SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

December 2023, Vol. 66 222206:1–222206:14
https://doi.org/10.1007/s11432-023-3829-7

Controllability of game-based multi-agent system

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Received 28 April 2023/Revised 29 May 2023/Accepted 7 June 2023/Published online 27 November 2023

Abstract We introduce a strategy matrix, a novel concept for ensuring controllability in game-based control systems (GBCSs). This graph-based condition is presented as an alternative to utilizing complex mathematical calculations through algebraic conditions. Moreover, to address these issues, one must first study the expression of Nash equilibrium actions. This expression yields a general formula of the game controllability matrix, which is always affected by the specific matrix (strategy matrix) comprising Nash equilibrium actions, and the matrix can not only be obtained by matrix calculation but can also be directly written through the topology, indicating the topology's specific influence on the GBCS. Finally, we build a new game-based multi-agent system and determine the controllability relationship between the system and the general system.

Keywords controllability, game-based control system, topological structure, equivalent partition

Citation Guo J H, Ji Z J, Liu Y G. Controllability of game-based multi-agent system. Sci China Inf Sci, 2023, 66(12): 222206, https://doi.org/10.1007/s11432-023-3829-7

1 Introduction

Game is ubiquitous in nature. Recent years have witnessed the emergence of game theory as a powerful tool for studying control systems [1, 2], with an emphasis on distributed control systems [3–6]. This resulted in the development and study of game-based control systems (GBCS) [7]. This system's decision structure is hierarchical, with one regulator and multiple agents. Although the study of GBCSs holds practical importance, when the game is defined on a large-scale system, each player's strategy typically depends on the structure of the underlying network. Therefore, developing a game control model based on topology structure is paramount. In the literature, many algebraic conditions for judging the controllability of the GBCS can be found. However, judging the controllability of a massive GBCS through mathematical calculations is extremely difficult. Therefore, a graph-based condition of the GBCS must be established. Although establishing this result is important, there exists a noticeable shortage of research in this field. With the development of networked systems [8, 9], the research of controllability based on the graph (Controllability & Graph), a game system based on the graph (Game & Graph), and controllability based on the game (Controllability & Game) has reached a mature stage; however, research into the controllability of a graph-based game system (Controllability & Graph & Game) is virtually in their infancy.

1.1 Literature review

• Controllability & Graph. Currently, the influence of network topology structure on the controllability of multi-agent systems (MASs) is primarily investigated by graph theory, as shown in [10–17]. It is evident that the controllability of the MASs is strongly linked to the underlying graph topology, and the graph division [18–22] describes the controllability of the MASs from the perspective of topology. Cardoso et al. [18] suggested that an almost equivalent partition is both a sufficient and necessary condition for ensuring controllability and clarified the relationship between the Laplacian matrix and the general Laplacian matrix. Qu et al. [23] investigated multi-agent system controllability under equivalent partition.

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• Game & Graph. In evolutionary game systems, when who-meets-whom is no longer random but rather based on social networks or spatial relationships, the study of "Game & Graph" emerges [24–31]. For example, the pursuit-evasion game problem [32–35], which has been extensively used in aircraft control and missile guidance, intelligent transportation system, and collision avoidance design of wireless sensor networks in military implementation, exemplifies this line of thinking. In [36], Bayesian games (games with incomplete information) involve an agent participating in an unspecified game where the true intentions of other players are unknown, and each player must adjust their goals accordingly. Lopez et al. [37] proposed two belief-updating methods that require no graphical topology knowledge. The first method involves the application of Bayesian rules, whereas the second involves the modification of non-Bayesian updates. Pirani et al. [38] employed structured systems theory and other graph theory concepts to analyze games, improve the detectability of network physical attacks, and investigate the optimal configuration of sensors in networked control systems.

• Controllability & Game. The game-based control system originates from a study conducted by Engwerda [39], which describes the regulator problem of linear time-invariant systems. That is, one goal may be the discovery of a control function $u(t)$ that drives the state of the system to a small neighborhood of zero at time T. In contrast, if the state of the system indicates a set of financially relevant economic variables, u indicates the investment behavior that serves to increase these variables, with the goal being able to control these variables to reach the desired level as quickly as possible. Zhang et al. [7] investigated a linear system with a hierarchical decision structure comprising a regulator and multiple agents. The regulator first makes a decision, and then each agent optimizes its own payoff function to reach the possible Nash equilibrium of the noncooperative dynamic game. Meanwhile, the underlying agents in the system are exchanging information with the regulator.

Despite the above research making considerable progress, the topological characterization of the controllability of game systems based on Nash equilibrium action remains largely unknown.

1.2 Contributions

In contrast to the traditional control theory framework, the GBCS will consider the strategic behavior of each agent, avoiding the unrealistic phenomenon of system dynamics caused by ignoring the agent's behavior. However, the current research on this system is only at the algebraic level. The interacting relationship among neighbor agents is one of the underlying logic that runs the system, which prompts us to judge the controllability of GBCSs based on topology and accordingly complex mathematical calculations can be avoid.

To shed light on this problem, this paper first examines the framework of a GBCS. We investigate the Nash equilibrium action expression of the system under the regulator's strategy. From this expression, we derive the general formula of the controllability matrix. We discover that the strategy matrix, which is crucial in the general formula, can not only be obtained algebraically but also be written directly according to the topology structure.

Furthermore, a graph-theoretic condition for the controllability of game-based control systems is obtained, and the conjecture that there is no limitation of equivalent partition in GBCS is proposed. Arguably, this is a surprising conjecture on the equivalent partition of graphs because only the limitation of equivalent partition in five-node graphs has previously been solved.

Finally, we build a new game control system that is closely related to the topology structure and can directly reflect the characteristics of the topology structure. When compared with the general system that fails to consider game factors, we obtain the fundamental conditions for the controllability difference between the two systems. Therefore, under certain conditions, the two systems are equivalent, which helps to characterize the graph theory of the new GBCS.

1.3 Notations and organization

In this paper, the players in the GBCS are represented as nodes on a graph G. The edges indicate their interactive relationship. The G consists of the node set $V(G)$ and edge set $E(G) \subset V(G) \times V(G)$. Assuming that the system has one regulator and H bottom agents in the system, then $V(G) = \{r_1, l_1, \ldots, l_H\}, E(G) = \{(i,j)|i,j \in V(G)\}$, and the neighbor of node l_i is defined as $N(l_i) = \{(l_i, l_j)|l_i, l_j \in E(G)\}$. Graph G is connected if there is a path between any two different nodes i and j in graph G. The adjacency matrix of graph G is defined as $A(G)$. The Laplacian matrix of the graph G is defined as $L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}([d_i]_{i=1}^n)$, d_i indicates the number of neighbors of node i. I_n is the $n \times n$-dimensional identity matrix, and $0_{n\times m}$ implies the null matrix of $n \times m$-dimension (or 0_n, if $n = m$). $1_n = [1, 1, \ldots, 1]^{\mathrm{T}}$ represents an n-dimensional column vector of n ones.

The rest of this paper is structured as follows. Section 2 introduces the game-based control system and analyzes the representation of the topology structure of the system. Section 3 expresses Nash equilibrium actions under the assumption that the regulator's control input is not zero. Section 4 describes the graph theory condition of a game-based control system. Section 5 investigates the controllability of a new GBCS. Section 6 concludes this paper.

2 Game-based control system

Based on the concept of a two-player zero-sum game, the following linear differential equation describes the dynamics of the game with H players [39]:

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + \cdots + B_Hu_H(t), \tag{1}$$

where $x(t)$ still represents the state of the system, and $u_i, i = 1, 2, \ldots, H$, represents the m_i-dimensional vector that can be manipulated by player i. Each player strives to reduce their quadratic cost function:

$$J_i(u_1, \ldots, u_H) = \int_0^{\mathrm{T}} \left\{ x^{\mathrm{T}}(t)Q_ix(t) + \sum_{j=1}^{H} u_j^{\mathrm{T}}(t)R_{ij}u_j(t) \right\} \mathrm{d}t + x^{\mathrm{T}}(T)Q_{iT}x(T),\ i = 1, 2, \ldots, H. \tag{2}$$

When the game-based control system in [7] is combined with the strategy of regulator in (1), the GBCS with one regulator and H agents can be expressed as

$$\dot{x}_r(t) =Ax_r(t) + \sum_{i=1}^{H} A_ix_i(t) + \sum_{i=1}^{H} D_iu_i(t) + Bu(t), \tag{3}$$

$$\dot{x}_i(t) = E_ix_r(t) + \sum_{j=1}^{H} F_{ij}x_i(t) + \sum_{j=1}^{H} B_{ij}u_j(t) + B_iu(t),\ x_i(0) = x_{i,0},\ i = 1, 2, \ldots, H, \tag{4}$$

where $x_r(t) \in \mathbb{R}^n, x_i(t) \in \mathbb{R}^{n_i}$ are the state of the regulator and agent i, respectively; and $A, A_i, D_i, B, E_i, F_{ij}, B_{ij}, B_i$ are matrices of corresponding dimensions. $u(t) \in \mathbb{R}^m$ is the regulator's strategy, and $u_i \in \mathbb{R}^{m_i}$ is the strategy of agent i.

In particular, to more clearly demonstrate the importance of topology structure on GBCSs, we only consider the case where $x(t), x_i(t), u_i(t), u(t)$ are one-dimensional in this paper, then

$$\dot{x}_r(t) = ax_r(t) + \sum_{i=1}^{H} a_ix_i(t) + \sum_{i=1}^{H} d_iu_i(t) + bu(t),$$

$$\dot{x}_i(t) = e_ix_r(t) + \sum_{i=1}^{H} f_{ij}x_i(t) + \sum_{i=1}^{H} b_{ij}u_j(t) + b_iu(t),\ x_r(0) = x_{r,0},\ x_i(0) = x_{i,0}, \tag{5}$$

where $a, a_i, d_i, b, e_i, f_{ij}, b_{ij}$ and b_i are all constant.

Each player has a quadratic cost function:

$$J_i(u_0, u_1, \ldots, u_H) =\frac{1}{2} \int_0^{\mathrm{T}} \left\{ X^{\mathrm{T}}(t)Q_iX(t) + \sum_{j=0}^{H} u_j^{\mathrm{T}}(t)R_{ij}u_j(t) \right\} \mathrm{d}t + \frac{1}{2}x^F(T)^{\mathrm{T}}Q_{iT}x^F(T),\ i = 1, 2, \ldots, H, \tag{6}$$

where

$$u_0 = u,\ x^F = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_H \end{pmatrix},\ X(t) = \begin{pmatrix} x_r(t) \\ x^F(t) \end{pmatrix},$$

Q_i, Q_{iT} are symmetric matrices, $R_{ij} > 0$. To get a compact form, let

$$\tilde{A} = \begin{pmatrix} a & a_1 & a_2 & \dots & a_H \\ e_1 & f_{11} & f_{12} & \dots & f_{1H} \\ e_2 & f_{21} & f_{22} & \dots & f_{2H} \\ \vdots & \vdots & \vdots & & \vdots \\ e_H & f_{H1} & f_{H2} & \dots & f_H \end{pmatrix}, \ \tilde{B}_i = \begin{pmatrix} d_i \\ b_{1i} \\ b_{2i} \\ \vdots \\ b_{Hi} \end{pmatrix}, \ \tilde{B} = \begin{pmatrix} b \\ b_1 \\ b_2 \\ \vdots \\ b_H \end{pmatrix}.$$

Then, Eq. (5) can be rewritten as follows:

$$\dot{X}(t) = \tilde{A}X(t) + \sum_{i=1}^{H} \tilde{B}_i u_i(t) + \tilde{B}u(t). \tag{7}$$

This model expresses the idea that the regulator initially develops the strategy, and the underlying agents reach a Nash equilibrium after receiving the strategy of the regulator, leading to the conclusion that the state of the system is primarily caused by the regulator's strategy. Therefore, the game strategy formed by H agents is not parallel with the regulator's strategy, and according to the regulator's strategy, Nash equilibrium action is reached.

3 Nash equilibrium action of the GBCS

For better differentiation, Eq. (5) can be written as

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{H} B_i u_i(t) + Cz(t), \tag{8}$$

where u_i represents the action taken by agent i following the noncooperative differential game. If we consider the topological relationship between agents (for instance, agents i and j are not neighbors), it can be seen from (5) that for $\dot{x}_i(t)$, the coefficient in front of u_j can be zero. Similarly, for $\dot{x}_j(t)$, the coefficient in front of u_i is zero. Therefore, the coefficient B_i in front of u_i is a crucial index reflecting the influence of the topology on the game. To avoid clumsy notation, we will limit the analyses to the $H = 1$ case in (8), that is, $\dot{x}(t) = Ax(t) + Bu(t) + Cz(t)$. And $g(t, x(t), u(t))$ and $f(t, x(t), u(t), z(t))$ correspond to the main parts of the cost function J and $\dot{x}(t)$, respectively.

Lemma 1. If there exists a strategy $u(t)$ that minimizes the payoff function (9) in the game-based control system:

$$J(x_0, u) = \int_0^{\mathrm{T}} g(t, x(t), u(t))\mathrm{d}t + h(x(T)), \tag{9}$$

$$\dot{x}(t) = f(t, x(t), u(t), z(t)), \ x(0) = x_0, \tag{10}$$

then for all $t \in [0, T]$,

$$\frac{\mathcal{H}(t, x^*(t), u^*(t), \lambda^*(t), z(t))}{\partial u} = 0, \ \dot{\lambda}^*(t) = -\frac{\partial \mathcal{H}}{\partial x}, \ \lambda^*(T) = \frac{\partial h(x^*(T))}{\partial x}, \tag{11}$$

where $\mathcal{H}(t, x(t), u(t), \lambda(t), z(t)) := g(t, x(t), u(t)) + \lambda(t) f(t, x(t), u(t), z(t))$, $u^*(t)$ is the action that minimizes (9) and $x^*(t), \lambda^*(t)$, and $z(t)$ are the corresponding states, costate variable, and regulator's strategy, respectively. Functions $g(t, x(t), u(t))$ and $f(t, x(t), u(t), z(t))$ are continuous and differentiable.

Lemma 1 is a version of the maximum principle, the specific proof method is similar to [39, P128], and we only provide a sketch of the proof.

Proof. According to (10), a Lagrange multiplier $\lambda(t)$ is selected arbitrarily, then

$$\int_0^{\mathrm{T}} \lambda(t)[f(t, x(t), u(t), z(t)) - \dot{x}(t)]\mathrm{d}t = 0. \tag{12}$$

Appending (12) to (9),

$$\bar{J}(x_0,u)=\int_0^{\mathrm{T}}\{\mathcal{H}(t,x(t),u(t),\lambda(t),z(t))-\lambda(t)\dot{x}(t)\}\mathrm{d}t+h(x(T)). \tag{13}$$

According to (13), the last three terms in $\bar{J}(x_0,u)$ are only related to the initial time, and are independent of t. Therefore, regardless of how the path of $\lambda(t)$ is chosen, it does not affect the value of $\bar{J}(x_0,u)$. That is, $\dot{x}(t)=\frac{\partial\mathcal{H}}{\partial\lambda}, t\in[0,T]$ forms the necessary condition for $\bar{J}$ to take an extreme value. Assume that u^* is the optimal strategy for minimizing $\bar{J}(x_0,u)$, and $x^*(t)$ is the corresponding optimal state trajectory. If u^* is slightly disturbed, then $u(t)=u^*(t)+\epsilon p(t)$, and if ϵ is sufficiently small, Eq. (13) becomes

$$\bar{J}(\epsilon)=\int_0^{\mathrm{T}}[\mathcal{H}(t,x(t,\epsilon,p),u^*(t)+\epsilon p,\lambda,z(t))]\mathrm{d}t+h(x(T,\epsilon,p))-\lambda(T)x(T,\epsilon,p)+\lambda(0)x_0. \tag{14}$$

Because $\bar{J}(\epsilon)$ minimizes at $\epsilon=0$ and f,g,h,z can be differentiated, $\bar{J}$ can be differentiated with respect to ϵ. Then, $\frac{\mathrm{d}\bar{J}}{\mathrm{d}\epsilon}=0$ when $\epsilon=0$,

$$\frac{\partial\bar{J}}{\partial\epsilon}=\int_0^{\mathrm{T}}\left\{\left[\frac{\partial\mathcal{H}}{\partial x}+\dot{\lambda}(t)\right]\frac{\mathrm{d}x}{\mathrm{d}\epsilon}+\frac{\partial\mathcal{H}}{\partial u}p(t)\right\}\mathrm{d}t+\left[\frac{\partial h(x^*)}{\partial x}-\lambda(T)\right]\frac{\mathrm{d}x(T,\epsilon,p)}{\mathrm{d}\epsilon}. \tag{15}$$

According to the boundary value conditions,

$$\int_0^{\mathrm{T}}\frac{\partial\mathcal{H}(t,x^*,u^*,\lambda,z)}{\partial u}p(t)\mathrm{d}t=0. \tag{16}$$

If $p(t)=\frac{\partial\mathcal{H}^{\mathrm{T}}}{\partial u}$, then

$$\frac{\partial\mathcal{H}(t,x^*,u^*,\lambda,z)}{\partial u}=0. \tag{17}$$

Assumption 1. For any initial state $x_0,x_{i,0}$, open-loop Nash equilibria exist in (5) and (6), and the following Riccati differential equations have symmetric solutions $K_i, i=1,2,\ldots,H$:

$$\dot{K}_i(t)=-\tilde{A}^{\mathrm{T}}K_i(t)-K_i(t)\tilde{A}+K_i(t)\tilde{S}_iK_i(t)-Q_i,\ K_i(T)=\tilde{Q}_{iT},\ i=1,2,\ \tilde{S}_i:=\tilde{B}_iR_{ii}^{-1}\tilde{B}_i^{\mathrm{T}}. \tag{18}$$

Lemma 2. Consider a linear system with H players,

$$J_i:=\int_0^{\mathrm{T}}\{x^{\mathrm{T}}(t)Q_ix(t)+u_1^{\mathrm{T}}R_{i1}^{-1}u_1+u_2^{\mathrm{T}}R_{i2}^{-1}u_2+\cdots+u_H^{\mathrm{T}}R_{iH}^{-1}u_H\}\mathrm{d}t+x^{\mathrm{T}}(T)Q_{iT}x(T), \tag{19}$$

$$\dot{x}=Ax+B_1u_1+B_2u_2+\cdots+B_Hu_H+Cz, \tag{20}$$

$$M=\begin{pmatrix} A & -S_1 & -S_2 & \cdots & -S_H \\ -Q_1 & -A^{\mathrm{T}} & 0 & \cdots & 0 \\ -Q_2 & 0 & -A^{\mathrm{T}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -Q_H & 0 & 0 & \cdots & -A^{\mathrm{T}} \end{pmatrix},\ S_i:=B_iR_{ii}^{-1}B_i^{\mathrm{T}}. \tag{21}$$

The existence of an open-loop Nash equilibrium action for each initial state is then required and sufficient if the matrix $\mathbb{H}(T)$ is invertible,

$$\mathbb{H}(T)=\begin{pmatrix} I & 0 & 0 & \cdots & 0\end{pmatrix}\mathrm{e}^{-MT}\begin{pmatrix} I \\ Q_{1T} \\ Q_{2T} \\ \vdots \\ Q_{HT}\end{pmatrix}. \tag{22}$$

Furthermore, if for every x_0, there is an open-loop Nash equilibrium action, then the action is unique, and $u_i = -R_{ii}^{-1}B_i^{\mathrm{T}}\varphi_i(t)$, where $\varphi_i(t)$ is costate variable.

Methods similar to [39, P266] can be employed to prove this lemma, and details are omitted.

According to Lemma 2, we can derive

$$\begin{cases} u_i^* = R_{ii}^{-1}B_i^{\mathrm{T}}\varphi_i(t), \\ \dot{\varphi}_i(t) = Q_iX(t) - \tilde{A}\varphi_i(t), \\ \varphi_i(T) = -\tilde{Q}_{iT}X(T),\ i = 1, 2, \ldots, H. \end{cases} \tag{23}$$

Then, Eq. (5) can be rewritten as follows:

$$\begin{cases} \dot{X}(t) = \tilde{A}X(t) + \sum_{i=1}^{H}\tilde{B}_iu_i^* + \tilde{B}u, \\ X(0) = X_0. \end{cases} \tag{24}$$

According to (23) and (24), it can be obtained

$$\begin{pmatrix} \dot{X}(t) \\ \dot{\varphi}(t) \end{pmatrix} = \bar{A}\begin{pmatrix} X(t) \\ \varphi(t) \end{pmatrix} + \bar{B}u, \tag{25}$$

where

$$\bar{A} = \begin{pmatrix} \tilde{A} & \tilde{B}_1R_{11}^{-1}\tilde{B}_1^{\mathrm{T}} & \cdots & \tilde{B}_HR_{HH}^{-1}\tilde{B}_H^{\mathrm{T}} \\ Q_1 & -\tilde{A}^{\mathrm{T}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ Q_H & 0 & \cdots & -\tilde{A}^{\mathrm{T}} \end{pmatrix},\ \bar{B} = \begin{pmatrix} \tilde{B} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Because the elements in $\bar{A}$ are mostly studied in the form of block matrix in the following, each block matrix $\tilde{A}, Q_i$, and $\tilde{B}_iR_{ii}^{-1}\tilde{B}_i^{\mathrm{T}}$ in $\bar{A}$ is regarded as a whole, and $\bar{A}_{ij}$ represents the block matrix in row i and column j.

According to [7], the controllability of system (7) and (8) is equivalent to that of the system (25), thus, the controllability of system (25) is investigated further. In particular, the controllability matrix of the game-based control system is as follows:

$$Q = \begin{pmatrix} \bar{B} & \bar{A}\bar{B} & \bar{A}^2\bar{B} & \cdots & \bar{A}^{(H+1)^2-1}\bar{B} \end{pmatrix}. \tag{26}$$

4 Graph-theoretic conditions for GBCS controllability

When we use algebraic conditions to judge the controllability of GBCSs, we must perform complex mathematical calculations, prompting us to seek graph-theoretic conditions to judge the controllability, and the above algebraic conditions validate the rationality of the graph theory conditions obtained in this paper.

According to the system (5) and (6), it can be seen that u_i plays an important role in the system, and the Nash equilibrium action considered in this paper is composed of the coefficient $\tilde{B}_i$ in front of u_i. To comprehensively analyze the key influence of topology structure on GBCSs, let $\tilde{A}$ be the identity matrix, $C = 0$. If agents i and j are neighbors, it is reflected in $\tilde{B}_i(j) = 1$, where $\tilde{B}_i(j)$ represents the j-th element in vector $\tilde{B}_i$, otherwise, $\tilde{B}_i(j) = 0$.

Definition 1. Strategy equivalence partition (SEP): If multiple agents receive the same number of strategies from agents in any cell, then these agents can be divided into the same cell. If $s(i)$ represents the number of strategies received by agent i, and C_p represents the p-th cell, then for any $i, j, p, (s(i), C_p) = (s(j), C_p)$, agents i, j can be divided into the same cell, where $(s(j), C_p)$ represents the total number of strategies of the cell C_p received by agent j. If the number of nodes in a cell exceeds one, the cell is considered nontrivial; otherwise, it is considered trivial.

SEP is a partition within the game-based control system, and it is similar to the concept of EP. Specifically, SEP is only applicable when the system has a single regulator. At this point, the strategy matrix can derive the following rules from the strategy equivalent partition: the row vectors of the strategy matrix corresponding to nodes in the same cell are equal (except for the diagonal position).

Theorem 1. Assume Assumption 1 is correct. If a nontrivial strategy equivalent partition exists, $C_i = \{i_1, i_2, \ldots, i_p\}$, and the i_1-th, i_2-th, $\ldots$, i_q-th row vectors of T are equal, where

$$T = [I_{(H+1)}\ 0]\mathrm{e}^{-\bar{A}T}\begin{pmatrix} I_{H+1} \\ -\tilde{Q}_{1T} \\ \vdots \\ -\tilde{Q}_{HT} \end{pmatrix}\begin{pmatrix} 0_{1\times(H)} \\ I_H \end{pmatrix},$$

then, the GBCS (25) is uncontrollable. Furthermore, the general formula of the controllability matrix (26) is

$$Q_{pq} = \begin{cases} Q_{1q} = \begin{cases} \tilde{B}, & q = 1, \\ \tilde{A}^2 Q_{1(q-2)} + \displaystyle\sum_{i=1}^{H} \tilde{B}_i R_{ii}^{-1} \tilde{B}_i^{\mathrm{T}} Q_i Q_{1(q-2)}, & q \text{ is odd}, \\ \tilde{A} Q_{1(q-1)}, & q \text{ is even}, \end{cases} \\ Q_{pq}(1 < p \leqslant (H+1)^2 - 1) \ = \begin{cases} 0, & q \text{ is odd}, \\ Q_{(p-1)} Q_{1(q-1)}, & q \text{ is even}. \end{cases} \end{cases}$$

Proof. To demonstrate the controllability condition of the game-based control system (25), we first demonstrate the general formula of the controllability matrix. We begin by calculating the first element of the three cases in Q_{1q}:

$$Q_{11} = \tilde{B},\ Q_{12} = \tilde{A}\tilde{B},\ Q_{13} = \tilde{A}^2\tilde{B} + \sum_{i=1}^{H} \tilde{B}_i R_{ii}^{-1} \tilde{B}_i^{\mathrm{T}} Q_i \tilde{B}.$$

(1) If $(H+1)^2$ is even, then $(H+1)^2 - 1$ is odd; that is, the last element in Q_{1q} is odd. Then we assume that

$$Q_{1[(H+1)^2-3]} = \tilde{A}^2 Q_{1[(H+1)^2-5]} + \sum_{i=1}^{H} \tilde{B}_i R_i^{-1} \tilde{B}_i^{\mathrm{T}} Q_i Q_{1[(H+1)^2-5]},\ Q_{1[(H+1)^2-4]} = \tilde{A} Q_{1[(H+1)^2-5]};$$

the calculations indicate that

$$Q_{21} = \cdots = Q_{H1} = 0,\ Q_{i2} = \tilde{A} Q_i,\ i = 2, \ldots, H,$$

then

$$Q_{23} = Q_1 Q_{12} - \tilde{A}^{\mathrm{T}} Q_1 \tilde{B}.$$

And because Q_1 is a symmetric matrix, $Q_{23} = 0$. Likewise, $Q_{33} = Q_{43} = \cdots = Q_{H3} = 0$. Assume that

$$Q_{2[(H+1)^2-3]} = Q_{3[(H+1)^2-3]} = \cdots = Q_{H[(H+1)^2-3]} = 0,$$

$$Q_{i[(H+1)^2-4]} = Q_{(i-1)} Q_{1[(H+1)^2-5]},\ i = 2, 3, \ldots, H.$$

Then,

$$Q_{1[(H+1)^2-2]} = \tilde{A} Q_{1[(H+1)^2-3]} + \sum_{i=1}^{H} \tilde{B}_i R_{ii}^{-1} \tilde{B}_i^{\mathrm{T}} \times 0,$$

$$Q_{i[(H+1)^2-2]} = Q_i Q_{1[(H+1)^2-3]} + (-\tilde{A}^{\mathrm{T}}) \times 0,$$

$$Q_{1[(H+1)^2-1]} = \tilde{A} Q_{1[(H+1)^2-2]} + \sum_{i=1}^{H} \tilde{B}_i R_{ii}^{-1} \tilde{B}_i^{\mathrm{T}} Q_i Q_{1[(H+1)^2-3]},$$

$$Q_{i[(H+1)^2-1]} = Q_i\tilde{A}Q_{1[(H+1)^2-3]} - \tilde{A}^{\mathrm{T}}Q_iQ_{1[(H+1)^2-3]}.$$

(2) If $(H+1)^2$ is odd, the last column of the controllability matrix is even. Then assume that

$$Q_{1[(H+1)^2-3]} = \tilde{A}Q_{1[(H+1)^2-4]},$$

$$Q_{1[(H+1)^2-4]} = \tilde{A}^2Q_{1[(H+1)^2-6]} + \sum_{i=1}^{H}\tilde{B}_iR_{ii}^{-1}\tilde{B}_i^{\mathrm{T}}Q_iQ_{1[(H+1)^2-6]}.$$

Its initial elements remain unchanged. In this case, the difference from (1) is the subscript of the element, and the proof details are omitted.

According to the aforementioned general formula of controllability matrix, it can be found that matrix$\sum_{i=1}^{H}\tilde{B}_iR_i^{-1}\tilde{B}_i^{\mathrm{T}}$ exists in every element of controllability matrix. To obtain a clearer view of this question within the context of topology structure, take $R_{ii}^{-1}=1$. The following focuses on the properties of matrix $\sum_{i=1}^{H}\tilde{B}_iR_{ii}^{-1}\tilde{B}_i^{\mathrm{T}}$ in the presence of strategy equivalent partition. Call $S=\sum_{i=1}^{H}\tilde{B}_i\tilde{B}_i^{\mathrm{T}}$ a strategy matrix, $S\in\mathbb{R}^{(H+1)\times(H+1)}$. Set the serial number of the system's regulator to r_1, and the serial numbers of the other agents as $r_2,r_3,\ldots,r_H$. Because the regulator has a neighbor relationship with each agent as well as with itself, the non-neighbor row of vector $\tilde{B}_i$ is 0; i.e., the nonzero elements in matrix $\tilde{B}_i\tilde{B}_i^{\mathrm{T}}$ are located in the neighbor rows and neighbor columns of agent i. Therefore, the element in row i and column j of matrix $\sum_{i=1}^{H}\tilde{B}_i\tilde{B}_i^{\mathrm{T}}$ represents the number of agents that contain both i and j in the neighbor set; i.e., S_{ij} is equal to the number of common neighbors of i and j. It should be noted that S_{ii} represents the number of neighbors of agent i, including i itself. S must be a symmetric matrix, and the first row and column are equal to the diagonal elements. If the system has a strategy equivalent partition in the system, it is assumed that there are p cells, $C_1,\ldots,C_p$. If $|C_i|=s_i$, the cell C_i contains s_i elements. The nodes in the cell are labeled with s_i consecutive numbers, and the strategy matrix S is divided into blocks based on the order of the cell:

$$S=\begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1p}\\ P_{21} & P_{22} & \cdots & P_{2p}\\ \vdots & \vdots & & \vdots\\ P_{p1} & P_{p2} & \cdots & P_{pp}\end{pmatrix},$$

where $P_{ij}\in\mathbb{R}^{s_i\times s_j}$. Let $i_1,i_2\ldots,i_q\in C_i$, then

$$\sum_{j=1}^{(H+1)}S_{i_1j}=\sum_{j=1}^{(H+1)}S_{i_2j}=\cdots=\sum_{j=1}^{(H+1)}S_{i_nj}.$$

That is, the elements in the same cell have the same sum. Or, to put it another way, lines $i_1,\ldots,i_q$ in $\sum_{i=1}^{H}\tilde{B}_iR_{ii}^{-1}\tilde{B}_i^{\mathrm{T}}\tilde{B}$ are equal. Therefore, in the matrix

$$[I_{(H+1)}\ 0]\bar{A}^{k+1}\bar{B},\ k=1,2,\ldots,[(H+1)^2-1],$$

the rows of the above matrix that correspond to the elements of the same cell are equal. If the entries in the same cell correspond to the same row vectors of the following matrix are also equal:

$$[I_{(H+1)}\ 0]\mathrm{e}^{-\bar{A}T}\begin{pmatrix} I_{H+1}\\ -\tilde{Q}_{1T}\\ \vdots\\ -\tilde{Q}_{HT}\end{pmatrix}\begin{pmatrix} 0_{1\times(H)}\\ I_H\end{pmatrix},$$

then the following matrix is not full row rank:

$$\left(\begin{matrix}[I_{(H+1)}\ 0]\bar{A}^{k+1}\bar{B} & [I_{(H+1)}\ 0]\mathrm{e}^{-\bar{A}T}\begin{pmatrix} I_{H+1}\\ -\tilde{Q}_{1T}\\ \vdots\\ -\tilde{Q}_{HT}\end{pmatrix}\begin{pmatrix} 0_{1\times(H)}\\ I_H\end{pmatrix}\end{matrix}\right).$$

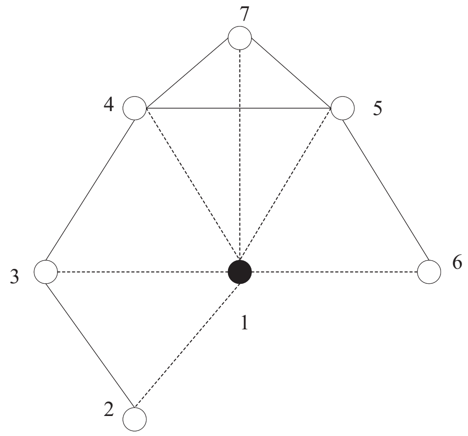

Figure 1 A system with 1 regulator and 6 agents.

According to ([7, Theorem 2]), the game-based control system is uncontrollable.

In the traditional control theoretical framework, as demonstrated in [40], the equivalence partition possesses limitations; that is, it is impossible to determine whether the system is controllable solely based on the equivalent partition. According to the model of [40], the presence of an equivalent partition can lead to an uncontrollable system, but it is not a necessary condition for uncontrollability. As illustrated in Figure 1, node 1 is designated as the regulator, while the remaining nodes are designated as followers. Although it is determined that there is no equivalent partition in the system, the system remains uncontrollable because the system matrix contains eigenvectors orthogonal to the 1_n vector. However, under the game-based control system, the system comprising this topology structure is controllable. Due to the limitation of the equivalence partition problem, this exists as the sole counterexample; however, this structure does not hold for GBCSs because the strategy matrix S lacks eigenvectors that are orthogonal to the 1_n vector. Therefore, we hypothesize that there is no limitation of strategy equivalent partition (SEP) in GBCSs, implying that Theorem 1 is sufficient.

5 Controllability analysis of a new control system based on leader-follower framework

Following the fundamentals of Section 4, in this section, we investigate the controllability of a new GBCS in the leader-follower framework based on Laplacian.

Assume there are $n+m$ agents in the system, with the first n acting as followers: $1,\dots,n$, and the last m acting as leaders: $n+1,\dots,n+m$; the Laplacian matrix can be divided into

$$\begin{array}{rcccccc} & \text{followers} & n+1 & n+2 & n+3 & \cdots & n+m \\ [I_n \quad 0]L = \big(& L_f & L_{fl1} & L_{fl2} & L_{fl3} & \cdots & L_{flm} \big). \end{array}$$

All players participating in the game are considered as leaders, u_i as the leader's strategy, and the variables x_i influenced by these players are considered follower i. The status of followers is influenced not only by the leader's post-game strategy, but also by interactions between followers and followers and between followers and leaders. Therefore, the system can be expressed as

$$\dot{x}(t) = -L_f x(t) - L_{fl1}u_1 - \cdots - L_{flm}u_m - L_{fl}u(t),\ x(0) = x_0. \tag{27}$$

This suggests that the noncooperative linear-quadratic difference game with m leaders corresponds to a fixed connected graph with a Laplacian matrix L describing the communication relationship between agents.

u_i is the open-loop strategy of leader i; $L_f, L_{fl1},\dots,L_{fln}$ represent the communication relations between agents that can be written directly according to the topology, and are known quantities. $x(t) = [x_1^{\mathrm{T}}(t),\dots,x_n^{\mathrm{T}}(t)]^{\mathrm{T}}$ denotes the follower's current state $i, i = 1,2,\dots,n$. m leaders can always reach Nash equilibrium after playing a non-cooperative game. $u(t)$ represents the external control signal dictated by the leaders and drives the follower's state to a desired value. Each u_i minimizes the following payoff:

$$J_i(u_i) = \frac{1}{2}\int_0^{\mathrm{T}} [x^{\mathrm{T}}(t)Q_i(t)x(t) + u_i^{\mathrm{T}}(t)R_{ii}(t)u_i(t)]\mathrm{d}t, \tag{28}$$

where $Q_i(t) = Q_i^{\mathrm{T}}(t) \geqslant 0$, $R_{ii}(t) = R_{ii}^{\mathrm{T}}(t) > 0$. Each leader's goal is to minimize its own cost function by selecting the optimal strategy for the linear dynamic system. Noncooperative implies that no leader will cooperate in achieving this goal. Each leader has an open-loop information structure, which means that each leader must formulate their actions at the beginning of the development of the system, and these actions cannot be changed once the system is operational. Therefore, each leader must minimize J_i using knowledge of the differential equation and its initial state. We will consider Nash equilibria reached by leaders in noncooperative differential games; that is, for a set of strategies $u^*(t) = [u_1^*(t), \ldots, u_m^*(t)]^{\mathrm{T}}$ of the leader if a leader i deviates from his strategy u_i^*, his payoff increases.

Remark 1. According to the system (27) and (28), the Nash equilibrium $u^*(t) = [u_1^*(t), \ldots, u_m^*(t)]^{\mathrm{T}}$ always exists and is unique. Because the system (27) always corresponds to a fixed topology with a Laplacian matrix, and the elements in the corresponding matrix and vector $L_f, L_{fl1}, \ldots, L_{flm}$ are fixed constants, then the corresponding m Riccati differential equations

$$\dot{K}_i(t) = L_f^{\mathrm{T}}(t)K_i(t) + K_i(t)L_f(t) + K_i(t)L_{fli}(t)R_{ii}L_{fli}^{\mathrm{T}}(t)K_i(t) - Q_i, \quad K_i(T) = 0, i = 1, 2, \ldots, m$$

has a $K(\cdot)$ solution on $[0, T]$. This can be obtained according to the fundamental existential-uniqueness theorem [39, P72]. The solution $K(\cdot)$ to the Riccati differential is symmetric because $K^{\mathrm{T}}(\cdot)$ also satisfies the corresponding Riccati differential by transposing both sides of Riccati differential equations, so $K(\cdot)$ is symmetric by the uniqueness of the solution. To summarize, there exists a unique Nash equilibrium u_i^* in the system (27) and (28), and this unique Nash equilibrium can be obtained using the maximum principle: $u_i^*(t) = R_{ii}^{-1}L_{fli}\varphi_i(t)$, where $\varphi_i(t)$ is adjoint variables.

Remark 2 states that the m leaders will always reach a unique Nash equilibrium. By substituting $u_i^*(t)$ into (27), we can obtain

$$\begin{cases} \dot{x}(t) = -L_f x(t) + L_{fl1}R_1^{-1}L_{fl1}^{\mathrm{T}}\varphi_1(t) + \cdots + L_{flm}R_m^{-1}L_{flm}^{\mathrm{T}}\varphi_m(t) - L_{fl}u(t), \\ \dot{\varphi}_i(t) = Q_i x(t) + L_f^{\mathrm{T}}\varphi_i(t), \\ x(0) = x_0, \varphi_i(T) = 0, i = 1, 2, \ldots, m, \end{cases} \tag{29}$$

where $x(t) = [x_1^{\mathrm{T}}(t), \ldots, x_n^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\varphi_i(t) = [\varphi_{i1}^{\mathrm{T}}(t), \ldots, \varphi_{in}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\varphi(t) = [\varphi_1^{\mathrm{T}}(t), \ldots, \varphi_m^{\mathrm{T}}(t)]^{\mathrm{T}}$, $X(t) = [x^{\mathrm{T}}(t) \;\; \varphi^{\mathrm{T}}(t)]^{\mathrm{T}}$, $L_{fl} = [L_{fl1}, \ldots, L_{flm}]$,

$$\hat{A} = \begin{pmatrix} -L_f & L_{fl1}R_1L_{fl1}^{\mathrm{T}} & \cdots & L_{flm}R_mL_{flm}^{\mathrm{T}} \\ Q_1 & L_f & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ Q_m & 0 & \cdots & L_f \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} L_{fl1} & L_{fl2} & \cdots & L_{flm} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then Eq. (29) can be expressed as

$$\dot{X}(t) = \hat{A}X(t) + \hat{B}u(t). \tag{30}$$

For the sake of simplicity, we will present the one-dimensional case, with $Q_i = I, R_{ii} = 1$, then $X(t) \in \mathbb{R}^{(m+1)n}$, $\hat{A} \in \mathbb{R}^{(m+1)n\times(m+1)n}$, $\hat{B} \in \mathbb{R}^{(m+1)n\times mn}$.

The necessary and sufficient conditions for driving the system state of a linear time-invariant system (30) to the desired state are

$$\operatorname{rank}((\, I_n \;\; 0 \,)(\, \hat{B} \;\; \hat{A}\hat{B} \;\; \hat{A}^2\hat{B} \;\cdots\; \hat{A}^{(m+1)n-1}\hat{B} \,)) = n. \tag{31}$$

Therefore, it is clear that whether the new GBCS can be controlled is dependent on the topology, because different topologies result in different matrices $\hat{A}, \hat{B}$. The specific proof process is similar to [7, Th2], with some details omitted.

Remark 2. A classical system for studying the controllability of topological structures is as follows:

$$\dot{x}_i = u_i, \tag{32}$$

$$u_i = \sum_{j\in N_i}(x_j - x_i). \tag{33}$$

Based on the neighbor relationship, the system can be transformed into the following form:

$$\dot{x} = -L_f x - L_{fl}u. \tag{34}$$

If we define the above equation by (L_f, L_{fl}), then the necessary and sufficient condition for the system's controllability is

$$\mathrm{rank}(Q_C) = n,$$

$$Q_C = [-L_{fl} \;\; L_f L_{fl} \;\; -L_f^2 L_{fl} \;\; \cdots \;\; -(-L_f)^{n-1}L_{fl}].$$

When leaders play noncooperative differential games, the controllability of the new GBCS has the following relationship with the controllability of (L_f, L_{fl}), which also reflects the influence of the game between leaders on the controllability of the system.

Theorem 2. If $\mathrm{rank}(T_1 \; T_2 \; T_3) = \dim(\mathrm{Im}(Q_C \; 0) \cap \mathrm{Im}(T_1 \; T_2 \; T_3)), T_1 = 0_{n\times 2}, T_2 = \hat{A}^i\hat{B}(1) + (-L_f)^i L_{fl}, i = 2,\ldots,(n-1), T_3 = [\hat{A}^n\hat{B} \; \hat{A}^{n+1}\hat{B} \; \cdots \; \hat{A}^{(m+1)n-1}\hat{B}]$, then, under the same topology and leader nodes, the necessary and sufficient condition for controllability of system (30) is that system (L_f, L_{fl}) is controllable, and the general formula of each element in controllability matrix $\hat{Q}_C$ of system (30) is

$$\hat{A}^k\hat{B}(1) = \begin{cases} L_{fl}, & k = 0, \\ (-L_f)\hat{A}^{k-1}\hat{B}(1), & k \text{ is an odd number}, \\ \sum\limits_{j=1}^{m} L_{fl}R_j L_{fl}^{\mathrm{T}}\hat{A}^{k-1}\hat{B}(j+1) + (-L_f)\hat{A}^{k-1}\hat{B}(1), & k \text{ is a nonzero even number}, \end{cases}$$

$$\hat{A}^k\hat{B}(j+1),\; j = 1,2,\ldots,(m+1) = \begin{cases} 0, & k \text{ is an even number}, \\ Q_j\hat{A}^{k-1}\hat{B}(1), & k \text{ is an odd number}. \end{cases}$$

Proof. Let

$$Q_C = [-L_{fl} \;\; L_f L_{fl} \;\; -L_f^2 L_{fl} \;\; \cdots \;\; -(-L_f)^{n-1}L_{fl}],$$

$$\hat{Q}_C = [\hat{B}(1) \;\; \hat{A}\hat{B}(1) \;\; \hat{A}^2\hat{B}(1) \;\; \cdots \;\; \hat{A}^{(m+1)n-1}\hat{B}(1)]$$

denote the controllability matrix of (L_f, L_{fl}) and system (30), respectively; then the above conclusion can be expressed as follows: if $\mathrm{rank}(T_1 \; T_2 \; T_3) = \dim(\mathrm{Im}(Q_C \; 0) \cap \mathrm{Im}(T_1 \; T_2 \; T_3))$, then the necessary and sufficient condition for $\mathrm{rank}(Q_C) = n$ is $\mathrm{rank}(\hat{Q}_C) = n$.

The matrix $A^k B$ is partitioned sequentially, where $\hat{A}^k\hat{B}(1)$ represents the first n rows of $\hat{A}^k\hat{B}$, namely the first block matrix of $\hat{A}^k\hat{B}$, and the remaining mn rows are sequentially divided into m block matrices.

We first assess the characteristics of each element in Q_C. When $k = 0$,

$$\hat{B}(1) = L_{fl},\; \hat{B}(j+1) = 0,\; j = 1,2,\ldots,m.$$

When $k = 1$,

$$\hat{A}\hat{B}(1) = (-L_f)\hat{B}(1),\; \hat{A}\hat{B}(j+1) = Q_j\hat{B}(1).$$

When $k = 2$,

$$\begin{aligned} \hat{A}^2\hat{B}(1) &= (-L_f)^2(-L_{fl}) + \sum_{j=1}^{m} L_{fl}R_j L_{fl}^{\mathrm{T}}Q_j L_{fl} \\ &= (-L_f)[\hat{A}\hat{B}(1)] + \sum_{j=1}^{m} L_{fl}R_j L_{fl}^{\mathrm{T}}[\hat{A}\hat{B}(j+1)]. \end{aligned}$$

By the symmetry of Q_j and L_f, $\hat{A}^2\hat{B}(j+1) = -Q_j L_f L_{fl} + L_f Q_j L_{fl} = 0$. Since $k = (m+1)n-1$, the parity of k is determined by the number of leaders m and the number of followers n. Therefore, the parity of (m,n) only occurs in the following four situations: (even, even), (odd, odd), (odd, even), (even, odd). The following cases are discussed:

(1) When (m,n) takes the first three cases, k is odd. In this case, suppose

$$\hat{A}^{k-2}\hat{B}(1) = (-L_f)[\hat{A}^{k-3}\hat{B}(1)],$$

$$\hat{A}^{k-2}\hat{B}(j+1) = Q_j\hat{A}^{k-3}\hat{B}(1),$$

then

$$\hat{A}^{k-1}\hat{B}(1) = \sum_{j=1}^{m} L_{fl}R_jL_{fl}^{\mathrm{T}}[\hat{A}^{k-2}\hat{B}(j+1)] - L_f[\hat{A}^{k-2}\hat{B}(1)],$$

$$\begin{aligned}\hat{A}^{k-1}\hat{B}(j+1) &= Q_j[\hat{A}^{k-2}\hat{B}(1)] + L_f[\hat{A}^{k-2}\hat{B}(j+1)]\\ &= Q_j[(-L_f)[\hat{A}^{k-3}\hat{B}(1)]] + L_fQ_j\hat{A}^{k-3}\hat{B}(1).\end{aligned}$$

We know from the symmetry of L_f and Q_i that $\hat{A}^{k-1}\hat{B}(j+1) = 0$, therefore

$$\begin{aligned}\hat{A}^{k}\hat{B}(1) &= (-L_f)\hat{A}^{k-1}\hat{B}(1) + \sum_{j=1}^{m} L_{fl}R_jL_{fl}^{\mathrm{T}}[\hat{A}^{k-1}\hat{B}(j+1)]\\ &= (-L_f)\hat{A}^{k-1}\hat{B}(1),\end{aligned}$$

$$\hat{A}^{k}\hat{B}(j+1) = Q_j[\hat{A}^{k-1}\hat{B}(1)] + L_f * 0 = Q_j[\hat{A}^{k-1}\hat{B}(1)].$$

(2) When the number of leaders m is even and the number of followers is odd, suppose that

$$\begin{cases}\hat{A}^{k-2}\hat{B}(1) = \sum_{j=1}^{m} L_{fl}R_jL_{fl}^{\mathrm{T}}[\hat{A}^{k-3}\hat{B}(j+1)] + (-L_f)\hat{A}^{k-3}\hat{B}(1),\\ \hat{A}^{k-2}\hat{B}(j+1) = 0,\end{cases}$$

then

$$\hat{A}^{k-1}\hat{B}(1) = (-L_f)[\hat{A}^{k-2}\hat{B}(1)],$$

$$\hat{A}^{k-1}\hat{B}(j+1) = Q_j\hat{A}^{k-2}\hat{B}(1).$$

Therefore,

$$\hat{A}^{k}\hat{B}(1) = (-L_f)[\hat{A}^{k-1}\hat{B}(1)] + \sum_{j=1}^{m} L_{fl}R_jL_{fl}^{\mathrm{T}}[\hat{A}^{k-1}\hat{B}(j+1)],$$

$$\begin{aligned}\hat{A}^{k}\hat{B}(j+1) &= Q_j[\hat{A}^{k-1}\hat{B}(1)] + L_f[\hat{A}^{k-1}\hat{B}(j+1)]\\ &= Q_j(-L_f)[\hat{A}^{k-2}\hat{B}(1)] + L_fQ_j[\hat{A}^{k-2}\hat{B}(1)]\\ &= 0.\end{aligned}$$

In summary, each element of $\hat{Q}_C$ can be directly given by the general formula, and every element $\hat{A}^{k}\hat{B}(1)$ in $\hat{Q}_C$ always contains the $(-L_f)^k(-L_{fl})$ term, then

$$\hat{Q}_C = [Q_C\ 0] + [T_1\ T_2\ T_3],$$

where $[Q_C\ 0] \in \mathbb{R}^{n\times(m+1)n}$. Since $\mathrm{Im}(\hat{Q}_C) = \mathrm{Im}(Q_C\ 0) + \mathrm{Im}(T_1\ T_2\ T_3)$, then

$$\dim[\mathrm{Im}(\hat{Q}_C)] + \dim[\mathrm{Im}(Q_C\ 0) + \mathrm{Im}(T_1\ T_2\ T_3)] = \dim[\mathrm{Im}(Q_C\ 0)] + \dim[\mathrm{Im}(T_1\ T_2\ T_3)].$$

If $\mathrm{rank}(T_1\ T_2\ T_3) = \dim[\mathrm{Im}(Q_C\ 0) \cap \mathrm{Im}(T_1\ T_2\ T_3)]$, then $\mathrm{rank}(\hat{Q}_C) = \mathrm{rank}(Q_C)$. Therefore, the necessary and sufficient condition for the system (30) to be controllable is that (L_f, L_{fl}) is controllable.

6 Conclusion

How does topology affect the GBCS? This paper contends that its primary influence is found in the Nash equilibrium action; that is, different topologies will produce different Nash equilibrium actions. We started with the expression of Nash equilibrium action under the assumption that the regulator's control strategy is nonzero. The general formula of the controllability matrix of a game-based control system was then obtained. Based on this, we discovered that the strategy matrix can be obtained not only through algebraic calculation but also through topological structure. The position of the strategy matrix in a game-based control system is comparable to that of the Laplacian matrix in a general control system. Through the preceding analysis, we obtained the graph theory condition based on the strategy equivalent partition, and proposed the hypothesis that the strategy equivalent partition does not possess limitations in the GBCS. Finally, we established a new GBCS and investigated the conditions under which the system is as controllable as the general system.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 62373205, 62033007), Taishan Scholars Project of Shandong Province of China (Grant Nos. tstp20230624, ts20190930), and Taishan Scholars Climbing Program of Shandong Province of China.

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