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Input-to-state stability analysis of stochastic delayed switching systems

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Abstract In this paper, the issues regarding the input-to-state stability (ISS) and integral input-to-state stability (iISS) of the stochastic delayed switching systems with Lévy noise and input control (SDSS-LN-IC) are proposed by employing the comparison theorem approach, mode-dependent average dwell time (MDADT) method, delay integral inequality (DII), and Lyapunov-Razumikhin (L-R) technique. Two switching situations, namely, synchronous switching and asynchronous switching, are considered. Applying integral inequalities, sufficient stability conditions in both cases are given. Moreover, the sequences of this paper allow for the coefficients for the upper-bound expectation estimation of the infinitesimal operator to be mode-dependent, regardless of the sign and time-varying function rather than a constant, as is the case in certain existing results, which manifests the condition of the Lyapunov-Razumikhin technique in this paper is looser and less conservative. Finally, the validity and correctness of the theoretical results are verified by two examples and some simulations.

Keywords input-to-state stability, synchronous switching, asynchronous switching, Lyapunov-Razumikhin technique, mode-dependent average dwell time, comparison theorem approach

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1 Introduction

Since their introduction in the late 1980s [1] and 2000s [2], respectively, input-to-state stability (ISS) and integral ISS (iISS) have attracted interest from researchers due to their remarkable role in stochastic system analysis [3, 4]. For systems with ISS properties, the norm estimation of the solution should consist of an upper bound of an initial state-dependent disappearing transient term plus a term that is proportional to the external input norm. Therefore, ISS illustrates that not only the systems without input are asymptotically stable but also retain the property of being bounded when their external input is bounded [5]. Lately, several invariants of ISS for different dynamical systems have been studied, such as iISS of discrete-time systems [6], local ISS of production networks [7], and exponential-weighted ISS of hybrid impulsive system [8].

For all we know, time delays often occur in many control applications, especially due to transmission phenomena and measurement, causing instability, performance degradation, and oscillations in the considered systems. A delayed system is a dynamic system in which its current state rate is affected by its bygone state. In recent decades, there have been increasing studies on time-delay systems (TDSs) [9, 10], especially ones focusing on the stability of TDSs [11]. To the best of our knowledge, the Krasovskii approach [12] and the Razumikhin method [13, 14] expanded the Lyapunov direct approach to TDSs. Both methods are highly applicable to the stabilization/stability analysis of TDSs, such as impulse TDSs [15], functional TDSs [16], and neutral TDSs [17]. In addition, adequate conditions for the stability of TDSs with time-invariant delays/time-varying delays are provided. When there exists a time delay in the subsystem of switching systems, the systems can be regarded as switching time delay systems (STDSs), and their stabilization/stability analysis has also been extensively studied in the past [18, 19]. The STDSs offer

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a consolidated criterion for the mathematical modeling of artificial systems or abundant physical systems displaying switching properties such as network control systems [20, 21] and flight control systems [22]. Different switching signals distinguish STDSs from general time-varying systems because the solutions of STDSs depend not only on the initial conditions of the systems but also on the switched signals; thus, research on STDS stability/stabilization has invariably posed considerable challenges. The Krasovskii approach is slightly conservative because it is not easy to discover a communal Lyapunov-Krasovskii (L-K) function, particularly with time delays. However, by utilizing multiple Lyapunov functions in [23], the switching systems are exponentially stable, provided that the dwell time (DT) τ is large enough. Moreover, certain studies identified a smaller bound of DT bound to ensure system stability [24, 25]. However, it is worth noting that in some circumstances, the development of DT switching is limited. In [26], average DT (ADT), which is more effective and flexible for system stability analysis, is proposed. The MDADT was shown in [27], indicating some easily verified sufficient conditions for system stability. Chen et al. [28] investigated the ISS/iISS of impulse TDSs whose impulse time series possess the fixed DT (FDT) characteristic.

The authors [29,30] always considered the switching signal or semi-Markovian switching in the systems. Two probable control scenarios exist in controlled STDSs. First, synchronous switching implies that input-controlled switching signals are consistent with controlled systems. Second, asynchronous switching indicates that controller switching may not be exactly the same as subsystem switching. Further, a system driven by a controller passes through a communication channel, and the present subsystem switches to the next subsystem, which takes some time. This delay occurs due to the time taken to recognize the activity of the subsystem and alter the controller from the present subsystem to the relevant subsystem; the time taken is called the switching delay, and in this status, the corresponding closed-loop system will undergo asynchronous switching [31, 32]. For this type of STDSs with asynchronous switching, using the L-K functional is difficult. In [33], the synchronous switching was considered by the L-K functional, which cannot be directly extended to TDSs depending on ADT with asynchronous switching. This can be attributed to the fact that the maximum increment of the current L-K functional in any mode-switching process must be shown. Moreover, the exponential decay bounds and the limits on the maximum growth rate about the functional need to be described in the case of match and mismatch, respectively. Moreover, Lévy noise can describe both continuous noise and discontinuous noise, which makes the SDSS-LN-IC noise more comprehensive and practical.

Based on the aforementioned instructions, ISS/iISS of TDSs under synchronous switching and asynchronous switching are explored in this article. Different from our previous studies [34–36], the DII will be utilized. Multiple Lyapunov functions, DII, comparison theorem approach, L-R technique, and MDADT are used for studying the ISS/iISS of TDSs under synchronous switching. For asynchronous switching, a novel hybrid switching signal is generated by merging switching signal technology. The contributions of this study are given as follows.

- Sufficient conditions for the ISS/iISS for stochastic TDSs under synchronous switching and asynchronous switching are gotten. In this work, the system with Brownian motion and Lévy noise is investigated, which increases the scope of the model studied.
- We consider the TDSs under asynchronous switching depend on MDADT, which reduces the conservatism of ADT and is more general than [37–39], wherein the synchronous switching and ADT were investigated. Moreover, a new merging switching approach is used to handle the problem of asynchronous switching.
- The upper bound estimate for the infinitesimal operator expectation is mode-dependent, has the indefinite sign, and is time-varying, providing a more accurate mathematical analysis.
- A mode-dependent hypothesis related to active time (i.e., MDADT in this paper) is proposed, which can preserve the different property subsystems of active time.

The article is structured as follows: Section 2 presents the preliminaries/definitions/models studied. Section 3 shows the investigation of the ISS/iISS of TDSs under synchronous switching/asynchronous switching. Examples and simulations are revealed in Section 4. Section 5 concludes the paper.

2 Problem statement and preliminaries

Notations. $|x|$ refers to the Euclidean norm for vector x ($x \in \mathbb{R}^n$) and its transpose is defined as x^{T}. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ on behalf of the complete probability space with some filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ meets the general conditions. Let $\mathcal{K}$ denote the class of strictly increasing and continuous function $\hat{\phi}:\mathbb{R}^+\to\mathbb{R}^+$ with $\hat{\phi}(0)=0$. Moreover, $\mathcal{K}_\infty$ is unbounded and represents the subset of $\mathcal{K}$ functions. $\mathcal{VK}(\mathcal{VK}_\infty)$ and $\mathcal{CK}(\mathcal{CK}_\infty)$ are convex and concave functions, respectively, which are also the subsets of $\mathcal{K}(\mathcal{K}_\infty)$ functions. Let $\mathcal{C}^{1,2}(\mathbb{R}^+\times\mathcal{PC}^b_{\mathcal{F}_t}([t_0-\tau,t_0];\mathbb{R}^n);\mathbb{R}^+)$ be the family of nonnegative functions $V(t,\psi)$ which is defined on $\mathbb{R}^+\times\mathcal{PC}^b_{\mathcal{F}_t}([t_0-\tau,t_0];\mathbb{R}^n)$ that are continuously twice differentiable in ψ and once in t. $\mathcal{PC}([a,b];\mathbb{R}^n)$ on behalf of the class of sectional-continuous functions, provided that the functions have no more than a limited amount of discontinuous jumps on $(a,b]$ and are continuous from the right for all points $[a,b)$. $\mathcal{PC}^b_{\mathcal{F}_t}([t_0-\tau,t_0];\mathbb{R}^n)$ $(\mathcal{PC}^b_{\mathcal{F}_0}([t_0-\tau,t_0];\mathbb{R}^n))$ means the family for all bounded $\mathcal{F}_t(\mathcal{F}_0)$-measurable, $\mathcal{PC}$-value stochastic variables ζ, which satisfies $\sup_{t_0-\tau\leqslant t\leqslant t_0}\mathbb{E}|\zeta(t)|^p<\infty$. Let $\bar{\varphi}(t^-)=\lim_{s\to0^-}\bar{\varphi}(t+s)$ and $D^+\bar{\varphi}(t)=\operatorname{limsup}_{s\to0^+}(\bar{\varphi}(t+s)-\bar{\varphi}(t))/s$ be called the Dini derivative about $\bar{\varphi}(t)$. $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ represents the n-dimensional (n-D) $\mathcal{F}_t$-adapted Brownian motion. $\Phi:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is defined as a class of $\mathcal{KL}$ provided that $\Phi(\cdot,t)$ is a class of $\mathcal{K}$ for arbitrarily fixed $t>0$ and $\Phi(\cdot,t)$ decreases to zero on $t\to\infty$. $\mathcal{N}$ signifies the set for positive integers. For any $\mathfrak{K}_1,\mathfrak{K}_2\in\mathbb{R}$, $\mathfrak{K}_1\vee\mathfrak{K}_2=\max\{\mathfrak{K}_1,\mathfrak{K}_2\}$, $\mathfrak{K}_1\wedge\mathfrak{K}_2=\min\{\mathfrak{K}_1,\mathfrak{K}_2\}$.

Study the following stochastic delayed switching systems with Lévy noise and input control (SDSS-LN-IC):

$$\begin{cases}\mathrm{d}x(t)=f_{\rho(t)}(t,x_t,u(t))\mathrm{d}t+g_{\rho(t)}(t,x_t,u(t))\mathrm{d}w(t)+\displaystyle\int_{\mathbb{R}}h_{\rho(t)}(t,x_t,u(t),\epsilon)N(\mathrm{d}t,\mathrm{d}\epsilon),\\ x_{t_0}=\vartheta(t+s),\quad-\tau\leqslant s\leqslant0,\end{cases}\tag{1}$$

where $t\geqslant t_0$, $u(t)\in\mathcal{PC}([t_0,+\infty);\mathbb{R}^n)$ means the external input control with input disturbance and switching signal. $\vartheta\in\mathcal{PC}^b_{\mathcal{F}_0}([t_0-\tau,t_0];\mathbb{R}^n)$, $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ stands for the system state, x_t is a $\mathcal{PC}$-value random process and $x_t=x(t+s),s\in[-\tau,0]$. Define an index set $\mathcal{N}=\{1,2,\ldots,\mathfrak{N}\}$. $\rho(t):\mathbb{R}^+\to\mathcal{N}$ represents the switching function and is supposed as a sectional-continuous constant function from the right side. The switching sequence of $\rho(t)$ is defined as $\{(\sigma_0,t_0),(\sigma_1,t_1),\ldots,(\sigma_k,t_k)\},\sigma_k\in\mathcal{N},k\in\mathcal{N}$, which indicates the σ_k-th subsystem is contributing on $t\in[t_k,t_{k+1})$. For any $\sigma_k\in\mathcal{N}$, $f_{\sigma_k}(t,\psi,u):\mathbb{R}^+\times\mathcal{PC}^b_{\mathcal{F}_t}([t_0-\tau,t_0];\mathbb{R}^n)\times\mathbb{R}^n\to\mathbb{R}^n$, $g_{\sigma_k}(t,\psi,u):\mathbb{R}^+\times\mathcal{PC}^b_{\mathcal{F}_t}([t_0-\tau,t_0];\mathbb{R}^n)\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$, $h_{\sigma_k}(t,\psi,u,\epsilon):\mathbb{R}^+\times\mathcal{PC}^b_{\mathcal{F}_t}([t_0-\tau,t_0];\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$. Suppose that $w(t),\rho(t),N(t,\epsilon)$ are mutually independent and presume that $f_{\sigma_k}(t,0,u)=0$, $g_{\sigma_k}(t,0,u)=0$, $h_{\sigma_k}(t,0,u,\epsilon)=0$, $t\in\mathbb{R}^+,u\in\mathbb{R}^n,\epsilon\in\mathbb{R}$, which declares the SDSS-LN-IC (1) has a trivial solution $x(t)\equiv0$. Moreover, the SDSS-LN-IC (1) satisfies the linear growth condition and Lipschtiz condition as standard hypothesis such that it possesses a unique solution which is defined as $x(t)=x(t;t_0,\vartheta,\rho(t_0))$ [40]. $\hat{\pi}$ is the Lévy measure, $\tilde{N}(\mathrm{d}t,\mathrm{d}\epsilon)=N(\mathrm{d}t,\mathrm{d}\epsilon)-\hat{\pi}(\mathrm{d}\epsilon)\mathrm{d}t$, and $\int_{\mathbb{R}}|\epsilon|^p\wedge1v(\mathrm{d}\epsilon)=\mathfrak{C}<\infty$.

For each $V_{\sigma_k}(t,\psi)\in\mathcal{C}^{1,2}(\mathbb{R}^+\times\mathcal{PC}^b_{\mathcal{F}_t}([t_0-\tau,t_0];\mathbb{R}^n);\mathbb{R}^+)$, we define an infinitesimal operator as follows:

$$\begin{aligned}\mathcal{L}V_{\sigma_k}(t,\psi,u)=&V_{\sigma_k,t}(t,\psi)+V_{\sigma_k,x}(t,\psi)f_{\sigma_k}(t,\psi,u)+\frac{1}{2}\mathrm{trace}[g^{\mathrm{T}}_{\sigma_k}(t,\psi,u)V_{\sigma_k,xx}(t,\psi)g_{\sigma_k}(t,\psi,u)]\\&+\int_{\mathbb{R}}[V_{\sigma_k}(t,\psi+h_{\sigma_k}(t,\psi,u,\epsilon))-V_{\sigma_k}(t,\psi)]\hat{\pi}(\mathrm{d}\epsilon),\end{aligned}$$

where

$$V_{\sigma_k,t}(t,\psi)=\frac{\partial V_{\sigma_k}(t,\psi)}{t},V_{\sigma_k,x}(t,\psi)=\Big(\frac{\partial V_{\sigma_k}(t,\psi)}{\partial x_1},\ldots,\frac{\partial V_{\sigma_k}(t,\psi)}{\partial x_n}\Big),V_{\sigma_k,xx}(t,\psi)=\Big(\frac{\partial^2V_{\sigma_k}(t,\psi)}{\partial x_k\partial x_l}\Big)_{n\times n}.$$

According to the above formula, one could quote the underlying generalized Itô formula,

$$V_{\rho(t)}(t,\psi(t))=V_{\rho(t_0)}(t_0,\psi(t_0))+M_t+\int_{t_0}^t\mathcal{L}V_{\rho(s)}(s,\psi(s),u(s))\mathrm{d}s,\tag{2}$$

where

$$\begin{aligned}M_t=&\int_{t_0}^tV_{\rho(s),x}(s,\psi(s))g_{\rho(s)}(s,\psi(s),u(s))\mathrm{d}w(s)\\&+\int_{t_0}^t\int_{\mathbb{R}}[V_{\rho(s)}(s,\psi(s)+h_{\rho(s)}(s,\psi(s),u(s),\epsilon))-V_{\rho(s)}(s,\psi(s))]\tilde{N}(\mathrm{d}s,\mathrm{d}\epsilon).\end{aligned}$$

Definition 1 ([41])**.** For $\forall t \in [t_0, T]$, let $N_{\rho(t)}(T, t)$ be the switching number for $\rho(t)$ on $(t, T]$, provided that

$$N_{\rho(t)}(T,t) \leqslant N_0 + \frac{T-t}{\mathfrak{T}_a},\ N_0 > 0,\ \mathfrak{T}_a > 0;$$

then N_0 and $\mathfrak{T}_a$ are called a chattering bound and ADT of $\rho(t)$.

Definition 2 ([41])**.** For a switching signal $\rho(t)$, $\forall t \in [t_0, T]$, $\sigma_k \in \mathcal{N}$, let $N_{\rho(t),\sigma_k}(T, t)$ be the switching amounts of the σ_k-th subsystem that is active on the interval $(t, T]$, and $T_{\sigma_k}(T, t)$ be the entire elapsed time of the σ_k-th subsystem on $(t, T]$, provided that

$$N_{\rho(t),\sigma_k}(T,t) \leqslant N_{0,\sigma_k} + \frac{T_{\sigma_k}(T,t)}{\mathfrak{T}_{a,\sigma_k}},\ N_{0,\sigma_k} > 0,\ \mathfrak{T}_{a,\sigma_k} > 0;$$

then N_{0,σ_k} and $\mathfrak{T}_{a,\sigma_k}$ represent the mode-dependent chattering bound and MDADT of $\rho(t)$.

Moreover, for $\forall i = \sigma_k \in \mathcal{N}$, $N_{\rho(t)}(T, t) = \sum_{i=1}^{\mathfrak{N}} N_{\rho(t),i}(T, t)$, $T - t = \sum_{i=1}^{\mathfrak{N}} T_i(T, t)$. Let $S_{\text{ave}}[\mathfrak{T}_{a,\sigma_k}, N_{0,\sigma_k}]$ be the family for switched signals which carry with the σ_k-th MDADT $\mathfrak{T}_{a,\sigma_k}$ and chattering bound N_{0,σ_k}.

Definition 3. The SDSS-LN-IC (1) is said to be

(i) ISS, provided that there are functions $\Phi \in \mathcal{KL}$, $\alpha, \omega \in \mathcal{K}_\infty$, such that for $t \geqslant t_0$,

$$\alpha(\mathbb{E}|x(t)|) \leqslant \Phi(\mathbb{E}||\vartheta||, t - t_0) + \sup_{t_0 \leqslant s \leqslant t} \omega(|u_{\rho(s)}(s)|);$$

(ii) iISS, provided that there are functions $\Phi \in \mathcal{KL}$, $\alpha, \omega \in \mathcal{K}_\infty$, such that for $t \geqslant t_0$,

$$\alpha(\mathbb{E}|x(t)|) \leqslant \Phi(\mathbb{E}||\vartheta||, t - t_0) + \int_{t_0}^{t} \omega(|u_{\rho(s)}(s)|)\mathrm{d}s.$$

3 Main results

This paper also investigates the switching signal $\rho(t)$ in the input controller, but we study two cases. (1) Synchronous: the switching (i.e., $\rho(t)$) available for the input control (i.e., $u(t)$) is isochronous with the $\rho(t)$ of the systems, such that the candidate input controller is shown as $u(t) = U_{\rho(t)}(t, x(t), \xi(t))$, and $\xi(t) : [t_0, +\infty) \to \mathbb{R}^n$ is the additional input disturbance. (2) Asynchronous: the switching of the input controller is inconsistent with the switching of the systems, such that the candidate input controller is shown as $u(t) = U_{\rho(t-\tau_s)}(t, x(t), \xi(t))$, where τ_s is called the switching delay and $0 < \tau_s < \inf\{t_{k+1} - t_k\}, k = 0, 1, 2, \ldots$.

3.1 The ISS/iISS of TDSs under synchronous switching

In this subsection, we consider the synchronous switching signal in the SDSS-LN-IC (1) first. The adequate conditions for the ISS and iISS of the SDSS-LN-IC (1) will be investigated by the comparison theorem approach, MDADT, integral inequality, and L-R technique.

Theorem 1. For $\forall \sigma_k \in \mathcal{N}$, let functions $\alpha_{1\sigma_k} \in \mathcal{VK}_\infty$, $\alpha_{2\sigma_k} \in \mathcal{CK}_\infty$, $\phi_{\sigma_k} \in \mathcal{K}_\infty$, and $\beta_{\sigma_k} \in \mathcal{PC}([t_0 - \tau, \infty); \mathbb{R})$. Presume that there exist Lyapunov functions $V(t, x) \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathcal{PC}^b_{\mathcal{F}_t}([t_0 - \tau, t_0]; \mathbb{R}^n); \mathbb{R}^+)$, some positive numbers $c_1, c_2, \gamma_{\sigma_k} \geqslant 1$ and $\delta > 1$, such that

(A1) $\alpha_{1\sigma_k}(|x|) \leqslant V_{\sigma_k}(t, x) \leqslant \alpha_{2\sigma_k}(|x|)$;

(A2) For $\forall t \in [t_k, t_{k+1})$,

$$\mathbb{E}\mathcal{L}V_{\sigma_k}(t, x_t) \leqslant \beta_{\sigma_k}(t)\mathbb{E}V_{\sigma_k}(t, x(t)) + \phi_{\sigma_k}(|u(t)|),$$

provided $\mathbb{E}V_{\rho(t+\varrho)}(t + \varrho, x(t + \varrho)) \leqslant \delta\mathbb{E}V_{\rho(t)}(t, x(t))$, where $\varrho \in [-\tau, 0]$;

(A3) For any $\sigma_k, \sigma_l \in \mathcal{N}(k \neq l)$, $\mathbb{E}V_{\sigma_k}(t, x) \leqslant \gamma_{\sigma_k}\mathbb{E}V_{\sigma_l}(t, x)$;

(A4) $\mathfrak{T}_{a,\sigma_k} > \mathfrak{T}^*_{a,\sigma_k} = \frac{\ln \gamma_{\sigma_k}}{c_2}$ and for any $s \in [t_0, t]$,

$$\int_s^t \bar{\beta}_{\rho(v)}(v)\mathrm{d}v \leqslant c_1 - c_2(t - s),$$

where $\bar{\beta}_{\rho(t)}(t) = (\frac{-\ln \delta}{\tau}) \vee \beta_{\rho(t)}(t)$. Then the SDSS-LN-IC (1) is ISS and iISS over S_{ave}.

Proof. Assume that $\bar{\alpha} = \sup_{\sigma_k \in \mathcal{N}}\{\alpha_{2\sigma_k}\} \in \mathcal{CK}_\infty$, $\underline{\alpha} = \inf_{\sigma_k \in \mathcal{N}}\{\alpha_{1\sigma_k}\} \in \mathcal{VK}_\infty$, and $L_1 = \bar{\alpha}(\mathbb{E}||\vartheta||)$ and let

$$
\begin{aligned}
&\beta_{\rho(t)}(t) = \beta_{\rho_0}(t_0),\ t \in [t_0 - \tau, t_0),\\
&W_{\rho(t)}(t) = V_{\rho(t)}(t, x(t)),\ t \geqslant t_0 - \tau,\\
&\Theta_{N(t,t_0)}(t) = L_1 \mathrm{e}^{\int_{t_0}^t \bar{\beta}_{\rho(s)}(s)\mathrm{d}s} + \prod_{p=1}^{N(t,t_0)} \gamma_{\sigma_p}^{-1} \int_{t_{N(t,t_0)}}^t \mathrm{e}^{\int_\varsigma^t \bar{\beta}_{\rho(v)}(v)\mathrm{d}v} \phi_{\rho(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma\\
&\qquad + \sum_{p=1}^{N(t,t_0)} \prod_{q=1}^{p-1} \gamma_{\sigma_q}^{-1} \mathrm{e}^{\int_{t_p}^t \bar{\beta}_{\rho(v)}(v)\mathrm{d}v} \int_{t_{p-1}}^{t_p} \mathrm{e}^{\int_\varsigma^t \bar{\beta}_{\rho(v)}(v)\mathrm{d}v} \phi_{\rho(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma,
\end{aligned}
$$

such that $\Theta_0(t) = L_1 = \bar{\alpha}(\mathbb{E}||\vartheta||)$, $t \in [t_0 - \tau, t_0)$. By (2), for $t \in [t_k, t_{k+1})$, one has

$$\mathrm{d}W_{\sigma_k}(t) = \mathcal{L}W_{\sigma_k}(t)\mathrm{d}t + V_x(t, x(t)) g_{\sigma_k}(t, x_t, u(t))\mathrm{d}w(t).$$

Then one can easily calculate that

$$\mathcal{L}W_{\sigma_k}(t) = \mathcal{L}V_{\sigma_k}(t, x_t),\ t \in [t_k, t_{k+1}).$$

Let $\tilde{t}$ be sufficiently small and $t + \tilde{t} \in [t_k, t_{k+1})$ and by Fubini theorem,

$$\mathbb{E}W_{\sigma_k}(t + \tilde{t}) - \mathbb{E}W_{\sigma_k}(t) = \int_t^{t+\tilde{t}} \mathbb{E}\mathcal{L}W_{\sigma_k}(s)\mathrm{d}s.$$

Therefore, one obtains for all $t \in [t_k, t_{k+1})$ that

$$D^+\mathbb{E}W_{\sigma_k}(t) = \mathbb{E}\mathcal{L}W_{\sigma_k}(t) = \mathbb{E}\mathcal{L}V_{\sigma_k}(t, x_t).$$

Next, the remaining proof will be written in two parts: in Part 1, the estimations are given for $\mathbb{E}W_{\sigma_k}(t)$, while in Part 2, the ISS and iISS will be shown based on the results in Part 1.

Part 1. We would demonstrate that

$$\mathbb{E}W_{\sigma_k}(t) \leqslant \prod_{p=1}^{N(t,t_0)} \gamma_{\sigma_p} \Theta_{N(t,t_0)}(t),\ \forall t \geqslant t_0 - \tau. \tag{3}$$

By condition (A1), one knows that for $t \in [t_0 - \tau, t_0)$,

$$\mathbb{E}W_{\sigma_k}(t) \leqslant \alpha_{2\sigma_k}(\mathbb{E}||\vartheta||) \leqslant \Theta_0(t) = L_1. \tag{4}$$

Then, inequality (3) is tenable for $t \in [t_0, t_1)$, namely

$$\mathbb{E}W_{\sigma_0}(t) \leqslant L_1 \mathrm{e}^{\int_{t_0}^t \bar{\beta}_{\sigma_0}(s)\mathrm{d}s} + \int_{t_0}^t \phi_{\sigma_0}(|u(s)|)\mathrm{e}^{\int_s^t \bar{\beta}_{\sigma_0}(v)\mathrm{d}v}\mathrm{d}s. \tag{5}$$

For any $\aleph > 0$, consider relevant comparison ODE,

$$\begin{cases} \dot{y}(t) = \bar{\beta}_{\sigma_0}(t)y(t) + \phi_{\sigma_0}(|u(t)|) + \aleph,\ t \in [t_0, t_1),\\ y(t_0) = L_1 + \aleph. \end{cases} \tag{6}$$

The solution of (6) is

$$y(t) = (L_1 + \aleph)\mathrm{e}^{\int_{t_0}^t \bar{\beta}_{\sigma_0}(s)\mathrm{d}s} + \int_{t_0}^t [\phi_{\sigma_0}(|u(s)|) + \aleph]\mathrm{e}^{\int_s^t \bar{\beta}_{\sigma_0}(v)\mathrm{d}v}\mathrm{d}s,\ t \in [t_0, t_1).$$

We claim that

$$\mathbb{E}W_{\sigma_0}(t) < y(t),\ t \in [t_0, t_1). \tag{7}$$

Apparently, $\mathbb{E}W_{\sigma_0}(t_0) < y(t_0) = L_1 + \aleph$, which implies that inequality (7) holds for $t = t_0$. Now, we presume that inequality (7) is not true; then there are some $t \in (t_0, t_1)$, such that $\mathbb{E}W_{\sigma_0}(t) \geqslant y(t)$. Let

$$t^\diamond = \inf\{t \in (t_0, t_1) : \mathbb{E}W_{\sigma_0}(t) \geqslant y(t)\}.$$

What needs to be pointed out is that $\mathbb{E}W_{\sigma_0}(t), y(t)$ are continuous on (t_0, t_1); one obtains $\mathbb{E}W_{\sigma_0}(t^\diamond) = y(t^\diamond)$ and $\mathbb{E}W_{\sigma_0}(t) \geqslant y(t)$ for $\forall t \in (t^\diamond, t^\diamond + \bar{t}) \subset (t_0, t_1)$, where $\bar{t}$ is a constant and sufficiently small. Hence, for $\forall t \in (t^\diamond, t^\diamond + \bar{t})$, one has

$$\frac{\mathbb{E}W_{\sigma_0}(t) - \mathbb{E}W_{\sigma_0}(t^\diamond)}{t - t^\diamond} \geqslant \frac{y(t) - y(t^\diamond)}{t - t^\diamond},$$

which signifies that

$$D^+\mathbb{E}W_{\sigma_0}(t^\diamond) \geqslant D^+ y(t^\diamond). \tag{8}$$

In addition, it may be checked that

$$\mathbb{E}W_{\rho(t^\diamond+\varrho)}(t^\diamond + \varrho) \leqslant \delta \mathbb{E}W_{\rho(t^\diamond)}(t^\diamond),\ t^\diamond \in (t_0, t_1), \tag{9}$$

which would be demonstrated by the following two cases.

Case 1. $t^\diamond + \varrho < t_0$, which implies that $t^\diamond - t_0 < -\varrho \leqslant \tau$. For $\bar{\beta}_{\rho(t)}(t) \geqslant -\frac{\ln\delta}{\tau}$, $t \geqslant t_0$, the following inequality holds:

$$\mathrm{e}^{\int_{t_0}^{t^\diamond} -\bar{\beta}_{\rho(s)}(s)\mathrm{d}s} \leqslant \delta;$$

then it follows from inequality (4) that when $t^\diamond + \varrho \in [t_0 - \tau, t_0)$,

$$\mathbb{E}W_{\rho(t^\diamond+\varrho)}(t^\diamond + \varrho) \leqslant L_1 = L_1 \mathrm{e}^{\int_{t_0}^{t^\diamond} \bar{\beta}_{\rho(s)}(s)\mathrm{d}s} \mathrm{e}^{\int_{t_0}^{t^\diamond} -\bar{\beta}_{\rho(s)}(s)\mathrm{d}s} \leqslant \delta L \mathrm{e}^{\int_{t_0}^{t^\diamond} \bar{\beta}_{\rho(s)}(s)\mathrm{d}s} < \delta y(t^\diamond) = \delta \mathbb{E}W_{\rho(t^\diamond)}(t^\diamond).$$

Case 2. $t^\diamond + \varrho \geqslant t_0$. Similarly, because $\bar{\beta}_{\rho(t)}(t) \geqslant -\frac{\ln\delta}{\tau}, t \geqslant t_0$, we get $\mathrm{e}^{\int_{t^\diamond+\varrho}^{t^\diamond} \bar{\beta}_{\rho(s)}(s)\mathrm{d}s} \geqslant \delta^{-1}$, and then

$$\begin{aligned}
y(t^\diamond) =& (L_1 + \aleph)\mathrm{e}^{\int_{t_0}^{t^\diamond} \bar{\beta}_{\sigma_0}(s)\mathrm{d}s} + \int_{t_0}^{t^\diamond} (\phi_{\sigma_0}(|u(s)|) + \aleph)\mathrm{e}^{\int_{t_0}^{t^\diamond} \bar{\beta}_{\sigma_0}(v)\mathrm{d}v}\mathrm{d}s \\
=& (L_1 + \aleph)\mathrm{e}^{\int_{t_0}^{t^\diamond+\varrho} \bar{\beta}_{\sigma_0}(s)\mathrm{d}s}\mathrm{e}^{\int_{t^\diamond+\varrho}^{t^\diamond} \bar{\beta}_{\sigma_0}(s)\mathrm{d}s} + \int_{t_0}^{t^\diamond} (\phi_{\sigma_0}(|u(s)|) + \aleph)\mathrm{e}^{\int_{t_0}^{t^\diamond+\varrho} \bar{\beta}_{\sigma_0}(s)\mathrm{d}s}\mathrm{e}^{\int_{t^\diamond+\varrho}^{t^\diamond} \bar{\beta}_{\sigma_0}(s)\mathrm{d}s}\mathrm{d}s \\
\geqslant& \delta^{-1}\left[(L_1 + \aleph)\mathrm{e}^{\int_{t_0}^{t^\diamond+\varrho} \bar{\beta}_{\sigma_0}(s)\mathrm{d}s} + \int_{t_0}^{t^\diamond} (\phi_{\sigma_0}(|u(s)|) + \aleph)\mathrm{e}^{\int_{s}^{t^\diamond+\varrho} \bar{\beta}_{\sigma_0}(v)\mathrm{d}v}\mathrm{d}s\right],
\end{aligned} \tag{10}$$

which indicates that

$$y(t^\diamond) \geqslant \delta^{-1}\left[(L + \aleph)\mathrm{e}^{\int_{t_0}^{t^\diamond+\varrho} \bar{\beta}_{\sigma_0}(s)\mathrm{d}s} + \int_{t_0}^{t^\diamond+\varrho} (\phi_{\sigma_0}(|u(s)|) + \aleph)\mathrm{e}^{\int_{t_0}^{t^\diamond+\varrho} \bar{\beta}_{\sigma_0}(v)\mathrm{d}v}\mathrm{d}s\right] = \delta^{-1} y(t^\diamond + \varrho), \tag{11}$$

which yields that Eq. (9) holds for the both cases. Then we obtain by inequality (9) and condition (A2) that

$$D^+\mathbb{E}W_{\sigma_0}(t^\diamond) \leqslant \bar{\beta}_{\sigma_0}(t^\diamond)\mathbb{E}W_{\sigma_0}(t^\diamond) + \phi_{\sigma_0}(|u(t^\diamond)|) < \bar{\beta}_{\sigma_0}(t^\diamond)y(t^\diamond) + \phi_{\sigma_0}(|u(t^\diamond)|) + \aleph = D^+ y(t^\diamond),$$

which contradicts to (8), and Eq. (7) holds, and then, we let $\aleph \to 0$ so that inequality (5) is obtained. Suppose that inequality (3) is true on $t \in [t_0, t_k)$, namely

$$\mathbb{E}W_{\sigma_k}(t) \leqslant \prod_{p=1}^{N(t,t_0)} \gamma_{\sigma_p}\Theta_{N(t,t_0)}(t),\ t \in [t_m, t_{m+1}), \tag{12}$$

for $m = 0, 1, 2, \ldots, k-1$. Next, we prove that inequality (3) is true on $t \in [t_k, t_{k+1})$. Letting $N(t, t_0) = k$ for $t \in [t_k, t_{k+1})$, inequality (12) is converted to

$$\mathbb{E}W_{\sigma_k}(t) \leqslant \prod_{p=1}^{k} \gamma_{\sigma_p} \Theta_k(t),\ t \in [t_k, t_{k+1}). \tag{13}$$

Combining (13) with condition (A3), one gets that

$$\mathbb{E}W_{\sigma_k}(t_k) \leqslant \prod_{p=1}^{k} \gamma_{\sigma_p} \Theta_k(t_k) \triangleq L_2.$$

For any $\aleph > 0$, consider another comparison ODE,

$$\begin{cases} \dot{y}(t) = \bar{\beta}_{\sigma_k}(t) y(t) + \phi_{\sigma_k}(|u(t)|) + \aleph,\ t \in [t_k, t_{k+1}), \\ y(t_k) = L_2 + \aleph. \end{cases} \tag{14}$$

By the variance-of-constant formula, we have

$$y(t) = (L_2 + \aleph)\mathrm{e}^{\int_{t_k}^{t} \bar{\beta}_{\sigma_k}(s)\mathrm{d}s} + \int_{t_k}^{t} (\phi_{\sigma_k}(|u(s)|) + \aleph)\mathrm{e}^{\int_{s}^{t} \bar{\beta}_{\sigma_k}(v)\mathrm{d}v}\mathrm{d}s,\ t \in [t_k, t_{k+1}).$$

One would claim that

$$\mathbb{E}W_{\sigma_k}(t) < y(t),\ t \in [t_k, t_{k+1}), \tag{15}$$

if Eq. (15) is invalid, one can discover that $\mathbb{E}W_{\sigma_k}(t) \geqslant y(t)$ on $t \in (t_k, t_{k+1})$. In addition, we can also easily know that $\mathbb{E}W_{\sigma_k}(t_k) < y(t_k)$ holds. Letting

$$t' = \inf\{t \in (t_k, t_{k+1}) : \mathbb{E}W_{\sigma_k}(t) \geqslant y(t)\},$$

we know that $\mathbb{E}W_{\sigma_k}(t) < y(t)$ on $t \in (t_k, t')$, $\mathbb{E}W_{\sigma_k}(t') = y(t')$, and $\mathbb{E}W_{\sigma_k}(t) \geqslant y(t), t \in (t', t' + \hat{t})$, where $\hat{t}$ is a small enough constant, which take account of the continuity for $\mathbb{E}W_{\sigma_k}(t), y(t)$ on (t_k, t_{k+1}). Then one has for $\forall t \in (t', t' + \hat{t})$,

$$\frac{\mathbb{E}W_{\sigma_k}(t) - \mathbb{E}W_{\sigma_k}(t')}{t - t'} \geqslant \frac{y(t) - y(t')}{t - t'},$$

which implies that

$$D^+\mathbb{E}W_{\sigma_k}(t') \geqslant D^+ y(t'). \tag{16}$$

Utilizing the similar proof process above,

$$\mathbb{E}W_{\rho(t'+\varrho)}(t' + \varrho) \leqslant \delta \mathbb{E}W_{\rho(t')}(t'),\ t' \in (t_k, t_{k+1}). \tag{17}$$

Combining (17) with condition (A2),

$$D^+\mathbb{E}W_{\sigma_k}(t') \leqslant \bar{\beta}_{\sigma_k}(t')\mathbb{E}W_{\sigma_0}(t') + \phi_{\sigma_k}(|u(t')|) < \bar{\beta}_{\sigma_k}(t') y(t') + \phi_{\sigma_k}(|u(t')|) + \aleph = D^+ y(t'),$$

which contradicts to (16) and then Eq. (15) is obtained. Letting $\aleph \to 0$ and by (15), one obtains for $t \in [t_k, t_{k+1})$,

$$\mathbb{E}W_{\sigma_k}(t) \leqslant L_2 \mathrm{e}^{\int_{t_k}^{t} \bar{\beta}_{\sigma_k}(s)\mathrm{d}s} + \int_{t_k}^{t} \phi_{\sigma_k}(|u(s)|)\mathrm{e}^{\int_{s}^{t} \bar{\beta}_{\sigma_k}(v)\mathrm{d}v}\mathrm{d}s. \tag{18}$$

Substitute L_2 into (18) and we calculate that

$$\mathbb{E}W_{\sigma_k}(t) \leqslant \prod_{p=1}^{k} \gamma_{\sigma_p} \Theta_k(t_k)\mathrm{e}^{\int_{t_k}^{t} \bar{\beta}_{\sigma_k}(s)\mathrm{d}s} + \int_{t_k}^{t} \phi_{\sigma_k}(|u(s)|)\mathrm{e}^{\int_{s}^{t} \bar{\beta}_{\sigma_k}(v)\mathrm{d}s}\mathrm{d}s$$

$$
\begin{aligned}
&=\prod_{p=1}^{k}\gamma_{\sigma_p}\mathrm{e}^{\int_{t_k}^{t}\bar{\beta}_{\sigma_k}(s)\mathrm{d}s}\Bigg(L_1\mathrm{e}^{\int_{t_0}^{t_k}\bar{\beta}_{\rho(s)}(s)\mathrm{d}s}+\prod_{p=1}^{k}\gamma_{\sigma_p}^{-1}\int_{t_k}^{t_k}\mathrm{e}^{\int_{v}^{t}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\phi_{\rho(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma\\
&\quad+\sum_{p=1}^{k}\prod_{m=1}^{p-1}\gamma_{\sigma_m}^{-1}\mathrm{e}^{\int_{t_n}^{t_k}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\int_{t_{n-1}}^{t_n}\mathrm{e}^{\int_{v}^{t}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\phi_{\rho(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma\Bigg)+\int_{t_k}^{t}\phi_{\sigma_k}(|u(s)|)\mathrm{e}^{\int_{s}^{t}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\mathrm{d}s\\
&=\prod_{p=1}^{k}\gamma_{\sigma_p}L_1\mathrm{e}^{\int_{t_0}^{t}\bar{\beta}_{\rho(s)}(s)\mathrm{d}s}+\sum_{p=1}^{k}\prod_{m=p}^{k}\gamma_{\sigma_m}\mathrm{e}^{\int_{t_p}^{t}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\int_{t_p-1}^{t_p}\mathrm{e}^{\int_{v}^{t}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\phi_{\rho(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma\\
&\quad+\int_{t_k}^{t}\phi_{\sigma_k}(|u(s)|)\mathrm{e}^{\int_{s}^{t}\bar{\beta}_{\sigma_k}(v)\mathrm{d}v}\mathrm{d}s\\
&=\prod_{p=1}^{k}\gamma_{\sigma_p}\Theta_k(t).
\end{aligned}
$$

According to the above proof process, the inequality (3) holds for $\forall t\geqslant t_0-\tau$.

Part 2. The iISS/ISS of the system (1) would be shown in this part. For convenience, let $\sigma_k=\iota\in\mathcal{N}$, which could be easily checked that

$$
\prod_{p=1}^{N(t,s)}\gamma_{\sigma_p}=\prod_{\iota=1}^{\mathfrak{N}}\gamma_{\iota}^{N_\iota(t,s)},\ s\in[t_0,t]. \tag{19}
$$

Then by combining inequality (3), (19), and condition (A1), we get for $t\geqslant t_0$,

$$
\begin{aligned}
\underline{\alpha}(\mathbb{E}|x(t)|)\leqslant&\prod_{\iota=1}^{\mathfrak{N}}\gamma_{\iota}^{N_\iota(t,t_0)}\bar{\alpha}(\mathbb{E}||\vartheta||)\mathrm{e}^{\int_{t_0}^{t}\bar{\beta}_{\rho(s)}(s)\mathrm{d}s}+\int_{t_{N(t,t_0)}}^{t}\mathrm{e}^{\int_{\varsigma}^{t}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\phi_{\rho(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma\\
&+\sum_{p=1}^{N(t,t_0)}\prod_{\iota=1}^{\mathfrak{N}}\gamma_{\iota}^{N_\iota(t,t_p)}\mathrm{e}^{\int_{t_p}^{t}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\int_{t_{p-1}}^{t_p}\mathrm{e}^{\int_{\varsigma}^{t}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\phi_{\rho(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma\\
\triangleq&J_1+J_2+J_3.
\end{aligned}\tag{20}
$$

It can also be derived from condition (A4) and Definition 2 that for $s\in[t_0,t]$,

$$
\begin{aligned}
\prod_{\iota=1}^{\mathfrak{N}}\gamma_{\iota}^{N_\iota(t,s)}\mathrm{e}^{\int_{s}^{t}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\leqslant&\mathrm{e}^{\sum_{\iota=1}^{\mathfrak{N}}\left(N_{0,\iota}+\frac{T_\iota(t,s)}{\mathfrak{T}_{a,\iota}}\right)\ln\gamma_\iota}\mathrm{e}^{c_1-c_2(t-s)}\\
\leqslant&\mathrm{e}^{c_1+\sum_{\iota=1}^{\mathfrak{N}}N_{0,\iota}\ln\gamma_\iota}\mathrm{e}^{\sum_{\iota=1}^{\mathfrak{N}}(\frac{\ln\gamma_\iota}{\mathfrak{T}_{a,\iota}}-c_2)T_\iota(t,s)}\\
=&c_4\mathrm{e}^{-c_3(t-s)},
\end{aligned}\tag{21}
$$

where $c_3=\min_{\iota\in\mathcal{N}}\{c_2-\frac{\ln\gamma_\iota}{\mathfrak{T}_{a,\iota}}\}>0, c_4=\mathrm{e}^{c_1+\sum_{\iota=1}^{\mathfrak{N}}N_{0,\iota}\ln\gamma_\iota}$. By (21), we have

$$
J_1\leqslant c_4\bar{\alpha}(\mathbb{E}||\vartheta||)\mathrm{e}^{-c_3(t-t_0)}, \tag{22}
$$

$$
\begin{aligned}
J_2+J_3\leqslant&c_4\sum_{p=1}^{N(t,t_0)}\int_{t_{p-1}}^{t_p}\mathrm{e}^{\int_{\varsigma}^{t}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\phi_{\rho(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma+\int_{t_{N(t,t_0)}}^{t}\mathrm{e}^{\int_{\varsigma}^{t}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\phi_{\rho(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma\\
\leqslant&c_4\int_{t_0}^{t}\mathrm{e}^{\int_{\varsigma}^{t}\bar{\beta}_{\rho(v)}(v)\mathrm{d}v}\phi_{\rho(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma\leqslant c_4\mathrm{e}^{c_1}\int_{t_0}^{t}\mathrm{e}^{-c_2(t-s)}\phi_{\rho(s)}(|u(s)|)\mathrm{d}s\\
\leqslant&c_4\mathrm{e}^{c_1}\int_{t_0}^{t}\phi_{\rho(s)}(|u(s)|)\mathrm{d}s.
\end{aligned}\tag{23}
$$

Substituting (22) and (23) into (20) implies that

$$
\underline{\alpha}(\mathbb{E}|x(t)|)\leqslant c_4\bar{\alpha}(\mathbb{E}||\vartheta||)\mathrm{e}^{-c_3(t-t_0)}+c_4\mathrm{e}^{c_1}\int_{t_0}^{t}\phi_{\rho(s)}(|u(s)|)\mathrm{d}s,
$$

which implies that the SDSS-LN-IC (1) is iISS over $S_{\rm ave}$. Moreover, we get from (23)

$$J_2+J_3\leqslant c_4{\rm e}^{c_1}\int_{t_0}^{t}{\rm e}^{-c_2(t-s)}\phi_{\rho(s)}(|u(s)|){\rm d}s\leqslant c_4{\rm e}^{c_1}c_2^{-1}\sup_{t_0\leqslant s\leqslant t}\{\phi_{\rho(s)}(|u(s)|)\};\tag{24}$$

then combining (22) and (24) with (20) indicates that

$$\underline{\alpha}(\mathbb{E}|x(t)|)\leqslant c_4\bar{\alpha}(\mathbb{E}||\vartheta||){\rm e}^{-c_3(t-t_0)}+c_4{\rm e}^{c_1}c_2^{-1}\sup_{t_0\leqslant s\leqslant t}\phi_{\rho(s)}(|u(s)|).$$

Therefore, the SDSS-LN-IC (1) is ISS over $S_{\rm ave}$.

Remark 1. We used Definition 1 in the following Corollary 2 and Definition 2 in the main Theorem 1; the difference lies in the mode dependence of Definition 2. Definition 1 indicates that for all systems modes, the ADT between arbitrarily two consecutive switched is no less than $\mathfrak{T}_a$. However, one only requires the average time among the intervals about the subsystem, which is bigger than $\mathfrak{T}_{a,k}$, and the intervals are not adjacent here provided that $\rho(t)$ possesses the MDADT characteristic. We can deduce that $\mathfrak{T}_{a,\sigma_k}\leqslant\mathfrak{T}_a$, $\forall\sigma_k\in\mathcal{N}$, in other words, mode-dependent features reduce the conservatism existing in Corollary 2. In reality, if $\mathfrak{T}_{a,\sigma_k}=\mathfrak{T}_a$, then $\sum_{\sigma_k\in\mathcal{N}}N_{\rho(t),\sigma_k}(T,t)\leqslant\sum_{\sigma_k\in\mathcal{N}}N_{0,\sigma_k}+\frac{T_{\sigma_k}(T,t)}{\mathfrak{T}_a},\forall T\geqslant t\geqslant 0$. Then there is $N_0=\sum_{\sigma_k\in\mathcal{N}}N_{0,\sigma_k}$ and $\mathfrak{T}_{a,\sigma_k}=\mathfrak{T}_a$; naturally, we have the usual situation $N_{\rho(t)}(T,t)\leqslant N_0+\frac{T-t}{\mathfrak{T}_a}$, $\forall T\geqslant t\geqslant 0$. Nevertheless, the switching signals cannot be defined in the set, provided that MDADT denotes the set of switching signals; then the consequences in [42, 43] are hard to use to get the stability for the switching system over the switched signal setting. Also, when $\gamma_{\sigma_k}=1$, the condition of MDADT is redundant.

Remark 2. The system (1) is asymptotically stable when $u(t)=0$, implying that there is no external input in the system. The upper-bound estimate for the infinitesimal operator expectation is mode-dependent and not only has an indefinite sign but is also time-varying. We are able to get Theorem 6.1 in [40] of stochastic continuous delay systems provided that $\beta_{\sigma_k}(t)=-\lambda(\lambda>0)$ in Theorem 1 in this paper. In [33], authors used the condition $\mathcal{L}U(t,x_t)\leqslant-\mathfrak{M}_1U(t,x(t))+\mathfrak{M}_2U(t,x_t)+\varphi(|u(t)|)$, $0<\mathfrak{M}_2<\mathfrak{M}_1$, but we can get this condition when $\beta_{\sigma_k}(t)=-\mathfrak{M}_3$ and $\delta={\rm e}^{\mathfrak{M}_3\tau}$ in Theorem 1, where the unique solution of $-\mathfrak{M}_1+\mathfrak{M}_2{\rm e}^{\mathfrak{M}_3\tau}+\mathfrak{M}_3=0$ is $\mathfrak{M}_3$ ($\mathfrak{M}_3>0$) and the ISS/iISS of the system (1) can be gotten likewise. Under these circumstances, we can also get the stability visualization criterion and the Lévy noise plays a positive role. If $g_{\rho(t)}(t,x_t,u(t))=0, h_{\rho(t)}(t,x_t,u(t),\epsilon)=0$, then our results can be reduced to those in [33]. Condition (A4) is shown that, for any $s\in[t_0,t]$, $\int_s^t\bar{\beta}_{\rho(v)}(v){\rm d}v\leqslant c_1-c_2(t-s)$, which indicates that $\bar{\beta}_{\rho(s)}(s)$ is said to be a uniformly exponentially stable function with a guaranteed decay rate c_2 [42].

Corollary 1. For $\forall\sigma_k\in\mathcal{N}$, allow the conditions (A1) and (A3) in Theorem 1 hold; let functions $\phi_{\sigma_k}\in\mathcal{K}_\infty$ and $\beta_{\sigma_k}\in\mathcal{PC}([t_0-\tau,\infty);\mathbb{R})$. Presume that there are Lyapunov functions $V(t,x)\in\mathcal{C}^{1,2}(\mathbb{R}^+\times\mathcal{PC}^b_{\mathcal{F}_t}([t_0-\tau,t_0];\mathbb{R}^n);\mathbb{R}^+)$ and some positive numbers $c_1,c_2,\gamma_{\sigma_k}\geqslant 1$, such that

(A2)* For $\forall t\in[t_k,t_{k+1})$,

$$\mathbb{E}\mathcal{L}V_{\sigma_k}(t,x(t))\leqslant\beta_{\sigma_k}(t)\mathbb{E}V_{\sigma_k}(t,x(t))+\phi_{\sigma_k}(|u(t)|);$$

(A4)* $\mathfrak{T}_{a,\sigma_k}>\frac{\ln\gamma_{\sigma_k}}{c_2}$ and for $s\in[t_0,t]$,

$$\int_s^t\beta_{\rho(v)}(v){\rm d}v\leqslant c_1-c_2(t-s);$$

then the SDSS-LN-IC (1) without delay is ISS/iISS over $S_{\rm ave}$.

Corollary 2. Study the general continuous switching system $\dot{x}(t)=f_{\lambda(t)}(x(t),u(t))$, $\lambda(t)\in\mathcal{N}$; let constants $\eta_1>0,\eta_2>1$ and functions $\alpha_1\in\mathcal{VK}_\infty$, $\alpha_2\in\mathcal{CK}_\infty$, $\psi\in\mathcal{K}_\infty$; assume that there are Lyapunov functions $V(t,x)\in\mathcal{C}^{1,2}(\mathbb{R}^+\times\mathbb{R}^n;\mathbb{R}^+)$, such that, $\forall i\in\mathcal{N}$

$$\alpha_1(|x|)\leqslant V_i(t,x)\leqslant\alpha_2(|x|),$$
$$\dot{V}_i(t,x)\leqslant-\eta_1V_i(t,x)+\psi_i(|u(t)|),$$

and $\forall(p,q)\in\mathcal{N}\times\mathcal{N}$, $p\neq q$,

$$V_p(t,x)\leqslant\eta_2V_q(t,x);$$

then the system is iISS/ISS with $\mathfrak{T}_a>\mathfrak{T}_a^*=\frac{\ln\eta_1}{\eta_2}$.

3.2 The ISS/iISS of TDSs under asynchronous switching

In the preceding theorem, the synchronous switching is investigated in the SDSS-LN-IC (1). Next, we will investigate the asynchronous switching and also study the ISS/iISS for the SDSS-LN-IC (1). For the sake of overcoming the difficulties generated by asynchronous switching, we consider the case of the merging switching approach, which can be seen in [44]. Define a new merging switching: $\bar{\rho}(t) = (\rho(t), \rho(t-\tau_s)) : [t_0, +\infty) \to \mathcal{N}\times\mathcal{N}$. Then let $\oplus$ denote the merging action, $\bar{\rho}(t) = \rho(t)\oplus\rho(t-\tau_s)$, and $\rho_1(t) = \rho(t), \rho_2(t) = \rho(t-\tau_s)$. According to this definition, the set of switched times of $\bar{\rho}(t) = \rho(t)\oplus\rho(t-\tau_s)$ is the union of the sets of switched times of $\rho(t)$ and $\rho(t-\tau_s)$. Before we manifest the main conclusion, we outline two lemmas worth noting.

Lemma 1 ([45]). Supposing that $\rho_1(t) \in S_{\text{ave}}[\mathfrak{T}_{a,\iota}, N_{0,\iota}]$, we have $\rho_2(t) \in S_{\text{ave}}[\mathfrak{T}_{a,\iota}, N_{0,\iota} + \frac{\tau_s}{\mathfrak{T}_{a,\iota}}]$, for $\forall\iota \in \mathcal{N}$.

Lemma 2 ([45]). For $\forall\iota \in \mathcal{N}$, let $H_\iota(t, t_0)$ be the total time of $\rho_1(t) = \rho_2(t) = \iota$ and $\bar{H}_\iota(t, t_0) = T_\iota(t, t_0) - H_\iota(t, t_0)$, provided

$$\tau_s(a_\iota + b_\iota) \leqslant (a_\iota - \bar{a}_\iota)\mathfrak{T}_{a,\iota},\ a_\iota, b_\iota > 0,\ \bar{a}_\iota \in [0, a_\iota);$$

then

$$-a_\iota H_\iota(t, t_0) + b_\iota\bar{H}_\iota(t, t_0) \leqslant d_\iota - \bar{a}_\iota T_\iota(t, t_0),\ t \geqslant t_0,$$

where $d_\iota = (a_\iota + b_\iota)N_{0,\iota}\tau_s$.

Remark 3. One can notice that for $k \in \mathcal{N}$, $\rho(t) = \sigma_k, t \in [t_k, t_{k+1})$, $\rho(t-\tau_s) = \sigma_k, t \in [t_k + \tau_s, t_{k+1})$, such that $\bar{\rho}(t) = (\sigma_k, \sigma_k), t \in [t_k + \tau_s, t_{k+1})$, $\bar{\rho}(t) = (\sigma_k, \sigma_{k-1}), t \in [t_k, t_k + \tau_s)$, which means that $\bar{\rho}(t)$ has the switching time instants $t_0, t_0 + \tau_s, t_1, t_1 + \tau_s, \ldots$, such that the sequence of $\bar{\rho}(t)$ can be shown as $\{(\rho_0, t_0), (\rho_1, t_0 + \tau_s), (\rho_2, t_1), \ldots, (\rho_{2k}, t_k), (\rho_{2k+1}, t_k + \tau_s)\}$.

Theorem 2. For $\forall\rho_k \in \mathcal{N}$, let functions $\alpha_{1\rho_k} \in \mathcal{VK}_\infty$, $\alpha_{2\rho_k} \in \mathcal{CK}_\infty$, $\phi_{\rho_k} \in \mathcal{K}_\infty$, and $\beta_{\rho_k} \in \mathcal{PC}([t_0 - \tau, \infty); \mathbb{R})$. Assume that there are Lyapunov functions $V(t, x) \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathcal{PC}^b_{\mathcal{F}_t}([t_0 - \tau, t_0]; \mathbb{R}^n); \mathbb{R}^+)$, and some positive numbers $c_1, c_2, c_{s,\rho_k}, c_{u,\rho_k}$, $\bar{c}_{s,\rho_k} \in \mathbb{R}, \bar{c}_{u,\rho_k} \in \mathbb{R}$, and $\hat{\delta} > 1$, such that

(B1) $\alpha_{1\rho_k}(|x|) \leqslant V_{\rho_k}(t, x) \leqslant \alpha_{2\rho_k}(|x|)$;

(B2) For $\forall t \in [t_k, t_{k+1})$,

$$\mathbb{E}\mathcal{L}V_{\rho_k}(t, x_t) \leqslant \beta_{\rho_k}(t)\mathbb{E}V_{\rho_k}(t, x(t)) + \phi_{\rho_k}(|u(t)|),$$

provided $\mathbb{E}V_{\bar{\rho}(t+\varrho)}(t + \varrho, x(t + \varrho)) \leqslant \hat{\delta}\mathbb{E}V_{\bar{\rho}(t)}(t, x(t))$, where $\varrho \in [-\tau, 0]$;

(B3) For any $(\rho_i, \rho_j, \rho_k, \rho_l) \in \mathcal{N}\times\mathcal{N}\times\mathcal{N}\times\mathcal{N}$, and $\rho_i \neq \rho_k, \rho_j \neq \rho_l$,

$$\mathbb{E}V_{\rho_i,\rho_j}(t, x) \leqslant \gamma_{\rho_i,\rho_j}\mathbb{E}V_{\rho_k,\rho_l}(t, x),$$

where $\gamma_{\rho_i,\rho_j} = \gamma_{\rho_i} > 1(\rho_i \neq \rho_k), \gamma_{\rho_i,\rho_j} = 1(\rho_i = \rho_k)$;

(B4) $\mathfrak{T}_{a,\rho_k} > \mathfrak{T}^*_{a,\rho_k} = \frac{\ln\gamma_{\rho_k}\mathrm{e}^{\bar{c}_{s,\rho_k}+(c_{s,\rho_k}+c_{u,\rho_k})\tau_s}}{c_{s,\rho_k}}$, and for any $t_k + \tau_s \leqslant s < t < t_{k+1}$,

$$\int_s^t \bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v \leqslant \bar{c}_{s,\rho_k} - c_{s,\rho_k}H_{\rho_k}(t, s);$$

for any $t_k \leqslant s < t < t_k + \tau_s$,

$$\int_s^t \bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v \leqslant \bar{c}_{u,\rho_k} + c_{u,\rho_k}\bar{H}_{\rho_k}(t, s),$$

where $\bar{\beta}_{\bar{\rho}(t)}(t) = (\frac{-\ln\hat{\delta}}{\tau}) \vee \beta_{\bar{\rho}(t)}(t)$. Then the SDSS-LN-IC (1) is ISS and iISS over S_{ave}.

Proof. Let $\mu_0, \mu_1, \mu_2, \ldots, \mu_{N_{\bar{\rho}(t)}(t,t_0)}$ denote the switching times of $\bar{\rho}(t)$ in (t_0, t), $\mu_0 = t_0$, and $\triangle t = t - \mu_{N_{\bar{\rho}(t)}(t,t_0)+1}$, such that $\mu_{N_{\bar{\rho}(t)}(t,t_0)+1} = t - \triangle t \triangleq T$, where $\triangle t < t_{k+1} - t_k$. Assume that $\hat{\alpha} = \sup_{\rho_k\in\mathcal{N}}\{\alpha_{2\rho_k}\} \in \mathcal{CK}_\infty$, $\check{\alpha} = \inf_{\rho_k\in\mathcal{N}}\{\alpha_{1\rho_k}\} \in \mathcal{VK}_\infty$, and $\hat{L}_1 = \hat{\alpha}(\mathbb{E}||\vartheta||)$ and let

$$\Upsilon_{1,\rho_k} = \gamma_{\rho_k}\mathrm{e}^{\bar{c}_{s,\rho_k}}, \Upsilon_{2,\rho_k} = \mathrm{e}^{\bar{c}_{u,\rho_k}},$$

$$
\begin{aligned}
&\beta_{\bar\rho(t)}(t)=\beta_{\rho_0}(\mu_0),\ t\in[\mu_0-\tau,\mu_0),\\
&W_{\bar\rho(t)}(t)=V_{\bar\rho(t)}(t,x(t)),\ t\geqslant\mu_0-\tau,\\
&\Omega_{N(t,\mu_0)}(t)=\hat L_1\mathrm{e}^{\int_{\mu_0}^{t}\bar\beta_{\bar\rho(s)}(s)\mathrm{d}s}+\prod_{p=1}^{N(t,\mu_0)}\gamma_{\rho_p}^{-1}\int_{t_{N(t,\mu_0)}}^{t}\mathrm{e}^{\int_{\varsigma}^{t}\bar\beta_{\bar\rho(v)}(v)\mathrm{d}v}\phi_{\bar\rho(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma\\
&\qquad+\sum_{p=1}^{N(t,\mu_0)}\prod_{q=1}^{p-1}\gamma_{\rho_q}^{-1}\mathrm{e}^{\int_{\mu_p}^{t}\bar\beta_{\bar\rho(v)}(v)\mathrm{d}v}\int_{\mu_{p-1}}^{\mu_p}\mathrm{e}^{\int_{\varsigma}^{t}\bar\beta_{\bar\rho(v)}(v)\mathrm{d}v}\phi_{\bar\rho(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma,
\end{aligned}
$$

such that $\Omega_0(t)=\hat L_1=\hat\alpha(\mathbb{E}||\vartheta||)$, $t\in[\mu_0-\tau,\mu_0)$.

Using (2), we get for $t\in[\mu_k,\mu_{k+1})$

$$\mathrm{d}W_{\rho_k}(t)=\mathcal{L}W_{\rho_k}(t)\mathrm{d}t+V_x(t,x(t))g_{\rho_k}(t,x_t,u(t))\mathrm{d}w(t).$$

Then, we can calculate

$$\mathcal{L}W_{\rho_k}(t)=\mathcal{L}V_{\rho_k}(t,x_t),\ t\in[\mu_k,\mu_{k+1});$$

let $\tilde\mu$ be sufficiently small and $t+\tilde\mu\in[\mu_k,\mu_{k+1})$, by the Fubini theorem,

$$\mathbb{E}W_{\rho_k}(t+\tilde\mu)-\mathbb{E}W_{\rho_k}(t)=\int_{\mu}^{t+\tilde\mu}\mathbb{E}\mathcal{L}W_{\rho_k}(s)\mathrm{d}s.$$

Hence, for $\forall t\in[\mu_k,\mu_{k+1})$, we have

$$D^+\mathbb{E}W_{\rho_k}(t)=\mathbb{E}\mathcal{L}W_{\rho_k}(t)=\mathbb{E}\mathcal{L}V_{\rho_k}(t,x_t).$$

Next, the proof will be also divided into two parts: in Part 1, we give the estimation for $\mathbb{E}W_{\rho_k}(t)$; the iISS/ISS of system (1) with asynchronous switching will be shown in Part 2.

Part 1. We demonstrate that

$$\mathbb{E}W_{\rho_k}(t)\leqslant\prod_{p=1}^{N(t,\mu_0)}\gamma_{\rho_p}\Omega_{N(t,\mu_0)}(t),\ \forall t\geqslant\mu_0-\tau.\tag{25}$$

By condition (B1) one can know that for $t\in[\mu_0-\tau,\mu_0)$,

$$\mathbb{E}W_{\rho_k}(t)\leqslant\alpha_{2\rho_k}(\mathbb{E}||\vartheta||)\leqslant\Omega_0(t)=\hat L_1.\tag{26}$$

Next, we would show Eq. (25) is true for $t\in[\mu_0,\mu_1)$, namely

$$\mathbb{E}W_{\rho_0}(t)\leqslant\hat L_1\mathrm{e}^{\int_{\mu_0}^{t}\bar\beta_{\rho_0}(s)\mathrm{d}s}+\int_{\mu_0}^{t}\phi_{\rho_0}(|u(s)|)\mathrm{e}^{\int_{s}^{t}\bar\beta_{\rho_0}(v)\mathrm{d}v}\mathrm{d}s.\tag{27}$$

For any $\wp>0$, consider correlative comparison ODE,

$$\begin{cases}\dot y(t)=\bar\beta_{\rho_0}(t)y(t)+\phi_{\rho_0}(|u(t)|)+\wp,\ t\in[\mu_0,\mu_1),\\ y(\mu_0)=\hat L_1+\wp.\end{cases}\tag{28}$$

The solution of (28) can be drawn as

$$y(t)=(\hat L_1+\wp)\mathrm{e}^{\int_{\mu_0}^{t}\bar\beta_{\rho_0}(s)\mathrm{d}s}+\int_{\mu_0}^{t}(\phi_{\rho_0}(|u(s)|)+\wp)\mathrm{e}^{\int_{s}^{t}\bar\beta_{\rho_0}(v)\mathrm{d}v}\mathrm{d}s,\ t\in[\mu_0,\mu_1).$$

It is shown that

$$\mathbb{E}W_{\rho_0}(t)<y(t),\ t\in[\mu_0,\mu_1).\tag{29}$$

Obviously, $\mathbb{E}W_{\rho_0}(\mu_0) < y(\mu_0) = \hat{L}_1 + \wp$, which means Eq. (29) holds when $t = \mu_0$. Assuming that Eq. (29) is not true, there exist some $t \in (\mu_0, \mu_1)$, such that $\mathbb{E}W_{\rho_0}(t) \geqslant y(t)$. Let

$$\mu^* = \inf\{t \in (\mu_0, \mu_1) : \mathbb{E}W_{\rho_0}(t) \geqslant y(t)\}.$$

What needs to be pointed out is that $\mathbb{E}W_{\rho_0}(t), y(t)$ are continuous on (μ_0, μ_1). One has $\mathbb{E}W_{\rho_0}(\mu^*) = y(\mu^*)$ and $\mathbb{E}W_{\rho_0}(t) \geqslant y(t)$ for $\forall t \in (\mu^*, \mu^* + \bar{\mu}) \subset (\mu_0, \mu_1)$, and $\bar{\mu}$ is a constant and sufficiently small. Therefore, for $\forall t \in (\mu^*, \mu^* + \bar{\mu})$, one has

$$\frac{\mathbb{E}W_{\rho_0}(t) - \mathbb{E}W_{\rho_0}(\mu^*)}{t - \mu^*} \geqslant \frac{y(t) - y(\mu^*)}{t - \mu^*},$$

which indicates that

$$D^+\mathbb{E}W_{\rho_0}(\mu^*) \geqslant D^+ y(\mu^*). \tag{30}$$

It may be checked in the same way as Theorem 1 that

$$\mathbb{E}W_{\bar{\rho}(\mu^*+\varrho)}(\mu^* + \varrho) \leqslant \delta \mathbb{E}W_{\bar{\rho}(\mu^*)}(\mu^*),\ \mu^* \in (\mu_0, \mu_1). \tag{31}$$

Then, by (31) and condition (B2), one obtains

$$D^+\mathbb{E}W_{\rho_0}(\mu^*) \leqslant \bar{\beta}_{\rho_0}(\mu^*)\mathbb{E}W_{\rho_0}(\mu^*) + \phi_{\rho_0}(|u(\mu^*)|) < \bar{\beta}_{\rho_0}(\mu^*)y(\mu^*) + \phi_{\rho_0}(|u(\mu^*)|) + \wp = D^+ y(\mu^*),$$

which contradicts (30), and then inequality (25) holds for $t \in [\mu_0, \mu_1)$. Assuming that inequality (25) holds for $t \in [\mu_0, \mu_k), k \in \mathcal{N}$, one gets

$$\mathbb{E}W_{\rho_k}(t) \leqslant \prod_{p=1}^{N(t,\mu_0)} \gamma_{\rho_p} \Omega_{N(t,\mu_0)}(t),\ t \in [\mu_m, \mu_{m+1}), \tag{32}$$

for $m = 0, 1, 2, \ldots, k-1$. Next, we show that Eq. (32) also holds for $t \in [\mu_k, \mu_{k+1})$. Let $N(t, \mu_0) = k$, for $t \in [\mu_k, \mu_{k+1})$, which is equivalent to proving that

$$\mathbb{E}W_{\rho_k}(t) \leqslant \prod_{p=1}^{k} \gamma_{\rho_p} \Omega_k(t),\ t \in [\mu_k, \mu_{k+1}). \tag{33}$$

Combining (33) with condition (B3),

$$\mathbb{E}W_{\rho_k}(\mu_k) \leqslant \prod_{p=1}^{k} \gamma_{\rho_p} \Omega_k(\mu_k) \triangleq \hat{L}_2.$$

For any $\wp > 0$, consider another comparison ODE,

$$\begin{cases} \dot{y}(t) = \bar{\beta}_{\rho_k}(t)y(t) + \phi_{\rho_k}(|u(t)|) + \wp,\ t \in [\mu_k, \mu_{k+1}), \\ y(\mu_k) = \hat{L}_2 + \wp. \end{cases} \tag{34}$$

Eq. (34) has the following solution:

$$y(t) = (\hat{L}_2 + \wp)\mathrm{e}^{\int_{\mu_k}^{t} \bar{\beta}_{\rho_k}(s)\mathrm{d}s} + \int_{\mu_k}^{t} (\phi_{\rho_k}(|u(s)|) + \wp)\mathrm{e}^{\int_s^t \bar{\beta}_{\rho_k}(v)\mathrm{d}v}\mathrm{d}s,\ t \in [\mu_k, \mu_{k+1}).$$

We may show that

$$\mathbb{E}W_{\rho_k}(t) < y(t),\ t \in [\mu_k, \mu_{k+1}). \tag{35}$$

If Eq. (35) does not hold, we have $\mathbb{E}W_{\rho_k}(t) \geqslant y(t)$ on $t \in [\mu_k, \mu_{k+1})$. Also, we can easily know that $\mathbb{E}W_{\rho_k}(\mu_k) < y(\mu_k)$ holds. Letting

$$\mu' = \inf\{t \in (\mu_k, \mu_{k+1}) : \mathbb{E}W_{\rho_k}(t) \geqslant y(t)\},$$

we know that $\mathbb{E}W_{\rho_k}(t) < y(t)$ on $t \in (\mu_k, \mu')$, $\mathbb{E}W_{\rho_k}(\mu') = y(\mu')$, and $\mathbb{E}W_{\rho_k}(t) > y(t), t \in (\mu', \mu' + \hat{\mu})$, where $\hat{\mu}$ is sufficiently small, which are all due to the continuity of $\mathbb{E}W_{\rho_k}(t), y(t)$ on (μ_k, μ_{k+1}). Then, for $t \in (\mu', \mu' + \hat{\mu})$, we obtain

$$\frac{\mathbb{E}W_{\rho_k}(t) - \mathbb{E}W_{\rho_k}(\mu')}{t - \mu'} \geqslant \frac{y(t) - y(\mu')}{t - \mu'},$$

which implies that

$$D^+\mathbb{E}W_{\rho_k}(\mu') \geqslant D^+ y(\mu'). \tag{36}$$

One can also get the following inequality by the identical approach to Theorem 1:

$$\mathbb{E}W_{\bar{\rho}(\mu'+\varrho)}(\mu' + \varrho) \leqslant \delta \mathbb{E}W_{\bar{\rho}(\mu')}(\mu'),\ \mu' \in (\mu_k, \mu_{k+1}). \tag{37}$$

Combining (37) with condition (B2), one has

$$D^+\mathbb{E}W_{\rho_k}(\mu') \leqslant \bar{\beta}_{\rho_k}(\mu')\mathbb{E}W_{\bar{\sigma}_0}(\mu') + \phi_{\rho_k}(|u(\mu')|) < \bar{\beta}_{\rho_k}(\mu')y(\mu') + \phi_{\rho_k}(|u(\mu')|) + \wp = D^+ y(\mu'),$$

which contradicts (36) and then, Eq. (35) is true. Using (35) and letting $\wp \to 0$, for $t \in [\mu_k, \mu_{k+1})$, we have

$$\mathbb{E}W_{\rho_k}(t) \leqslant \hat{L}_2 \mathrm{e}^{\int_{\mu_k}^t \bar{\beta}_{\rho_k}(s)\mathrm{d}s} + \int_{\mu_k}^t \phi_{\rho_k}(|u(s)|)\mathrm{e}^{\int_s^t \bar{\beta}_{\rho_k}(v)\mathrm{d}v}\mathrm{d}s. \tag{38}$$

Substituting $\hat{L}_2$ into (38), one calculates that

$$\begin{aligned}
\mathbb{E}W_{\rho_k}(t) \leqslant{}& \prod_{p=1}^{k} \gamma_{\rho_p} \Omega_k(\mu_k) \mathrm{e}^{\int_{\mu_k}^t \bar{\beta}_{\rho_k}(s)\mathrm{d}s} + \int_{\mu_k}^t \phi_{\rho_k}(|u(s)|)\mathrm{e}^{\int_s^t \bar{\beta}_{\rho_k}(v)\mathrm{d}s}\mathrm{d}s \\
={}& \prod_{p=1}^{k} \gamma_{\rho_p} \mathrm{e}^{\int_{\mu_k}^t \bar{\beta}_{\rho_k}(s)\mathrm{d}s} \Bigg(\hat{L}_1 \mathrm{e}^{\int_{\mu_0}^{\mu_k} \bar{\beta}_{\bar{\rho}(s)}(s)\mathrm{d}s} + \prod_{p=1}^{k} \gamma_{\rho_p}^{-1} \int_{\mu_k}^{\mu_k} \mathrm{e}^{\int_\varsigma^t \bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v} \phi_{\bar{\rho}(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma \\
&+ \sum_{p=1}^{k} \prod_{m=1}^{p-1} \gamma_{\rho_m}^{-1} \mathrm{e}^{\int_{\mu_p}^{\mu_k} \bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v} \int_{\mu_{p-1}}^{\mu_p} \mathrm{e}^{\int_\varsigma^t \bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v} \phi_{\bar{\rho}(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma \Bigg) \\
&+ \int_{\mu_k}^t \phi_{\rho_k}(|u(s)|)\mathrm{e}^{\int_s^t \bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v}\mathrm{d}s \\
={}& \prod_{p=1}^{n} \gamma_{\rho_p}^{-1} \hat{L}_1 \mathrm{e}^{\int_{\mu_0}^t \bar{\beta}_{\bar{\rho}(s)}(s)\mathrm{d}s} + \int_{\mu_k}^t \phi_{\rho_k}(|u(s)|)\mathrm{e}^{\int_s^t \bar{\beta}_{\rho_k}(v)\mathrm{d}v}\mathrm{d}s \\
&+ \sum_{p=1}^{k} \prod_{m=p}^{k} \gamma_{\rho_m} \mathrm{e}^{\int_{\mu_p}^t \bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v} \int_{\mu_p - 1}^{\mu_p} \mathrm{e}^{\int_\varsigma^t \bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v} \phi_{\bar{\rho}(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma \\
={}& \prod_{p=1}^{k} \gamma_{\rho_p} \Omega_k(t),
\end{aligned}$$

such that inequality (33) holds for $t \geqslant \mu_0 - \tau$.

Part 2. The iISS and ISS of the system (1) are presented here. For simplicity, let $\ell = \bar{\rho}(t) \in \mathcal{N}$, $t \in [\mu_k, \mu_{k+1})$, and it can be checked that

$$\prod_{p=1}^{N(t,s)} \gamma_{\rho_p} = \prod_{\iota=1}^{\mathfrak{N}} \gamma_\ell^{N_\ell(t,s)},\ s \in [\mu_0, t]. \tag{39}$$

Then, combining (25), (39), and condition (B1), for $t \geqslant \mu_0$, we have

$$\breve{\alpha}(\mathbb{E}|x(t)|) \leqslant \prod_{\ell=1}^{\mathfrak{N}} \gamma_\ell^{N_\ell(t,\mu_0)} \hat{\alpha}(\mathbb{E}\|\vartheta\|)\mathrm{e}^{\int_{\mu_0}^t \bar{\beta}_{\bar{\rho}(s)}(s)\mathrm{d}s} + \int_{\mu_{N(t,\mu_0)}}^t \mathrm{e}^{\int_\varsigma^t \bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v} \phi_{\bar{\rho}(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma$$

$$+\sum_{p=1}^{N(t,\mu_0)}\prod_{\ell=1}^{\mathfrak{N}}\gamma_\ell^{N_\ell(t,\mu_p)}\mathrm{e}^{\int_{\mu_p}^t\bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v}\int_{\mu_{p-1}}^{\mu_p}\mathrm{e}^{\int_\varsigma^t\bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v}\phi_{\bar{\rho}(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma$$
$$\triangleq I_1+I_2+I_3. \tag{40}$$

For condition (B4), there are $c_\ell\in(0,c_{s,\ell})$ satisfying $\frac{\ln\gamma_\ell+\bar{c}_{s,\ell}}{\mathfrak{T}_{a,\iota}}<c_\ell<c_{s,\ell}-\frac{(c_{s,\ell}+c_{u,\ell})\tau_s}{\mathfrak{T}_{a,\ell}}$, $\ell\in\mathcal{N}$. It can also be derived from Definition 2 and Lemma 2 that for $s\in[\mu_0,t]$,

$$\begin{aligned}
&\prod_{\ell=1}^{\mathfrak{N}}\gamma_\ell^{N_\ell(t,s)}\mathrm{e}^{\int_s^t\bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v}\\
&\quad\leqslant\prod_{\ell=1}^{\mathfrak{N}}\Upsilon_{2,\ell}^{N_{\rho_1(t),\ell}(t,s)}\prod_{\ell=1}^{\mathfrak{N}}\Upsilon_{1,\ell}^{N_{\rho_2(t),\ell}(t,s)}\mathrm{e}^{\sum_{\ell=1}^{\mathfrak{N}}[-c_{s,\ell}H_\ell(t,s)+c_{u,\ell}\bar{H}_\ell(t,s)]}\\
&\quad\leqslant\mathrm{e}^{\sum_{\ell=1}^{\mathfrak{N}}N_{\rho_1(t),\ell}(t,s)\ln\Upsilon_{2,\ell}}\mathrm{e}^{\sum_{\ell=1}^{\mathfrak{N}}N_{\rho_2(t),\ell}(t,s)\ln\Upsilon_{1,\ell}}\mathrm{e}^{\sum_{\ell=1}^{\mathfrak{N}}[-c_{s,\ell}H_\ell(t,s)+c_{u,\ell}\bar{H}_\ell(t,s)]}\\
&\quad\leqslant\mathrm{e}^{\sum_{\ell=1}^{\mathfrak{N}}[N_{0,\ell}+\frac{T_\ell(t,s)}{\mathfrak{T}_{a,\ell}}]\bar{c}_{u,\ell}}\mathrm{e}^{\sum_{\ell=1}^{\mathfrak{N}}[N_{0,\ell}+\frac{\tau_s}{\mathfrak{T}_{a,\ell}}+\frac{T_\ell(t,s)}{\mathfrak{T}_{a,\ell}}]\ln\gamma_\ell\mathrm{e}^{\bar{c}_{s,\ell}}}\mathrm{e}^{\sum_{\ell=1}^{\mathfrak{N}}[-c_{s,\ell}H_\ell(t,s)+c_{u,\ell}\bar{H}_\ell(t,s)]}\\
&\quad\leqslant\triangle_1\,\mathrm{e}^{\sum_{\ell=1}^{\mathfrak{N}}[\frac{\ln\gamma_\ell\mathrm{e}^{\bar{c}_{s,\ell}}}{\mathfrak{T}_{a,\ell}}-c_\ell]}T_\ell(t,s)=\triangle_1\,\mathrm{e}^{-\hat{c}(t-s)},
\end{aligned}\tag{41}$$

where $\triangle_1=\mathrm{e}^{\sum_{\ell=1}^{\mathfrak{N}}N_{0,\ell}[\bar{c}_{u,\ell}+\ln\gamma_\ell\mathrm{e}^{\bar{c}_{s,\ell}}]+\sum_{\ell=1}^{\mathfrak{N}}\frac{\tau_s}{\mathfrak{T}_{a,\ell}}\ln\gamma_\ell\mathrm{e}^{\bar{c}_{s,\ell}}}\times\mathrm{e}^{\sum_{\ell=1}^{\mathfrak{N}}(c_{s,\ell}+c_{u,\ell})N_{0,\ell}\tau_s}$, $\hat{c}=\min_{\ell\in\mathcal{N}}\{c_\ell-\frac{\ln\gamma_\ell\mathrm{e}^{\bar{c}_{s,\ell}}}{\mathfrak{T}_{a,\ell}}\}$. By (41), we have

$$I_1\leqslant\triangle_1\,\hat{\alpha}(\mathbb{E}||\vartheta||)\mathrm{e}^{-\hat{c}(t-\mu_0)}, \tag{42}$$

$$\begin{aligned}
I_2+I_3&\leqslant\triangle_1\sum_{p=1}^{N(t,\mu_0)}\int_{\mu_{p-1}}^{\mu_p}\mathrm{e}^{\int_\varsigma^t\bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v}\phi_{\bar{\rho}(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma+\int_{t_{N(t,\mu_0)}}^t\mathrm{e}^{\int_\varsigma^t\bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v}\phi_{\bar{\rho}(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma\\
&\leqslant\triangle_1\int_{\mu_0}^t\mathrm{e}^{\int_\varsigma^t\bar{\beta}_{\bar{\rho}(v)}(v)\mathrm{d}v}\phi_{\bar{\rho}(\varsigma)}(|u(\varsigma)|)\mathrm{d}\varsigma\\
&\leqslant\triangle_2\int_{\mu_0}^t\phi_{\bar{\rho}(s)}(|u(s)|)\mathrm{d}s,
\end{aligned}\tag{43}$$

where $\triangle_2=\triangle_1\,\mathrm{e}^{\bar{c}_{s,\ell}+\bar{c}_{u,\ell}}$. Substituting (42)–(43) to (40) implies that

$$\breve{\alpha}(\mathbb{E}|x(t)|)\leqslant\triangle_1\,\hat{\alpha}(\mathbb{E}||\vartheta||)\mathrm{e}^{-\hat{c}(t-\mu_0)}+\triangle_2\int_{\mu_0}^t\phi_{\bar{\rho}(s)}(|u(s)|)\mathrm{d}s,$$

which implies that the SDSS-LN-IC (1) is iISS over S_{ave}. Moreover, we can get from (43)

$$I_2+I_3\leqslant\triangle_2\int_{\mu_0}^t\mathrm{e}^{-\hat{c}(t-s)}\phi_{\bar{\rho}(s)}(|u(s)|)\mathrm{d}s\leqslant\triangle_2\,\hat{c}^{-1}\sup_{\mu_0\leqslant s\leqslant t}\{\phi_{\bar{\rho}(s)}(|u(s)|)\}; \tag{44}$$

then, combining (42), (44) with (40) indicates that

$$\breve{\alpha}(\mathbb{E}|x(t)|)\leqslant\triangle_1\,\hat{\alpha}(\mathbb{E}||\vartheta||)\mathrm{e}^{-\hat{c}(t-\mu_0)}+\triangle_2\,\hat{c}^{-1}\sup_{\mu_0\leqslant s\leqslant t}\phi_{\bar{\rho}(s)}(|u(s)|).$$

Thus, the SDSS-LN-IC (1) is ISS over S_{ave}.

Remark 4. In Theorem 2, we have reordered the switching serial (i.e., asynchronous switching), and a new switching signal $\bar{\rho}(t)$ is proposed. The constants $\bar{c}_{s,\rho_k},\bar{c}_{u,\rho_k}$ are incorporated into $\Upsilon_{1,\rho_k},\Upsilon_{2,\rho_k}$, which are distinct from Theorem 1, and $\Upsilon_{1,\rho_k}>1,\Upsilon_{2,\rho_k}>1$ without any restriction. Condition (B2) is also a more lenient L-R stability condition, which is related to the asynchronous switching.

Remark 5. The conditions (B1) and (B3) generally exist in some pre-existing studies [40,44,46]. Due to the input control in the SDSS-LN-IC (1) being $u(t)=U_{\rho(t-\tau_s)}(t,x(t),\xi(t))$, the Itô operator is shown as the criterion (B2) with the coefficient $\beta_{\rho_k}(t)$ rather than the condition (A2). In addition, we suppose that the ordering of the delayed switching signal $\rho(t-\tau_s)$ is the same as the ordering of the correlative switching times of $\rho(t)$ whenever τ_s is time-varying.

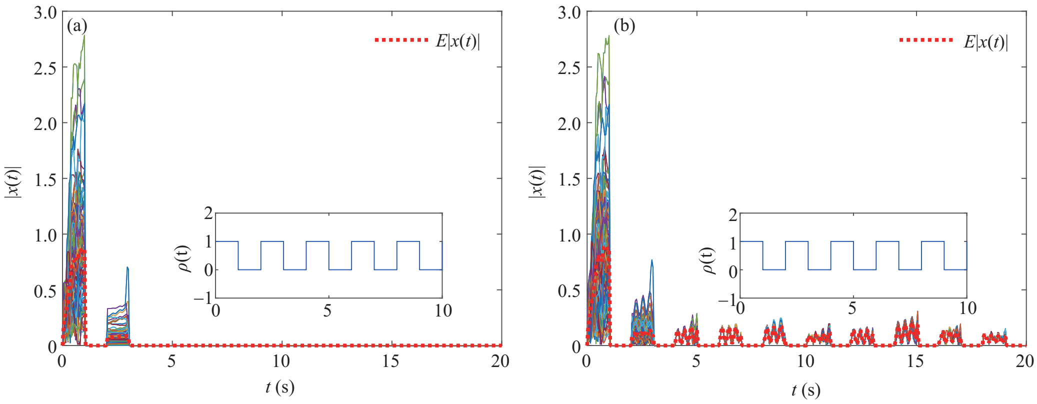

Figure 1 (Color online) The trajectories of the considered system (45) (a) without external control and (b) with external control $u(t) = (\mathrm{sin}t, \mathrm{cos}t)^{\mathrm{T}}$.

4 Numerical simulations

In this section, two examples are provided to highlight the feasibility and theoretical rationality of the major theorems. Example 1 shows the results of system (1) under synchronous switching, whereas Example 2 shows those under asynchronous switching.

Example 1. Consider the TDSs with Lévy noise and input control under synchronous switching as follows:

$$\mathrm{d}x(t) = [f_{\rho(t)}(t, x_t) + u(t)]\mathrm{d}t + g_{\rho(t)}(t, x_t)\mathrm{d}w(t) + \int_{\mathbb{R}} h_{\rho(t)}(t, x_t, \epsilon)N(\mathrm{d}t, \mathrm{d}\epsilon) \tag{45}$$

with $t \geqslant t_0$, where $\rho(t) \in \mathcal{N} = \{1, 2\}$, $x(t) = (x_1(t), x_2(t))^{\mathrm{T}}$, $w(t) = (w_1(t), w_2(t))^{\mathrm{T}}$, $w_1(t), w_2(t)$ are 1-D Brownian motion, $\hat{\pi}(\mathrm{d}\epsilon) = \acute{a}\breve{\pi}(\mathrm{d}\epsilon)$, $\acute{a}$ is the strength for the Poisson distribution, the probability intensity of standard normally distributed variable ϵ is $\breve{\pi}$, and

$$f_1 = \begin{bmatrix} -\frac{3}{4}x_1(t)\mathrm{sin}t + x_2(t-\tau)|\mathrm{cos}t| \\ \frac{3}{2}x_1(t-\tau)\mathrm{sin}t + \frac{1}{2}x_2(t)|\mathrm{cos}t| \end{bmatrix},\ g_1 = \begin{bmatrix} \sqrt{|\mathrm{cos}t|}x_1(t) & \sqrt{|\mathrm{cos}t|}x_2(t) \end{bmatrix}^{\mathrm{T}},\ h_1 = \sqrt{|\mathrm{cos}t|}x_1(t)\epsilon,$$

$$f_2 = \begin{bmatrix} x_1(t-\tau)\mathrm{sin}t + x_2(t)\mathrm{sin}t \\ x_1(t)|\mathrm{cos}t| + x_2(t-\tau)\mathrm{sin}t \end{bmatrix},\ g_2 = \begin{bmatrix} \sqrt{|\mathrm{cos}t|}x_1(t) & \frac{\sqrt{2|\mathrm{cos}t|}}{2}x_2(t) \end{bmatrix}^{\mathrm{T}},\ h_2 = \sqrt{|\mathrm{cos}t|}x_2(t)\epsilon.$$

Then for subsystem 1, letting $V_1 = x_1^2(t) + x_2^2(t)$, $\delta = 2$, $\alpha_{11} = 0.9$, $\alpha_{21} = 1$, and $\acute{a} = 1/4, \mathfrak{C} = 2$, the condition (A2) of Theorem 1 satisfies $\mathbb{E}\mathcal{L}V_1(t, x_t) \leqslant (3|\mathrm{cos}t| + \frac{3}{2}\mathrm{sin}t + \frac{5}{4})\mathbb{E}V_1(t, x(t)) + 2\tilde{u}_1^2(t)$, which means $\beta_1(t) = 3|\mathrm{cos}t| + \frac{3}{2}\mathrm{sin}t + \frac{5}{4}, u_1(t) = 2\tilde{u}_1^2(t)$, $\beta_1(\frac{2k+1}{2}\pi) = -\frac{1}{4} < 0, k = 2n+1, n = 0, 1, 2, \ldots$, and $\beta_1(k\pi) = \frac{15}{4} > 0, k = 0, 1, 2, \ldots$.

For subsystem 2, let $V_2 = \frac{1}{2}(x_1^2(t) + x_2^2(t))$, $\delta = 2$, $\alpha_{21} = 0.9, \alpha_{22} = 1$, $\acute{a} = 1/2$, and $\mathfrak{C} = 1$. One can get the condition (A2) of Theorem 1 satisfies $\mathbb{E}\mathcal{L}V_2(t, x_t) \leqslant (4\mathrm{sin}t + 3|\mathrm{cos}t| + \frac{5}{2})\mathbb{E}V_2(t, x(t)) + 2\tilde{u}_2^2(t)$, which implies that $\beta_2(t) = 4\mathrm{sin}t + 3|\mathrm{cos}t| + \frac{5}{2}, u_2(t) = 2\tilde{u}_2^2(t)$, $\beta_2(\frac{2k+1}{2}\pi) = -\frac{3}{2} < 0$, $k = 2n+1, n = 0, 1, 2, \ldots$, and $\beta_2(k\pi) = \frac{13}{2} > 0, k = 0, 1, 2, \ldots$.

Taking $\tau = 0.25$, $\gamma_1 = e$, and $\gamma_2 = e^3$, we can get $c_2 = 1/4, \mathfrak{T}^*_{a,1} = 4$ in subsystem 1 and $c_2 = 3/2, \mathfrak{T}^*_{a,2} = 3$ in subsystem 2 can be obtained. To sum up, the above example satisfies all the conditions of Theorem 1; that is, the system (45) is ISS/iISS in the mean square under synchronous switching. Figures 1(a) and (b) respectively show the state trajectories of 100 sample paths for system (45) without input control and with input control $u(t) = (\mathrm{sin}t, \mathrm{cos}t)^{\mathrm{T}}$, where $(x_1(t), x_2(t))^{\mathrm{T}} = [-1 \ \ 1]$ and the red dotted line presents the trajectory of $\mathbb{E}|x(t)|$.

Example 2. Consider the TDSs with Lévy noise and input control under asynchronous switching as follows:

$$\mathrm{d}x(t) = [A_{\bar{\rho}(t)}x(t) + F_{\bar{\rho}(t)}(t, x_t, u(t))]\mathrm{d}t + G_{\bar{\rho}(t)}(t, x_t)\mathrm{d}w(t) + \int_{\mathbb{R}} H_{\bar{\rho}(t)}(t, x_t, \epsilon)N(\mathrm{d}t, \mathrm{d}\epsilon) \tag{46}$$

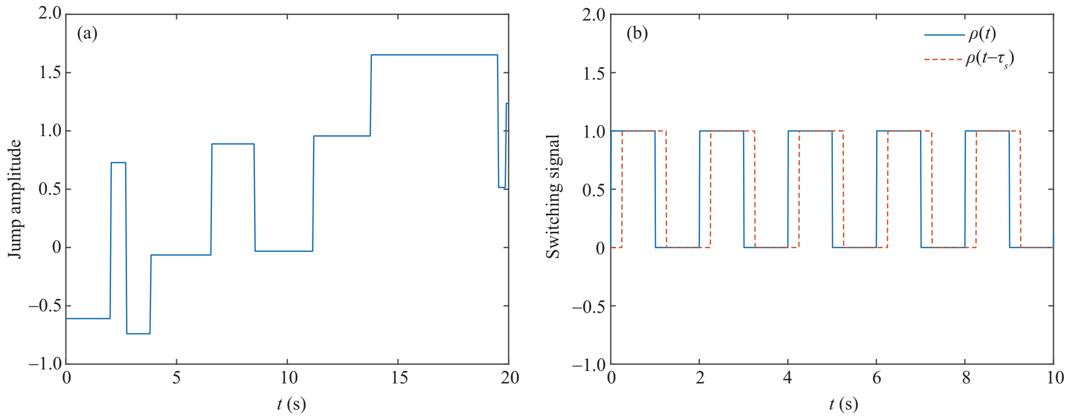

Figure 2 (Color online) The trajectories of (a) poisson process, (b) switching signals $\rho(t)$ and $\rho(t-\tau_s)$.

with $t \geqslant t_0$, where $x(t), w(t), \hat{\pi}, \acute{a}$ have the same definitions as Example 1, $\bar{\rho}(t) = \rho_1(t) \oplus \rho_2(t)$, $\rho_1(t) = \rho(t)$, and $\rho_2(t) = \rho(t-\tau_s)$. We set $V = |x(t)|^2$ with $\alpha_{11} = \alpha_{21} = \alpha_{12} = \alpha_{22} = 1, \gamma_1 = \gamma_2 = 1.001, \hat{\delta} = 4, \tau_s = 7/3, \acute{a} = 1/3, \mathfrak{C} = 2$, and

$$
\begin{aligned}
&A_1 = \begin{bmatrix} 1/8 & 1 \\ -1 & -7/8 \end{bmatrix}, A_2 = \begin{bmatrix} 1/6 & 1 \\ -1 & 7/8 \end{bmatrix}, \\
&H_1(t, x_t) = \frac{1}{2} x_1(t)\epsilon, \; H_2(t, x_t) = \frac{1}{2} x_2(t)\epsilon, \\
&G_1(t, x_t) = \begin{bmatrix} 0 & -x_1(t)\mathrm{cos}t + \frac{1}{2} x_2(t)\mathrm{cos}t \end{bmatrix}^{\mathrm{T}}, \\
&G_2(t, x_t) = \begin{bmatrix} 0 & -x_1(t-\tau)\mathrm{sin}t - \frac{1}{2} x_2(t-\tau)\mathrm{sin}t \end{bmatrix}^{\mathrm{T}}, \\
&F_1(t, x_t, u(t)) = \begin{bmatrix} 0 \\ -\frac{1}{2} x_2(t)\mathrm{cos}^2 t + x_2(t-\tau)\mathrm{cos}^2 t - u(t) \end{bmatrix}, \\
&F_2(t, x_t, u(t)) = \begin{bmatrix} 0 \\ -x_1(t-\tau)\mathrm{sin}^2 t + \frac{1}{2} x_2(t-\tau)\mathrm{sin}^2 t - u(t) \end{bmatrix}.
\end{aligned}
$$

Then for subsystem 1, when $t \in [t_k, t_k + \tau_s)$, $u(t) = \frac{177}{26} x_2(t) - \frac{185}{52} x_2(t-\tau) + \xi(t)\mathrm{cos}^2 t$; then condition (B2) of Theorem 2 satisfies $\mathbb{E}\mathcal{L}V_{11}(t, x_t) \leqslant (\frac{1}{4}\mathrm{cos}2t - \frac{1}{8})\mathbb{E}V_{11}(t, x_t) + (\mathrm{sin}^4 t + \frac{169}{52}\mathrm{sin}^2 t)\xi^2(t)$, which means $\beta_{11}(t) = \frac{1}{4}\mathrm{cos}2t - \frac{1}{8}, \phi_{11}(t) = (\mathrm{sin}^4 t + \frac{169}{52}\mathrm{sin}^2 t)\xi^2(t)$. When $t \in [t_k + \tau_s, t_{k+1})$, $u(t) = -\frac{161}{26} x_1(t) + \frac{153}{52} x_1(t-\tau) + \xi(t)\mathrm{cos}^2 t$, similarly, $\mathbb{E}\mathcal{L}V_{12}(t, x_t) \leqslant (\frac{15}{8}\mathrm{cos}2t + 1)\mathbb{E}V_{12}(t, x_t) + (\mathrm{sin}^4 t - \frac{169}{52}\mathrm{sin}^2 t)\xi^2(t)$, which indicates $\beta_{12}(t) = \frac{15}{8}\mathrm{cos}2t + 1, \phi_{12}(t) = (\mathrm{sin}^4 t - \frac{169}{52}\mathrm{sin}^2 t)\xi^2(t)$.

For subsystem 2, when $t \in [t_k, t_k + \tau_s)$, $u(t) = \frac{11}{4} x_2(t) - \frac{27}{8} x_2(t-\tau) + \xi(t)\mathrm{sin}^2 t$; then the condition (B2) of Theorem 2 satisfies $\mathbb{E}\mathcal{L}V_{21}(t, x_t) \leqslant (\mathrm{cos}2t - \frac{1}{2})\mathbb{E}V_{21}(t, x_t) + (\mathrm{sin}^4 t - \frac{5}{8}\mathrm{sin}^2 t)\xi^2(t)$, which indicates $\beta_{21}(t) = \mathrm{cos}2t - \frac{1}{2}, \phi_{21}(t) = (\mathrm{sin}^4 t - \frac{5}{8}\mathrm{sin}^2 t)\xi^2(t)$. When $t \in [t_k + \tau_s, t_{k+1})$, $u(t) = \frac{89}{12} x_1(t) - \frac{169}{24} x_1(t-\tau) + \xi(t)\mathrm{sin}^2 t$, similarly, $\mathbb{E}\mathcal{L}V_{22}(t, x_t) \leqslant (-\frac{1}{4}\mathrm{cos}2t + \frac{1}{8})\mathbb{E}V_{22}(t, x_t) + (\mathrm{sin}^4 t + \frac{3}{8}\mathrm{sin}^2 t)\xi(t)$, which indicates $\beta_{22}(t) = -\frac{1}{4}\mathrm{cos}2t + \frac{1}{8}, \phi_{22}(t) = (\mathrm{sin}^4 t + \frac{3}{8}\mathrm{sin}^2 t)\xi^2(t)$. Based on above results we can get $\bar{c}_{s,1} = \bar{c}_{u,1} = 1/2, \bar{c}_{s,2} = \bar{c}_{u,2} = -1/8, c_{s,1} = c_{u,1} = 1/3, c_{s,2} = c_{u,2} = 1/8, \mathfrak{T}^*_{a,1} = 23/6$, and $\mathfrak{T}^*_{a,2} = 11/3$. Thus, all the conditions in Theorem 2 are satisfied and state that the TDSs (46) under asynchronous switching are ISS/iISS in the mean square. Figure 2(a) gives the Poisson process, Figure 2(b) shows the switching signal $\rho(t)$ and $\rho(t-\tau_s)$, and the system states without input and with $u(t) = (\mathrm{sin}t, \mathrm{cos}t)^{\mathrm{T}}$ are respectively shown in Figures 3(a) and (b), showing the trajectories of 100 sample paths, where $(x_1(t), x_2(t))^{\mathrm{T}} = [-1 \;\; 1]$ and the red dotted line refers to the trajectory of $\mathbb{E}|x(t)|$.

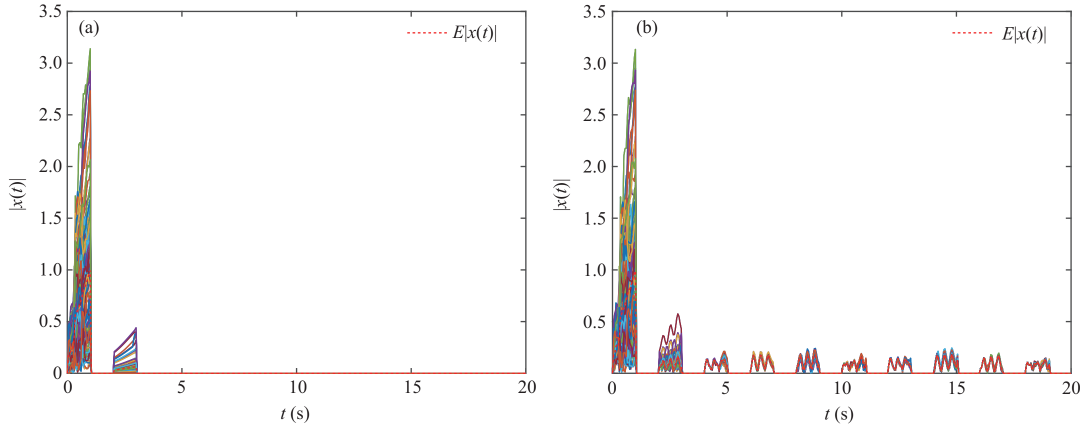

Figure 3 (Color online) The trajectories of the considered system (46) (a) without external control and (b) with external control $u(t) = (\sin t, \cos t)^{\mathrm{T}}$.

5 Conclusion

In this work, the ISS/iISS of SDSS-LN-IC (1) under synchronous switching and asynchronous switching were studied using the L-R method, DII, MDADT, and comparison theorem approach. In terms of asynchronous switching, a new merging switching approach was used. In addition, to overcome the difficulty of switching and randomness, we fully implemented the comparison theorem approach and DII. The upper bound of the Lyapunov function expectation was time-varying and mode-dependent regardless of the sign, which can be well reflected in the above two examples. These results are less conservative and more widely applied compared with previous studies [33,40]. In practice, the MDADT used in this article is less restrictive than usual ADT, and the DT under asynchronous mode is considered, which is different from general MDADT. Subsequently, we will consider the stability of nonlinear switching delayed systems with adaptive controllers and attempt to generalize the hypothesis in the main conclusion. In addition, for general TDSs, it is imperative to explore the ISS-type character of a Lévy noise or impulse driven system when discrete dynamics subsystems cannot be ISS. In this regard, neutral systems, particularly the challenging task of handling neutral operators, are worth considering.

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