SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

December 2023, Vol. 66 222202:1–222202:16
https://doi.org/10.1007/s11432-022-3781-4

Modified graph systems for distributed optimization

Zicong XIA[1,2], Yang LIU[1,2*], Dongyun WANG[3] & Weihua GUI[4]

[1]*Key Laboratory of Intelligent Education Technology and Application of Zhejiang Province, Jinhua* 321004*, China;*
[2]*School of Mathematical Sciences, Zhejiang Normal University, Jinhua* 321004*, China;*
[3]*College of Engineering, Zhejiang Normal University, Jinhua* 321004*, China;*
[4]*School of Information Science and Engineering, Central South University, Changsha* 410083*, China*

Received 29 December 2022/Revised 5 March 2023/Accepted 18 April 2023/Published online 28 November 2023

Abstract In distributed optimization theory, network topology graphs are important in communications among multiple agents. However, distributed optimization approaches cannot solve optimization problems well if the graphs are infeasible or tampered. To this end, this paper develops two types of modified graph systems for modifying or recovering the communication graphs among agents employed in distributed optimization. Two optimization problems for obtaining feasible graphs are formulated. Based on the two optimization problems, two modified graph systems are derived accordingly and their convergence to the optimal solution is proven. Via a coordination mechanism consisting of a distributed optimization approach and a modified graph system, we can modify an infeasible communication graph into a feasible one or recover a tampered graph, and the distributed optimization approach can resume its solver capability with the modified graphs. Several examples are provided to demonstrate the efficiency of the main results.

Keywords distributed optimization, modified graph system, neurodynamic optimization, communication graph

Citation Xia Z C, Liu Y, Wang D Y, et al. Modified graph systems for distributed optimization. Sci China Inf Sci, 2023, 66(12): 222202, https://doi.org/10.1007/s11432-022-3781-4

1 Introduction

With the development of multiagent system (MAS) techniques and parallel computing approaches, distributed optimization has become a research hotspot in the area of control and computer technology. The desirable distinguishing features of distributed optimization are summarized as: distributed optimization approach reduces communication consumption since it merely needs local information rather than global information; it has strong fault-tolerance and is safer; and the scalability of optimization approaches is increased. Thanks to the above advantages, various distributed optimization approaches are developed and applied in many areas, including sensor networks [1], resource allocation [2, 3], smart grid [4, 5], machine learning [6, 7], and so on [8–12].

As an optimization approach via multiagent systems, a noteworthy feature of distributed optimization is that the communication of optimization information is realized by the graph of multiple agents. Specifically, in the basic structure of distributed optimization, a node denotes an agent and each node (agent) has a local objective function. The global objective function is the sum of local objective functions. Every node cooperates to achieve the global optimization objective through information interaction with neighbor nodes. Therefore, the study of communication graphs becomes a hot point in distributed optimization and many types of graphs have been designed and applied to distributed optimization approaches: undirected graphs [4, 6, 7, 10, 13–19], strongly connected and weight-balanced graphs [2, 3, 11, 20–22], uniformly jointly connected graphs [23, 24], and so on [9, 25]. For example, in the studies of distributed optimization over undirected graphs, the undirected graphs applied to distributed optimization approaches should be connected to guarantee optimal solutions in consensus constraints. In [2, 20], digraphs are adopted in distributed optimization approaches. To obtain the optimal solutions, the digraphs should be strongly

* Corresponding author (email: liuyang@zjnu.edu.cn)

connected and weight-balanced. In [21], the weights of communication graphs are adaptive and the graphs are weight-balanced. In [25], although it only needs weight-unbalanced graphs, the considered optimization problem has a weaker constraint, called approximate consensus constraint, than consensus constraints in the problems considered in distributed optimization. In [23], an accelerated convergence algorithm with time-varying general directed graphs is designed for distributed optimization and the time-varying digraphs should be B_0-strongly connected. The aforementioned studies provide several sufficient conditions of communication graphs for distributed optimization. If communication graphs are tampered or destroyed so that they do not satisfy these conditions, we may hardly obtain the optimal solutions for distributed optimization problems with these graphs. Thus, a feasible graph is important in a distributed optimization method.

The research on communication graphs shows that if an undirected connected graph or a strongly connected and weight-balanced graph is applied to distributed optimization approaches, it can guarantee the consensus constraints in distributed optimization [20] or the consensus of gradient-value in distributed resource allocation [3, 26]. Based on the consensus, the optimization problems can be solved in a distributed manner. However, if communication graphs are not feasible for the optimization approach, or the initially feasible graphs are tampered or destroyed later, it is hard for existing distributed optimization approaches to obtain the optimal solutions.

To this end, we develop two modified graph systems (MGSs) to modify infeasible communication graphs or recover the communication graphs which are feasible initially but are tampered or destroyed later. The contributions are summarized as follows:

(1) For developing MGSs, two new optimization problems (Problems (7) and (14)) are established for obtaining an undirected graph and a weight-balanced graph, respectively. Two new optimization problems are transformed into vector forms (Problems (13) and (16)) in which a matrix for constructing undirected graphs and a matrix for constructing weight-balanced graphs are derived (see Lemmas 4 and 6). Besides, several properties of the proposed matrices are provided (see Lemmas 5 and 7).

(2) Based on the new optimization problems, two MGSs (MGSs (17) and (18)) are derived. The equilibrium points of the MGSs are proven to be the optimal solutions to the proposed optimization problems (Theorem 1) and their convergence is proven (Theorems 2 and 3).

(3) A coordination mechanism consisting of a distributed optimization approach and an MGS is proposed (see in (1) of Remark 2). The simulations show that, via the coordination mechanism, an infeasible graph or a tampered or destroyed graph can be modified into a feasible one. Besides, the MAS for solving optimization problems can run normally via the coordination mechanism.

The rest of this paper is organized as follows: In Section 2, the notations and problem formulation are provided. In Section 3, two MGSs are developed and their convergence is proven. Besides, a coordination mechanism consisting of a distributed optimization approach and a modified graph system is developed. In Section 4, a numerical example is elaborated to illustrate the validity of the main results. In Section 5, a conclusion is made.

2 Preliminaries and problem formulation

2.1 Preliminaries

Notations. Let $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^n$ denote the set of all real numbers, the set of all positive numbers, and the set of all n-dimension real vectors, respectively. $\times$ denotes the Cartesian product operator and $\otimes$ denotes the Kronecker product operator. A denotes an $n \times m$ matrix and $A(i,j)$ denotes the (i,j)-th element of A for $i = 1,\dots,n$ and $j = 1,\dots,m$. $\mathrm{Row}_i(A)$ denotes the i-th row of A and $\mathrm{Col}_i(A)$ denotes the i-th column of A. $\mathrm{Vec}(A) := [\mathrm{Row}_1(A),\dots,\mathrm{Row}_N(A)]^{\mathrm{T}} \in \mathbb{R}^{nm\times 1}$. For a group of vectors: $x_1, x_2, \dots, x_N$, $\mathrm{col}[x_1, x_2, \dots, x_N] := (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_N^{\mathrm{T}})^{\mathrm{T}}$. For an $n \times n$ matrix M and $\forall\ x \in \mathbb{R}^n$, $\|x\|_M := \sqrt{x^{\mathrm{T}}Mx}$. $\|\cdot\|$ denotes the Euclidean norm. $\boldsymbol{I}_n$ denotes an n-dimension identity matrix, and $\mathbf{1}_n(v)$ denotes an n-dimension vector with the v-th component being 1 and other components being 0.

Graph theory fundamentals. $\mathcal{G}(\mathcal{V},\mathcal{E},A)$ denotes a graph with N nodes. $\mathcal{V} = \{1,\dots,N\}$ is the set of all nodes in graph $\mathcal{G}$. $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is the set of all edges in graph $\mathcal{G}$. $A \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix of $\mathcal{G}$. $A(i,j) > 0$ if $(i,j) \in \mathcal{E}$. Otherwise, $A(i,j) = 0$. Let $\mathcal{N}_i = \{j : A(i,j) \neq 0\}$ be the set of the neighbors of node i. L denotes the Laplacian matrix of graph $\mathcal{G}$, where $L(i,i) = \sum_{j=1}^{n} A(i,j)$ for $\forall i = 1,\dots,N$ and $L(i,j) = -A(i,j)$ for $i \neq j$. A directed graph is said to be strongly connected if there exists a path from i to j for $\forall i, j \in \mathcal{V}$. Besides, $\mathcal{G}$ is undirected if and only if $A(i,j) = A(j,i)$ for $\forall i, j = 1, \ldots, N$, and an undirected graph is said to be connected if there exists a path from i to j for $\forall i, j \in \mathcal{V}$. $\mathcal{G}$ is said to be weight-balanced if $\sum_{j=1}^{N} A(i,j) = \sum_{j=1}^{N} A(j,i),\ \forall i = 1, \ldots, N$.

Definition 1. $\bar{x}$ is said to be an equilibrium point of the system $\dot{x} = f(x)$ $(f : \mathbb{R}^n \to \mathbb{R}^n)$ if $\mathbf{0}_n = f(\bar{x})$.

Lemma 1 ([27])**.** For projection operation $P_\Omega(x) = \arg\min_{y\in\Omega} \|x - y\|$ with $\Omega \subset \mathbb{R}^n$, it holds that

$$(y - P_\Omega(y))^{\mathrm{T}}(P_\Omega(y) - x) \geqslant 0,\ \forall y \in \mathbb{R}^n,\ \forall x \in \Omega, \tag{1}$$

$$\|P_\Omega(y) - P_\Omega(x)\| \leqslant \|y - x\|,\ \forall x, y \in \mathbb{R}^n, \tag{2}$$

$$\|P_\Omega(y) - P_\Omega(x)\|^2 \leqslant (y - x)^{\mathrm{T}}\left(P_\Omega(y) - P_\Omega(x)\right),\ \forall x, y \in \mathbb{R}^n. \tag{3}$$

Lemma 2 (KKT conditions [28])**.** Consider a constrained convex optimization problem as follows:

$$\min\ f(x)\ \text{ s.t. } Mx = \mathbf{0}_m,\ x \in \Omega, \tag{4}$$

with convex objective function $f : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$, $M \in \mathbb{R}^{m\times n}$, and convex set $\Omega \subset \mathbb{R}^n$, the following two statements hold:

(1) x^* is an optimal solution to Problem (4) if and only if there exists $\lambda^* \in \mathbb{R}^n$ such that $P_\Omega(x^* - \nabla f(x^*) + M^{\mathrm{T}}\lambda^*) - x^* = \mathbf{0}_n$ and $Mx^* = \mathbf{0}_m$.

(2) If M is of a row full rank, then x^* is an optimal solution to Problem (4) if and only if there exists y^* such that $\mathbf{0}_n = \tilde{M}P_\Omega(y^*) + (\boldsymbol{I} - \tilde{M})(y^* - P_\Omega(y^*) + \nabla f((\boldsymbol{I} - \tilde{M})P_\Omega(y^*)))$ with $\tilde{M} = M^{\mathrm{T}}(MM^{\mathrm{T}})^{-1}M$ and $x^* = P_\Omega(y^*)$.

2.2 Problem formulation

Consider an optimization problem with N agents as follows:

$$\min\ \textstyle\sum_{i=1}^{N} f_i(x_i)\ \text{ s.t. } x_i = x_j \in \mathbb{R}^n,\ \forall i, j\ = 1, \ldots, N, \tag{5}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ is convex for $\forall i = 1, \ldots, N$ and the communication graph of agents is defined as $\mathcal{G}(\mathcal{V}, \mathcal{E}, A)$.

To solve Problem (5), we introduce an MAS developed based on the KKT conditions as follows:

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = -\nabla f(\boldsymbol{x}) - \mathbb{L}\mu - \alpha\mathbb{L}\boldsymbol{x}, \\ \frac{\mathrm{d}\mu(t)}{\mathrm{d}t} = \mathbb{L}\boldsymbol{x}, \end{cases} \tag{6}$$

where $\boldsymbol{x} = \mathrm{col}[x_1, \ldots, x_N]$, $f(\boldsymbol{x}) = \sum_{i=1}^{N} f_i(x_i)$, $\mathbb{L} = L \otimes \boldsymbol{I}_n$, and L is the Laplacian matrix of graph $\mathcal{G}$. μ is a Lagrangian multiplier for the constraint $x_i = x_j$ for $i, j = 1, \ldots, N$. α is a positive parameter and its selection principle can be found in [20]. In addition, the design method of MAS (6) can be found in [13, Theorem 2].

A lemma is introduced to show the sufficient conditions such that MAS (6) solves Problem (5).

Lemma 3 ([7, 20])**.** If $\mathcal{G}$ is undirected and connected or $\mathcal{G}$ is strongly connected and weight-balanced, then, from any initial state, MAS (6) converges to an optimal solution to Problem (5).

According to Lemma 3, in this paper, graph $\mathcal{G}$ is said to be feasible for MAS (6) if it is an undirected and connected graph or a strongly connected and weight-balanced graph. Besides, graph $\mathcal{G}$ is said to be infeasible for MAS (6) if MAS (6) cannot converge to the optimal solution to Problem (5) with graph $\mathcal{G}$. To this end, two cases need to be tackled for MAS (6).

Case 1. Graph $\mathcal{G}$ is infeasible for MAS (6).

Case 2. Graph $\mathcal{G}$ is feasible for MAS (6) initially, but it is tampered or destroyed during the operation of MAS (6). The considered tampering behaviors or destroying behaviors are the tampering of the elements of the adjacency matrix.

For two cases, especially for Case 2, if we do not know when or where the tampering behaviors happen to the graph, we need to develop a dynamic mechanism to recover or modify the tampered graph to a feasible one constantly during the operation of MAS (6). According to Lemma 3, MAS (6) is a dynamic mechanism and it is globally stable at an optimal solution to Problem (5). Inspired by this idea, we define new optimization problems with proper objective functions and constraints such that the optimal solutions to the defined optimization problems are feasible graphs for MAS (6). Then, by the similar idea of developing MAS (6), we can develop dynamic mechanisms for solving the defined optimization problems. Since the developed dynamic mechanisms can modify the graph for MAS (6), they are said to be MGSs in this paper. Besides, the existing distributed optimization approaches (e.g., [2–8, 10, 11, 13–17, 20, 21, 23, 24, 29]) cannot tackle two cases since most of them choose a feasible graph before running the systems.

3 Main results

In this section, we develop two types of MGSs to modify graph $\mathcal{G}$ for MAS (6). In detail, two new optimization problems are established for obtaining undirected graphs and weight-balanced graphs. The proposed problems are transformed into vector forms as Problem (4). According to new vector-form problems, two MGSs are developed and their convergence is proven.

3.1 Optimization problems for undirected graphs

In this subsection, we consider an optimization problem as follows:

$$\begin{aligned} &\min_{A}\ C(A)\\ &\text{s.t. } A(i,j)=A(j,i),\ \forall i,j=1,\ldots,N,\\ &\qquad A(i,j)\in[\underline{A}(i,j),\overline{A}(i,j)],\ \forall i,j=1,\ldots,N,\ i\neq j,\\ &\qquad A(i,i)=0,\ \forall i=1,\ldots,N, \end{aligned} \tag{7}$$

where $A\in\mathbb{R}^{N\times N}$. $\underline{A}(i,j)$ and $\overline{A}(i,j)$ denote the given lower bound and upper bound of $A(i,j)$ with $0\leqslant\underline{A}(i,j)\leqslant\overline{A}(i,j)$. Note that A is a weighted adjacency matrix of an undirected graph when it satisfies the constraints of Problem (7). $C(A)$ is an objective function and it can be adjusted to satisfy some requirements.

The decision variable of Problem (7) is a matrix. To adopt the KKT conditions in Lemma 2 better, we need to reformulate it as the form of Problem (4). Now, we provide a lemma to transform the first equality constraint of Problem (7).

Lemma 4. Let $M_u\in\mathbb{R}^{(N(N-1)/2)\times N^2}$ with

$$M_u(v,w)=\begin{cases}1, & v\in[\underline{v}(k),\bar{v}(k)],\ w=w_1(v,k),\ k=1,\ldots,N-1,\\ -1, & v\in[\underline{v}(k),\bar{v}(k)],\ w=w_2(v,k),\ k=1,\ldots,N-1,\\ 0, & \text{otherwise,}\end{cases} \tag{8}$$

where

$$\underline{v}(k)=\sum_{p=1}^{k}(N-p)-N+k+1, \tag{9}$$

$$\bar{v}(k)=\sum_{p=1}^{k}(N-p), \tag{10}$$

$$w_1(v,k)=(v-\underline{v}(k))+(k-1)(N+1)+2, \tag{11}$$

$$w_2(v,k)=(k-1)(N+1)+1+N(w_1(v,k)-((k-1)(N+1)+1)). \tag{12}$$

Then, $A(i,j)=A(j,i),\ \forall i,j=1,\ldots,N$, if and only if $M_u\mathrm{Vec}(A)=\mathbf{0}_{N(N-1)/2}$.

Proof. Note that $\mathrm{Row}_w(\mathrm{Vec}(A))=A(i,j)$ if and only if $w=N(i-1)+j$. Assume that w_1 satisfies $\mathrm{Row}_{w_1}(\mathrm{Vec}(A))=A(i,j)$ for $i<j$ and $\forall\ i=1,\ldots,N-1$. The proof begins with two parts: (i) If there exists w_2 such that $w_2=(i-1)(N+1)+1+N(w_1-((i-1)(N+1)+1))$, then $\mathrm{Row}_{w_2}(\mathrm{Vec}(A))=A(j,i)$; (ii) For $\forall\ k=1,\ldots,N-1$ and $\forall\ v\in[\underline{v}(k),\bar{v}(k)]$, $\mathrm{Row}_{w_1(v,k)}(\mathrm{Vec}(A))=A(k,k+v-\underline{v}(k)+1)$.

Now, we prove (i). It can be obtained that $w_1=N(i-1)+j$. Moreover, $w_2=(i-1)(N+1)+1+N((N(i-1)+j)-((i-1)(N+1)+1))=N(j-1)+i$ which implies that $\mathrm{Row}_{w_2}(\mathrm{Vec}(A))=A(j,i)$.

Next, we prove (ii). From (11), $w_1(v,k) - (N(k-1)+k+v-\underline{v}(k)+1) = (v-\underline{v}(k)) + (k-1)(N+1)+2-N(k-1)-k-v+\underline{v}(k)-1 = 0$. Thus, $w_1(v,k) = N(k-1)+k+v-\underline{v}(k)+1$ which implies $\mathrm{Row}_{w_1(v,k)}(\mathrm{Vec}(A)) = A(k, k+v-\underline{v}(k)+1)$.

Based on (ii), (9), and (10), $v-\underline{v}(k)+1 \in [1, N-k]$ holds for $\forall\ k = 1,\ldots,N-1$. Hence, for $\forall\ k=1,\ldots,N-1$, $\{A(k,k+v-\underline{v}(k)+1)\}_{v\in[\underline{v}(k),\bar{v}(k)]} = \{A(k,k+1), A(k,k+2),\ldots,A(k,N)\}$. Besides, $\underline{v}(k+1)-\bar{v}(k) = \sum_{p=1}^{k+1}(N-p)-N+k+2-\sum_{p=1}^{k}(N-p) = 1$, i.e., $[\underline{v}(k),\bar{v}(k)]\cap[\underline{v}(k+1),\bar{v}(k+1)] = \emptyset$ for $\forall\ k=1,\ldots,N-2$. Let $\bar{B} := \{\mathrm{Row}_{w_1(v,k)}(\mathrm{Vec}(A))\}_{k=1,\ldots,N-1,v\in[\underline{v}(k),\bar{v}(k)]}$ and let B denote the set of entries in the upper triangle of A. Thus, $\bar{B} = B$ and $\mathrm{Row}_v(M_u)\mathrm{Vec}(A) = \mathrm{Row}_{w_1(v,k)}(\mathrm{Vec}(A)) - \mathrm{Row}_{w_2(v,k)}(\mathrm{Vec}(A))$ hold for $\forall\ k=1,\ldots,N-1$.

Suppose that $M_u\mathrm{Vec}(A) = \mathbf{0}_{N(N-1)/2}$ holds; then for $\forall\ v\in[1,N(N-1)/2]$, we have that $\mathrm{Row}_v(M_u)$ $\mathrm{Vec}(A) = 0$ which implies $\mathrm{Row}_{w_1(v,k)}(\mathrm{Vec}(A)) = \mathrm{Row}_{w_2(v,k)}(\mathrm{Vec}(A))$. Based on (i) and $\bar{B}=B$, $A(i,j) = A(j,i),\ i<j,\ \forall i,j=1,\ldots,N$ holds, which is equal to $A(i,j)=A(j,i),\ \forall i,j=1,\ldots,N$. Conversely, suppose that $A(i,j)=A(j,i),\ \forall i,j=1,\ldots,N$ holds; then $A(i,j)=A(j,i),\ i<j,\ \forall i,j=1,\ldots,N$ holds. Based on (i), $\mathrm{Row}_{w_1(v,k)}(\mathrm{Vec}(A)) = \mathrm{Row}_{w_2(v,k)}(\mathrm{Vec}(A))$ for $\forall\ k=1,\ldots,N-1$ and $\forall\ v\in[\underline{v}(k),\bar{v}(k)]$. Owing to $\bar{B}=B$, then $\mathrm{Row}_v(\mathrm{Vec}(A)) = 0$ for $\forall\ v\in[1,N(N-1)/2]$, which implies $M_u\mathrm{Vec}(A) = \mathbf{0}_{N(N-1)/2}$. The proof is complete.

In (2) of Lemma 2, the matrix of the linear equality constraint should be of a row full rank, thus, the following lemma is provided.

Lemma 5. M_u is of a row full rank, i.e., $\mathrm{rank}(M_u) = N(N-1)/2$.

Proof. Based on (8), note that $[\underline{v}(k),\bar{v}(k)]\cap[\underline{v}(k+1),\bar{v}(k+1)] = \emptyset$ for $k=1,\ldots,N-2$ (according to the proof of Lemma 4) and $\sum_{k=1}^{N-1}(\bar{v}(k)-\underline{v}(k)+1) = \sum_{k=1}^{N-1}k = N(N-1)/2$. Besides, we have that $\mathrm{Col}_{w_1(v,k)}(M_u) = \mathbf{1}(v)$ and $\mathrm{Col}_{w_2(v,k)}(M_u) = -\mathbf{1}(v)$ for $\forall\ k=1,\ldots,N-1$ and $\forall\ v\in[\underline{v}(k),\bar{v}(k)]$. Since $\bar{B}=B$, we have that $\mathrm{Col}_w(M_u) = \mathbf{0}_{N(N-1)/2}$, $w=\{1,\ldots,N^2\}\setminus(\{w_1(v,k),w_2(v,k)\}_{k=1,\ldots,N-1}, v\in[\underline{v}(k+1),\bar{v}(k+1)])$. Then,

$$\begin{aligned}
\mathrm{rank}(M_u) = \mathrm{rank}([&\mathrm{Col}_{w_1(\underline{v}(1),1)}(M_u), \mathrm{Col}_{w_1(\underline{v}(1)+1,1)}(M_u),\ldots,\mathrm{Col}_{w_1(\bar{v}(1),1)}(M_u),\\
&\mathrm{Col}_{w_1(\underline{v}(2),2)}(M_u), \mathrm{Col}_{w_1(\underline{v}(2)+1,2)}(M_u),\ldots,\mathrm{Col}_{w_1(\bar{v}(2),2)}(M_u),\\
&\ldots,\mathrm{Col}_{w_1(\underline{v}(N-1),N-1)}(M_u),\ldots,\mathrm{Col}_{w_1(\bar{v}(N-1),N-1)}(M_u),\\
&\mathrm{Col}_{w_2(\underline{v}(1),1)}(M_u), \mathrm{Col}_{w_2(\underline{v}(1)+1,1)}(M_u),\ldots,\mathrm{Col}_{w_2(\bar{v}(1),1)}(M_u),\\
&\mathrm{Col}_{w_2(\underline{v}(2),2)}(M_u), \mathrm{Col}_{w_2(\underline{v}(2)+1,2)}(M_u),\ldots,\mathrm{Col}_{w_2(\bar{v}(2),2)}(M_u),\\
&\ldots,\mathrm{Col}_{w_2(\underline{v}(N-1),N-1)}(M_u),\ldots,\mathrm{Col}_{w_2(\bar{v}(N-1),N-1)}(M_u)])\\
= \mathrm{rank}([&\mathrm{Col}_{w_1(\underline{v}(1),1)}(M_u), \mathrm{Col}_{w_1(\underline{v}(1)+1,1)}(M_u),\ldots,\mathrm{Col}_{w_1(\bar{v}(1),1)}(M_u),\\
&\mathrm{Col}_{w_1(\underline{v}(2),2)}(M_u), \mathrm{Col}_{w_1(\underline{v}(2)+1,2)}(M_u),\ldots,\mathrm{Col}_{w_1(\bar{v}(2),2)}(M_u),\\
&\ldots,\mathrm{Col}_{w_1(\underline{v}(N-1),N-1)}(M_u),\ldots,\mathrm{Col}_{w_1(\bar{v}(N-1),N-1)}(M_u)])\\
= \mathrm{rank}&\left(\boldsymbol{I}_{\sum_{k=1}^{N-1}(\bar{v}(k)-\underline{v}(k)+1)}\right) = \mathrm{rank}\left(\boldsymbol{I}_{\sum_{k=1}^{N-1}k}\right) = \sum_{k=1}^{N-1}k = \frac{N(N-1)}{2},
\end{aligned}$$

which completes the proof.

Example 1. An example of M_u with $N=3$ is provided as follows:

$$M_u = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

If $A(i,j)=A(j,i)$ for any $i,j=1,2,3$, then we can obtain that $M_u\mathrm{Vec}(A) = \mathbf{0}_3$. Vice versa. In addition, $\mathrm{rank}(M_u)=3$.

Let $\boldsymbol{a}^{\mathrm{Vec}} = \mathrm{Vec}(A)\in\mathbb{R}^{N^2}$ and $\tilde{C}\left(\boldsymbol{a}^{\mathrm{Vec}}\right) = C(A)$; an optimization problem is shown in the following:

$$\begin{aligned}
&\min\ \tilde{C}\left(\boldsymbol{a}^{\mathrm{Vec}}\right)\\
&\text{s.t. } M_u\boldsymbol{a}^{\mathrm{Vec}} = \mathbf{0}_{N(N-1)/2},\ \forall i\neq j,\ i,j=1,\ldots,N,\\
&\quad\ \boldsymbol{a}^{\mathrm{Vec}}\in\Omega,
\end{aligned}\tag{13}$$

where $\Omega = \prod_{i=1}^{N^2} \Omega_i$ in which $\Omega_i = \{0\}$ if $i = 1 + (k-1)(N+1),\ k = 1,\ldots,N$ and $\Omega_{N(i-1)+j} = [\underline{A}(j,i), \overline{A}(j,i)],\ \forall i,j = 1,\ldots,N,\ i \neq j$.

Corollary 1. A_u^* is an optimal solution to Problem (7) if and only if $\boldsymbol{a}_u^{\mathrm{Vec}*} = \mathrm{Vec}(A_u^*)$ is an optimal solution to Problem (13).

According to Lemma 4, we have that $A(i,j) = A(j,i), \forall i,j = 1,\ldots,N$, if and only if $M_u \boldsymbol{a}^{\mathrm{Vec}} = \mathbf{0}_{N(N-1)/2}$. In addition, $A(i,j) \in [\underline{A}(i,j), \overline{A}(i,j)], \forall i,j = 1,\ldots,N, i \neq j$, and $A(i,i) = 0, \forall i = 1,\ldots,N$ if and only if $\boldsymbol{a}^{\mathrm{Vec}} \in \Omega$. Therefore, Corollary 1 holds obviously. According to Corollary 1, Problem (7) is converted into Problem (13).

3.2 Optimization problems for weight-balanced graphs

Similar to Problem (7), we consider an optimization problem whose decision variable is a weighted adjacency matrix of a weight-balanced graph:

$$\begin{aligned} &\min_A C(A) \\ &\text{s.t.} \ \sum_{j=1}^N A(i,j) = \sum_{j=1}^N A(j,i),\ \forall i = 1,\ldots,N, \\ &\qquad A(i,j) \in [\underline{A}(i,j), \overline{A}(i,j)],\ \forall i,j = 1,\ldots,N,\ i \neq j, \\ &\qquad A(i,i) = 0, \forall i = 1,\ldots,N. \end{aligned} \tag{14}$$

Similar to the transformation of Problem (7), in the following lemma, the first equality constraint of Problem (14) is reformulated in a vector form.

Lemma 6. Let $M_d \in \mathbb{R}^{N \times N^2}$ with

$$\mathrm{Row}_k(M_d) = \begin{cases} \sum_{q_1=\underline{v}(k)}^{\bar{v}(k)} \mathrm{Row}_{q_1}(M_u) - \sum_{q_2=1}^{k-1} \mathrm{Row}_{\bar{v}(q_2)}(M_u), & k = 1,\ldots,N-1, \\ -\sum_{q_2=1}^{k-1} \mathrm{Row}_{\bar{v}(q_2)}(M_u), & k = N, \end{cases} \tag{15}$$

where $\underline{v}(k)$, $\bar{v}(k)$, and M_u are from Lemma 4. Then, $\sum_{j=1}^N A(i,j) = \sum_{j=1}^N A(j,i),\ \forall i = 1,\ldots,N$, if and only if $M_d \mathrm{Vec}(A) = \mathbf{0}_N$.

Proof. We just need to prove the statement: $\forall i = 1,\ldots,N$, $\mathrm{Row}_i(M_d)\mathrm{Vec}(A) = 0$ if and only if $\sum_{j=1}^N A(i,j) = \sum_{j=1}^N A(j,i)$. Suppose that $\mathrm{Row}_i(M_d)\mathrm{Vec}(A) = 0$; then one can obtain that

$$\left[\sum_{q_1=\underline{v}(i)}^{\bar{v}(i)} \mathrm{Row}_{q_1}(M_u) - \sum_{q_2=1}^{i-1} \mathrm{Row}_{\bar{v}(q_2)}(M_u)\right] \mathrm{Vec}(A) = 0.$$

For $i = 1,\ldots,N-1$, combining (9) with (10), it holds that $\sum_{q_1=\underline{v}(i)}^{\bar{v}(i)} \mathrm{Row}_{q_1}(M_u)\mathrm{Vec}(A) = \sum_{j=i+1}^N A(i,j) - \sum_{i=j+1}^N A(j,i)$ and $\sum_{q_2=1}^{i-1} \mathrm{Row}_{\bar{v}(q_2)}(M_u)\mathrm{Vec}(A) = -\sum_{j=1}^{i-1} A(i,j) + \sum_{j=1}^{i-1} A(j,i)$. Thus, $\sum_{j=i+1}^N A(i,j) - \sum_{i=j+1}^N A(j,i) - (-\sum_{j=1}^{i-1} A(i,j) + \sum_{j=1}^{i-1} A(j,i)) = \sum_{j=1,j\neq i}^N A(i,j) - \sum_{j=1,j\neq i}^N A(j,i) = 0$. Thus, $\sum_{j=1,j\neq i}^N A(i,j) = \sum_{j=1,j\neq i}^N A(j,i)$ which implies $\sum_{j=1}^N A(i,j) = \sum_{j=1}^N A(j,i)$. Conversely, suppose that $\sum_{j=1}^N A(i,j) = \sum_{j=1}^N A(j,i)$ holds; then $\mathrm{Row}_i(M_d)\mathrm{Vec}(A) = 0$ follows in the same way.

Similar to Lemma 5, the following lemma provides the ranks of M_d and its augmented form $\tilde{M}_d$.

Lemma 7. (1) $\mathrm{rank}(M_d) = N - 1$;

(2) Letting $M_s = \begin{bmatrix} 1 & \mathbf{0}_{1\times(N^2-1)} \\ \mathbf{0}_{(N-1)\times 1} & \mathbf{0}_{(N-1)\times(N^2-1)} \end{bmatrix} \in \mathbb{R}^{N \times N^2}$ and $\tilde{M}_d = M_d + M_s$, then $\tilde{M}_d$ is of a row full rank, i.e., $\mathrm{rank}(\tilde{M}_d) = N$.

Proof. The proof of (1) begins with three statements:

(i) $\mathrm{Col}_w(M_d) = \mathbf{0}_N$ for $w = (k-1)(N+1)+1,\ k = 1,\ldots,N$;

(ii) $\mathrm{Col}_w(M_d) = \mathbf{1}_N(k) - \mathbf{1}_N(k + v - \underline{v}(k) + 1)$ for $v \in [\underline{v}(k), \bar{v}(k)],\ w = w_1(v,k),\ k = 1,\ldots,N-1$;

(iii) $\mathrm{Col}_{w_1(v,k)}(M_d) + \mathrm{Col}_{w_2(v,k)}(M_d) = \mathbf{0}_N$ for $v \in [\underline{v}(k), \bar{v}(k)],\ k = 1,\ldots,N-1$.

Combining (15) with (8), (11), and (12), (i) and (iii) hold. For (ii), based on (15) and (8), we have $M_d(k, w_1(v,k)) = 1$ for $\forall v \in [\underline{v}(k), \bar{v}(k)], k = 1,\ldots,N-1$. Then, comparing (15) at different

$k = 1,\ldots,N-1$, $M_d(k+v-\underline{v}(k)+1, w_1(v,k)) = -1$ holds for $v \in [\underline{v}(k), \bar{v}(k)], k = 1,\ldots,N-1$ which leads to (iii). To sum up with (i), (ii), and (iii), the rank of M_d is equal to the following matrix M_d':

$$M_d' = \begin{bmatrix} \mathbf{1}_{N-1}^{\mathrm{T}} & \mathbf{0}_{1\times(N-2)} & \cdots & \mathbf{0}_{n-1\times(N-n)} & \cdots & \mathbf{0}_{N-1} \\ & \mathbf{1}_{N-2}^{\mathrm{T}} & & \mathbf{1}_{N-n}^{\mathrm{T}} & & 1 \\ -\boldsymbol{I}_{N-1} & -\boldsymbol{I}_{N-2} & \cdots & -\boldsymbol{I}_{N-n} & \cdots & -1 \end{bmatrix}.$$

Thus, $\mathrm{rank}(M_d) = \mathrm{rank}(M_d') = N-1$. The proof of (2) can be obtained easily: $\mathrm{rank}(\tilde{M}_d) = \mathrm{rank}(M_d + M_s) = \mathrm{rank}([\mathbf{1}_N(1)\ M_d']) = N$.

Example 2. Based on M_u in Example 1 and (15), an example of M_d with $N = 3$ is provided as follows:

$$M_d = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & 0 \end{bmatrix}.$$

Note that $M_d\mathrm{Vec}(A) = \mathbf{0}_3$ if $\sum_{j=1}^{3} A(i,j) = \sum_{j=1}^{3} A(j,i),\ \forall i = 1,2,3$. Vice versa. In addition, we can calculate $\mathrm{rank}(M_u) = 2$. Meanwhile, M_d' and $\tilde{M}_d$ can be calculated as follows:

$$M_d' = \begin{bmatrix} 1 & 1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 1 & 1 \end{bmatrix},$$

$$\tilde{M}_d = \begin{bmatrix} 1 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & 0 \end{bmatrix}.$$

Then, we can obtain that $\mathrm{rank}(M_d') = N-1$ and $\mathrm{rank}(\tilde{M}_d) = N$.

To proceed, with the similar settings in Problem (13), an optimization problem is proposed as follows:

$$\begin{aligned} &\min\ \tilde{C}(\boldsymbol{a}^{\mathrm{Vec}}) \\ &\text{s.t. } \tilde{M}_d\boldsymbol{a}^{\mathrm{Vec}} = \mathbf{0}_N,\ \forall i \neq j, \\ &\qquad \boldsymbol{a}^{\mathrm{Vec}} \in \Omega,\ \forall i,j = 1,\ldots,N. \end{aligned} \tag{16}$$

Corollary 2. A_d^* is an optimal solution to Problem (14) if and only if $\boldsymbol{a}_d^{\mathrm{Vec}^*} = \mathrm{Vec}(A_d^*)$ is an optimal solution to Problem (16).

According to Corollary 2, Problem (14) is converted into Problem (16).

Remark 1. Solving Problems (13) and (16) can provide an undirected graph and a weight-balanced graph, respectively. However, the obtained graphs can hardly be applicable to the conditions in Lemma 3 since they may not be connected or strongly connected. As mentioned in Subsection 3.1, the objective function can be adjusted to satisfy some requirements. Thus, we can adjust a suitable objective function to guarantee the connectivity. For example, a connected graph (or a strongly connected graph) $\hat{A}$ is provided in advance. Let $\hat{\boldsymbol{a}}^{\mathrm{Vec}} := \mathrm{Vec}(\hat{A})$. To guarantee the connectivity of the optimal solution, we can set $C(A) = \sum_{i=1}^{N}\sum_{j=1}^{N}(A(i,j)-\hat{A}(i,j))^2$, i.e., $\tilde{C}(\boldsymbol{a}^{\mathrm{Vec}}) = \|\boldsymbol{a}^{\mathrm{Vec}} - \hat{\boldsymbol{a}}^{\mathrm{Vec}}\|^2$. From the example in Section 4, the given objective function can lead to a connected optimal solution or a strongly connected optimal solution.

3.3 Modified graph systems

In this subsection, we focus on developing MGSs for solving Problems (13) and (16). Based on the invertibility of M_u or $\tilde{M}_d$, let $P_u = (M_u)^{\mathrm{T}}(M_u(M_u)^{\mathrm{T}})^{-1}M_u$ and $P_d = (\tilde{M}_d)^{\mathrm{T}}(\tilde{M}_d(\tilde{M}_d)^{\mathrm{T}})^{-1}\tilde{M}_d$.

Now, we propose two systems to solve Problems (13) and (16) as follows:

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{a}^{\mathrm{Vec}}}{\mathrm{d}t} = -\boldsymbol{a}^{\mathrm{Vec}} + P_\Omega(\boldsymbol{a}^{\mathrm{Vec}} - \nabla\tilde{C}(\boldsymbol{a}^{\mathrm{Vec}}) + M^{\mathrm{T}}\lambda), \\ \frac{\mathrm{d}\lambda}{\mathrm{d}t} = -M\boldsymbol{a}^{\mathrm{Vec}}, \end{cases} \tag{17}$$

where $M \in \{M_u, M_d\}$, $\boldsymbol{a}^{\mathrm{Vec}} \in \mathbb{R}^{N^2}$, and $\lambda \in \{\lambda_u, \lambda_d\}$ with $\lambda_u \in \mathbb{R}^{N(N-1)/2}$ and $\lambda_d \in \mathbb{R}^N$.

$$\begin{cases} \boldsymbol{a}^{\mathrm{Vec}} = P_\Omega(\boldsymbol{a}), \\ \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = -P\boldsymbol{a}^{\mathrm{Vec}} - (\boldsymbol{I} - P)(\boldsymbol{a} - \boldsymbol{a}^{\mathrm{Vec}} + \nabla\tilde{C}\left((\boldsymbol{I} - P)\boldsymbol{a}^{\mathrm{Vec}}\right)), \end{cases} \tag{18}$$

where $P \in \{P_u, P_d\}$ and $\boldsymbol{a}, \boldsymbol{a}^{\mathrm{Vec}} \in \mathbb{R}^{N^2}$.

Then, we present three theorems (Theorems 1–3) to prove the convergence of MGS (17) or MGS (18).

Theorem 1. If $\tilde{C}\left(\boldsymbol{a}^{\mathrm{Vec}}\right)$ is convex, then the following four statements hold:

(1) If $M = M_u$, let col$[\boldsymbol{a}_u^{\mathrm{Vec}*}, \lambda_u^*]$ be an equilibrium point of MGS (17); then $\boldsymbol{a}_u^{\mathrm{Vec}*}$ is an optimal solution to Problem (13).

(2) If $M = M_d$, let col$[\boldsymbol{a}_d^{\mathrm{Vec}*}, \lambda_d^*]$ be an equilibrium point of MGS (17); then $\boldsymbol{a}_d^{\mathrm{Vec}*}$ is an optimal solution to Problem (16).

(3) If $P = P_u$, let $\boldsymbol{a}_u^*$ be an equilibrium point of MGS (18); then $\boldsymbol{a}_u^{\mathrm{Vec}*} = P_\Omega(\boldsymbol{a}_u^*)$ is an optimal solution to Problem (13).

(4) If $P = P_d$, let $\boldsymbol{a}_d^*$ be an equilibrium point of MGS (18); then $\boldsymbol{a}_d^{\mathrm{Vec}*} = P_\Omega(\boldsymbol{a}_d^*)$ is an optimal solution to Problem (16).

Proof. For (1), based on Definition 1, $\boldsymbol{0}_{N^2} = -\boldsymbol{a}_u^{\mathrm{Vec}*} + P_\Omega(\boldsymbol{a}_u^{\mathrm{Vec}*} - \nabla\tilde{C}(\boldsymbol{a}_u^{\mathrm{Vec}*}) + M^{\mathrm{T}}\lambda_u^*)$ and $M^{\mathrm{T}}\boldsymbol{a}_u^{\mathrm{Vec}*} = \boldsymbol{0}_{N(N-1)/2}$ hold. Then according to (1) in Lemma 2, $\boldsymbol{a}_u^{\mathrm{Vec}*}$ is an optimal solution to Problem (13). In a similar way, Eq. (2) holds. For (3), based on Definition 1, $\boldsymbol{a}_u^{\mathrm{Vec}*} = P_\Omega(\boldsymbol{a}_u^*)$ and $\boldsymbol{0}_{N^2} = -P\boldsymbol{a}_u^{\mathrm{Vec}*} - (\boldsymbol{I}_{N^2} - P)(\boldsymbol{a}_u^* - \boldsymbol{a}_u^{\mathrm{Vec}*} + \nabla\tilde{C}((\boldsymbol{I}_{N^2} - P)\boldsymbol{a}_u^{\mathrm{Vec}*}))$ hold. Then according to (2) in Lemma 2, $\boldsymbol{a}_u^*$ is an optimal solution to Problem (16). In a similar way, Eq. (4) holds.

Assumption 1. $\tilde{C}(\boldsymbol{a}^{\mathrm{Vec}})$ is differentiable and strictly convex.

Theorem 2. Under Assumption 1, if $\tilde{C}(\boldsymbol{a}^{\mathrm{Vec}})$ is twice differentiable, then the following two statements hold:

(1) If $M = M_u$, then from any initial state col$[\boldsymbol{a}^{\mathrm{Vec}}(0), \lambda_u(0)] \in \Omega \times \mathbb{R}^{N(N-1)/2}$, MGS (17) converges to an optimal solution to Problem (13).

(2) If $M = M_d$, then from any initial state col$[\boldsymbol{a}^{\mathrm{Vec}}(0), \lambda_d(0)] \in \Omega \times \mathbb{R}^N$, MGS (17) converges to an optimal solution to Problem (16).

Proof. For the proof of (1), according to [30, Lemma 3], it can be obtained that from any initial state col$[\boldsymbol{a}^{\mathrm{Vec}}(0), \lambda_u(0)] \in \Omega \times \mathbb{R}^{N(N-1)/2}$, col$[\boldsymbol{a}^{\mathrm{Vec}}(t), \lambda_u(t)] \in \Omega \times \mathbb{R}^{N(N-1)/2}$. Let $M = M_u$ and col$[\boldsymbol{a}_u^{\mathrm{Vec}*}, \lambda_u^*]$ be an equilibrium point of MGS (17). According to (1) in Theorem 1, $\boldsymbol{a}_u^{\mathrm{Vec}*}$ is an optimal solution to Problem (13). A Lyapunov function is constructed as follows:

$$\begin{aligned} V\left(\boldsymbol{a}^{\mathrm{Vec}}, \lambda_u\right) = {} & \frac{1}{2}\left(\left\|\boldsymbol{a}^{\mathrm{Vec}} - \boldsymbol{a}_u^{\mathrm{Vec}*}\right\|^2 + \|\lambda_u - \lambda_u^*\|^2\right) + \left\|M_u\boldsymbol{a}^{\mathrm{Vec}}\right\|^2 \\ & - \left(\nabla\tilde{C}(\boldsymbol{a}^{\mathrm{Vec}}) - M_u^{\mathrm{T}}\lambda_u\right)^{\mathrm{T}}\left(P_\Omega\left(\boldsymbol{a}^{\mathrm{Vec}} - \nabla\tilde{C}(\boldsymbol{a}^{\mathrm{Vec}}) + M_u^{\mathrm{T}}\lambda_u\right) - \boldsymbol{a}^{\mathrm{Vec}}\right) \\ & - \frac{1}{2}\left(\left\|P_\Omega(\boldsymbol{a}^{\mathrm{Vec}} - \nabla\tilde{C}(\boldsymbol{a}^{\mathrm{Vec}}) + M_u^{\mathrm{T}}\lambda_u) - \boldsymbol{a}^{\mathrm{Vec}}\right\|^2 + \left\|M_u\boldsymbol{a}^{\mathrm{Vec}}\right\|^2\right). \end{aligned}$$

Let $\Delta = \nabla\tilde{C}(\boldsymbol{a}^{\mathrm{Vec}}) - M_u^{\mathrm{T}}\lambda_u$. According to Lemma 1, we have

$$\begin{aligned} & -\Delta^{\mathrm{T}}\left(P_\Omega\left(\boldsymbol{a}^{\mathrm{Vec}} - \Delta\right) - \boldsymbol{a}^{\mathrm{Vec}}\right) + \left\|M_u\boldsymbol{a}^{\mathrm{Vec}}\right\|^2 - \frac{1}{2}\left(\left\|P_\Omega\left(\boldsymbol{a}^{\mathrm{Vec}} - \Delta\right) - \boldsymbol{a}^{\mathrm{Vec}}\right\|^2 + \left\|M_u\boldsymbol{a}^{\mathrm{Vec}}\right\|^2\right) \\ & \geqslant -\Delta^{\mathrm{T}}\left(P_\Omega\left(\boldsymbol{a}^{\mathrm{Vec}} - \Delta\right) - \boldsymbol{a}^{\mathrm{Vec}}\right) + \left\|M_u\boldsymbol{a}^{\mathrm{Vec}}\right\|^2 - \left(\left\|P_\Omega\left(\boldsymbol{a}^{\mathrm{Vec}} - \Delta\right) - \boldsymbol{a}^{\mathrm{Vec}}\right\|^2 + \left\|M_u\boldsymbol{a}^{\mathrm{Vec}}\right\|^2\right) \\ & = -\Delta^{\mathrm{T}}\left(P_\Omega\left(\boldsymbol{a}^{\mathrm{Vec}} - \Delta\right) - \boldsymbol{a}^{\mathrm{Vec}}\right) - \left\|P_\Omega\left(\boldsymbol{a}^{\mathrm{Vec}} - \Delta\right) - \boldsymbol{a}^{\mathrm{Vec}}\right\|^2 \\ & = -\left(P_\Omega\left(\boldsymbol{a}^{\mathrm{Vec}} - \Delta\right) - \left(\boldsymbol{a}^{\mathrm{Vec}} - \Delta\right)\right)^{\mathrm{T}}\left(P_\Omega\left(\boldsymbol{a}^{\mathrm{Vec}} - \Delta\right) - \boldsymbol{a}^{\mathrm{Vec}}\right) \geqslant 0. \end{aligned}$$

Thus, $V\left(\boldsymbol{a}^{\mathrm{Vec}}, \lambda_u\right) \geqslant 0$. Besides, if $\boldsymbol{a}^{\mathrm{Vec}} \neq \boldsymbol{a}_u^{\mathrm{Vec}*}$ or $\lambda_u \neq \lambda_u^*$, then $\|\boldsymbol{a}^{\mathrm{Vec}} - \boldsymbol{a}_u^{\mathrm{Vec}*}\|^2 + \|\lambda_u - \lambda_u^*\|^2 > 0$. Hence, $V(\boldsymbol{a}^{\mathrm{Vec}}, \lambda_u) > 0$ when $\boldsymbol{a}^{\mathrm{Vec}} \neq \boldsymbol{a}_u^{\mathrm{Vec}*}$ or $\lambda_u \neq \lambda_u^*$.

Next, ∇V and $\dot{V}(\boldsymbol{a}^{\mathrm{Vec}}, \lambda_u)$ are derived as follows:

$$\nabla V = G_1(\boldsymbol{a}^{\mathrm{Vec}}, \lambda_u) - (\nabla G_1(\boldsymbol{a}^{\mathrm{Vec}}, \lambda_u) - \boldsymbol{I})G_2(\boldsymbol{a}^{\mathrm{Vec}}, \lambda_u) + \mathrm{col}[\boldsymbol{a}^{\mathrm{Vec}^{\mathrm{T}}}, \lambda_u] - \mathrm{col}[\boldsymbol{a}_u^{\mathrm{Vec}*}, \lambda_u^*]$$

with

$$G_1(\boldsymbol{a}^{\rm Vec},\lambda_u)=\begin{pmatrix}\Delta\\ M_u\boldsymbol{a}^{\rm Vec}\end{pmatrix},$$

$$\nabla G_1(\boldsymbol{a}^{\rm Vec},\lambda_u)=\begin{pmatrix}\nabla^2\tilde{C}\left(\boldsymbol{a}^{\rm Vec}\right) & -M_u^{\rm T}\\ M_u & \mathbf{0}\end{pmatrix},$$

and

$$G_2(\boldsymbol{a}^{\rm Vec},\lambda_u)=\begin{pmatrix}P_\Omega\left(\boldsymbol{a}^{\rm Vec}-\Delta\right)-\boldsymbol{a}^{\rm Vec}\\ -M_u\boldsymbol{a}^{\rm Vec}\end{pmatrix}.$$

$$\begin{aligned}\dot{V}(\boldsymbol{a}^{\rm Vec},\lambda_u)&=\nabla V^{\rm T}{\rm col}[\dot{\boldsymbol{a}}^{\rm Vec},\dot{\lambda}_u]\\&=(G_1(\boldsymbol{a}^{\rm Vec},\lambda_u)+{\rm col}[\boldsymbol{a}^{\rm Vec},\lambda_u]-{\rm col}[\boldsymbol{a}_u^{\rm Vec*},\lambda_u^*])^{\rm T}G_2(\boldsymbol{a}^{\rm Vec},\lambda_u)\\&\quad+\|G_2(\boldsymbol{a}^{\rm Vec},\lambda_u)\|^2-G_2(\boldsymbol{a}^{\rm Vec},\lambda_u)^{\rm T}\nabla G_1(\boldsymbol{a}^{\rm Vec},\lambda_u)G_2(\boldsymbol{a}^{\rm Vec},\lambda_u).\end{aligned}\tag{19}$$

Then, based on (1), we can obtain that

$$\begin{aligned}&\left(G_2(\boldsymbol{a}^{\rm Vec},\lambda_u)+{\rm col}[\boldsymbol{a}^{\rm Vec},\lambda_u]-{\rm col}[\boldsymbol{a}_u^{\rm Vec*},\lambda_u^*]\right)^{\rm T}(G_1(\boldsymbol{a}^{\rm Vec},\lambda_u)+G_2(\boldsymbol{a}^{\rm Vec},\lambda_u))\\&=\begin{pmatrix}P_\Omega(\boldsymbol{a}^{\rm Vec}-\Delta)-\boldsymbol{a}_u^{\rm Vec*}\\ -M_u\boldsymbol{a}^{\rm Vec}+\lambda_u-\lambda_u^*\end{pmatrix}^{\rm T}\begin{pmatrix}P_\Omega(\boldsymbol{a}^{\rm Vec}-\Delta)-(\boldsymbol{a}^{\rm Vec}-\Delta)\\ \mathbf{0}_{N(N-1)/2}\end{pmatrix}\\&=(P_\Omega(\boldsymbol{a}^{\rm Vec}-\Delta)-\boldsymbol{a}_u^{\rm Vec*})^{\rm T}(P_\Omega(\boldsymbol{a}^{\rm Vec}-\Delta)-(\boldsymbol{a}^{\rm Vec}-\Delta))\leqslant 0,\end{aligned}$$

which leads to

$$\begin{aligned}&\left(G_1(\boldsymbol{a}^{\rm Vec},\lambda_u)+{\rm col}[\boldsymbol{a}^{\rm Vec},\lambda_u]-{\rm col}[\boldsymbol{a}_u^{\rm Vec*},\lambda_u^*]\right)^{\rm T}G_2(\boldsymbol{a}^{\rm Vec},\lambda_u)\\&\leqslant -G_1(\boldsymbol{a}^{\rm Vec},\lambda_u)^{\rm T}\left({\rm col}[\boldsymbol{a}^{\rm Vec},\lambda_u]-{\rm col}[\boldsymbol{a}_u^{\rm Vec*},\lambda_u^*]\right)-\|G_2(\boldsymbol{a}^{\rm Vec},\lambda_u)\|^2.\end{aligned}$$

Thus, Eq. (19) is converted into

$$\begin{aligned}\dot{V}\left(\boldsymbol{a}_u^{\rm Vec},\lambda_u\right)\leqslant&-G_1(\boldsymbol{a}^{\rm Vec},\lambda_u)^{\rm T}\left({\rm col}[\boldsymbol{a}_u^{\rm Vec},\lambda_u]-{\rm col}[\boldsymbol{a}_u^{\rm Vec*},\lambda_u^*]\right)\\&-G_2(\boldsymbol{a}^{\rm Vec},\lambda_u)^{\rm T}\nabla G_1(\boldsymbol{a}^{\rm Vec},\lambda_u)G_2(\boldsymbol{a}^{\rm Vec},\lambda_u).\end{aligned}\tag{20}$$

The convexity of $\tilde{C}(\boldsymbol{a}^{\rm Vec})$ yields $-G_1(\boldsymbol{a}^{\rm Vec},\lambda_u)^{\rm T}({\rm col}[\boldsymbol{a}_u^{\rm Vec},\lambda_u]-{\rm col}[\boldsymbol{a}_u^{\rm Vec*},\lambda_u^*])\leqslant 0$. In addition, with the convexity of $\tilde{C}(\boldsymbol{a}^{\rm Vec})$ and the form of $\nabla G_1(\boldsymbol{a}^{\rm Vec},\lambda_u)$, we obtain

$$-G_2(\boldsymbol{a}^{\rm Vec},\lambda_u)^{\rm T}\nabla G_1(\boldsymbol{a}^{\rm Vec},\lambda_u)G_2(\boldsymbol{a}^{\rm Vec},\lambda_u)\leqslant 0.$$

To sum up, $\dot{V}(\boldsymbol{a}^{\rm Vec},\lambda_u)\leqslant 0$.

Finally, let $\bar{\boldsymbol{a}}^{\rm Vec}$ and $\bar{\lambda}_u$ satisfy $\dot{V}\left(\bar{\boldsymbol{a}}^{\rm Vec},\bar{\lambda}_u\right)=0$. We obtain that

$$0=\begin{pmatrix}\nabla\tilde{C}(\bar{\boldsymbol{a}}^{\rm Vec})-\nabla\tilde{C}(\boldsymbol{a}^{\rm Vec*})-M_u^{\rm T}\left(\bar{\lambda}_u-\lambda_u{}^*\right)\\ M_u(\bar{\boldsymbol{a}}^{\rm Vec}-\boldsymbol{a}^{\rm Vec*})\end{pmatrix}^{\rm T}({\rm col}[\bar{\boldsymbol{a}}^{\rm Vec},\bar{\lambda}_u]-{\rm col}[\boldsymbol{a}_u^{\rm Vec*},\lambda_u^*])\tag{21}$$

and $\|P_\Omega(\bar{\boldsymbol{a}}^{\rm Vec}-\nabla\tilde{C}\left(\bar{\boldsymbol{a}}^{\rm Vec}\right)+M_u^{\rm T}\bar{\lambda}_u)-\bar{\boldsymbol{a}}^{\rm Vec}\|^2_{\nabla^2\tilde{C}(\bar{\boldsymbol{a}}^{\rm Vec})}=0$.

Based on [30], Eq. (21) yields $\dot{\lambda}_u=\mathbf{0}_{N(N-1)/2}$, i.e., $M_u^{\rm T}\bar{\boldsymbol{a}}^{\rm Vec}=\mathbf{0}_{N(N-1)/2}$. Owing to the strict convexity of $\tilde{C}(\boldsymbol{a}^{\rm Vec})$, $\nabla^2\tilde{C}(\boldsymbol{a}^{\rm Vec})$ is positive definite. Hence, $P_\Omega(\bar{\boldsymbol{a}}^{\rm Vec}-\nabla\tilde{C}(\bar{\boldsymbol{a}}^{\rm Vec})+M_u^{\rm T}\bar{\lambda}_u)-\bar{\boldsymbol{a}}^{\rm Vec}=\mathbf{0}_{N^2}$. Further, based on (1) in Theorem 1, it can be obtained that $\dot{V}(\boldsymbol{a}^{\rm Vec},\lambda_u)<0$ if and only if $\boldsymbol{a}^{\rm Vec}\neq\boldsymbol{a}^{\rm Vec*}=\bar{\boldsymbol{a}}^{\rm Vec}$ and $\lambda_u\neq\lambda_u{}^*=\bar{\lambda}_u$. Thus, $\boldsymbol{a}^{\rm Vec}(t)$ and $\lambda_u(t)$ are bounded. Let $k(t)={\rm col}[\boldsymbol{a}^{\rm Vec}(t),\lambda_u(t)]$. By the invariant set theorem, MGS (17) converges to a largest invariant set $\mathcal{M}=\{k(t)|{\rm d}V(k(t))/{\rm d}t=0\}$. Now, we need to prove that ${\rm d}V(k(t))/{\rm d}t=0$ if and only if ${\rm d}k(t)/{\rm d}t=0$. It is easy to obtain that if ${\rm d}k(t)/{\rm d}t=0$, then ${\rm d}V(k(t))/{\rm d}t=0$. Suppose that ${\rm d}V(k(t))/{\rm d}t=0$ with $\hat{k}(t)={\rm col}[\hat{\boldsymbol{a}}^{\rm Vec},\hat{\lambda}]\in\mathcal{M}$.

Based on (20), since $G_1(\hat{k})^{\rm T}(\hat{k}-k^*) \geqslant 0$, $G_1(\hat{k})^{\rm T}(\hat{k}-k^*) + G_2^{\rm T}(\hat{k})\nabla G_1(\hat{k})G_2(\hat{k}) = 0$ holds. Therefore, $(G_1(\hat{k}) - G_1(k^*))^{\rm T}(\hat{k}-k^*) = 0$ and $G_2(\hat{k})^{\rm T}\nabla G_1(\hat{k})G_2(\hat{k}) = 0$. Equation $(G_1(\hat{k}) - G_1(k^*))^{\rm T}(\hat{k}-k^*) = 0$ implies that $\|(\hat{\boldsymbol{a}}^{\rm Vec} - \boldsymbol{a}^{\rm Vec*})\|^2_{\nabla^2\tilde{C}(\boldsymbol{a}_\sigma^{\rm Vec})} = 0$ where $\boldsymbol{a}_\sigma^{\rm Vec} = \hat{\boldsymbol{a}}^{\rm Vec} + \sigma(\boldsymbol{a}^{\rm Vec*} - \hat{\boldsymbol{a}}^{\rm Vec})$. Thus, $\hat{\boldsymbol{a}}^{\rm Vec} = \boldsymbol{a}^{\rm Vec*}$ which implies ${\rm d}\lambda_u/{\rm d}t = 0$. $G_2(\hat{k})^{\rm T}\nabla G_1(\hat{k})G_2(\hat{k}) = 0$ leads to $(\hat{\boldsymbol{a}}^{\rm Vec} - \nabla\tilde{C}(\hat{\boldsymbol{a}}^{\rm Vec}) + M_u^{\rm T}\hat{\lambda}_u)^+ = \hat{\boldsymbol{a}}^{\rm Vec}$. Thus, ${\rm d}\boldsymbol{a}^{\rm Vec}/{\rm d}t = 0$. Finally, we have that ${\rm d}V(k(t))/{\rm d}t = 0$ if and only if ${\rm d}k(t)/{\rm d}t = 0$, i.e., MGS (17) converges to an equilibrium point. According to Theorem 1, the proof of (1) completes. The proof of (2) is similar to the proof of (1) when we adjust the dimensions accordingly and substitute λ_d for λ_u and M_d for M_u, respectively.

Theorem 3. Under Assumption 1, the following two statements hold:

(1) If $P = P_u$, then MGS (18) converges to its equilibrium point $\boldsymbol{a}_u^*$ from any initial state $\boldsymbol{a}(0)$.

(2) If $P = P_d$, then MGS (18) converges to its equilibrium point $\boldsymbol{a}_d^*$ from any initial state $\boldsymbol{a}(0)$.

Proof. For the proof of (1), consider a Lyapunov function as follows:

$$\tilde{V}(\boldsymbol{a}) = \frac{1}{2}\left(\|\boldsymbol{a} - P_\Omega(\boldsymbol{a}_u^*)\|^2 - \|\boldsymbol{a} - P_\Omega(\boldsymbol{a})\|^2\right) + \frac{1}{2}\|\boldsymbol{a} - \boldsymbol{a}_u^*\|^2_{P_u}.$$

Based on (1), there is

$$\left(\|\boldsymbol{a} - P_\Omega(\boldsymbol{a}_u^*)\|^2 - \|\boldsymbol{a} - P_\Omega(\boldsymbol{a})\|^2\right) - \|P_\Omega(\boldsymbol{a}) - P_\Omega(\boldsymbol{a}_u^*)\|^2 = 2(\boldsymbol{a} - P_\Omega(\boldsymbol{a}))^{\rm T}(P_\Omega(\boldsymbol{a}) - P_\Omega(\boldsymbol{a}_u^*)) \geqslant 0.$$

Thus, we can derive that $\tilde{V}(\boldsymbol{a}) \geqslant \|P_\Omega(\boldsymbol{a}) - P_\Omega(\boldsymbol{a}_u^*)\|^2/2 + \|\boldsymbol{a} - \boldsymbol{a}_u^*\|^2_{P_u}/2 \geqslant 0$. Note that $\tilde{V}(\boldsymbol{a}) > 0$ when $\boldsymbol{a}_u^{\rm Vec^*} \neq \boldsymbol{a}^{\rm Vec}$. Next, $\nabla\tilde{V}$ and $\dot{\tilde{V}}(\boldsymbol{a})$ are derived as follows. Based on [28, Lemma 4], $\nabla\tilde{V} = P_u(\boldsymbol{a} - \boldsymbol{a}_u^*) + \boldsymbol{a}^{\rm Vec} - \boldsymbol{a}_u^{\rm Vec^*}$. Thus, it holds

$$\begin{aligned}\dot{\tilde{V}}(\boldsymbol{a}) &= \nabla\tilde{V}^{\rm T}\dot{\boldsymbol{a}} \leqslant (P_u(\boldsymbol{a} - \boldsymbol{a}_u^*) + \boldsymbol{a}^{\rm Vec} - \boldsymbol{a}_u^{\rm Vec^*})^{\rm T}(\delta(\boldsymbol{a}) - \delta(\boldsymbol{a}_u^*)) \\ &\leqslant -\|\boldsymbol{a}^{\rm Vec} - \boldsymbol{a}_u^{\rm Vec^*}\|^2_{P_u} - (\boldsymbol{a}^{\rm Vec} - \boldsymbol{a}_u^{\rm Vec^*})^{\rm T}(\boldsymbol{a} - \boldsymbol{a}_u^*) + \|\boldsymbol{a}^{\rm Vec} - \boldsymbol{a}_u^{\rm Vec^*}\|^2_{\boldsymbol{I}_{N^2} - P_u} + \phi(\boldsymbol{a}),\end{aligned}$$

where $\delta(\boldsymbol{a}) = -P_u\boldsymbol{a}^{\rm Vec} - (\boldsymbol{I}_{N^2} - P_u)(\boldsymbol{a} - \boldsymbol{a}^{\rm Vec} + \nabla\tilde{C}((\boldsymbol{I}_{N^2} - P_u)\boldsymbol{a}^{\rm Vec}))$ and $\phi(\boldsymbol{a}) = -(\boldsymbol{a}^{\rm Vec} - \boldsymbol{a}_u^{\rm Vec^*})^{\rm T}(\boldsymbol{I}_{N^2} - P_u)(\nabla\tilde{C}((\boldsymbol{I}_{N^2} - P_u)\boldsymbol{a}^{\rm Vec}) - \nabla\tilde{C}((\boldsymbol{I}_{N^2} - P_u)\boldsymbol{a}_u^{\rm Vec^*}))$.

Then, based on (2), (3), and $\|\boldsymbol{I}_{N^2} - P_u\| \leqslant 1$, it can be obtained that $\|\boldsymbol{a}^{\rm Vec} - \boldsymbol{a}_u^{\rm Vec^*}\|^2_{\boldsymbol{I}_{N^2} - P_u} \leqslant (\boldsymbol{a}^{\rm Vec} - \boldsymbol{a}_u^{\rm Vec^*})^{\rm T}(\boldsymbol{a} - \boldsymbol{a}_u^*)$. Besides, according to the convexity of $\tilde{C}$, $\phi(\boldsymbol{a}) \leqslant 0$. Hence, $\dot{\tilde{V}}(\boldsymbol{a}) = -\|\boldsymbol{a}^{\rm Vec} - \boldsymbol{a}_u^{\rm Vec^*}\|^2_{P_u} + \phi(\boldsymbol{a}) \leqslant 0$. Thus, $\boldsymbol{a}(t)$ is bounded, then $\|\dot{\boldsymbol{a}}(t)\|$ is bounded, and we set the boundedness to be U. Let $O(\boldsymbol{a}) = \|\boldsymbol{a}^{\rm Vec} + \boldsymbol{a}_u^{\rm Vec^*}\|^2_{P_u} + \phi(\boldsymbol{a})$. According to [28], we have that $\boldsymbol{a}^{\rm Vec} = P_\Omega(\boldsymbol{a})$ is an optimal solution if and only if $O(\boldsymbol{a}) = 0$, i.e., $O(\boldsymbol{a}^*) = 0$. Let $\{\tau_\kappa\}$ be an increasing time sequence with $\lim_{\kappa\to\infty}\tau_\kappa = +\infty$ and let $\hat{\boldsymbol{a}} = \lim_{\kappa\to+\infty}\boldsymbol{a}(\tau_k)$. Now, we prove that $O(\hat{\boldsymbol{a}}) = 0$ by contradiction. Assume that $O(\hat{\boldsymbol{a}}) > 0$, then $\exists\,\delta_0 > 0$ and $\exists\,\epsilon > 0$ such that $O(\hat{\boldsymbol{a}}) > \epsilon$ for all $\boldsymbol{a} \in \{\boldsymbol{a} \mid \|\boldsymbol{a} - \hat{\boldsymbol{a}}\| \leqslant \delta_0\}$. Since $\lim_{\kappa\to+\infty}\boldsymbol{a}(\tau_\kappa) = \hat{\boldsymbol{a}}$, there exists an integer $N_0 > 0$ such that $\|\boldsymbol{a}(\tau_\kappa) - \hat{\boldsymbol{a}}\| \leqslant \delta_0/2$ for $\forall\,\kappa \geqslant N_0$. Then for $t \in [\tau_\kappa - \delta_0/4U, \tau_\kappa + \delta_0/4U]$ with $\kappa \geqslant N_0$, $\|\boldsymbol{a}(t) - \hat{\boldsymbol{a}}\| \leqslant \|\boldsymbol{a}(t) - \boldsymbol{a}(\tau_\kappa)\| + \|\boldsymbol{a}(\tau_k) - \hat{\boldsymbol{a}}\| \leqslant \delta_0/2 + G|t - \tau_\kappa| \leqslant \delta_0$ holds. It implies that $O(\boldsymbol{a}(t)) > 0$ for $\forall\,t \in [\tau_\kappa - \delta_0/4U, \tau_k + \delta_0/4U]$. Therefore, $\int_{t_0}^{+\infty} O(\boldsymbol{a}(t)) = +\infty$. Furthermore, because $V(\boldsymbol{a}) \geqslant 0$ and $\dot{V}(\boldsymbol{a}) \leqslant 0$, there exists $\hat{V} = \lim_{t\to+\infty} V(\boldsymbol{a}(t))$. We can obtain that $\int_{t_0}^{+\infty} O(\boldsymbol{a}(t)) = -\lim_{\zeta\to+\infty}\int_{t_0}^{\zeta}\dot{V}(\boldsymbol{a}(t)) = -(\lim_{\zeta\to+\infty} V(\boldsymbol{a}(\zeta)) - V(\boldsymbol{a}(t_0))) \leqslant +\infty$ which leads to a contradiction. Thus, $\lim_{t\to+\infty}\boldsymbol{a}(t) = \lim_{\kappa\to+\infty}\boldsymbol{a}(\tau_\kappa) = \hat{\boldsymbol{a}} = \boldsymbol{a}^*$ and $\boldsymbol{a}_u^{\rm Vec*} = P_\Omega(\boldsymbol{a}^*)$. Thus, the proof of (1) completes.

The proof of (2) is similar to the proof of (1) if we adjust the dimensions of the matrices and vectors accordingly and substitute P_d for P_u.

Remark 2. MAS (6) is a distributed optimization approach for solving Problem (5), and MGS (17) or (18) is a centralized dynamic mechanism for recovering or modifying the graph employed MAS by (6). They are not at the same level, but in a coordination mechanism with two levels: an upper level and a lower level. At the upper level, MGSs recover or modify the tampered graph to a feasible one constantly. At the lower level, MAS (6) solves Problem (5) in a distributed manner. An illustrated example is shown in Figure 1.

If $(i,j) \notin \mathcal{E}$, how to guarantee $A(i,j) = 0$ during the optimization process should be discussed.

Theorem 4. (1) In (17), for any $i,j = 1,\ldots,N$, if $\boldsymbol{a}^{\rm Vec}_{N(i-1)+j}(0) = 0$ and $\Omega_{N(i-1)+j} = \{0\}$, then $\boldsymbol{a}^{\rm Vec}_{N(i-1)+j}(t) = 0$ for $t \geqslant 0$.

(2) In (18), for any $i,j = 1,\ldots,N$, if $\Omega_{N(i-1)+j} = \{0\}$, then $\boldsymbol{a}^{\rm Vec}_{N(i-1)+j}(t) = 0$ for $t \geqslant 0$.

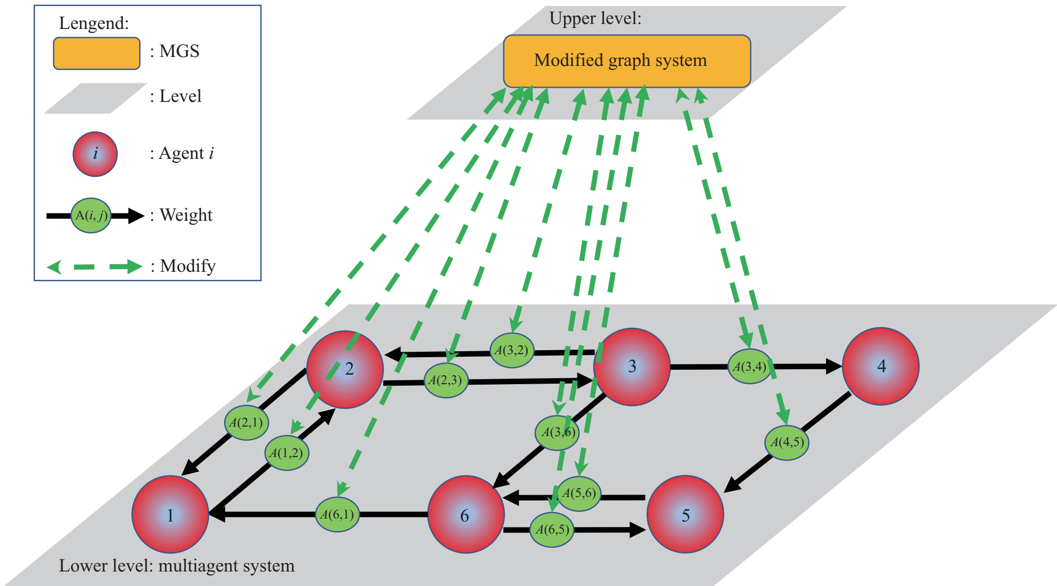

Figure 1 (Color online) A coordination mechanism consisting of MAS (6) and an MGS (the explanation is provided in (1) of Remark 2).

Proof. For (1), in (17), we have that

$$\dot{\boldsymbol{a}}^{\mathrm{Vec}}_{N(i-1)+j} = -\boldsymbol{a}^{\mathrm{Vec}}_{N(i-1)+j} + P_{\Omega_{N(i-1)+j}}(*),$$

where $*$ is the $(N(i-1)+j)$-th component of vector $\boldsymbol{a}^{\mathrm{Vec}} - \nabla\tilde{C}\left(\boldsymbol{a}^{\mathrm{Vec}}\right)$. Since $\Omega_{N(i-1)+j} = \{0\}$, we have that

$$\dot{\boldsymbol{a}}^{\mathrm{Vec}}_{N(i-1)+j} = -\boldsymbol{a}^{\mathrm{Vec}}_{N(i-1)+j} + P_{\{0\}}(*) = -\boldsymbol{a}^{\mathrm{Vec}}_{N(i-1)+j} + 0 = -\boldsymbol{a}^{\mathrm{Vec}}_{N(i-1)+j}.$$

Thus, $\boldsymbol{a}^{\mathrm{Vec}}_{N(i-1)+j}(t) = \boldsymbol{a}^{\mathrm{Vec}}_{N(i-1)+j}(0)\mathrm{e}^{-t}$. Since $\boldsymbol{a}^{\mathrm{Vec}}_{N(i-1)+j}(0) = 0$, $\boldsymbol{a}^{\mathrm{Vec}}_{N(i-1)+j}(t) = \boldsymbol{a}^{\mathrm{Vec}}_{N(i-1)+j}(0)\mathrm{e}^{-t} = 0$ for $t \geqslant 0$, which completes the proof.

For (2), in (18), we have that $\boldsymbol{a}^{\mathrm{Vec}}_{N(i-1)+j}(t) = P_{\Omega_{N(i-1)+j}}(\boldsymbol{a}_{N(i-1)+j}(t))$. Since $\Omega_{N(i-1)+j} = \{0\}$, $\boldsymbol{a}^{\mathrm{Vec}}_{N(i-1)+j}(t) = P_{\{0\}}(\boldsymbol{a}_{N(i-1)+j}(t)) = 0$ for $t \geqslant 0$, which completes the proof.

From Theorem 4, if $(i,j) \notin \mathcal{E}$, let $\boldsymbol{a}^{\mathrm{Vec}}_{N(i-1)+j}(0) = 0$ and $\Omega_{N(i-1)+j} = \{0\}$ in (17); $A(i,j) = 0$ can be guaranteed during the optimization process there is no communication channel between i and j. If $(i,j) \notin \mathcal{E}$, let $\Omega_{N(i-1)+j} = \{0\}$ in (18); then $A(i,j) = 0$ can be guaranteed during the optimization process when there is no communication channel between i and j.

Remark 3. According to Theorem 2, we can tackle two cases mentioned in Subsection 2.2 by MGS (17). For Case 1, the graph is infeasible for MAS (6) initially. We can regard this graph as an initial state of MGS (17) with $M = M_u$. Then by running MGS (17) with $M = M_u$ and MAS (6) at the same time, we can modify the infeasible graph into a feasible undirected graph by MGS (17). According to Lemma 3, with the feasible undirected graph which is modified by MGS (17) with $M = M_u$, MAS (6) can converge to the optimal solution to Problem (5). For Case 2, a feasible undirected graph is provided for MAS (6) initially, but it is tampered or destroyed during the operation of MAS (6). By running MGS (17) with $M = M_u$ and MAS (6) at the same time, we can recover the graph which is tampered or destroyed to be an infeasible graph into a feasible undirected graph constantly. With the feasible undirected graph recovered by MGS (17) with $M = M_u$, MAS (6) can converge to the optimal solution. Similarly, by running MGS (17) with $M = M_d$ and MAS (6) at the same time, we can also tackle Cases 1 and 2. Different from MGS (17) with $M = M_u$, MGS (17) with $M = M_d$ can modify or recover an infeasible graph into a feasible weight-balanced graph rather than a feasible undirected graph. Similarly, according to Theorem 3, MGS (18) can also tackle two cases. For a clear understanding, Figure 2 is provided to explain the functions of MGSs (17) and (18) in the coordination mechanism.

Remark 4. We give the differences between MGS (17) and MGS (18): The convergence of MGS (17) is from the initial states being in Ω. While the convergence of MGS (18) is from any initial states. Thus, MGS (17) can only recover the graph which is tampered or destroyed but is still in Ω. However, if the

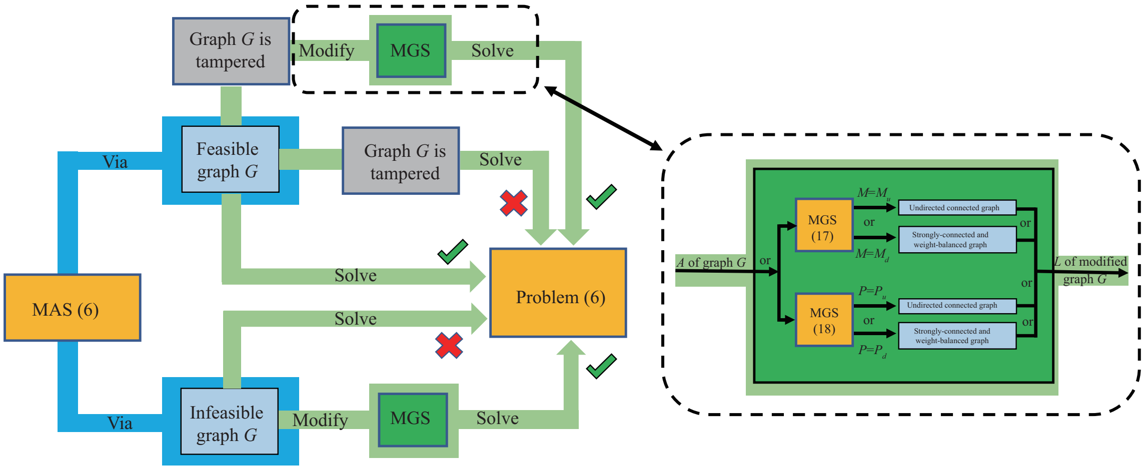

Figure 2 (Color online) Schematic diagram of the usage of MGSs (17) and (18) in the coordination mechanism.

graph is tampered or destroyed in any large range (especially outside of Ω), then MGS (18) can also recover it theoretically.

Remark 5. The proposed methods are developed under the notion of time-invariant graphs. Although the weights of the graph modified by MGSs are time-varying, the distributed optimization approach (system (6)) can achieve the optimal objective only if the states of the MGSs are stable (i.e., the graph is time-invariant if the MGS is stable). However, just like [23, 24], the assumed graphs are time-varying. Developing MGSs for distributed optimization approaches over time-varying graphs is challenging. The challenge is mainly from two aspects: (1) If we need to recover time-varying graphs (they are a sequence of graphs with respect to time t), the final states of the proposed MGSs should satisfy the corresponding time-varying attributes (e.g., "B_0-strongly connected" in [23] and "uniformly jointly strongly connected" in [24]), which may not be stable. (2) The distributed optimization approaches in [23, 24] are discrete-time. Thus, the MGSs should be also developed in a discrete-time manner. In sum, developing MGSs for distributed optimization approaches over time-varying graphs is challenging, but it deserves deep research and is interesting.

4 An illustrated example

In this section, an optimization problem is provided to illustrate the validity of the main results. Besides, the systems are implemented and simulated in MATLAB R2017b and run on Intel(R) Core(TM) i5-8257U CPU @ 1.40 GHz, Intel Iris Plus Graphics 645 1536 MB, 8 GB 2133 MHz LPDDR3, macOS 10.15.7.

Example 3. Consider an optimization problem as follows:

$$\min\ f(\boldsymbol{x}) = \sum_{i=1}^{N} f_i(x_i) \ \ \text{s.t.}\ \ x_i = x_j \in \mathbb{R}^n,\ \ i, j\ = 1, \ldots, N, \tag{22}$$

where $\boldsymbol{x} = \mathrm{col}[x_1, x_2, x_3, x_4]$, $f_1(x_1) = (x_1 - 77)^2$, $f_2(x_2) = (x_2 - 7)^2$, $f_3(x_3) = (x_3 + 112)^2$, and $f_4(x_4) = (x_4 - 2)^2$.

To tackle Case 1. We can obtain the optimal solution by MAS (6) with an undirected and connected $\mathcal{G}$ or a strongly connected and weight-balanced $\mathcal{G}$ (see Lemma 3). Now, we use the system below which consists of MAS (6) and MGS (17) or MGS (18):

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = -\nabla f(\boldsymbol{x}) - \mathbb{L}\mu - \alpha\mathbb{L}\boldsymbol{x}, \\ \frac{\mathrm{d}\mu}{\mathrm{d}t} = \mathbb{L}\boldsymbol{x}, \\ (17) \text{ or } (18). \end{cases} \tag{23}$$

$\mathbb{L}$ in MAS (6) is fixed, but $\mathbb{L}$ in MAS (23) is determined by $\boldsymbol{a}^{\mathrm{Vec}}(t)$. In detail, in MAS (23), $\mathrm{Vec}(A(t)) = \boldsymbol{a}^{\mathrm{Vec}}(t)$ and $\mathbb{L}(t) = L(t) \otimes \boldsymbol{I}_n$ where $L(t)(i,i) = \sum_{j=1, j\neq i}^{n} A(t)(i,j)$ for $\forall i = 1, \ldots, N$ and $L(t)(i,j) = -A(t)(i,j)$ for $i \neq j$.

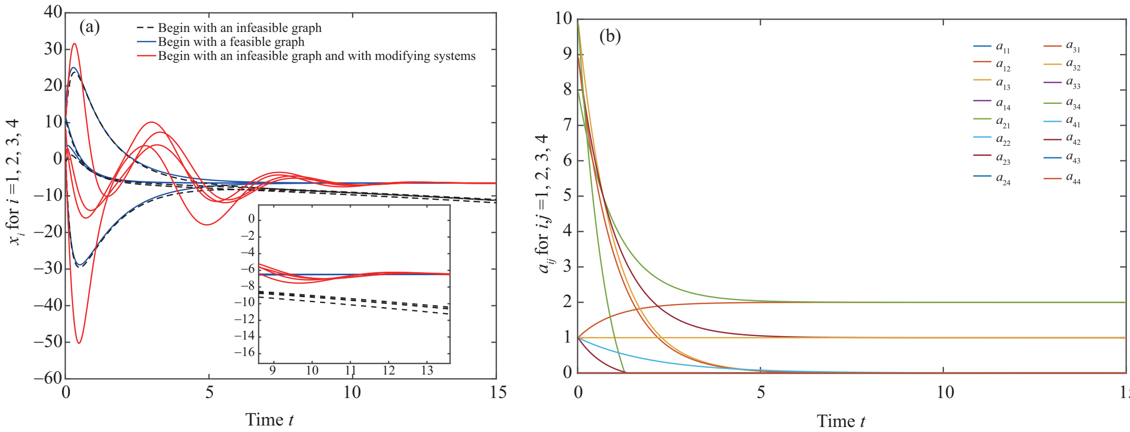

Figure 3 (Color online) Transient states of (a) x_i for $i = 1, 2, 3, 4$ and (b) $\boldsymbol{a}^{\text{Vec}}$ in Case 1.

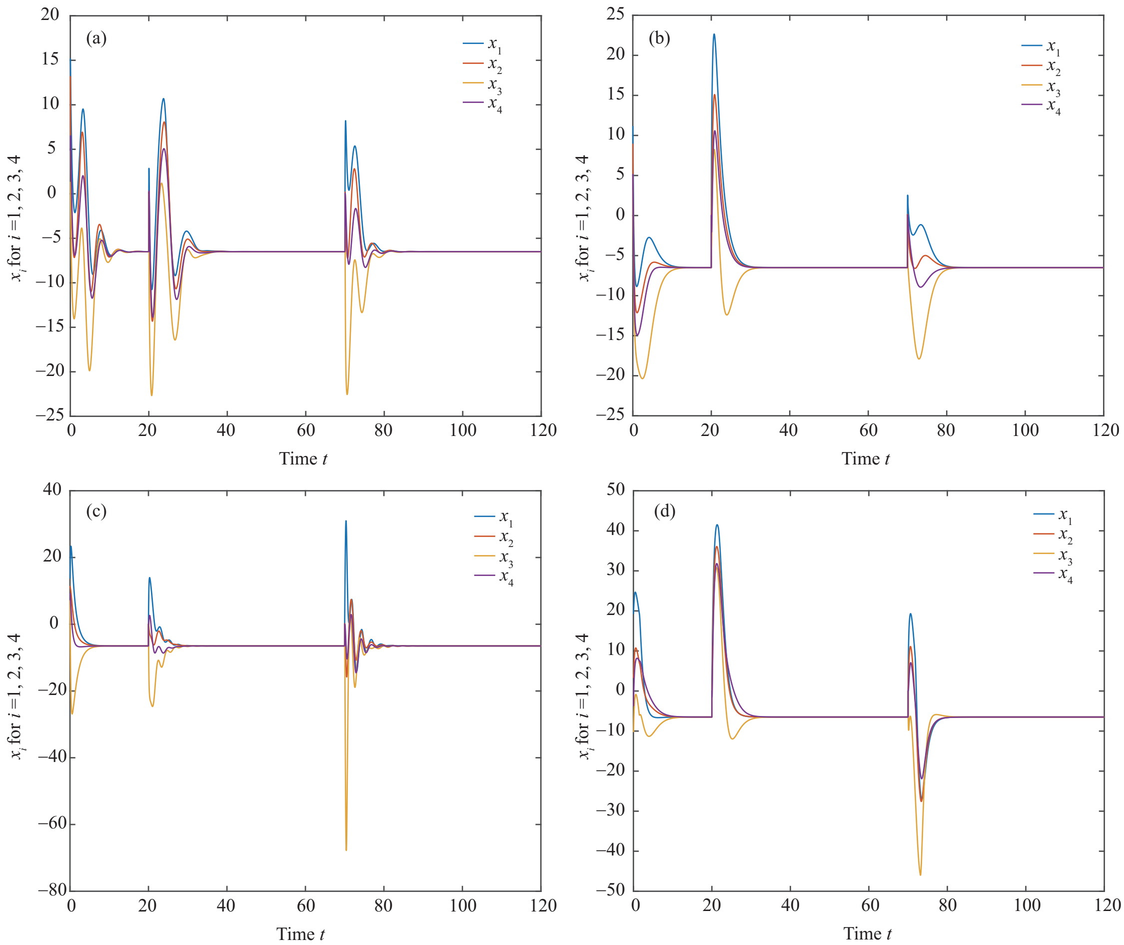

Figure 4 (Color online) Transient states of x_i for $i = 1, 2, 3, 4$ in Case 2. (a) MGS (17) with M_u; (b) MGS (17) with M_d; (c) MGS (18) with P_u; (d) MGS (18) with P_d.

Let the objective function $\tilde{C}(\boldsymbol{a}^{\text{Vec}}) = (\boldsymbol{a}^{\text{Vec}} - s)^2$ with $s = [0, 2, 0, 1, 2, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0]^{\text{T}}$. Note that $\tilde{C}(\boldsymbol{a}^{\text{Vec}})$ guarantees with Assumption 1. By MATLAB, we run MAS (6) with an undirected and connected graph and an infeasible graph, respectively. Besides, we run (23) (by (17) and $M = M_u$) with an infeasible graph. In Figure 3(a), the blue solid line shows that MAS (6) with an undirected and connected graph can solve Problem (22), and the optimal solution is $x_1 = x_2 = x_3 = x_4 = -6.5$. The black dotted line shows that MAS (6) with an infeasible graph cannot realize the consensus and cannot obtain the optimal solution to Problem (22). The red solid line implies that Eq. (23) beginning with

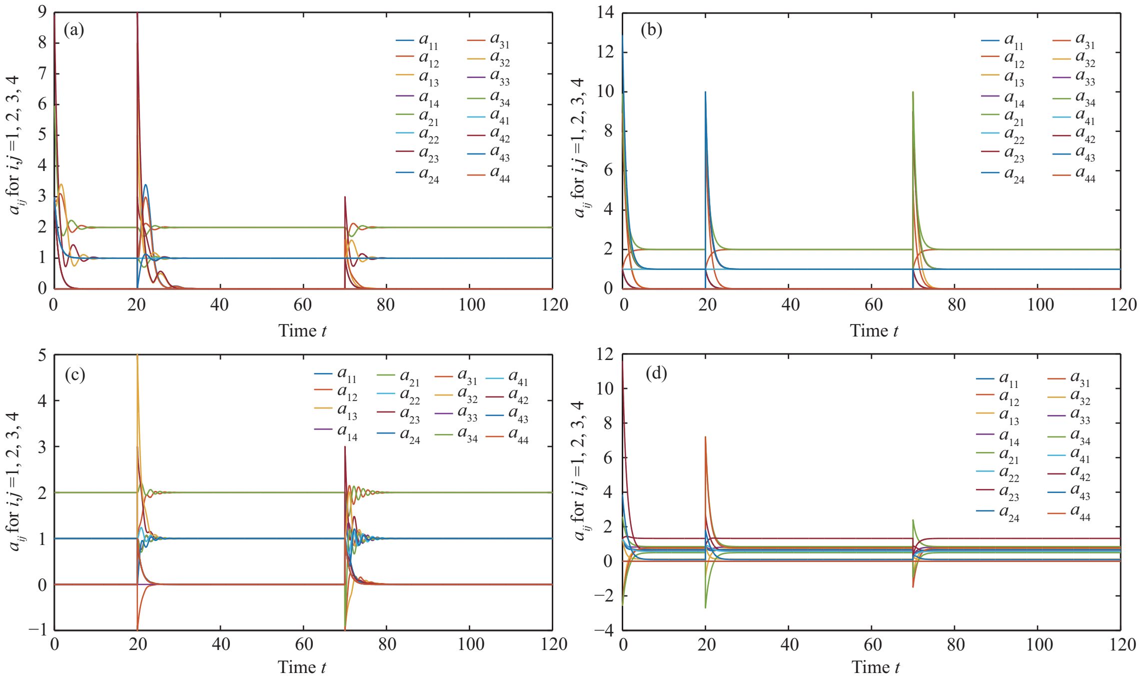

Figure 5 (Color online) Transient states of $\boldsymbol{a}^{\mathrm{Vec}}$ in Case 2. (a) MGS (17) with M_u; (b) MGS (17) with M_d; (c) MGS (18) with P_u; (d) MGS (18) with P_d.

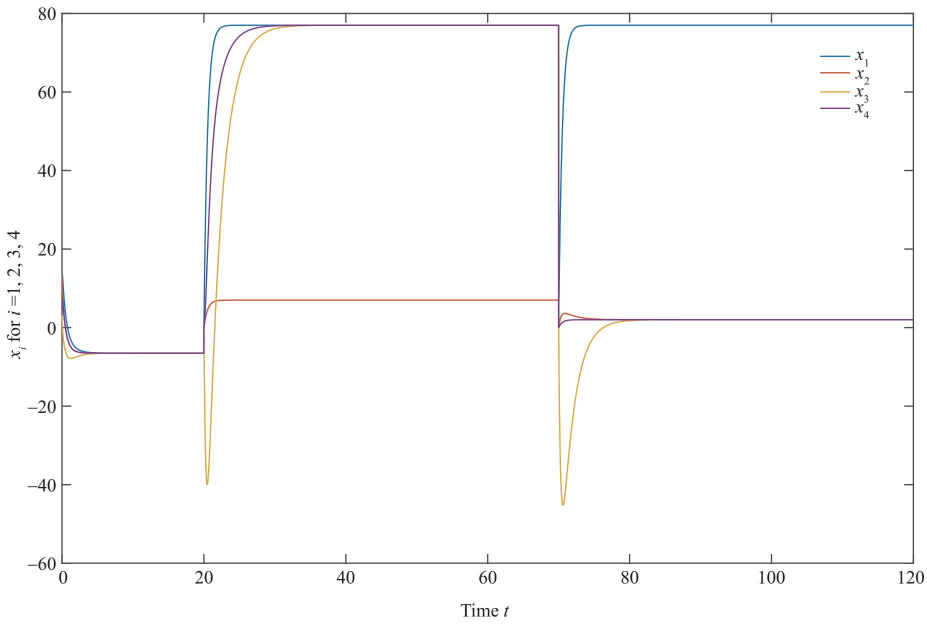

Figure 6 (Color online) Transient states of x_i for $i = 1, 2, 3, 4$ in [20, System (11)] in Case 2.

an infeasible graph can also solve Problem (22). Figure 3(b) depicts the transient states of $A(i, j)$, and it shows that the infeasible graph A becomes an undirected and connected graph, which illustrates the validity of the proposed MGSs for tackling Case 1. Note that MAS (6) without any MGS is System (3) in [29]. Compared with System (3) in [29], MAS (6) with MGS (17) tackles Case 1 well.

To tackle Case 2. In this case, the graph is feasible initially, but it is tampered or destroyed during the operation of MAS (6). We also use the MAS (23). There are four forms in MAS (23): (1) MGS (17) with M_u; (2) MGS (17) with M_d; (3) MGS (18) with P_u; (4) MGS (18) with P_d. We simulate the tampering and the destroying at $t = 20$ and $t = 70$ by changing the entries of $\boldsymbol{a}^{\mathrm{Vec}}(t)$ randomly. Now, we run four forms by using MATLAB, and we can obtain Figures 4 and 5. Figure 4 shows that four forms can all tackle Case 2 and solve Problem (22). Figure 5 implies that four forms can recover the graph which is tampered or destroyed during the operation of MAS (6). Besides, comparing (17) (Figures 5(a) and (b)) with (18) (Figures 5(c) and (d)), we can obtain that Eq. (17) can only recover the graph which is tampered or destroyed but is still in Ω, and Eq. (18) can also recover the graphs which are tampered or destroyed in any large range (especially outside of Ω). Now, we compare MAS (6) (with MGS (18)) with [20, System (11)] by tackling Case 2. We run [20, System (11)] under the same tampering in Figure 4(d). Compared with Figure 4(d), Figure 6 shows that Ref. [20, System (11)] cannot converge to the optimal solution, which implies that MAS (6) with an MGS tackles Case 2 well.

5 Conclusion

In this paper, to obtain the feasible graphs or to recover the tampered or destroyed graphs, two MGSs are designed for the distributed optimization approaches. The MGSs are derived from the designed optimization problems and their convergence is proven. An example is given to demonstrate the efficiency of the main results. The simulations show that we can effectively modify an infeasible communication graph into a feasible one or recover the tampered or destroyed graph which is feasible initially by a coordination mechanism consisting of a distributed optimization approach and a modified graph system. Future work may focus on the MGSs for distributed optimization over time-varying graphs.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 62173308), Natural Science Foundation of Zhejiang Province of China (Grant Nos. LR20F030001, LD19A010001), and Jinhua Science and Technology Project (Grant No. 2022-1-042).

References

1 Wan P, Lemmon M D. Event-triggered distributed optimization in sensor networks. In: Proceedings of the 8th International Conference on Information Processing in Sensor Networks, 2009. 49–60
2 Liang S, Zeng X L, Hong Y G. Distributed sub-optimal resource allocation over weight-balanced graph via singular perturbation. Automatica, 2018, 95: 222–228
3 Deng Z H, Liang S, Hong Y G. Distributed continuous-time algorithms for resource allocation problems over weight-balanced digraphs. IEEE Trans Cybern, 2018, 48: 3116–3125
4 Yu W W, Li C J, Yu X H, et al. Economic power dispatch in smart grids: a framework for distributed optimization and consensus dynamics. Sci China Inf Sci, 2018, 61: 012204
5 Mao S, Dong Z W, Schultz P, et al. A finite-time distributed optimization algorithm for economic dispatch in smart grids. IEEE Trans Syst Man Cybern Syst, 2021, 51: 2068–2079
6 Zeng X L, Yi P, Hong Y G. Distributed continuous-time algorithm for constrained convex optimizations via nonsmooth analysis approach. IEEE Trans Automat Contr, 2017, 62: 5227–5233
7 Xia Z C, Liu Y, Lu J Q, et al. Penalty method for constrained distributed quaternion-variable optimization. IEEE Trans Cybern, 2021, 51: 5631–5636
8 Tang C B, Li X, Wang Z, et al. Cooperation and distributed optimization for the unreliable wireless game with indirect reciprocity. Sci China Inf Sci, 2017, 60: 110205
9 Yang T, Yi X L, Wu J F, et al. A survey of distributed optimization. Annu Rev Control, 2019, 47: 278–305
10 Wang X Y, Wang G D, Li S H. A distributed fixed-time optimization algorithm for multi-agent systems. Automatica, 2020, 122: 109289
11 Wang D, Wang Z, Wu Z J, et al. Distributed convex optimization for nonlinear multi-agent systems disturbed by a second-order stationary process over a digraph. Sci China Inf Sci, 2022, 65: 132201
12 Liu Y, Xia Z C, Gui W H. Multi-objective distributed optimization via a predefined-time multi-agent approach. IEEE Trans Automat Contr, 2023. doi: 10.1109/TAC.2023.3244122
13 Xia Z C, Liu Y, Qiu J L, et al. An RNN-based algorithm for decentralized-partial-consensus constrained optimization. IEEE Trans Neural Netw Learn Syst, 2023, 34: 534–542
14 Xia Z C, Liu Y, Kou K I, et al. Clifford-valued distributed optimization based on recurrent neural networks. IEEE Trans Neural Netw Learn Syst, 2022. doi: 10.1109/TNNLS.2021.3139865
15 Wang X Y, Wang G D, Li S H. Distributed finite-time optimization for integrator chain multiagent systems with disturbances. IEEE Trans Automat Contr, 2020, 65: 5296–5311
16 Liu Q S, Yang S F, Wang J. A collective neurodynamic approach to distributed constrained optimization. IEEE Trans Neural Netw Learn Syst, 2017, 28: 1747–1758
17 Ning B D, Han Q L, Zuo Z Y. Distributed optimization for multiagent systems: an edge-based fixed-time consensus approach. IEEE Trans Cybern, 2019, 49: 122–132
18 Xia Z C, Liu Y, Wang J. A collaborative neurodynamic approach to distributed global optimization. IEEE Trans Syst Man Cybern Syst, 2023, 53: 3141–3151
19 Huang B H, Liu Y, Glielmo L, et al. Fixed-time distributed robust optimization for economic dispatch with event-triggered intermittent control. Sci China Technol Sci, 2023, 66: 1385–1396
20 Gharesifard B, Cortes J. Distributed continuous-time convex optimization on weight-balanced digraphs. IEEE Trans Automat Contr, 2014, 59: 781–786
21 Yue D D, Baldi S, Cao J D, et al. Distributed adaptive optimization with weight-balancing. IEEE Trans Automat Contr, 2022, 67: 2068–2075
22 Xia Z C, Liu Y, Lu W L, et al. Matrix-valued distributed stochastic optimization with constraints. Front Inform Technol Electron Eng, 2022. doi: 10.1631/FITEE.2200381

23 Li H Q, Lü Q G, Liao X F, et al. Accelerated convergence algorithm for distributed constrained optimization under time-varying general directed graphs. IEEE Trans Syst Man Cybern Syst, 2020, 50: 2612–2622

24 Yu W W, Liu H Z, Zheng W X, et al. Distributed discrete-time convex optimization with nonidentical local constraints over time-varying unbalanced directed graphs. Automatica, 2021, 134: 109899

25 Jiang X R, Qin S T, Xue X P. Continuous-time algorithm for approximate distributed optimization with affine equality and convex inequality constraints. IEEE Trans Syst Man Cybern Syst, 2021, 51: 5809–5818

26 Lakshmanan H, de Farias D P. Decentralized resource allocation in dynamic networks of agents. SIAM J Optim, 2008, 19: 911–940

27 Ruszczyski A P. Nonlinear Optimization. New Jersey: Princeton University Press, 2006

28 Liu Q S, Wang J. A one-layer projection neural network for nonsmooth optimization subject to linear equalities and bound constraints. IEEE Trans Neural Netw Learn Syst, 2013, 24: 812–824

29 Wang J, Elia N. Control approach to distributed optimization. In: Proceedings of the 48th Annual Allerton Conference on Communication, Control, and Computings, Monticello, 2011. 557–561

30 Xia Y S, Wang J. A recurrent neural network for solving nonlinear convex programs subject to linear constraints. IEEE Trans Neural Netw, 2005, 16: 379–386