1,\\ -4.7, & |\chi_1(t)|\geqslant 1,\end{cases}\quad b_{22}(\chi_2(t))=\begin{cases}3.0, & |\chi_2(t)|<1,\\ 3.1, & |\chi_2(t)|\geqslant 1,\end{cases}$$

$$d_{11}(\chi_1(t-\tilde{h}(t)))=\begin{cases}-1.6, & |\chi_1(t-\tilde{h}(t))|<1,\\ -1.5, & |\chi_1(t-\tilde{h}(t))|\geqslant 1,\end{cases}\quad d_{12}(\chi_2(t-\tilde{h}(t)))=\begin{cases}-0.12, & |\chi_2(t-\tilde{h}(t))|<1,\\ -0.10, & |\chi_2(t-\tilde{h}(t))|\geqslant 1,\end{cases}$$

$$d_{21}(\chi_1(t-\tilde{h}(t)))=\begin{cases}0.45, & |\chi_1(t-\tilde{h}(t))|<1,\\ 0.50, & |\chi_1(t-\tilde{h}(t))|\geqslant 1,\end{cases}\quad d_{22}(\chi_2(t-\tilde{h}(t)))=\begin{cases}-2.6, & |\chi_2(t-\tilde{h}(t))|<1,\\ -2.4, & |\chi_2(t-\tilde{h}(t))|\geqslant 1.\end{cases}$$

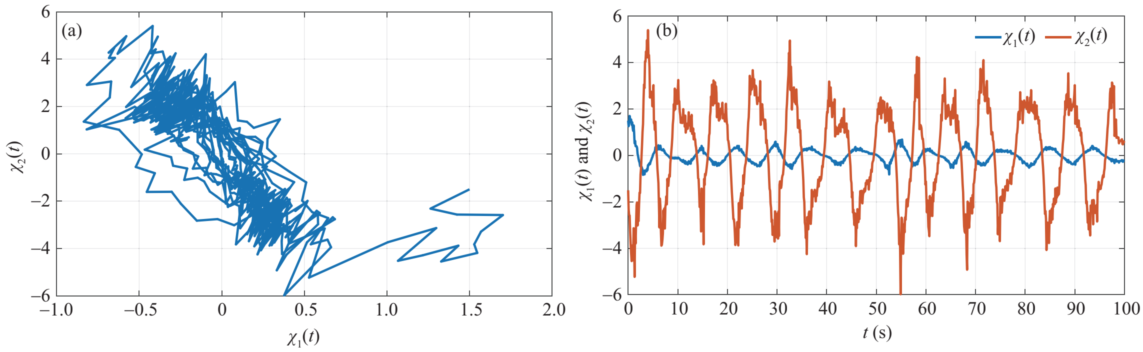

Figure 1 (Color online) The system (3) without the controller. (a) Phase plane diagram; (b) state trajectories diagram.

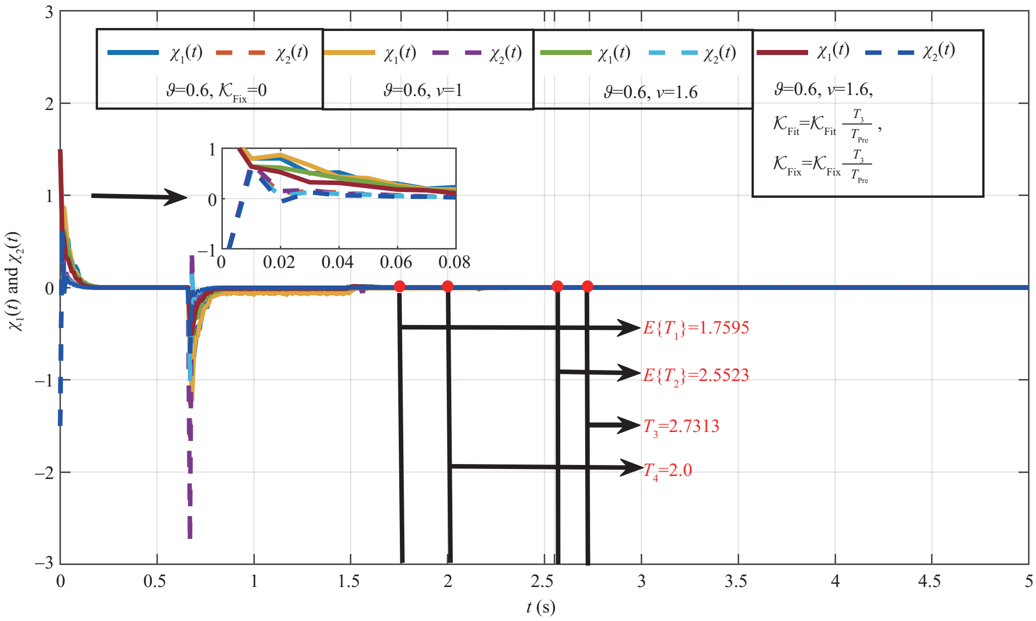

Figure 2 (Color online) General diagram of state trajectories of the system (3) with the PI controller (7).

The time-varying delay was $\tilde{h}(t) = (\frac{\mathrm{e}^{\mathrm{T}}}{1+\mathrm{e}^{\mathrm{T}}})$. The noise function matrix is expressed as

$$\delta(t, \chi(t)) = \begin{bmatrix} 0.5|\chi_1(t)| & 0 \\ 0 & 0.5|\chi_2(t)| \end{bmatrix}.$$

Subsequently, we can obtain $l_1 = l_2 = 1$, $\Phi = I$. The initial condition of the state trajectories is set as $\chi(s) = [1.5, -0.5]^{\mathrm{T}}, s \in [-1, 0]$. Figure 1(a) shows state trajectories of the phase plane of the system (3), which can be observed to generate chaos in the absence of a control condition. Figure 1(b) shows a graph of the state trajectories of the system (3), which reveals that the system is oscillatory; that is, the system (3) is unstable under uncontrolled application.

The feasible solution calculated by MATLAB toolbox according to LMI (8) is as follows:

$$Z = \begin{bmatrix} 0.3201 & 0.0000 \\ 0.0000 & 0.3080 \end{bmatrix}, M = \begin{bmatrix} 3.1239 & 0.0000 \\ 0.0000 & 3.2468 \end{bmatrix}, Y_P = \begin{bmatrix} 6.9592 & -8.5454 \\ -8.5454 & 34.5059 \end{bmatrix}, Y_{\rm Fit} = \begin{bmatrix} 0.7747 & 0.0000 \\ 0.0000 & 0.7777 \end{bmatrix},$$

$$\mathcal{K}_P = \begin{bmatrix} 21.7402 & -27.7452 \\ -26.6953 & 112.0334 \end{bmatrix}, \mathcal{K}_{\rm Fit} = \begin{bmatrix} 2.4200 & 0.0000 \\ 0.0000 & 2.5250 \end{bmatrix}, \mathcal{K}_I = \begin{bmatrix} 0.85 & 0.00 \\ 0.00 & 0.85 \end{bmatrix}, \mathcal{K}_{\rm Fix} = \begin{bmatrix} 0.0/2.5 & 0.0 \\ 0.0 & 0.0/2.5 \end{bmatrix},$$

$\xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_5 = 1.7094$.

If the terms of Theorem 1 are satisfied, the system (3) achieves MCSFTS under the PI controller (7). As shown in Figure 2, the state trajectories of the system (3) achieve MCSFTS with the PI controller (7)

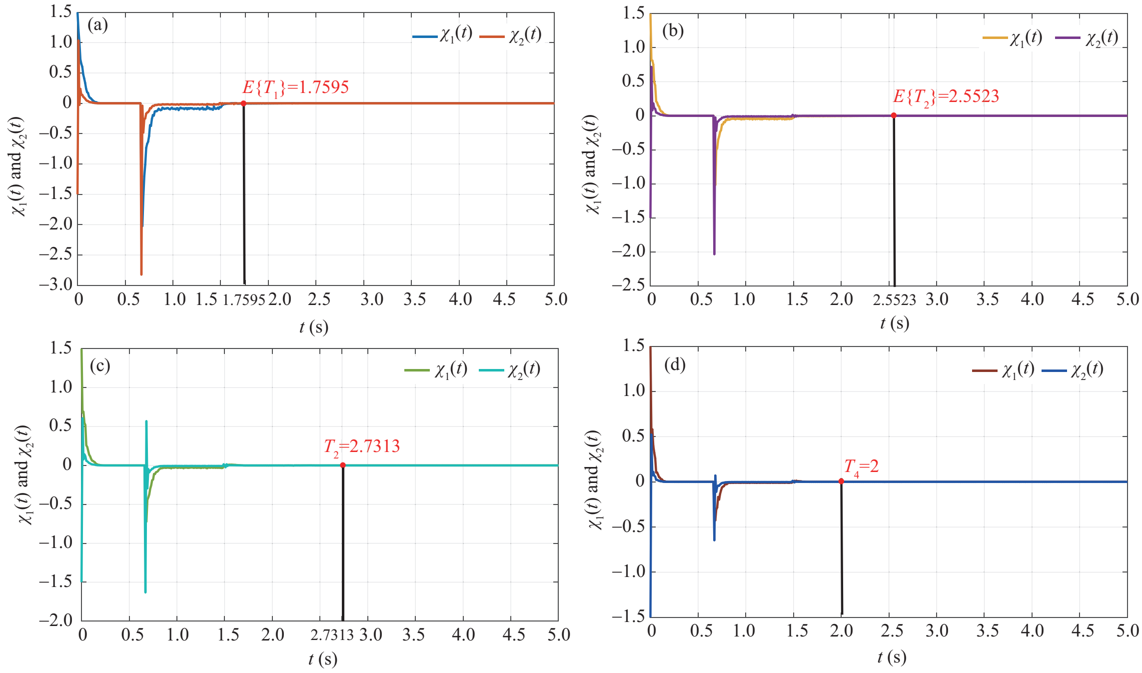

Figure 3 (Color online) State trajectories of the system (3) with the PI controller (7). (a) SFTS with $\vartheta = 0.6, \mathcal{K}_{\rm Fix} = {\rm diag}(0,0), \mathcal{K}_I = {\rm diag}(0,0)$; (b) SFTS with $\vartheta = 0.6, \nu = 1$; (c) SFXTS with $\vartheta = 0.6, \nu = 1.6$; (d) SPTS with $\vartheta = 0.6, \nu = 1.6$, $T_{\rm Pre} = 2$.

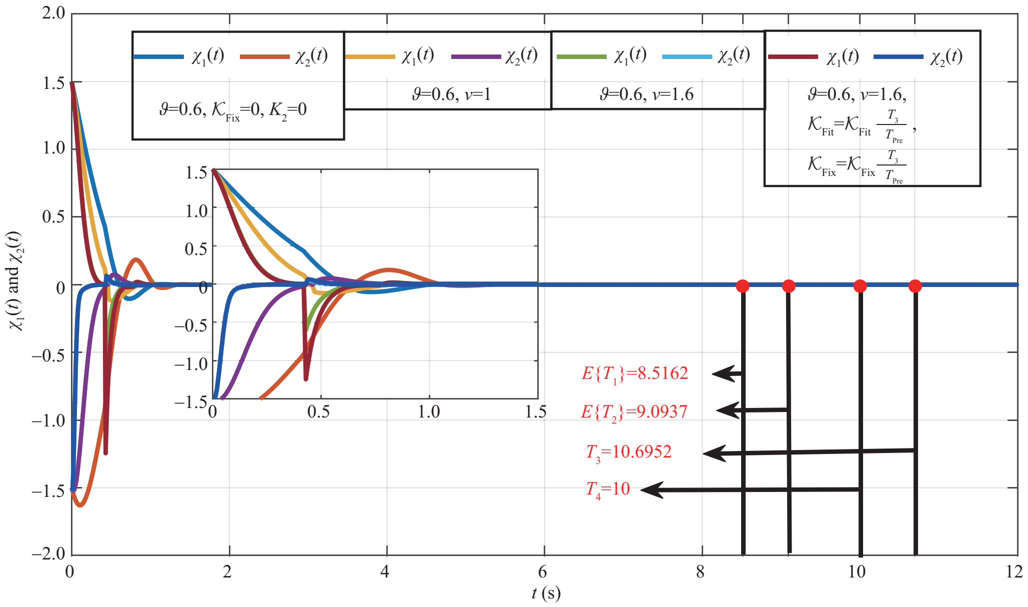

Figure 4 (Color online) General diagram of state trajectories of the system (3) with the adaptive PI controller (24).

under different conditions. For each specific condition discussed separately, each case achieves a specific stability of the system (3), as shown in Figure 3.

Case 1: Choose $\vartheta = 0.6, \mathcal{K}_{\rm Fix} = \left[\begin{smallmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{smallmatrix}\right], \mathcal{K}_I = \left[\begin{smallmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{smallmatrix}\right]$. From Theorem 1, the system (3) achieves SFTS with the PI controller (7), and the ST $E\{T_1\}$ is 1.7595, as shown in Figure 3(a).

Case 2: Choose $\vartheta = 0.6, \nu = 1$. From Theorem 1, the system (3) achieves SFTS with the PI controller (7), and the ST $E\{T_2\}$ is 2.5523, as shown in Figure 3(b).

Case 3: Choose $\vartheta = 0.6, \nu = 1.6$. From Theorem 1, the system (3) achieves SFXTS with the PI controller (7), and the ST T_3 is 2.7313, as shown in Figure 3(c).

Table 1 Some upper bounds for the settling time functions

	$E\{T_1\}$	$E\{T_2\}$	T_3	T_4
Theorem 1	1.7595	2.5523	2.7313	2
Theorem 2	8.5162	9.0937	10.6952	10

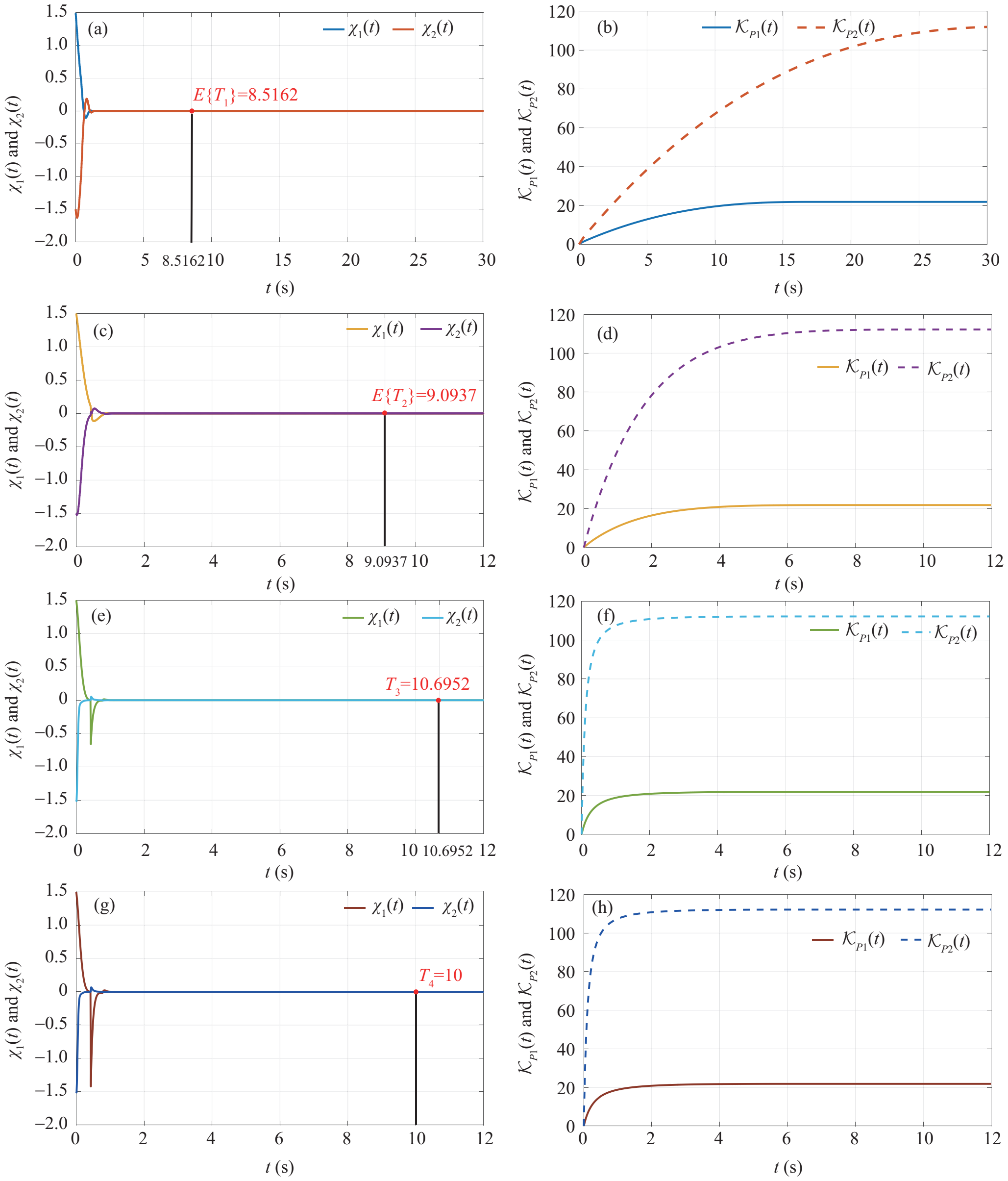

Figure 5 (Color online) State trajectories of the system (3) with the adaptive PI controller (24). (a) SFTS with $\vartheta = 0.6, \mathcal{K}_{\rm Fix} = {\rm diag}(0,0), \mathcal{K}_I = {\rm diag}(0,0), K_2 = {\rm diag}(0,0)$; (b) controller gain variable $\mathcal{K}_P(t)$ of (a); (c) SFTS with $\vartheta = 0.6, \nu = 1$; (d) controller gain variable $\mathcal{K}_P(t)$ of (c); (e) SFXTS with $\vartheta = 0.6, \nu = 1.6$; (f) controller gain variable $\mathcal{K}_P(t)$of (e); (g) SPTS with $\vartheta = 0.6, \nu = 1.6, T_{\rm Pre} = 10$; (h) controller gain variable $\mathcal{K}_P(t)$ of (g).

Case 4: Choose $\vartheta = 0.6, \nu = 1.6, \mathcal{K}_{\rm Fit} = \mathcal{K}_{\rm Fit}\frac{T_3}{T_{\rm pre}} = \left[\begin{smallmatrix}2.4200 & 0.0000\\ 0.0000 & 2.5250\end{smallmatrix}\right] \times 1.3656, \mathcal{K}_{\rm Fix} = \mathcal{K}_{\rm Fix}\frac{T_3}{T_{\rm pre}} = \left[\begin{smallmatrix}2.5000 & 0.0000\\ 0.0000 & 2.5000\end{smallmatrix}\right] \times 1.3656$. From Theorem 1, the system (3) achieves SPTS with the PI controller (7), and the ST is $T_4 = T_{\rm pre} = 2$, as shown in Figure 3(d).

Example 2. The system parameters in Example 2 are the same as those in Example 1, however, the difference is that we improved the controller by imposing the adaptive PI control protocol (24) and (25). Calculate the solution of LMI (26) using the MATLAB toolbox, and the feasible solutions are given below:

$$Z = \begin{bmatrix} 0.3201 & 0.0000 \\ 0.0000 & 0.3080 \end{bmatrix}, M = \begin{bmatrix} 3.1239 & 0.0000 \\ 0.0000 & 3.2468 \end{bmatrix}, Y_P = \begin{bmatrix} 6.9592 & -8.5454 \\ -8.5454 & 34.5059 \end{bmatrix}, Y_{\mathrm{Fit}} = \begin{bmatrix} 0.7747 & 0.0000 \\ 0.0000 & 0.7777 \end{bmatrix},$$

$$\mathcal{K}_{\diamond} = \begin{bmatrix} 21.7402 & -27.7452 \\ -26.6953 & 112.0334 \end{bmatrix}, \mathcal{K}_P(t) = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix} \sim \begin{bmatrix} 21.7402 & 0.0000 \\ 0.0000 & 112.0334 \end{bmatrix},$$

$$\mathcal{K}_{\mathrm{Fit}} = \begin{bmatrix} 2.4200 & 0.0000 \\ 0.0000 & 2.5250 \end{bmatrix}, \mathcal{K}_I = \begin{bmatrix} 0.25 & 0.00 \\ 0.00 & 0.25 \end{bmatrix}, \mathcal{K}_{\mathrm{Fix}} = \begin{bmatrix} 0.0/0.5 & 0.0 \\ 0.0 & 0.0/0.5 \end{bmatrix}, K_1 = K_2 = \begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 0.5 \end{bmatrix},$$

$$\xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_5 = 1.7094.$$

If the terms of Theorem 2 are satisfied, the system (3) achieves MCSFTS by imposing an adaptive PI controller (24). From Figure 4, the state trajectories of the system (3) achieve MCSFTS with the adaptive PI controller (24) under different conditions. As shown in Figure 4, the system saves some control costs under adaptive PI control, which leads to a larger estimated ST. For a comparison of the ST for the two theorems, see Table 1. Specific conditions are detailed discussion below; see Figure 5.

Case 1: Choose $\vartheta = 0.6, \mathcal{K}_{\mathrm{Fix}} = \left[\begin{smallmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{smallmatrix}\right], \mathcal{K}_I = \left[\begin{smallmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{smallmatrix}\right], K_2 = \left[\begin{smallmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{smallmatrix}\right]$. From Theorem 2, the system (3) achieves SFTS with the adaptive PI controller (24), and the ST $E\{T_1\}$ is 8.5162, as shown in Figure 5(a). The corresponding controller gain $\mathcal{K}_P(t)$ updating laws are illustrated in Figure 5(b).

Case 2: Choose $\vartheta = 0.6, \nu = 1$. From Theorem 2, the system (3) achieves SFTS with the adaptive PI controller (24), and the ST $E\{T_2\}$ is 9.0937, as shown in Figure 5(c). The corresponding controller gain $\mathcal{K}_P(t)$ updating laws are illustrated in Figure 5(d).

Case 3: Choose $\vartheta = 0.6, \nu = 1.6$. From Theorem 2, the system (3) achieves SFXTS with the adaptive PI controller (24), and the ST T_3 is 10.6952, as shown in Figure 5(e). The corresponding controller gain $\mathcal{K}_P(t)$ updating laws are illustrated in Figure 5(f).

Case 4: Choose $\vartheta = 0.6, \nu = 1.6, \mathcal{K}_{\mathrm{Fit}} = \mathcal{K}_{\mathrm{Fit}} \frac{T_3}{T_{\mathrm{pre}}} = \left[\begin{smallmatrix} 2.4200 & 0.0000 \\ 0.0000 & 2.5250 \end{smallmatrix}\right] \times 1.0695, \mathcal{K}_{\mathrm{Fix}} = \mathcal{K}_{\mathrm{Fix}} \frac{T_3}{T_{\mathrm{pre}}} = \left[\begin{smallmatrix} 0.5000 & 0.0000 \\ 0.0000 & 0.5000 \end{smallmatrix}\right] \times 1.0695$. From Theorem 2, the system (3) achieves SPTS with the adaptive PI controller (24), and the ST is $T_4 = T_{\mathrm{pre}} = 10$, as shown in Figure 5(g). The corresponding controller gain $\mathcal{K}_P(t)$ updating laws are illustrated in Figure 5(h).

5 Conclusion

This paper discusses the multicase stochastic finite-time stabilizations of SDMNNs in a unified control framework. Subsequently, a controller with PI control protocol and a controller with adaptive PI control are designed. By tuning the controller factors, the SFTS, SFXTS, and SPTS of SDMNNs were implemented to obtain the corresponding ST estimates. Finally, the reliability of the obtained results is demonstrated using two examples, implicitly indicating that adaptive PI controllers can save on the cost of control but lose some ST.

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