SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

December 2023, Vol. 66 222205:1–222205:13
https://doi.org/10.1007/s11432-023-3787-y

Fault detectability of Boolean control networks via nonaugmented methods

Rong ZHAO, Caixia WANG, Yongyuan YU* & Jun-E FENG

School of Mathematics, Shandong University, Jinan 250100*, China*

Received 10 March 2023/Revised 11 May 2023/Accepted 25 May 2023/Published online 27 November 2023

Abstract This study is concerned with the fault detectability of Boolean control networks (BCNs) by two nonaugmented methods. Firstly, the equivalent system-based approach is considered, and the equivalence of BCNs is applied to analyze weak active fault detectability. Further, an iterative matrix set-based approach is proposed, by which, several novel criteria for strong and weak active fault detectability are presented. Meanwhile, effective algorithms are designed to check strong and weak active fault detectability and generate all feasible input sequences with the minimum length. In comparison, our results reduce the computational complexity dramatically than the existing studies.

Keywords Boolean control network, fault detectability, equivalent system, nonaugmented method, semi-tensor product

Citation Zhao R, Wang C X, Yu Y Y, et al. Fault detectability of Boolean control networks via nonaugmented methods. Sci China Inf Sci, 2023, 66(12): 222205, https://doi.org/10.1007/s11432-023-3787-y

1 Introduction

One powerful model to characterize gene regulatory networks (GRNs) is the Boolean network (BN), which is discrete-time and discrete-state [1]. In a BN, the genes of a GRN are viewed as either active (ON/TRUE/1) or inhibited (OFF/FALSE/0), and the interactions among genes are expressed in terms of logical update rules [1, 2]. Further, a BN with inputs and outputs is called a Boolean control network (BCN) [3]. In the past few decades, BNs and BCNs have been widely applied in systems biology to capture the coarse-grained dynamics of various GRNs and confirmed to provide a nice approximation of continuous dynamic processes [4–6]. However, compared with the theory of continuous models, the control methods for discrete models are still in their infancy.

Inspired by the Morgan's problem of control systems, Cheng [7] developed the theory of semi-tensor product (STP), which breaks the restriction on the dimensionality of traditional matrix product. By means of STP, an algebraic state-space representation approach was proposed to deal with discrete (finite-valued) dynamical systems involving BCNs, finite games, cryptography, finite automata, smart homes, and so on [8, 9]. In this framework, substantial progress has been made in the study of BCNs, related research results including but not limited to controllability and stabilization [10–12], observability and detectability [13–15], and optimal control [16–18].

Fault detection technology is a hot topic in the field of control systems, which is of great practical significance as it is the basis for ensuring system safety and reliability. In terms of biological systems, some systemic diseases, such as cancer, originating from random mutations in somatic cells [19], cause model failures and alter their expected dynamics, which often result in undesirable outputs, the so-called disease phenotypes. In GRNs, previous studies have shown that genetic alterations can be properly modeled by different faulty models in the BCN paradigm [20]. By studying the dynamics of the original BCN and the faulty BCN, it is possible to design appropriate intervention strategies to drive the system from a diseased state to a healthy or less harmful state. That is, the fault detection plays an important role in the detection and treatment of diseases [21, 22].

* Corresponding author (email: yyyu@sdu.edu.cn)

In a BN, the faults can be broadly divided into two types: stuck-at faults and bridging faults [20]. A stuck-at fault refers to that a node in the network is stuck at a specific value, which can be regarded as a genetic mutation; i.e., a specific gene is permanently turned on or off. A bridging fault means that an old interconnection is broken or a new interconnection is merged into the network. However, the faults considered in [20, 21] were only "stuck-at faults". Utilizing STP, Fornasini and Valcher [23, 24] generalized the results of [20,21] and proposed the general concept of fault detectability for BCNs. Then, Leifeld et al. [25] extended the idea of [24] to probabilistic BCNs. The problems discussed in [23, 24] are mainly related to on-line fault detection. Afterwards, the off-line fault detection of BCNs was studied in [26–28] successively. Fornasini and Valcher [26] provided a graphical criterion and an augmented algebraic criterion for off-line fault detectability. Zhang et al. [27] transformed the fault detection problem into a dead-beat stabilization problem. Li et al. [28] obtained a verification matrix by indistinguishability for off-line fault detectability. In our recent paper [29], passive fault detectability and four kinds of active fault detectability, corresponding to online and offline cases, respectively, were investigated using the reachability analysis of an augmented system. Associated fault detection problems were also applied to discrete event systems [30], nuclear plants [31, 32], and combinational circuits [33, 34].

It should be pointed out that the method in [26, 27, 29] is cascading the faulty BCN and non-faulty BCN to obtain an augmented system, which is computationally expensive. And the method of [28] following that of [35] is constructing a transferable matrix to show the control-transferability among indistinguishable states. In this article, we are dedicated to further enriching and improving the research of fault detectability and putting forward some new criteria. In detail, two nonaugmented methods are exploited to analyze the fault detectability. One is the equivalent system-based approach, and the other is the iterative matrix set-based approach. These approaches can provide new perspectives for describing different fault detectability, while avoiding the disadvantage of the high computational complexity of the augmented method. The main contributions of this paper are summarized as follows. (i) With knowledge of convergent sequences and limits, new criteria for the equivalence of BCNs are presented, which are successfully applied to the weak active fault detectability. (ii) By means of a series of iterative matrix sets, necessary and sufficient conditions for strong and weak active fault detectability are provided. (iii) Efficient algorithms are developed to determine the fault detectability, and all possible input sequences of the shortest length for active fault detection are obtained. Compared with the existing studies, our results can reduce the computational complexity significantly.

The remainder of this paper is organized as follows. Section 2 contains some preliminaries. Section 3 includes the equivalent system-based approach for weak active fault detectability. Section 4 is the iterative matrix set-based approach for strong and weak fault detectability. Section 5 is an example. Section 6 is the concluding remarks.

2 Preliminaries

In this section, some preliminaries including STP and fault detectability of BCNs are reviewed. First, some notations are listed below.

- $\mathbb{N}$: the set of non-negative integers.
- $\lceil \cdot \rceil$: the upward rounding function.
- $[M]_{i,j}$: the (i, j)th element of matrix M.
- $\mathrm{Col}_i(M)$: the ith column of matrix M.
- $\mathrm{Row}_i(M)$: the ith row of matrix M.
- $\mathcal{B} := \{0, 1\}$.
- I_n: identity matrix in $\mathbb{R}^{n\times n}$.
- $\delta_n^i := \mathrm{Col}_i(I_n)$.
- $\Delta_n := \{\delta_n^i | 1 \leqslant i \leqslant n\}$.
- $\mathbf{1}_n := \sum_{i=1}^{n} \delta_n^i$.
- $\mathbf{0}_{m\times n}$: an $m \times n$ matrix with all elements being zero.
- $|\cdot|$: cardinality of a set.
- $\mathcal{L}^{m\times n}$: the set of $m \times n$ logical matrices.
- $\mathcal{B}^{m\times n}$: the set of $m \times n$ Boolean matrices.
- $\otimes$ $(*)$: Kronecker product (Khatri-Rao product).
- $\bigwedge$ $(\bigvee)$: conjunction (disjunction).

Besides, logical operations of Boolean matrices used in the sequel are defined element wise, for example, $\mathbf{1}_n \bigwedge \delta_n^i = \delta_n^i$.

Definition 1 ([7])**.** Let $M \in \mathbb{R}^{a\times b}$ and $N \in \mathbb{R}^{c\times d}$. The STP of M and N is defined as

$$M \ltimes N = (M \otimes I_{\frac{e}{b}})(N \otimes I_{\frac{e}{c}}),$$

where e is the least common multiple of b and c.

Throughout this paper, the matrix multiplication $\ltimes$ used by default is Boolean, i.e., $AB = \mathrm{Sgn}(A \ltimes B)$, where $\mathrm{Sgn}(\cdot)$ is the sign function.

For a Boolean variable $\bar{x} \in \mathcal{B}$, identify $\bar{x}$ as $\delta_2^{2-\bar{x}}$. Then the following lemma is used to convert a Boolean function into its algebraic form.

Lemma 1 ([36])**.** Given a Boolean function $f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) : \mathcal{B}^n \to \mathcal{B}$, there exists a unique matrix $M_f \in \mathcal{L}^{2\times 2^n}$, called the structure matrix of $f(\cdot)$, such that $\delta_2^{2-f(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)} = M_f \ltimes_{i=1}^n x_i$, with $x_i = \delta_2^{2-\bar{x}_i} \in \Delta_2$.

Generally, a BCN with n state nodes, m input nodes, and p output nodes is of the form:

$$\begin{cases} \bar{x}_i^{t+1} = f_i(\bar{x}_1^t, \bar{x}_2^t, \ldots, \bar{x}_n^t, \bar{u}_1^t, \bar{u}_2^t, \ldots, \bar{u}_m^t), \\ \bar{y}_j^t = h_j(\bar{x}_1^t, \bar{x}_2^t, \ldots, \bar{x}_n^t), \\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant p, \end{cases} \tag{1}$$

where $f_i : \mathcal{B}^{m+n} \to \mathcal{B}$, $h_j : \mathcal{B}^n \to \mathcal{B}$ are Boolean functions and $\bar{x}_i^t, \bar{u}_k^t, \bar{y}_j^t \in \mathcal{B}$ are the state, input, and output, respectively. By Lemma 1, the algebraic form of BCN (1) is described as [36]

$$\begin{cases} x(t+1) = L_1 * L_2 * \cdots * L_n u(t)x(t) = Lu(t)x(t), \\ y(t) = H_1 * H_2 * \cdots * H_p x(t) = Hx(t), \end{cases} \tag{2}$$

where $x(t) = \ltimes_{i=1}^n x_i(t) = \ltimes_{i=1}^n \delta_2^{2-\bar{x}_i^t} \in \Delta_{2^n}$, $u(t) = \ltimes_{k=1}^m u_k(t) = \ltimes_{k=1}^m \delta_2^{2-\bar{u}_k^t} \in \Delta_{2^m}$, $y(t) = \ltimes_{j=1}^p y_j(t) = \ltimes_{j=1}^p \delta_2^{2-\bar{y}_j^t} \in \Delta_{2^p}$, and $L_i \in \mathcal{L}^{2\times 2^{m+n}}$ ($H_j \in \mathcal{L}^{2\times 2^n}$) is the structure matrix of $f_i(\cdot)$ ($h_j(\cdot)$). Since Eq. (2) can be determined by structure matrices L, H uniquely, BCN (2) is defined as $\mathcal{B}(L, H)$. Let $x(x_0, u(t)) = \ltimes_{i=1}^n x_i(x_0, u(t))$ and $y(x_0, u(t)) = \ltimes_{j=1}^p y_j(x_0, u(t))$ be the state and output of (2) at time $t \in \mathbb{N}$ starting from initial state $x(0) = x_0$ under input sequence $\{u(t)\}|_{t=0}^{+\infty}$, respectively.

Assume BCN $\mathcal{B}(\tilde{L}, \tilde{H})$ is the faulty model of $\mathcal{B}(L, H)$.

Definition 2 ([27–29])**.** Set that $x_0, \tilde{x}_0 \in \Delta_{2^n}$ are initial states of $\mathcal{B}(L, H)$ and $\mathcal{B}(\tilde{L}, \tilde{H})$, respectively.

(i) BCN $\mathcal{B}(\tilde{L}, \tilde{H})$ is said to be weakly active fault-detectable, if for any $x_0, \tilde{x}_0 \in \Delta_{2^n}$, there exist an integer $T \in \mathbb{N}$ and an input sequence $\{u(t)\}|_{t=0}^{+\infty}$, both of which are dependent on $x_0, \tilde{x}_0$, such that

$$\begin{aligned} &(y(x_0, u(0)), y(x_0, u(1)), \ldots, y(x_0, u(T))) \\ &\quad \neq (\tilde{y}(\tilde{x}_0, u(0)), \tilde{y}(\tilde{x}_0, u(1)), \ldots, \tilde{y}(\tilde{x}_0, u(T))). \end{aligned} \tag{3}$$

(ii) BCN $\mathcal{B}(\tilde{L}, \tilde{H})$ is said to be strongly active fault-detectable, if there exists an integer $T \in \mathbb{N}$, such that Eq. (3) holds for any input sequence $\{u(t)\}|_{t=0}^{+\infty}$ and any $x_0, \tilde{x}_0 \in \Delta_{2^n}$.

Clearly, strong active fault detectability is sufficient for the weak active fault detectability, but the converse is not necessarily true. Besides, it is crucial to emphasize that weak active fault detectability is equivalent to active fault detectability[1)], which has been proven in [29]. Thus, in the sequel, we mainly focus on the weak and strong active fault detectability.

3 Equivalent system-based approach

In this section, the equivalence of BCNs is considered, which provides a new viewpoint to study the fault detectability.

1) BCN $\mathcal{B}(\tilde{L}, \tilde{H})$ is said to be actively fault-detectable, if there exist an integer $T \in \mathbb{N}$ and an input sequence $\{u(t)\}|_{t=0}^{+\infty}$, such that Eq. (3) holds for any $x_0, \tilde{x}_0 \in \Delta_{2^n}$.

Definition 3 ([37]). Given two BCNs, they are said to be equivalent if for any point x_0 of one network there is a point $\tilde{x}_0$ of the other network such that for the same input $\{u(t)\}|_{t=0}^{+\infty}$ with initial values x_0 and $\tilde{x}_0$, respectively, the output $\{y(t)\}|_{t=0}^{+\infty}$ are the same.

Remark 1. It is worth stating that the equivalence of BCNs given in Definition 3 is consistent with the indistinguishability of sequential machines in [38], which also shows the interconnectivity between BCNs and finite automata.

Consider two BCNs: $\mathcal{B}(L,H)$ and $\mathcal{B}(\tilde{L},\tilde{H})$ with $L \in \mathcal{L}^{2^n\times 2^{n+m}}$, $\tilde{L} \in \mathcal{L}^{2^s\times 2^{s+m}}$, $H \in \mathcal{L}^{2^p\times 2^n}$, $\tilde{H} \in \mathcal{L}^{2^p\times 2^s}$, and set $F_k = L\delta_{2^m}^k, \tilde{F}_k = \tilde{L}\delta_{2^m}^k$, $k=1,2,\ldots,2^m$. Then construct a Boolean vector sequence

$$\vartheta_{i+1} = \bigwedge_{k=1}^{2^m} \vartheta_i(F_k \otimes \tilde{F}_k) \wedge \vartheta_i \in \mathcal{B}^{1\times 2^{n+s}}, \tag{4}$$

where $\vartheta_0 = M_e(H\otimes\tilde{H})$, and $M_e \in \mathcal{B}^{1\times 2^{2p}}$ with $M_e\delta_{2^p}^i\delta_{2^p}^j = 1$ if $i=j$, otherwise $M_e\delta_{2^p}^i\delta_{2^p}^j = 0$, $1 \leqslant i,j \leqslant 2^q$. Immediately, $\{\vartheta_i\}|_{i\in\mathbb{N}}$ is finite-time convergent, since ϑ_i is non-increasing element-wise on i. Set $\vartheta^* := \lim_{i\to\infty}\vartheta_i$ and $R \in \mathcal{B}^{2^n\times 2^s}$ with

$$\mathrm{Row}_i(R) = \vartheta^*\delta_{2^n}^i. \tag{5}$$

Based on the above vector sequence, we give the following condition on the equivalence of BCNs.

Proposition 1. BCNs $\mathcal{B}(L,H)$ and $\mathcal{B}(\tilde{L},\tilde{H})$ are equivalent, if and only if matrix R has no zero rows and columns.

Proof. (Necessity) If $\mathcal{B}(L,H)$ and $\mathcal{B}(\tilde{L},\tilde{H})$ are equivalent, then for any initial state x of $\mathcal{B}(L,H)$, there exists state $\tilde{x}$ of $\mathcal{B}(\tilde{L},\tilde{H})$, satisfying $y(x,u(t)) \equiv \tilde{y}(\tilde{x},u(t))$, for any $\{u(t)\}|_{t=0}^{+\infty}$, where $\tilde{y}(\tilde{x},u(t))$ is the output of $\mathcal{B}(\tilde{L},\tilde{H})$ under initial state $\tilde{x}$ and input signal $\{u(t)\}|_{t=0}^{+\infty}$ at time $t\in\mathbb{N}$. Furthermore, we can see that

$$\begin{cases} Hx = \tilde{H}\tilde{x}, \\ HF_{k_0}x = \tilde{H}\tilde{F}_{k_0}\tilde{x}, \\ HF_{k_1}F_{k_0}x = \tilde{H}\tilde{F}_{k_1}\tilde{F}_{k_0}\tilde{x}, \\ \vdots \end{cases} \tag{6}$$

hold for any $1\leqslant k_0,k_1,\ldots\leqslant 2^m$. In (6), $Hx=\tilde{H}\tilde{x}$ and $HF_{k_0}x = \tilde{H}\tilde{F}_{k_0}\tilde{x}$ can be rewritten as $M_e(H\otimes\tilde{H})x\tilde{x}=1$ and

$$M_e(HF_{k_0}x \ltimes \tilde{H}\tilde{F}_{k_0}\tilde{x}) = M_e(H\otimes\tilde{H})(F_{k_0}\otimes\tilde{F}_{k_0})x\tilde{x} = 1,$$

respectively. Thus, it can be obtained that $\vartheta_0x\tilde{x}=1$ and $\vartheta_1x\tilde{x}=1$, since $\vartheta_1 = \bigwedge_{k=1}^{2^m}\vartheta_0(F_k\otimes\tilde{F}_k)\wedge\vartheta_0$. Moreover, $HF_{k_1}F_{k_0}x = \tilde{H}\tilde{F}_{k_1}\tilde{F}_{k_0}\tilde{x}$ can be rewritten as

$$\begin{aligned} &M_e(HF_{k_1}F_{k_0}x \ltimes \tilde{H}\tilde{F}_{k_1}\tilde{F}_{k_0}\tilde{x}) \\ &\quad = M_e(H\otimes\tilde{H})(F_{k_1}\otimes\tilde{F}_{k_1})(F_{k_0}\otimes\tilde{F}_{k_0})x\tilde{x} = 1, \end{aligned}$$

which means $\vartheta_2x\tilde{x}=1$, since

$$\begin{aligned} \vartheta_2 &= \bigwedge_{k=1}^{2^m}\vartheta_1(F_k\otimes\tilde{F}_k)\wedge\vartheta_1 \\ &= \bigwedge_{k_2=1}^{2^m}\bigwedge_{k_1=1}^{2^m}\Big(\vartheta_0(F_{k_2}\otimes\tilde{F}_{k_2})(F_{k_1}\otimes\tilde{F}_{k_1})\wedge\vartheta_0(F_{k_1}\otimes\tilde{F}_{k_1})\wedge\vartheta_0\Big). \end{aligned}$$

Repeating the above processes gets $\vartheta_ix\tilde{x}=1$, $i\in\mathbb{N}$. Because x is arbitrary, we can set $x=\delta_{2^n}^i$. Thus, for any given $\delta_{2^n}^i\in\Delta_{2^n}$, there exists $\tilde{x}$, such that $\vartheta^*\delta_{2^n}^i\tilde{x}\equiv 1$ and $(\delta_{2^n}^i)^\top R\tilde{x}=1$, which implies that matrix R has no zero rows. Similarly, if $\mathcal{B}(L,H)$ and $\mathcal{B}(\tilde{L},\tilde{H})$ are equivalent, then for any initial state $\tilde{x}$ of $\mathcal{B}(\tilde{L},\tilde{H})$, there exists state x of $\mathcal{B}(L,H)$ such that Eq. (6) holds for any $1\leqslant k_0,k_1,\ldots\leqslant 2^m$. That is to say, for any $\delta_{2^s}^j\in\Delta_{2^s}$, there exists x such that $\vartheta^*x\delta_{2^s}^j\equiv 1$, i.e., $x^\top R\delta_{2^s}^j=1$, which means R has no zero columns as well.

Inverting the above processes completes the proof of sufficiency.

From the above proof, one can see the meaning of elements in ϑ^*. $\vartheta^*\delta_{2^n}^i\delta_{2^s}^j = 1$, i.e., $[R]_{i,j} = 1$ if and only if initial states $\delta_{2^n}^i$ of $\mathcal{B}(L,H)$ and $\delta_{2^s}^j$ of $\mathcal{B}(\tilde{L},\tilde{H})$ have same output sequences for any same input $\{u(t)\}|_{t=0}^{+\infty}$.

Proposition 1 is intuitive and easy to understand but is not convenient to obtain matrix R, and then we provide an alternative way to get it.

Proposition 2. $\lim_{i\to\infty}\mathcal{M}_i = R$, where $\mathcal{M}_0 = H^\top\tilde{H}$ and

$$\mathcal{M}_{i+1} = \bigwedge_{k=1}^{2^m} F_k^\top \mathcal{M}_i \tilde{F}_k \wedge \mathcal{M}_i,\ i \in \mathbb{N}. \tag{7}$$

Proof. Firstly, note that $\mathcal{M}_i$ is non-increasing element-wise on i and $\mathcal{M}_k \in \mathcal{B}^{2^n\times 2^s}$. $\mathcal{M}_k$ must be finite-time convergent, since $\mathcal{M}_{i+t} = \mathcal{M}_i$, $t \geqslant 0$, if $\mathcal{M}_{i+1} = \mathcal{M}_i$. Secondly, BCNs $\mathcal{B}(L,H)$ and $\mathcal{B}(\tilde{L},\tilde{H})$ are equivalent if and only if for any $x \in \Delta_{2^n}$ ($\tilde{x} \in \Delta_{2^s}$), there exists $\tilde{x} \in \Delta_{2^s}$ ($x \in \Delta_{2^n}$) such that Eq. (6) holds for any $1 \leqslant i_0, i_1, \ldots \leqslant 2^m$. That is,

$$\begin{cases} x^\top H^\top \tilde{H}\tilde{x} = 1, \\ x^\top \prod_{t=0}^{l} F_{i_t}^\top H^\top \tilde{H} \prod_{t=l}^{0} \tilde{F}_{i_t}\tilde{x} = 1,\ l \geqslant 0 \end{cases} \tag{8}$$

hold for any $1 \leqslant i_0, i_1, \ldots \leqslant 2^m$. Let $T^* = \min\{i|\mathcal{M}_i = \mathcal{M}_{i+1}\}$, then $\lim_{i\to\infty}\mathcal{M}_i = \mathcal{M}_{T^*}$. In addition, it follows from (7) that

$$\begin{aligned} \mathcal{M}_i &= \bigwedge_{k_1=1}^{2^m} F_{k_1}^\top \mathcal{M}_{i-1}\tilde{F}_{k_1} \wedge \mathcal{M}_{i-1} \\ &= \bigwedge_{k_1=1}^{2^m}\bigwedge_{k_2=1}^{2^m} (F_{k_1}^\top F_{k_2}^\top \mathcal{M}_{i-2}\tilde{F}_{k_2}\tilde{F}_{k_1} \wedge F_{k_1}^\top \mathcal{M}_{i-2}\tilde{F}_{k_1} \wedge \mathcal{M}_{i-2}) \\ &= \bigwedge_{k_1=1}^{2^m}\bigwedge_{k_2=1}^{2^m}\bigwedge_{k_3=1}^{2^m} (F_{k_1}^\top F_{k_2}^\top F_{k_3}^\top \mathcal{M}_{i-3}\tilde{F}_{k_3}\tilde{F}_{k_2}\tilde{F}_{k_1} \wedge F_{k_1}^\top F_{k_2}^\top \mathcal{M}_{i-3}\tilde{F}_{k_2}\tilde{F}_{k_1} \wedge F_{k_1}^\top \mathcal{M}_{i-3}\tilde{F}_{k_1} \wedge \mathcal{M}_{i-3}) \\ &= \cdots \\ &= \bigwedge_{k_1=1}^{2^m}\bigwedge_{k_2=1}^{2^m}\cdots\bigwedge_{k_i=1}^{2^m}\Big(\prod_{t=1}^{i} F_{k_t}^\top \mathcal{M}_0 \prod_{t=i}^{1}\tilde{F}_{k_t} \wedge \prod_{t=1}^{i-1} F_{k_t}^\top \mathcal{M}_0 \prod_{t=i-1}^{1}\tilde{F}_{k_t} \wedge \cdots \wedge F_{k_1}^\top \mathcal{M}_0\tilde{F}_{k_1} \wedge \mathcal{M}_0\Big). \end{aligned} \tag{9}$$

Combining (8) with (9), for any $x \in \Delta_{2^n}$ ($\tilde{x} \in \Delta_{2^s}$), there exists $\tilde{x} \in \Delta_{2^s}$ ($x \in \Delta_{2^n}$) satisfying $x^\top \mathcal{M}_{T^*}\tilde{x} = 1$. Then $[\mathcal{M}_{T^*}]_{i,j} = 1$ if and only if $\delta_{2^n}^i$ of $\mathcal{B}(L,H)$ and $\delta_{2^s}^j$ of $\mathcal{B}(\tilde{L},\tilde{H})$ have the same output sequences for any same input $\{u(t)\}|_{t=0}^{+\infty}$. Hence, $\mathcal{M}_{T^*} = R$, which completes the proof.

Propositions 1 and 2 provide new criteria for the equivalence of $\mathcal{B}(L,H)$ and $\mathcal{B}(\tilde{L},\tilde{H})$. In order to discuss the fault detectability, let $s = n$ in $\mathcal{B}(L,H)$ and $\mathcal{B}(\tilde{L},\tilde{H})$. In view of the analysis of equivalent systems, the following theorem is obtained for weak active fault detectability.

Theorem 1. BCN $\mathcal{B}(\tilde{L},\tilde{H})$ is weakly active fault-detectable if and only if $R = \mathbf{0}_{2^n\times 2^n}$.

Proof. The existence of $1 \leqslant i,j \leqslant 2^n$ such that $[R]_{i,j} = 1$, is equivalent to that the output sequences of $\mathcal{B}(L,H)$ and $\mathcal{B}(\tilde{L},\tilde{H})$ starting from initial states $x_0 = \delta_{2^n}^i$ and $\tilde{x}_0 = \delta_{2^n}^j$, respectively, are the same under any identical input sequence. Hence, $R \neq \mathbf{0}_{2^n\times 2^n}$ if and only if $\mathcal{B}(\tilde{L},\tilde{H})$ is not weakly active fault-detectable. The proof is completed.

From Theorem 1, we can see that if BCN $\mathcal{B}(\tilde{L},\tilde{H})$ is weakly active fault-detectable, then $\mathcal{B}(\tilde{L},\tilde{H})$ and $\mathcal{B}(L,H)$ are not equivalent. Conversely, the equivalence of $\mathcal{B}(\tilde{L},\tilde{H})$ and $\mathcal{B}(L,H)$ implies that $\mathcal{B}(\tilde{L},\tilde{H})$ is neither weakly nor strongly active fault-detectable.

4 Iterative matrix set-based approach

Motivated by the equivalent system approach, an alternative approach is given to depict the three kinds of fault detectability uniformly. We first construct a sequence of Boolean matrix sets iteratively:

$$S^{i+1}(\mathcal{M}) = \{F_k^\top \mathcal{M}\tilde{F}_k \wedge H^\top\tilde{H}|\mathcal{M} \in S^i(\mathcal{M}), 1 \leqslant k \leqslant 2^m\}, \tag{10}$$

where $S^0(\mathcal{M})=\{H^\top\tilde{H}\}$, $i\in\mathbb{N}$, based on which, the following criteria are deduced.

Theorem 2. (i) BCN $\mathcal{B}(\tilde{L},\tilde{H})$ is weakly active fault-detectable if and only if there exists an integer $T\in\mathbb{N}$ such that

$$\bigwedge_{\mathcal{M}\in S^T(\mathcal{M})}\mathcal{M}=\mathbf{0}_{2^n\times 2^n}. \tag{11}$$

(ii) BCN $\mathcal{B}(\tilde{L},\tilde{H})$ is strongly active fault-detectable if and only if there exists an integer $T\in\mathbb{N}$ such that $\{\mathbf{0}_{2^n\times 2^n}\}=S^T(\mathcal{M})$.

Proof. If for any $x,\tilde{x}\in\Delta_{2^n}$, $Hx\neq\tilde{H}\tilde{x}$, then $H^\top\tilde{H}=\mathbf{0}_{2^n\times 2^n}$. In this case, $S^i(\mathcal{M})=\{\mathbf{0}_{2^n\times 2^n}\}$, $i\in\mathbb{N}$, and thus (i) and (ii) of Theorem 2 hold clearly. Without loss of generality, we assume that there exist $x,\tilde{x}\in\Delta_{2^n}$ such that $Hx=\tilde{H}\tilde{x}$.

(i) (Necessity) Assume that BCN $\mathcal{B}(\tilde{L},\tilde{H})$ is weakly active fault-detectable. Then for any $x=\delta_{2^n}^i$ and $\tilde{x}=\delta_{2^n}^j$, there exists an input sequence $\{u(t)\}|_{t=0}^{T_{i,j}}:=\{\delta_{2^m}^{k_t}\}|_{t=0}^{T_{i,j}}$ such that

$$\begin{cases} Hx=\tilde{H}\tilde{x},\\ HF_{k_0}x=\tilde{H}\tilde{F}_{k_0}\tilde{x},\\ HF_{k_1}F_{k_0}x=\tilde{H}\tilde{F}_{k_1}\tilde{F}_{k_0}\tilde{x},\\ \quad\vdots\\ H\prod_{t=T_{i,j}-1}^{0}F_{k_t}x=\tilde{H}\prod_{t=T_{i,j}-1}^{0}\tilde{F}_{k_t}\tilde{x},\\ H\prod_{t=T_{i,j}}^{0}F_{k_t}x\neq\tilde{H}\prod_{t=T_{i,j}}^{0}\tilde{F}_{k_t}\tilde{x}, \end{cases} \tag{12}$$

where $T_{i,j}\in\mathbb{N}$ is the minimum integer satisfying (12), which means

$$x^\top H^\top\tilde{H}\tilde{x}=1, \tag{13}$$

$$x^\top\prod_{t=0}^{l}F_{k_t}^\top H^\top\tilde{H}\prod_{t=l}^{0}\tilde{F}_{k_t}\tilde{x}=1,\ 0\leqslant l\leqslant T_{i,j}-1, \tag{14}$$

$$x^\top\prod_{t=0}^{T_{i,j}}F_{k_t}^\top H^\top\tilde{H}\prod_{t=T_{i,j}}^{0}\tilde{F}_{k_t}\tilde{x}=0. \tag{15}$$

Consequently, we can see

$$x^\top\left(\bigwedge_{l=0}^{\rho}\prod_{t=0}^{l}F_{k_t}^\top H^\top\tilde{H}\prod_{t=l}^{0}\tilde{F}_{k_t}\wedge H^\top\tilde{H}\right)\tilde{x}=0,\ \rho\geqslant T_{i,j}. \tag{16}$$

Besides, $\bar{\mathcal{M}}_\rho:=\bigwedge_{l=0}^{\rho}\prod_{t=0}^{l}F_{k_t}^\top H^\top\tilde{H}\prod_{t=l}^{0}\tilde{F}_{k_t}\wedge H^\top\tilde{H}\in S^{\rho+1}(\mathcal{M})$. That is, for any $x=\delta_{2^n}^i$ and $\tilde{x}=\delta_{2^n}^j$, there exist an integer $T_{i,j}$ and $\bar{\mathcal{M}}_t\in S^{t+1}(\mathcal{M})$ such that $[\bar{\mathcal{M}}_t]_{i,j}=0$, $t\geqslant T_{i,j}$. Let $T_1=\max\{T_{i,j}|1\leqslant i,j\leqslant 2^n\}$ yield

$$\bigwedge_{\mathcal{M}\in S^{T_1+1}(\mathcal{M})}\mathcal{M}=\mathbf{0}_{2^n\times 2^n}.$$

(Sufficiency) Assume that Eq. (11) holds. Note that the general form of $\mathcal{M}\in S^T(\mathcal{M})$ is

$$\mathcal{M}=\bigwedge_{l=0}^{T-1}\prod_{t=0}^{l}F_{k_t}^\top H^\top\tilde{H}\prod_{t=l}^{0}\tilde{F}_{k_t}\wedge H^\top\tilde{H},$$

where $k_t\in\{1,2,\ldots,2^m\}$. Then Eq. (11) means that for any $x=\delta_{2^n}^i$ and $\tilde{x}=\delta_{2^n}^j$, there exist $k_0^{i,j},k_1^{i,j},\ldots,k_{T-1}^{i,j}\in\{1,2,\ldots,2^m\}$ such that

$$\left[\bigwedge_{l=0}^{T-1}\prod_{t=0}^{l}F_{k_t^{i,j}}^\top H^\top\tilde{H}\prod_{t=l}^{0}\tilde{F}_{k_t^{i,j}}\wedge H^\top\tilde{H}\right]_{i,j}=0.$$

Thus, for any $x = \delta_{2^n}^i$ and $\tilde{x} = \delta_{2^n}^j$, under input sequence $\{u(t)\}|_{t=0}^{T-1} := \{\delta_{2^m}^{k_t^{i,j}}\}|_{t=0}^{T-1}$, there exists $T_{i,j} \leqslant T$ satisfying

$$\left[\prod_{t=0}^{T_{i,j}} F_{k_t^{i,j}}^\top H^\top \tilde{H} \prod_{t=T_{i,j}}^{0} \tilde{F}_{k_t^{i,j}}\right]_{i,j} = 0,$$

which further implies $y(\delta_{2^n}^i, u(T_{i,j})) \neq \tilde{y}(\delta_{2^n}^j, u(T_{i,j}))$; i.e., $\mathcal{B}(\tilde{L}, \tilde{H})$ is weakly active fault-detectable.

(ii) To prove (ii), we firstly show that BCN $\mathcal{B}(\tilde{L}, \tilde{H})$ is actively fault-detectable if and only if there exists an integer $T \in \mathbb{N}$ such that $\mathbf{0}_{2^n \times 2^n} \in S^T(\mathcal{M})$.

Assume that BCN $\mathcal{B}(\tilde{L}, \tilde{H})$ is actively fault detectable. Then there exists an input sequence $\{u(t)\}|_{t=0}^{+\infty} := \{\delta_{2^m}^{k_t}\}|_{t=0}^{+\infty}$ such that for any $x = \delta_{2^n}^i$ and $\tilde{x} = \delta_{2^n}^j$, there is a minimum integer $T_{i,j} \geqslant 0$ satisfying (13)–(15). Hence, for any $\rho \geqslant T_{i,j}$,

$$x^\top \left(\bigwedge_{l=0}^{\rho} \prod_{t=0}^{l} F_{k_t}^\top H^\top \tilde{H} \prod_{t=l}^{0} \tilde{F}_{k_t} \wedge H^\top \tilde{H}\right) \tilde{x} = \left[\bigwedge_{l=0}^{\rho} \prod_{t=0}^{l} F_{k_t}^\top H^\top \tilde{H} \prod_{t=l}^{0} \tilde{F}_{k_t} \wedge H^\top \tilde{H}\right]_{i,j} = 0.$$

Take $T_2 = \max\{T_{i,j} | 1 \leqslant i, j \leqslant 2^n\}$. From the arbitrariness of i and j, it is obvious that

$$\mathbf{0}_{2^n \times 2^n} = \bigwedge_{l=0}^{T_2} \prod_{t=0}^{l} F_{k_t}^\top H^\top \tilde{H} \prod_{t=l}^{0} \tilde{F}_{k_t} \wedge H^\top \tilde{H} \in S^{T_2+1}(\mathcal{M}).$$

Conversely, if $\mathbf{0}_{2^n \times 2^n} \in S^T(\mathcal{M})$, then there exist $\bar{k}_t \in \{1, 2, \ldots, 2^m\}$, $t = 0, 1, \ldots, T-1$ such that $\bigwedge_{l=0}^{T-1} \prod_{t=0}^{l} F_{\bar{k}_t}^\top H^\top \tilde{H} \prod_{t=l}^{0} \tilde{F}_{\bar{k}_t} \wedge H^\top \tilde{H} = \mathbf{0}_{2^n \times 2^n}$. It means that under input sequence $\{u(t)\}|_{t=0}^{T-1} := \{\delta_{2^m}^{\bar{k}_t}\}|_{t=0}^{T-1}$, for any $x = \delta_{2^n}^i$ and $\tilde{x} = \delta_{2^n}^j$, there is an integer $T_{i,j} \leqslant T$ satisfying

$$\left[\prod_{t=0}^{T_{i,j}} F_{\bar{k}_t}^\top H^\top \tilde{H} \prod_{t=T_{i,j}}^{0} \tilde{F}_{\bar{k}_t}\right]_{i,j} = 0.$$

In other words, under input sequence $\{u(t)\}|_{t=0}^{T-1} = \{\delta_{2^m}^{\bar{k}_t}\}|_{t=0}^{T-1}$, we have that $y(\delta_{2^n}^i, u(T_{i,j})) \neq \tilde{y}(\delta_{2^n}^j, u(T_{i,j}))$ holds for any $x = \delta_{2^n}^i$ and $\tilde{x} = \delta_{2^n}^j$, which implies the active fault detectability.

Notice that BCN $\mathcal{B}(\tilde{L}, \tilde{H})$ is strongly active fault-detectable if and only if Eqs. (13)–(15) hold for any input sequence $\{u(t)\}|_{t=0}^{+\infty} := \{\delta_{2^m}^{k_t}\}|_{t=0}^{+\infty}$. Together with the proof above, we can conclude that $\mathcal{B}(\tilde{L}, \tilde{H})$ is strongly active fault-detectable if and only if there exists an integer $T \geqslant 0$ such that $\{\mathbf{0}_{2^n \times 2^n}\} = S^T(\mathcal{M})$.

Combining with Proposition 2, one sees that Theorem 1 and Theorem 2-(i) are equivalent. Thus, the iterative matrix set-based approach can be thought of as the extension and refinement of the equivalent system-based approach. The equivalent system-based approach is only applicable to weak active fault detectability, while the iterative matrix set-based approach is suitable for three kinds of active fault detectability. Theorem 2 provides a unified framework to investigate fault detectability via the iterative matrix sets (10), which can avoid starting again when the verified result is not consistent with the expected fault detectability. Moreover, similar to Theorem 1, the strong active fault detectability can also be determined by Boolean matrix sequences.

Corollary 1. BCN $\mathcal{B}(\tilde{L}, \tilde{H})$ is strongly active fault-detectable if and only if

$$\lim_{i \to \infty} \hat{\mathcal{M}}_i = \mathbf{0}_{2^n \times 2^n}, \tag{17}$$

where $\hat{\mathcal{M}}_{i+1} = \bigvee_{k=1}^{2^m} F_k^\top \hat{\mathcal{M}}_i \tilde{F}_k \wedge \hat{\mathcal{M}}_i$, $\hat{\mathcal{M}}_0 = H^\top \tilde{H}$, $i \in \mathbb{N}$.

Proof. A simple induction shows that

$$\begin{aligned}\hat{\mathcal{M}}_i &= \bigvee_{k_1=1}^{2^m} F_{k_1}^\top \hat{\mathcal{M}}_{i-1} \tilde{F}_{k_1} \wedge \hat{\mathcal{M}}_{i-1} \\ &= \bigvee_{k_1=1}^{2^m} \bigvee_{k_2=1}^{2^m} (F_{k_1}^\top F_{k_2}^\top \hat{\mathcal{M}}_{i-2} \tilde{F}_{k_2} \tilde{F}_{k_1} \wedge F_{k_1}^\top \hat{\mathcal{M}}_{i-2} \tilde{F}_{k_1} \wedge \hat{\mathcal{M}}_{i-2})\end{aligned}$$

$$= \cdots$$
$$= \bigvee_{k_1=1}^{2^m} \bigvee_{k_2=1}^{2^m} \cdots \bigvee_{k_i=1}^{2^m} \left(\prod_{t=1}^{i} F_{k_t}^\top \hat{\mathcal{M}}_0 \prod_{t=i}^{1} \tilde{F}_{k_t} \wedge \prod_{t=1}^{i-1} F_{k_t}^\top \hat{\mathcal{M}}_0 \prod_{t=i-1}^{1} \tilde{F}_{k_t} \wedge \cdots \wedge F_{k_1}^\top \hat{\mathcal{M}}_0 \tilde{F}_{k_1} \wedge \hat{\mathcal{M}}_0 \right),$$

which indicates $\hat{\mathcal{M}}_i = \bigvee_{\mathcal{M} \in S^i(\mathcal{M})} \mathcal{M}$. Therefore, from (ii) of Theorem 2, $\mathcal{B}(\tilde{L}, \tilde{H})$ is strongly active fault-detectable if and only if $\lim_{i\to\infty} \hat{\mathcal{M}}_i = \mathbf{0}_{2^n \times 2^n}$.

Remark 2. (Upper bound analysis)

(i) Let a new BCN $\mathcal{B}(\bar{L}, \bar{H})$ with $\bar{L} \in \mathcal{L}^{2^{n+1} \times 2^{m+n+1}}$ and $\bar{H} \in \mathcal{L}^{2^{p+1} \times 2^{n+1}}$, called the direct sum of $\mathcal{B}(L, H)$ and $\mathcal{B}(\tilde{L}, \tilde{H})$, where

$$\bar{L}\delta_{2^m}^k = \begin{bmatrix} F_k & \mathbf{0}_{2^n \times 2^n} \\ \mathbf{0}_{2^n \times 2^n} & \tilde{F}_k \end{bmatrix}, \ \bar{H} = \begin{bmatrix} H & \mathbf{0}_{2^p \times 2^n} \\ \mathbf{0}_{2^p \times 2^n} & \tilde{H} \end{bmatrix}.$$

Let $\bar{y}(\bar{x}_0, u(t))$ denote the output of $\mathcal{B}(\bar{L}, \bar{H})$ under initial state $\bar{x}_0 \in \Delta_{2^{n+1}}$ and input signal $\{u(t)\}|_{t=0}^{+\infty}$ at time $t \in \mathbb{N}$. Then, by Definition 2, $\mathcal{B}(\tilde{L}, \tilde{H})$ is weakly active fault-detectable if and only if for any $x_0 = \delta_{2^n}^i, \tilde{x}_0 = \delta_{2^n}^j \in \Delta_{2^n}$, there exist an integer $T \in \mathbb{N}$ and an input sequence $\{u(t)\}|_{t=0}^{+\infty}$, such that

$$\begin{aligned}&(y(\delta_{2^n}^i, u(0)), y(\delta_{2^n}^i, u(1)), \ldots, y(\delta_{2^n}^i, u(T)))\\ &\quad \neq (\tilde{y}(\delta_{2^n}^j, u(0)), \tilde{y}(\delta_{2^n}^j, u(1)), \ldots, \tilde{y}(\delta_{2^n}^j, u(T))),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&(\bar{y}(\delta_{2^{n+1}}^i, u(0)), \bar{y}(\delta_{2^{n+1}}^i, u(1)), \ldots, \bar{y}(\delta_{2^{n+1}}^i, u(T)))\\ &\quad \neq (\bar{y}(\delta_{2^{n+1}}^{2^n+j}, u(0)), \bar{y}(\delta_{2^{n+1}}^{2^n+j}, u(1)), \ldots, \bar{y}(\delta_{2^{n+1}}^{2^n+j}, u(T)));\end{aligned}$$

i.e., each pair of states $(\delta_{2^{n+1}}^i, \delta_{2^{n+1}}^{2^n+j})$, $1 \leqslant i, j \leqslant 2^n$ of $\mathcal{B}(\bar{L}, \bar{H})$ is distinguishable[2)]. Actually, it follows from Theorem 6 [38] that the smallest upper bound of steps to verify the distinguishability of two states of $\mathcal{B}(\bar{L}, \bar{H})$ is $2^{n+1} - 1$. Therefore, the minimum steps to verify the weak active fault detectability satisfy $T < 2^{n+1} - 1$.

(ii) To give a unified upper bound for Theorem 2, recall the sequence of Boolean matrix sets (10). Number the matrices in $S^i(\mathcal{M})$ by $S^i(\mathcal{M}) = \{\mathcal{M}_l^i | l = 1, 2, \ldots, 2^{im}\}$ with

$$\mathcal{M}_l^i = F_{l-(\lceil \frac{l}{2^m} \rceil - 1)2^m}^\top \mathcal{M}_{\lceil \frac{l}{2^m} \rceil}^{i-1} \tilde{F}_{l-(\lceil \frac{l}{2^m} \rceil - 1)2^m} \wedge H^\top \tilde{H}.$$

And let $m_l^i = \mathbf{1}_{2^n}^\top \mathcal{M}_l^i \mathbf{1}_{2^n} \geqslant 0$, the number of 1 in the l-th matrix in $S^i(\mathcal{M})$. Clearly, $m_l^i \geqslant m_t^{i+1}$ holds for any $(l-1)2^m + 1 \leqslant t \leqslant l2^m$. Hence, $S^i(\mathcal{M})$ must converge within m_l^0 steps; i.e., there is $i_0 \leqslant m_1^0$ such that $S^{i_0+t}(\mathcal{M}) = S^{i_0}(\mathcal{M})$, $t \geqslant 0$. Accordingly, the steps to verify the three kinds of fault detectability satisfy $T \leqslant m_1^0$. Besides, partition $\{1, 2, \ldots, 2^{2n}\}$ into two subsets:

$$\begin{aligned}\Theta_1 &= \{(i-1)2^n + j | [H^\top \tilde{H}]_{i,j} = 1, 1 \leqslant i, j \leqslant 2^n\},\\ \Theta_2 &= \{(i-1)2^n + j | [H^\top \tilde{H}]_{i,j} = 0, 1 \leqslant i, j \leqslant 2^n\}.\end{aligned}$$

Since $m_1^0 = \mathbf{1}_{2^n}^\top H^\top \tilde{H} \mathbf{1}_{2^n}$, we get $m_1^0 = |\Theta_1|$ as well.

In light of Corollary 1, Algorithm 1 is devised to determine the strong active fault detectability.

If $\mathcal{B}(\tilde{L}, \tilde{H})$ is not strongly active fault-detectable, then we aim to determine whether it is weakly active fault-detectable by Algorithm 2, which is based on Theorem 1/Theorem 2-(i).

When $\mathcal{B}(\tilde{L}, \tilde{H})$ is weakly active fault-detectable but not strongly active fault-detectable, we are also interested in designing all feasible input sequences with the shortest length. This is implemented in Algorithm 3 which is based on Theorem 2.

2) Consider $\mathcal{B}(\bar{L}, \bar{H})$. States $x_1 \neq x_2 \in \Delta_{2^{n+1}}$ are said to be distinguishable, if there exist an integer $T \in \mathbb{N}$ and an input $\{u(t)\}|_{t=0}^{T}$ such that $\bar{y}(x_1, u(T)) \neq \bar{y}(x_2, u(T))$.

Algorithm 1 Checking the strong active fault detectability of $\mathcal{B}(\tilde{L}, \tilde{H})$

Require: $F_k, \tilde{F}_k, H, \tilde{H}, k = 1, 2, \ldots, 2^m$.
Ensure: "Yes", if $\mathcal{B}(\tilde{L}, \tilde{H})$ is strongly active fault-detectable, and "No", otherwise.
Initialize $t = 0$, $M = H^\top \tilde{H}$, $N = P = \mathbf{0}_{2^n \times 2^n}$, $v_{k,i} = [1, 2, \ldots, 2^n]\mathrm{Col}_i(F_k)$, $\tilde{v}_{k,i} = [1, 2, \ldots, 2^n]\mathrm{Col}_i(\tilde{F}_k)$, $i = 1, 2 \ldots, 2^n$, $k = 1, 2, \ldots, 2^m$;
while $t \leqslant |\Theta_1|$ and $N \neq M$ **do**
  $N = M$, $t = t + 1$;
  for $i = 1 : 2^n, j = 1 : 2^n$ **do**
    for $k = 1 : 2^m$ **do**
      $[P]_{i,j} = [P]_{i,j} \vee [M]_{v_{k,i}, \tilde{v}_{k,j}}$;
    end for
    $[M]_{i,j} = [M]_{i,j} \wedge [P]_{i,j}$;
  end for
end while
if $M = \mathbf{0}_{2^n \times 2^n}$ **then**
  return "Yes";
else
  return "No";
end if

Algorithm 2 Checking the weak active fault detectability of $\mathcal{B}(\tilde{L}, \tilde{H})$

Require: $F_k, \tilde{F}_k, H, \tilde{H}, k = 1, 2, \ldots, 2^m$.
Ensure: "Yes", if $\mathcal{B}(\tilde{L}, \tilde{H})$ is weakly active fault-detectable, and "No", otherwise.
Initialize $t = 0$, $M = H^\top \tilde{H}$, $N = P = \mathbf{0}_{2^n \times 2^n}$, $v_{k,i} = [1, 2, \ldots, 2^n]\mathrm{Col}_i(F_k)$, $\tilde{v}_{k,i} = [1, 2, \ldots, 2^n]\mathrm{Col}_i(\tilde{F}_k)$, $i = 1, 2 \ldots, 2^n$, $k = 1, 2, \ldots, 2^m$;
while $t < 2^{n+1} - 1$ and $N \neq M$ **do**
  $N = M$, $t = t + 1$;
  for $i = 1 : 2^n, j = 1 : 2^n, k = 1 : 2^m$ **do**
    $[M]_{i,j} = [M]_{i,j} \wedge [M]_{v_{k,i}, \tilde{v}_{k,j}}$;
  end for
end while
if $M = \mathbf{0}_{2^n \times 2^n}$ **then**
  return "Yes";
else
  return "No";
end if

Algorithm 3 Generating all input sequences of the shortest length for active fault detection of $\mathcal{B}(\tilde{L}, \tilde{H})$

Require: $F_k, \tilde{F}_k, H, \tilde{H}, k = 1, 2, \ldots, 2^m$;
Ensure: There are a total of c shortest input sequences of length τ denoted by $\{u^i(t)\}|_{t=0}^{\tau-1} = \{\delta_{2^m}^{[N]_{i,t+1}}\}|_{t=0}^{\tau-1}$, $i = 1, 2 \ldots, c$.
Initialize $c = \tau = 0$, $M_1^0 = H^\top \tilde{H}$, $v_{k,i} = [1, 2, \ldots, 2^n]\mathrm{Col}_i(F_k)$, $\tilde{v}_{k,i} = [1, 2, \ldots, 2^n]\mathrm{Col}_i(\tilde{F}_k)$, $i = 1, 2 \ldots, 2^n$, $k = 1, 2, \ldots, 2^m$;
while $c = 0$ and $\tau \leqslant |\Theta_1|$ **do**
  $\tau = \tau + 1$;
  for $l = 1 : 2^{\tau m}$ **do**
    $r = \lceil \frac{l}{2^m} \rceil$, $s = l - (r-1)2^m$;
    for $i = 1 : 2^n, j = 1 : 2^n$ **do**
      $[M_l^\tau]_{i,j} = [M_r^{\tau-1}]_{v_{s,i}, \tilde{v}_{s,j}} \wedge [H^\top \tilde{H}]_{i,j}$;
    end for
    if $M_l^\tau = \mathbf{0}_{2^n \times 2^n}$ **then**
      $c = c + 1$;
      for $i = 1 : \tau$ **do**
        if $i = 1$ **then**
          $[N]_{c,i} = \lceil \frac{l}{2^{(\tau-1)m}} \rceil$;
        else
          $[N]_{c,i} = \lceil \frac{l}{2^{(\tau-i)m}} \rceil - \sum_{j=1}^{i-1}([N]_{c,j} - 1)2^{(i-j)m}$;
        end if
      end for
    end if
  end for
end while

Remark 3. (Computational complexity analysis) From Algorithms 1 and 2, it is straightforward that the time complexities for checking the strong and weak active fault detectability of $\mathcal{B}(\tilde{L}, \tilde{H})$ are $O(2^{2n+m}|\Theta_1|)$ and $O(2^{3n+m+1} - 2^{2n+m})$, respectively, and the space complexity of both is $O(2^{2n} + 2^{n+m})$. Besides, obtaining all shortest input sequences in Algorithm 3 requires time and space complexities $O(2^{2n+|\Theta_1|m}|\Theta_1|)$, $O(2^{2n+(|\Theta_1|+1)m})$, respectively. On the other hand, the complexities of existing results are shown in Table 1. Compared with [24, 26–29], our results have lower computational complexity

Table 1 Complexity comparison

Problem	Result	Time complexity	Space complexity
Strong active fault detectability	Algorithm 1	$O(2^{2n+m}\|\Theta_1\|)$	$O(2^{2n}+2^{n+m})$
	Theorem 4 [29]	$O(2^{8n})$	$O(2^{m+4n})$
(Weak) active fault detectability	Algorithm 2	$O(2^{3n+m+1}-2^{2n+m})$	$O(2^{2n}+2^{n+m})$
	Proposition 4 [24,26]	$O(2^{4n})$	$O(2^{4n})$
	Theorem 1 [27]	$\gg O((2^{2n}-\|\Theta_2\|)^3)$	$O(2^{m+4n}-\|\Theta_2\|2^{m+2n})$
	Algorithm 1 [28]	$O(2^{6n})$	$O(2^{m+4n})$
	Theorem 5 [29]	$O(2^{2^{2n+1}m+6n})$	$O(2^{2^{2n}m+4n})$
Obtaining the shortest input sequences	Algorithm 3	$O(2^{2n+\|\Theta_1\|m}\|\Theta_1\|)$	$O(2^{2n+(\|\Theta_1\|+1)m})$
	Theorem 5 [29]	$O(2^{2^{2n+1}m+6n})$	$O(2^{2^{2n}m+4n})$

evidently.

5 An example

Consider a Boolean model of oxidative stress response pathways [21]. In this model, 7 entries are included: ARE represents the family of antioxidant genes, ROS stands for reactive oxidative species, PKC, Keap1, Nrf2, and Bach1 are all proteins, and the Stress is the input signal, which are expressed as $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_6, \bar{u}$, respectively. The updating and output equations are as follows:

$$\begin{cases} \bar{x}_1^{t+1} = \bar{u}^t \wedge \neg \bar{x}_6^t, \\ \bar{x}_2^{t+1} = \neg \bar{x}_1^t, \\ \bar{x}_3^{t+1} = \neg \bar{x}_1^t \wedge (\bar{x}_3^t \vee \bar{x}_5^t), \\ \bar{x}_4^{t+1} = \bar{x}_1^t \wedge \neg \bar{x}_6^t, \\ \bar{x}_5^{t+1} = \bar{x}_4^t \vee \neg \bar{x}_3^t, \\ \bar{x}_6^{t+1} = \bar{x}_5^t \wedge (\neg \bar{x}_2^t \vee \neg \bar{x}_6^t), \\ \bar{y}_1^t = \bar{x}_2^t,\ \bar{y}_2^t = \bar{x}_3^t,\ \bar{y}_3^t = \bar{x}_5^t. \end{cases} \tag{18}$$

Then Eq. (18) can be converted into its algebraic form (2) with

$$\begin{aligned} L = \delta_{64}[&62,25,62,26,64,27,64,28,62,25,62,26,62,25,\\ &62,26,61,25,62,26,63,27,64,28,61,25,62,26,61,\\ &25,62,26,38,5,38,6,40,7,40,8,38,5,46,14,38,5,\\ &46,14,37,5,38,6,39,7,40,8,37,5,46,14,37,5,46,\\ &14,62,57,62,58,64,59,64,60,62,57,62,58,62,57,\\ &62,58,61,57,62,58,63,59,64,60,61,57,62,58,61,\\ &57,62,58,38,37,38,38,40,39,40,40,38,37,46,46,\\ &38,37,46,46,37,37,38,38,39,39,40,40,37,37,46,\\ &46,37,37,46,46] \in \mathcal{L}^{64\times 128},\\ H = \delta_8[&1,1,2,2,1,1,2,2,3,3,4,4,3,3,4,4,5,5,6,6,5,5,\\ &6,6,7,7,8,8,7,7,8,8,1,1,2,2,1,1,2,2,3,3,4,4,3,\\ &3,4,4,5,5,6,6,5,5,6,6,7,7,8,8,7,7,8,8] \in \mathcal{L}^{8\times 64}. \end{aligned}$$

Assume that the faulty model $\mathcal{B}(\tilde{L},\tilde{H})$ is derived from

$$\begin{cases} \mathrm{Col}_8(\tilde{L}) = \delta_{64}^{16}, \mathrm{Col}_{16}(\tilde{L}) = \delta_{64}^{8}, \mathrm{Col}_{40} = 64, \mathrm{Col}_{25}(\tilde{H}) = \delta_8^1, \mathrm{Col}_{64}(\tilde{H}) = \delta_8^7, \\ \mathrm{Col}_i(\tilde{L}) = \mathrm{Col}_i(L), \mathrm{Col}_j(\tilde{H}) = \mathrm{Col}_j(H),\ i \neq 8,16,40 \text{ and } j \neq 25,64. \end{cases}$$

By Algorithm 1, it is verified that $\mathcal{B}(\tilde{L},\tilde{H})$ is not strongly active fault-detectable. Furthermore, Algorithm 2 shows that $\mathcal{B}(\tilde{L},\tilde{H})$ is weakly active fault-detectable. Meanwhile, according to Algorithm 3, we

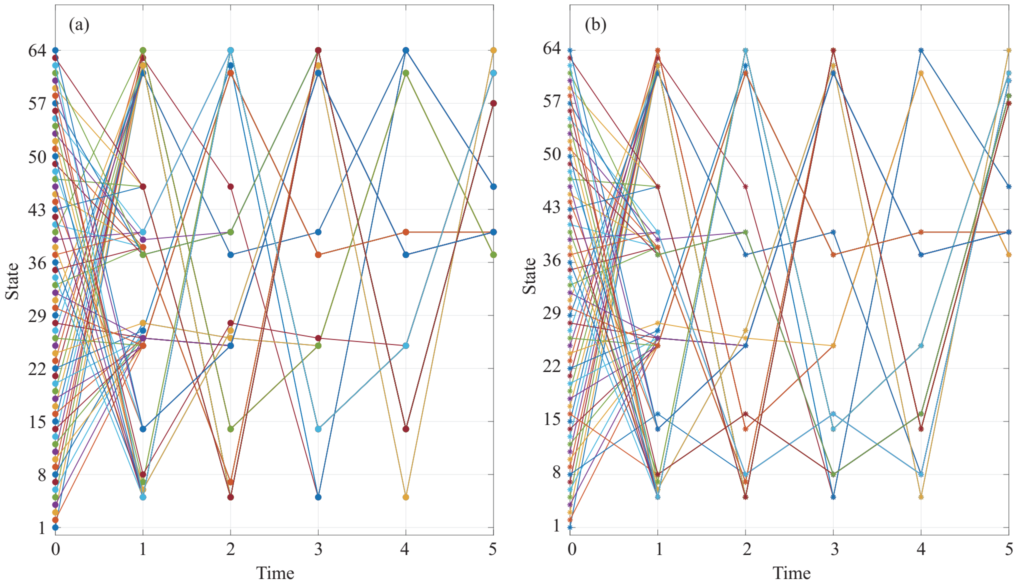

Figure 1 (Color online) The state trajectories of (a) the original and (b) faulty BCNs under input sequence $\{u(t)\}|_{t=0}^{4} = (\delta_2^1, \delta_2^1, \delta_2^1, \delta_2^1, \delta_2^2)$.

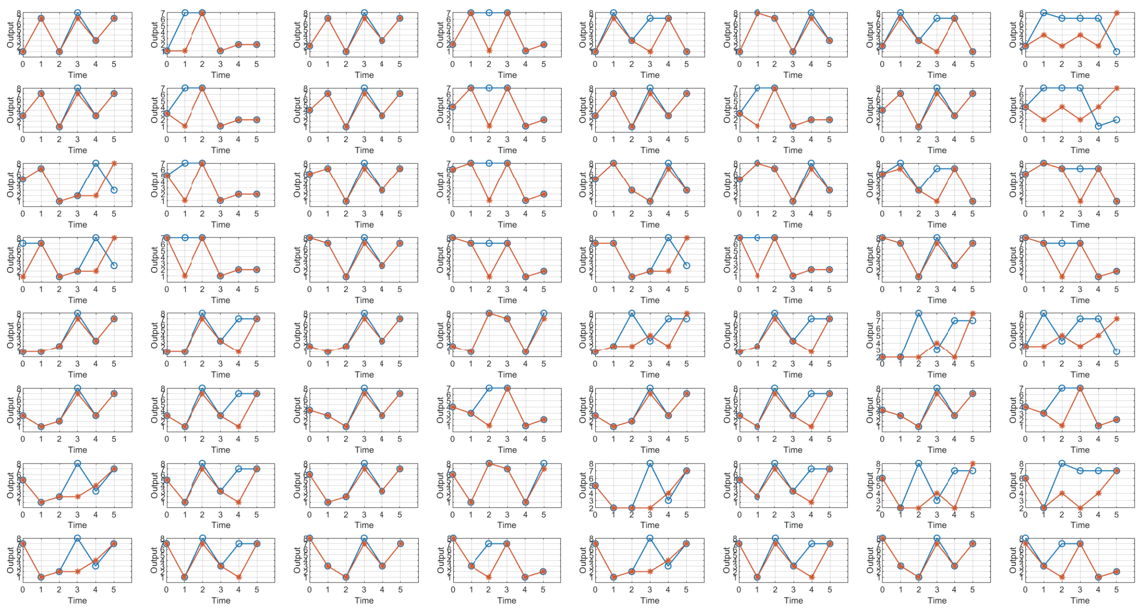

Figure 2 (Color online) The output trajectories of the original and faulty BCNs under input sequence $\{u(t)\}|_{t=0}^{4} = (\delta_2^1, \delta_2^1, \delta_2^1, \delta_2^1, \delta_2^2)$ starting from $(x_0, \tilde{x}_0) = (\delta_{64}^i, \delta_{64}^i)$, $i = 1, 2, \ldots, 64$ (in order from left to right, from top to bottom). The blue and red lines represent the outputs of the original and faulty BCNs, respectively.

get that there are 3 shortest input sequences of length 5 for active fault detection:

$$\{u(t)\}|_{t=0}^{4} = \begin{cases} (\delta_2^1, \delta_2^1, \delta_2^1, \delta_2^1, \delta_2^1), \\ (\delta_2^1, \delta_2^1, \delta_2^1, \delta_2^1, \delta_2^2), \\ (\delta_2^1, \delta_2^1, \delta_2^1, \delta_2^2, \delta_2^2). \end{cases}$$

Take $\{u(t)\}|_{t=0}^{4} = (\delta_2^1, \delta_2^1, \delta_2^1, \delta_2^1, \delta_2^2)$, and then Figure 1 shows the state trajectories of original and faulty BCNs. Corresponding to different initial states, 64^2 pairs of output sequences can be obtained, where each pair of output sequences is distinguishable. Due to the limitation of the space, we only show 64 pairs of output sequences in Figure 2.

6 Concluding remarks

In this paper, strong and weak active fault detectability of BCNs has been investigated by two new approaches. Firstly, the equivalent system-based method has been discussed, and necessary and sufficient conditions have been derived for weak active fault detectability. Secondly, the iterative matrix set-based method provides a unified framework to characterize strong and weak active fault detectability. In addition, corresponding algorithms have been developed for verifying fault detectability and generating all shortest input sequences. Compared with the existing results, our methods can significantly reduce the computational cost. Additionally, our methods can also be extended to various BCNs such as probabilistic, time-delayed, and disturbed BCNs. Another interesting topic, the fault isolation of BCNs, which was preliminarily studied by an augmented method in [39], can also be further investigated by our nonaugmented methods. On the other hand, to overcome the high computational complexity, the network structures have been taken into account for analysis and control of large-scale BNs [40, 41], which maintain the time complexity at a relatively low level. It is also interesting to investigate the fault detectability combined with the method of [40, 41] in the future research.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 62273201, 62203264), Research Fund for the Taishan Scholar Project of Shandong Province of China (Grant No. TSTP20221103), and Natural Science Fund of Shandong Province (Grant No. ZR2022QF061).

References

1 Kauffman S A. Metabolic stability and epigenesis in randomly constructed genetic nets. J Theor Biol, 1969, 22: 437–467
2 Glass L, Kauffman S A. The logical analysis of continuous, non-linear biochemical control networks. J Theor Biol, 1973, 39: 103–129
3 Ideker T, Galitski T, Hood L. A new approach to decoding life: systems biology. Annu Rev Genom Hum Genet, 2001, 2: 343–372
4 Kauffman S, Peterson C, Samuelsson B, et al. Random Boolean network models and the yeast transcriptional network. Proc Natl Acad Sci USA, 2003, 100: 14796–14799
5 Saez-Rodriguez J, Simeoni L, Lindquist J A, et al. A logical model provides insights into T cell receptor signaling. Plos Comput Biol, 2007, 3: e163
6 Veliz-Cuba A, Kumar A, Josić K. Piecewise linear and Boolean models of chemical reaction networks. Bull Math Biol, 2014, 76: 2945–2984
7 Cheng D Z. Semi-tensor product of matrices and its application to Morgen's problem. Sci China Ser F-Inf Sci, 2001, 44: 195–212
8 Li H T, Zhao G D, Meng M, et al. A survey on applications of semi-tensor product method in engineering. Sci China Inf Sci, 2018, 61: 010202
9 Yan Y Y, Cheng D Z, Feng J-E, et al. Survey on applications of algebraic state space theory of logical systems to finite state machines. Sci China Inf Sci, 2023, 66: 111201
10 Weiss E, Margaliot M, Even G. Minimal controllability of conjunctive Boolean networks is NP-complete. Automatica, 2018, 92: 56–62
11 Zhou R P, Guo Y Q, Liu X Z, et al. Stabilization of Boolean control networks with state-triggered impulses. Sci China Inf Sci, 2022, 65: 132202
12 Zhong J, Ho D W C, Lu J. A new approach to pinning control of Boolean networks. IEEE Trans Control Netw Syst, 2021, 9: 415–426
13 Zhang K, Zhang L, Su R. A weighted pair graph representation for reconstructibility of Boolean control networks. SIAM J Control Optim, 2016, 54: 3040–3060
14 Yu Y, Meng M, Feng J, et al. Observability criteria for Boolean networks. IEEE Trans Automat Contr, 2021, 67: 6248–6254
15 Liu Y, Zhong J, Ho D W C, et al. Minimal observability of Boolean networks. Sci China Inf Sci, 2022, 65: 152203
16 Wu Y, Sun X M, Zhao X, et al. Optimal control of Boolean control networks with average cost: a policy iteration approach. Automatica, 2019, 100: 378–387
17 Yao Y, Sun J. Optimal control of multi-task Boolean control networks via temporal logic. Syst Control Lett, 2021, 156: 105007
18 Wu Y H, Zhang J Y, Shen T L. A logical network approximation to optimal control on a continuous domain and its application to HEV control. Sci China Inf Sci, 2022, 65: 212203
19 Martincorena I, Campbell P J. Somatic mutation in cancer and normal cells. Science, 2015, 349: 1483–1489
20 Layek R, Datta A. Fault detection and intervention in biological feedback networks. J Biol Syst, 2012, 20: 441–453
21 Sridharan S, Layek R, Datta A, et al. Boolean modeling and fault diagnosis in oxidative stress response. BMC Genomics, 2012, 13: S4
22 Deshpande A, Layek R K. Fault detection and therapeutic intervention in gene regulatory networks using SAT solvers. Biosystems, 2019, 179: 55–62
23 Fornasini E, Valcher M E. Fault detection of Boolean control networks. In: Proceedings of the 53rd IEEE Conference on Decision and Control, Los Angeles, 2014. 6542–6547
24 Fornasini E, Valcher M E. Fault detection analysis of Boolean control networks. IEEE Trans Automat Contr, 2015, 60: 2734–2739
25 Leifeld T, Zhang Z, Zhang P. Fault detection for probabilistic Boolean networks. In: Proceedings of European Control Conference (ECC), Aalborg, 2016. 740–745
26 Fornasini E, Valcher M E. Fault detection problems for Boolean networks and Boolean control networks. In: Proceedings of the 34th Chinese Control Conference (CCC), Hangzhou, 2015. 1–8
27 Zhang Z, Leifeld T, Zhang P. Active fault detection of Boolean control networks. In: Proceedings of Annual American Control Conference (ACC), Milwaukee, 2018. 5001–5006

28 Dou W, Zhao G, Li H, et al. Off-line fault detection of logical control networks. Int J Syst Sci, 2022, 53: 478–487
29 Zhao R, Feng J, Wang B. Passive-active fault detection of Boolean control networks. J Franklin Institute, 2022, 359: 7196–7218
30 Chen Z, Zhou Y, Zhang Z, et al. Semi-tensor product of matrices approach to the problem of fault detection for discrete event systems (DESs). IEEE Trans Circuits Syst II, 2020, 67: 3098–3102
31 Dong Z. Boolean network-based sensor selection with application to the fault diagnosis of a nuclear plant. Energies, 2017, 10: 2125
32 Dong Z, Pan Y, Huang X. Parameter identifiability of Boolean networks with application to fault diagnosis of nuclear plants. Nucl Eng Tech, 2018, 50: 599–605
33 Li H, Wang Y. Boolean derivative calculation with application to fault detection of combinational circuits via the semi-tensor product method. Automatica, 2012, 48: 688–693
34 Liu Z, Wang Y, Li H. New approach to derivative calculation of multi-valued logical functions with application to fault detection of digital circuits. IET Control Theor Appl, 2014, 8: 554–560
35 Cheng D, Qi H, Liu T, et al. A note on observability of Boolean control networks. Syst Control Lett, 2016, 87: 76–82
36 Cheng D, Qi H. Controllability and observability of Boolean control networks. Automatica, 2009, 45: 1659–1667
37 Cheng D, Li Z, Qi H. Realization of Boolean control networks. Automatica, 2010, 46: 62–69
38 Moore E F. Gedanken-experiments on sequential machines. Autom Studies, 1956, 34: 129–153
39 Li Y, Li H. Fault isolation of logical control networks via set controllability of augmented system. IEEE Trans Circuits Syst II, 2023, 70: 1029–1033
40 Zhu S, Lu J, Azuma S, et al. Strong structural controllability of Boolean networks: polynomial-time criteria, minimal node control, and distributed pinning strategies. IEEE Trans Automat Contr, 2023, 68: 5461–5476
41 Zhu S, Lu J, Sun L, et al. Distributed pinning set stabilization of large-scale Boolean networks. IEEE Trans Automat Contr, 2023, 68: 1886–1893