SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

November 2023, Vol. 66 212204:1–212204:16
https://doi.org/10.1007/s11432-022-3737-4

Adaptive estimation and control for uncertain nonlinear systems and full actuation control

Fei YAN[1], Mingyuan ZHANG[1] & Guoxiang GU[2*]

[1]*School of Information Science and Technology, Southwest Jiaotong University, Chengdu* 611756, *China;*
[2]*School of Electrical Engineering and Computer Science, Louisiana State University, Baton Rouge* LA 70803, *USA*

Received 24 October 2022/Revised 5 January 2023/Accepted 24 February 2023/Published online 19 October 2023

Abstract We study adaptive control for a family of nonlinear systems, involving unknown and uncertain parameters. The proposed control law estimates the system parameters adaptively and stabilizes the closed-loop system asymptotically for the initial state over any given bounded set of the state-space. Moreover, reconstruction filters are designed to obtain error residue signals and to enable the use of the least-squares algorithm for estimating the parameters, in order to achieve the convergence based on the persistent excitation condition and asymptotic linearization. The proposed methods are applicable to full actuation and under actuation control systems. Simulation studies are carried out for a pendulum system and for a third-order vehicle model, as well as control of vehicle platoons, validating the theoretical results presented in this paper.

Keywords adaptive control, asymptotic stabilization, full actuation and under actuation, least-squares, nonlinear systems

Citation Yan F, Zhang M Y, Gu G X. Adaptive estimation and control for uncertain nonlinear systems and full actuation control. Sci China Inf Sci, 2023, 66(11): 212204, https://doi.org/10.1007/s11432-022-3737-4

1 Introduction

Almost all physical systems are nonlinear, and they pose tremendous challenges to the control community. Many different methods are developed to transform nonlinear systems into linear ones so that abundant design tools for linear systems can be used to design control systems with satisfactory performance. Exact linearization via state feedback [1] (Subsection 4.3) represents an analytic method to remove the nonlinearities, while the dynamic inversion [2–4] provides a more direct method for linearizing nonlinear systems. It is well-known that neural networks are universal approximators [5–9], and thus they also provide a way to cope with nonlinearities via approximation and inversion to cancel the nonlinearities. In the meantime, nonlinear control methods have been developed as well, including the dissipation [10], passivity [11], singular perturbation [12], and backstepping [13], to name a few. Fractional order control methods also play an important role in the nonlinear control field [14, 15].

The paper is primarily motivated by a series of studies [16–20] from Duan. In these studies, Duan formerly proposed feedback linearization using direct cancelation for fully actuated nonlinear systems, including high order nonlinear systems [16], generalized strict-feedback systems [17], adaptive and back-stepping control systems [18], and discrete-time delay systems [19, 20], to cite a few. While people have argued [1, 13] that feedback linearization through cancelation may not be a good approach as this method also discards good nonlinearities, in practice, direct cancelation has been widely employed, including dynamic inversion as discussed earlier, vehicle motion control systems [21, 22], and more recently quantized control systems [23, 24]. Its reason lies in the fact that many design tools are available for linearized control systems in achieving satisfactory transient responses and robustness against disturbances and modeling errors, and optimizing performance indices in both time and frequency domains.

Uncertainties are inevitable and exist in almost all system models. Because the mathematical nonlinear models are mostly obtained based on the physics principles, unmodeled dynamic uncertainties are less

* Corresponding author (email: ggu@lsu.edu)

likely. Often, parameter uncertainties dominate due to tear-wear and aging, frequently encountered in practice, posing significant challenges to the feedback linearization. The most effective methods in dealing with parameter uncertainties are adaptive control. Primary examples include the well-known adaptive backstepping control [13] and adaptive neural network control [25, 26]. Fuzzy control methods as proposed in [27, 28] provide a different perspective to mitigate uncertainties involved in nonlinear control systems. See also [23, 24, 29–33]. However, these adaptive control methods are focused on the asymptotic stabilization, rather than linearization. For this reason, abundant design tools for linear control systems are not truly applicable, which motivates the research to be reported in this paper.

We are interested in a family of uncertain nonlinear systems, commonly seen in engineering practice in which the control input also involves uncertain parameters. Such uncertain nonlinear systems are less studied. For instance, the fully actuated control systems studied by Duan in [34, 35] take the uncertain parameters into consideration; yet the control input does not involve the uncertain parameters. Even though Refs. [34, 35] aimed at feedback linearization through direct cancelation, the lack of convergence of the parameter estimates to the true one fails to achieve asymptotic linearization. This fact provides us the impetus in carrying out the study on a family of uncertain nonlinear systems in establishing not only asymptotic feedback stability but also asymptotic linearization. The major contributions and merits of this paper are summarized as follows:

- We propose a novel adaptive control method to achieve the asymptotic stabilization over any given bounded region in the state-space. The proposed adaptive control law applies to a family of uncertain nonlinear systems, in which the control input also involves uncertain parameters, possibly in the nonlinear form. The merits of the proposed adaptive control lie in the use of the Lyapunov function candidate, derived from the quadratic linear regulator (LQR) for the ideal linearized system, enabling the linear design tools and taking the desired time domain performance into consideration. Therefore, the proposed method compliments the existing work, such as the backstepping method that often obscures the physical meaning of the control law, buried in its recursive design process.
- We propose a secondary estimator, making use of the well-known least-squares (LS) algorithm to re-estimate the system parameter vector based on the error residue of the adaptive control system. The merit of the proposed estimator is exemplified by the convergence of the parameter estimates under certain persistent excitation (PE) condition, and by the asymptotic linearization via direct cancelation. In fact, the secondary estimator is not limited to the LS, and includes all other convergence estimators, including the ones proposed in [36].
- The theoretical results obtained in this paper are successfully applied to a pendulum system, and more importantly to the motion control problem arising from the control of autonomous vehicle platoons using the vehicle model developed in [37]. See [21] for using the feedback linearization via direct cancelation, and [22] for cooperative adaptive cruise control (CACC) based on the linearized model in [21]. In addition, the string stability of the vehicle platoon dictates the $\mathcal{H}_\infty$ control as emphasized in [21, 22, 38], implying that exact linearization is essential for successful vehicle platooning control.

The rest of the paper is organized as follows. Section 2 is focused on developing the proposed adaptive control method in achieving the asymptotic feedback stabilization for a family of commonly seen uncertain nonlinear systems, in which the control input also involves uncertain parameters and can be of nonlinear form. Asymptotic linearization is studied via the secondary LS estimator. Section 3 considers the asymptotic stabilization and linearization for a family of full actuation nonlinear systems. The main results of this paper are illustrated in Section 4 with simulation studies for a pendulum system [1, 23], vehicle system [37], and autonomous vehicle platoons [22, 39]. The paper is concluded in Section 5.

The notation is fairly standard, with $\mathbb{R}/\mathbb{R}^n$ standing for the set of real numbers/n-dimensional real vectors, respectively. For a vector $v \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times m}$, v' and A' denote transpose of v and A, respectively. Moreover, $\sigma_i(A)$ denotes the ith singular value with $\overline{\sigma}(A)$ and $\underline{\sigma}(A)$, the maximum and minimum singular value, respectively. Let $G(s)$ be a transfer matrix of dimension $p \times m$. Its $\mathcal{H}_\infty$-norm is defined to be $\|G\|_{\mathcal{H}_\infty} := \sup_{\mathrm{Re}\{s\}>0} \overline{\sigma}[G(s)]$. Other notations will be clarified as we proceed.

2 Adaptive control and asymptotic linearization

In this section, we study asymptotic stabilization and asymptotic linearization for a family of uncertain nonlinear systems. An adaptive control and LS estimation approach is developed, applicable to many different nonlinear control systems, including autonomous vehicle platooning systems.

2.1 Asymptotic stabilization

The family of uncertain nonlinear systems under consideration is described by

$$\dot{x} = Ax + B\left[J(x,\theta)u + g(x) + H(x)\theta\right], \quad x(0) = x_0, \tag{1}$$

with $x \in \mathbb{R}^n$ being the system state, $u \in \mathbb{R}^m$ being the control input, and $\theta \in \mathbb{R}^\eta$ being the uncertain parameter vector. Moreover, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Hence, nonlinear matrix functions $J(x,\theta)$ and $H(x)$ have dimension $m \times m$ and $m \times \eta$, respectively, and nonlinear vector function $g(x)$ has dimension m.

Assumption 1. (1) The origin of the state space is an equilibrium point of the autonomous system, i.e., $g(0) = 0 \in \mathbb{R}^m$ and $H(0) = 0 \in \mathbb{R}^{m\times\eta}$. (2) (A, B) is controllable, $\text{rank}\{B\} = m$, and $n > m$. (3) $\{g(x), H(x)\}$ are continuous functions of $x \in \mathbb{R}^n$ and satisfy local Lipschitz condition $\|g(x)\| \leqslant \ell_g\|x\|$ and $\overline{\sigma}[H(x)] \leqslant \ell_H\|x\|$ in small neighborhoods of the origin of $\mathbb{R}^n$ for some constants $\ell_g > 0$ and $\ell_H > 0$.

The family of nonlinear systems described in (1) is quite common. For instance, a system described by

$$\dot{x}_i = x_{i+1},\ 1 \leqslant i < n, \quad \dot{x}_n = u + g(x) + h(x)\theta, \quad m = 1, \tag{2}$$

belongs to this family by taking $J(x,\theta) \equiv 1$, $H(x) = h(x)$,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Such systems are addressed in [23, 40]. In fact, it is shown in [1, 41, 42] that various practically important nonlinear systems can be transformed to the form in (2), which is a special case of (1).

Assumption 2. Parameter vector $\theta \in \mathcal{S}_\theta$ that is a known bounded convex subset in $\mathbb{R}^\eta$, and $J(x,\theta) = J_0(x) + [\, J_1(x)\theta \ \cdots \ J_m(x)\theta \,]$. There holds $\det[J(x,\hat{\theta})] \neq 0$ for all $x \in \mathbb{R}^n$ and $\hat{\theta} \in \mathcal{S}_\theta$.

Remark 1. In practice, θ consists of physical parameters of the system and their ranges are often known based on the physics of the system. If parameter vector θ is known, then under Assumption 1, feedback linearization can be achieved via state feedback control, and many effective linear system design methods can be employed to design the feedback control system. However, often system parameters are unknown. Even if they are known, tear-wear and aging can cause slow changes. As a result, system parameters are not only unknown, but also uncertain. Note that in [34, 35], $J(x,\theta) = J(x)$, and thus control signal u does not involve uncertain parameters. In the later part of this subsection, we will consider more general $J(x,\theta)$ that can be a nonlinear function of θ.

Let $\hat{\theta} \in \mathcal{S}_\theta$ be adaptively estimated parameter vector based on state vector measurements in realtime. We propose the following adaptive control law:

$$u = J(x,\hat{\theta})^{-1}\left[Fx - g(x) - H(x)\hat{\theta}\right], \quad \hat{\theta} \in \mathcal{S}_\theta. \tag{3}$$

Assumption 3. Let $\Delta_J = I - J(x,\theta)J(x,\hat{\theta})^{-1}$ with $\hat{\theta} \in \mathcal{S}_\theta$. There holds $\overline{\sigma}(\Delta_J) \leqslant \delta$ over all $x \in \mathbb{R}^n$ and $\hat{\theta} \in \mathcal{S}_\theta$, with $\delta \in (0,\ 1)$ known. Recall that $\overline{\sigma}(\cdot)$ denotes the maximum singular value.

Note that $\Delta_J = \Delta_J(x,\theta,\hat{\theta})$ is a matrix function of x, θ, and $\hat{\theta}$, but its arguments are omitted to avoid cumbersome notation. By the expression of Δ_J, there holds $J(x,\theta) = (I - \Delta_J)J(x,\hat{\theta})$, and thus Δ_J represents a multiplicative or relative type of errors. It is conventional to assume such type of errors to be strictly bounded by 1. Assumptions 1–3 are assumed to hold true throughout this section, except when we consider different $J(x,\theta)$ later. The following lemma will be useful.

Lemma 1. Let $v, w \in \mathbb{R}^n$. Under Assumption 3,

$$f(v,w) := \delta^2\|v\|^2 \pm 2v'\Delta_J w \geqslant -\|w\|^2. \tag{4}$$

Proof. Let v_i and w_i be the ith entry of $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$, respectively. There holds

$$f(v,w) = \left(\sum_{i=1}^{n} \delta^2 v_i^2\right) \pm 2v'\Delta_J w.$$

Under Assumption 1, we arrive at

$$f(v,w) \geqslant \sum_{i=1}^{n} \left(\delta^2 v_i^2 - 2\delta|v_i w_i| + w_i^2 - w_i^2\right) = \sum_{i=1}^{n} \left[(\delta|v_i| - |w_i|)^2 - w_i^2\right] \geqslant -\|w\|^2.$$

Hence, inequality (4) holds true, thereby concluding the proof.

Let the linear state feedback gain F be given by $F = -R^{-1}B'X$ with $X > 0$ being the stabilizing solution to the following algebraic Riccati equation (ARE):

$$A'X + XA - XB(R^{-1} - \Pi)B'X + Q = 0, \tag{5}$$

where matrices $R > 0$, $\Pi > 0$, and $Q > 0$ can all be designed, subject to $R^{-1} - \Pi > 0$. The above ARE can be rewritten into the following form of the Lyapunov equation:

$$A_F'X + XA_F + XB(R^{-1} + \Pi)B'X + Q = 0, \quad A_F = A + BF. \tag{6}$$

Let $\mathcal{S}_{\rm RoA}$ be the set of all initial condition $x_0 \in \mathbb{R}^n$ such that the solution trajectory $x(t) \to 0$ as $t \to \infty$. Such a set is termed as region of attraction (RoA).

Theorem 1. Let $\Gamma \in \mathbb{R}^{\eta\times\eta}$ be positive definite. Consider adaptive estimator $\dot{\hat{\theta}} = \Gamma H(x)'B'Xx$ where $\hat{\theta}(0) = \hat{\theta}_0 \in \mathcal{S}_\theta$ and $X > 0$ is the stabilizing solution to ARE (5). Suppose that $\hat{\theta} \in \mathcal{S}_\theta$. Then, $\mathcal{S}_{\rm RoA}$ is nonempty and includes the origin of $\mathbb{R}^n$, under adaptive control law (3) with $F = -R^{-1}B'X$.

Proof. The closed-loop system under adaptive control law (3) is described by

$$\dot{x} = A_F x + B\left[H(x)\tilde{\theta} - \Delta_J f(x,\hat{\theta})\right], \quad f(x,\hat{\theta}) = Fx - g(x) - H(x)\hat{\theta}, \tag{7}$$

where $\tilde{\theta} = \theta - \hat{\theta} \in \mathbb{R}^\eta$, representing the parameter estimation error. Define Lyapunov function candidate

$$V = \frac{1}{2}\left(x'Xx + \tilde{\theta}'\Gamma^{-1}\tilde{\theta}'\right). \tag{8}$$

Taking derivative along the state trajectory of the closed system described in (7) yields

$$\begin{aligned}
\dot{V} &= x'X\dot{x} - \tilde{\theta}'\Gamma^{-1}\dot{\hat{\theta}} = x'X\left\{A_F x + B\left[H(x)\tilde{\theta} - \Delta_J f(x,\hat{\theta})\right]\right\} - \tilde{\theta}'H(x)'B'Xx \\
&= \frac{1}{2}x'(A_F'X + XA_F)x - x'XB\Delta_J f(x,\hat{\theta}) \\
&= -\frac{1}{2}x'[XB(R^{-1} + \Pi)B'X + Q]x - x'XB\Delta_J f(x,\hat{\theta}).
\end{aligned} \tag{9}$$

The last equality in the above makes use of (6). Upon substitution of $f(x,\hat{\theta}) = Fx - g(x) - H(x)\hat{\theta}$ and $F = -R^{-1}B'X$ into the expression of $\dot{V}$ with rearrangement yields

$$\begin{aligned}
\dot{V} &= -\frac{1}{2}x'XB(R^{-1} + \Pi)B'Xx + x'XB\Delta_J R^{-1}B'Xx + \frac{1}{2}\delta^2 x'XBB'Xx + r(x,\hat{\theta}) \\
&\leqslant -\frac{1}{2}x'XB\left(R^{-1} + \Pi - 2\delta R^{-1} - \delta^2 I\right)B'Xx + r(x,\hat{\theta}),
\end{aligned}$$

by Assumption 3 where $r(x,\hat{\theta}) = -\frac{1}{2}x'Qx - \frac{1}{2}\delta^2 x'XBB'Xx + x'XB\Delta_J[g(x) + H(x)\hat{\theta}]$. Furthermore, we can always choose matrices $R > 0$, $\Pi > 0$, and a scaler $\nu > 0$ to satisfy

$$\text{(i) } R^{-1} > (\nu + \delta^2)/[2(1-\delta)]I, \quad \text{(ii) } 0 < R^{-1} - \nu I \leqslant \Pi < R^{-1}. \tag{10}$$

Indeed, by $\delta \in (0,\ 1)$, we can set $R = [2(1-\delta)/(\nu+\delta^2) - \epsilon]I > 0$ for some sufficiently small $\epsilon > 0$ and $\nu > 0$, which ensure both (i) and (ii). Inequality (ii) ensures that ARE (5) admits the unique stabilizing solution $X > 0$. Moreover, the left inequality in (ii) implies that

$$R^{-1} + \Pi - 2\delta R^{-1} - \delta^2 I \geqslant 2(1-\delta)R^{-1} - (\nu+\delta^2)I > 0.$$

The last strict inequality follows from (i). We can thus conclude that $\dot{V} < r(x, \hat{\theta})$. That is,

$$\dot{V} < -\frac{1}{2}x'Qx - \frac{1}{2}\delta^2 x'XBB'Xx + x'XB\Delta_J\left[g(x) + H(x)\hat{\theta}\right].$$

Let $v = B'Xx$ and $w = g(x) + H(x)\hat{\theta}$. Applying Lemma 1 leads to

$$\begin{aligned}\dot{V} &< -\frac{1}{2}\left[x'Qx + \left(\delta^2\|v\|^2 - 2v'\Delta_J w\right)\right] \leqslant -\frac{1}{2}\left(x'Qx - \|w\|^2\right)\\ &= -\frac{1}{2}\left[x'Qx - \|g(x) + H(x)\hat{\theta}\|^2\right].\end{aligned} \tag{11}$$

For ease of analysis, let

$$\mu_Q(x) := x'Qx, \quad \nu_{\mathcal{N}}(x, \hat{\theta}) := \|g(x) + H(x, \hat{\theta})\|^2.$$

Since $Q > 0$, there holds $\underline{\sigma}(Q)\|x\|^2 \leqslant \mu_Q(x) \leqslant \overline{\sigma}(Q)\|x\|^2$. Define a subset of $\mathbb{R}^n$ over which $\dot{V} < 0$ via

$$\mathcal{S}_- := \left\{x \in \mathbb{R}^n :\ \mu_Q(x) > \nu_{\mathcal{N}}(x, \hat{\theta})\right\}. \tag{12}$$

In light of the local Lipschitz condition in Assumption 1, set $\mathcal{S}_-$ is nonempty. Next, define sets

$$\mathcal{S}_r := \{x \in \mathbb{R}^n :\ x'Xx \leqslant r\}\setminus\{0\}, \quad \mathcal{S}_{r_{\max}} := \{x \in \mathbb{R}^n :\ x'Xx < r_{\max}\}\setminus\{0\} \tag{13}$$

with $r_{\max}$ defined by

$$r_{\max} := \sup\{r :\ \mathcal{S}_r \subseteq \mathcal{S}_-\} > 0. \tag{14}$$

We can now conclude that, if initial condition $x_0 \in \mathcal{S}_{r_{\max}}$, then $x_0 \in \mathcal{S}_-$ and thus the state vector x remains in $\mathcal{S}_{r_{\max}} \subseteq \mathcal{S}_-$ $\forall t > 0$, concluding the asymptotic stability of the close-loop system under adaptive control law (3), thereby completing the proof of the theorem.

Remark 2. The hypothesis on $\hat{\theta} \in \mathcal{S}_\theta$ in Theorem 1 can be removed. In choosing initial condition $\hat{\theta}_0 \in \mathcal{S}_\theta$ for the adaptive estimator, we assume certain prior knowledge on the unknown θ, highlighted as in Assumptions 2 and 3. Hence, $d_0^2 := \tilde{\theta}'\Gamma^{-1}\tilde{\theta}$ can be assessed. Define set $\mathcal{S}_{\hat{\theta}_0}(d_0) := \{\theta :\ \tilde{\theta}'\Gamma^{-1}\tilde{\theta} = d_0^2\}$. By assuming $\mathcal{S}_{\hat{\theta}_0}(d_0) \subseteq \mathcal{S}_\theta$, the adaptively estimated $\hat{\theta} \in \mathcal{S}_\theta$ for $t \geqslant 0$, although it requires careful design of $\Gamma > 0$ and large enough bounded convex set of $\mathcal{S}_\theta$, which can be made true in practice.

We comment that while $\mathcal{S}_{r_{\max}} \subseteq \mathcal{S}_{\rm RoA}$, $\mathcal{S}_{r_{\max}} \neq \mathcal{S}_{\rm RoA}$, in general. There are incentives to enlarge the region of attractions through selection of $Q > 0$. Recall $\mathcal{S}_{r_{\max}} \subseteq \mathcal{S}_-$ by (14). The following result holds.

Corollary 1. Under the same hypotheses as those of Theorem 1 and $\mathcal{S}_{\hat{\theta}_0}(d_0) \subset \mathcal{S}_\theta$, the adaptive control law in (3) can be designed so that the region of attraction covers any given bounded set in $\mathbb{R}^n$.

Proof. By Remark 2, the hypothesis on $\mathcal{S}_{\hat{\theta}_0}(d_0) \subset \mathcal{S}_\theta$ implies that $\hat{\theta} \in \mathcal{S}_\theta$ for $t \geqslant 0$. Next, let $R_\Pi = (R^{-1} - \Pi)^{-1}$ and set $Q = \rho^2 Q_0$ for some $Q_0 > 0$. ARE (5) can be written as

$$A'X + XA - XBR_\Pi^{-1}B'X + \rho^2 Q_0 = 0. \tag{15}$$

Note that $R_\Pi > 0$ is fixed once matrices $R > 0$, $\Pi > 0$, and scaler $\nu > 0$ are chosen to satisfy inequalities in (10) as in the proof of Theorem 1. Therefore, for a given bounded set $\mathcal{S}_{\rm B} \subset \mathbb{R}^n$, setting $\mathcal{S}_-$ as defined in (12) can be made to include $\mathcal{S}_{\rm B}$ by taking $\rho > 0$ sufficiently large. In addition, setting $\mathcal{S}_{r_{\max}}$ as defined in (13) can also be made to include $\mathcal{S}_{\rm B}$ by taking sufficiently large $\rho > 0$. As a result, both sets of $\mathcal{S}_-$ and $\mathcal{S}_{r_{\max}}$ can be made as large as possible to cover the given bounded set $\mathcal{S}_{\rm B} \subset \mathbb{R}^n$. Since set $\mathcal{S}_{\rm B}$ can be arbitrary, and $\mathcal{S}_{r_{\max}} \subseteq \mathcal{S}_{\rm RoA}$, the corollary holds, which completes the proof.

Remark 3. Large value of $\rho > 0$ in the proof of Corollary 1 inevitably leads to high gain control, and thus high control costs. Hence, there is a tradeoff between the size of the region of attraction and the associated control cost. Since in practice, the required stability region is often bounded, the control gain can be constrained from being too high. We also comment that ARE in (15) signifies the LQR nature for the linearized feedback system, and thus it helps the control system design in achieving satisfactory time domain performance.

In practice, often the uncertain parameter vector does not show up linearly. The vehicle model in [21,37] provides such an example, motivating the following family of uncertain nonlinear systems:

$$\dot{x} = Ax + BJ(x,\theta)\left[u + g(x) + H(x)\theta\right], \quad x(0) = x_0. \tag{16}$$

The above is very similar to (1), but the unknown parameter vector θ does not have linear form on the right of the differential equation in (16). In fact, we do not even assume that $J(x,\theta)$ is a linear function of θ as assumed in Assumption 2. Nevertheless, the adaptive feedback stabilization in the Lyapunov sense again holds. To be specific, consider the following adaptive feedback control law:

$$u = J(x,\hat{\theta})^{-1}Fx - g(x) - H(x)\hat{\theta}, \tag{17}$$

with $\hat{\theta} = \hat{\theta}(t)$ being adaptively estimated in realtime. Then, the close-loop system can be described by

$$\dot{x} = A_F x + BJ(x,\hat{\theta})H(x)\tilde{\theta} - B\Delta_J\left[Fx + J(x,\hat{\theta})H(x)\tilde{\theta}\right]. \tag{18}$$

Recall that $\tilde{\theta} = \theta - \hat{\theta}$, $\Delta_J = I - J(x,\theta)J(x,\hat{\theta})^{-1}$, $A_F = A + BF$, and Assumptions 1–3 hold throughout the section, although Assumption 2 can be relaxed for $J(x,\theta)$ to include nonlinear terms of θ. The following asymptotic feedback stability result holds.

Theorem 2. Let $\Gamma \in \mathbb{R}^{\eta\times\eta}$ be positive definite. Consider adaptive estimator $\dot{\hat{\theta}} = \Gamma H(x)'J(x,\hat{\theta})'B'Xx$ where $\hat{\theta}(0) = \hat{\theta}_0 \in \mathcal{S}_\theta$ and $X > 0$ is the stabilizing solution to ARE (5). Suppose that $\hat{\theta} \in \mathcal{S}_\theta$ and $F = -R^{-1}B'X$. Then, the adaptive control law in (17) can be designed so that the region of attraction covers any given bounded set in $\mathbb{R}^n$.

Proof. The closed-loop system under adaptive control law (18) can be written in the following form:

$$\dot{x} = A_F x + B\left[J(x,\hat{\theta})H(x)\tilde{\theta} - \Delta_J\tilde{f}(x,\hat{\theta},\theta)\right], \quad \tilde{f}(x,\hat{\theta},\tilde{\theta}) = Fx + J(x,\hat{\theta})H(x)\tilde{\theta}.$$

The above is very similar to (7), except that $H(x)$ and $f(x,\hat{\theta})$ in (7) are replaced by $J(x,\hat{\theta})H(x)$ and $\tilde{f}(x,\hat{\theta},\tilde{\theta})$ in the above equation, respectively. Hence, setting the Lyapunov function candidate V having the same expression as in (8) leads to

$$\dot{V} = \frac{1}{2}x'\left(A_F'X + XA_F\right)x - x'B\Delta_J\tilde{f}(x,\hat{\theta},\tilde{\theta}),$$

which is the same as (9), if $\hat{f}(x,\hat{\theta},\tilde{\theta})$ is replaced by $f(x,\hat{\theta})$. Consequently, following the same steps as in the proof of Theorem 1 from (9) to (11) arrives at

$$\dot{V} < -\frac{1}{2}\left[x'Qx - \|J(x,\hat{\theta})H(x)\tilde{\theta}\|^2\right].$$

The rest of the proof for Theorem 1 and proof for Corollary 1 can help to prove that the adaptive control law in (17) can be designed such that the region of attraction covers any given bounded set in $\mathbb{R}^n$.

We comment that the asymptotic stability of the closed-loop system in Theorem 2 does not require that $J(x,\theta)$ be a linear function of θ. Subsection 2.2 will be focused on asymptotic linearization, which covers asymptotic stabilization in a broader sense.

2.2 Asymptotic linearization

Even though asymptotic stabilization over the region of attraction can be ensured for the two families of uncertain nonlinear systems under adaptive control, there are no guarantees that $\hat{\theta}$ converges to the true parameter vector, because $\dot{V}$ does not involve parameter estimation error $\tilde{\theta}$. In other words, while parameter estimation error does not increase, measured by $\tilde{\theta}'\Gamma^{-1}\tilde{\theta}$, it does not decrease to zero either, in general. In fact, there exist invariant sets for $\hat{\theta}$ and there can be multiple or even infinite number of limits for $\hat{\theta}$. For this reason, the closed-loop system is not asymptotically linearized, losing the time domain performance, designed using the LQR method or other linear methods. Consequently, change of the operating point and presence of disturbances may lead to poor transient responses. To mitigate this issue, we propose to employ a secondary estimator for θ, as presented next.

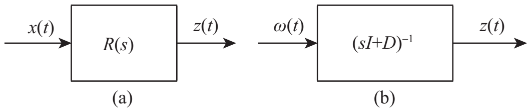

Figure 1 Reconstruction filter and its input-output (a) from $x(t)$ to $z(t)$ and (b) from $\alpha(t)$ to $z(t)$.

We will be focusing on the family of nonlinear systems described in (1). Under adaptive control law (3), the closed-loop system as described in (7) can be rewritten as

$$\dot{x} = A_F x + B\omega, \quad \omega = H(x)\tilde{\theta} - \Delta_J \left[Fx - g(x) - H(x)\hat{\theta}\right]. \tag{19}$$

Note that, if $\hat{\theta} = \theta$, then $\omega = 0$, and thus ω is referred to as the error residual signal. By Assumption 2, $J(x,\theta) = J_0(x) + [\, J_1(x)\theta \ \cdots \ J_m(x)\theta \,]$. Let

$$\ell(x,\hat{\theta}) = J(x,\hat{\theta})^{-1}\left[Fx - g(x) - H(x)\hat{\theta}\right] \implies \omega = \varphi(x,\hat{\theta}) + \Psi(x,\hat{\theta})\theta, \tag{20}$$

$$\varphi(x,\hat{\theta}) = g(x) - Fx + J_0(x)\ell(x,\hat{\theta}), \quad \Psi(x,\hat{\theta}) = H(x) + \sum_{i=1}^{m} J_i(x)\ell_i(x,\hat{\theta}),$$

with $\ell_i(x,\hat{\theta})$, the ith entry of $\ell(x,\hat{\theta})$.

Let $D = D' > 0$. The secondary parameter estimator is based on the following reconstruction filter:

$$R(s) = (sI + D)^{-1}[(sI + D)T_F(s)]^{-1}(sI + D)F, \quad T_F(s) = F(sI - A_F)^{-1}B. \tag{21}$$

The reconstruction filter $R(s)$ has input $x = x(t)$ and output $z = z(t)$, as illustrated in Figure 1. Clearly, $R(s)$ is internally stable by $D > 0$ and Hurwitz stability of $A_F = A + BF$. Let (A_g, B_g, C_g, D_g) be realization of $G(s)$. We will adopt the convention that

$$\left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array}\right] := G(s) = D_g + C_g(sI - A_g)^{-1}B_g.$$

Moreover, the following identities will be used to derive realization of the receiver $R(s)$:

$$\text{(i)}\ sG(s) = \left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & 0 \end{array}\right] = \left[\begin{array}{c|c} A_g & B_g \\ \hline C_gA_g & C_gB_g \end{array}\right], \quad \text{(ii)}\ \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ \hline D_2C_1 & C_2 & D_2D_1 \end{array}\right].$$

If D_g is square and nonsingular, there holds

$$\text{(iii)}\ \left[\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array}\right]^{-1} = G(s)^{-1} = \left[\begin{array}{c|c} A_g - B_gD_g^{-1}C_g & B_gD_g^{-1} \\ \hline -D_g^{-1}C_g & D_g^{-1} \end{array}\right] = \left[\begin{array}{c|c} A_g - B_gD_g^{-1}C_g & -B_gD_g^{-1} \\ \hline D_g^{-1}C_g & D_g^{-1} \end{array}\right].$$

Let $C_F = DF + FA_F$. By the expression of $T_F(s)$ in (21), an application of (i) yields

$$(sI + D)T_F(s) = FB + (DF + FA_F)(sI - A_F)^{-1}B =: \left[\begin{array}{c|c} A_F & B \\ \hline C_F & FB \end{array}\right].$$

Since $F = -R^{-1}B'X$ and $X > 0$, $\det(FB) \neq 0$. Let $A_c = A_F - B(FB)^{-1}C_F$. By (iii), there holds

$$[(sI + D)T_F(s)]^{-1} = \left[\begin{array}{c|c} A_c & B(FB)^{-1} \\ \hline -(FB)^{-1}C_F & (FB)^{-1} \end{array}\right].$$

Let (A_R, B_R, C_R, D_R) be a realization of $R(s)$, i.e., $R(s) = D_R + C_R(sI - A_R)^{-1}B_R$. Let $B_c = [(FB)^{-1}D - D(FB)^{-1}]F$. Using (i) and (ii) gives the following expressions:

$$A_R = \begin{bmatrix} A_c & 0 \\ -(FB)^{-1}C_F & -D \end{bmatrix}, \quad B_R = \begin{bmatrix} B(FB)^{-1}DF + A_cB(FB)^{-1}F \\ B_c - (FB)^{-1}C_FB(FB)^{-1}F \end{bmatrix}, \quad C_R = \begin{bmatrix} 0 & D \end{bmatrix}, \tag{22}$$

and $D_R = (FB)^{-1}F$. In the case of $m = 1$, $F = f \in \mathbb{R}^{1\times n}$, $B = b \in \mathbb{R}^n$, and $D = d > 0$ is a scaler. Realization of $R(s)$ is simplified to

$$A_R = A_f := A + bf, \quad B_R = b(fb)^{-1}f, \quad C_R = -(fb)^{-1}(df + fA_f), \quad D_R = (fb)^{-1}f. \tag{23}$$

By Figure 1(b), and expression of $\omega = \omega(t)$ in (20), there holds

$$\dot{z} + Dz - \varphi(x, \hat{\theta}) = \Psi(x, \hat{\theta})\theta. \tag{24}$$

Since $z = z(t)$ is known, which is the output of the reconstruction filter $R(s)$, integration on both sides of (24) from initial time t_i to t for $t > t_i \geqslant 0$ yields

$$y(t) := z(t) - z(t_i) + \int_{t_i}^{t} Dz(\tau) - \varphi(x, \hat{\theta})\mathrm{d}\tau = \Phi(t)\theta, \quad \Phi(t) := \int_{t_i}^{t} \Psi[x(\tau), \hat{\theta}(\tau)]\mathrm{d}\tau.$$

We propose the LS algorithm as the secondary estimator. Specifically, the LS algorithm over time interval $[t_0,\ t]$ with $t > 0$ is given by

$$\widehat{\theta}_{\mathrm{LS}}(t) = \left[\int_{t_0}^{t} \Phi(\tau)'\Phi(\tau)\mathrm{d}\tau\right]^{-1} \left[\int_{t_0}^{t} \Phi(\tau)'y(\tau)\mathrm{d}\tau\right], \quad t > t_0, \tag{25}$$

assuming that the inverse in the above expression exists. Note that the initial time $t_0 = 0$ can be taken as in (1). The following result is well-known.

Lemma 2. Suppose that random disturbances are present from time to time, and $\Phi(t)$ is PE, i.e., $\int_0^t \Phi(\tau)'\Phi(\tau)\mathrm{d}\tau > 0$ for $t > 0$. Then, $\widehat{\theta}_{\mathrm{LS}}(t) \to \theta$ as $t \to \infty$.

Remark 4. The PE condition is crucial for the LS estimate to converge to the true parameter vector. If the PE condition fails to hold, a commonly used method adds a small but persistent exciting signal to the control input to help restore the PE condition. It is important to point out that other estimation methods such as those proposed in [36] can also be used to warrantee the convergence of the parameter estimate under the PE condition. The use of the LS algorithm in this paper is owing to its wide popularity and its simplicity, compared with other existing estimation methods.

By incorporation of the LS estimate into the adaptive control algorithm in Theorem 1, we are led to the following asymptotic linearization result, in addition to the result of asymptotic stabilization.

Theorem 3. Partition time into consecutive intervals $[t_i,\ t_{i+1}]$ with $L_i = t_{i+1} - t_i > 0$ and $t_0 = 0$. Under the same hypotheses as those in Theorem 1, Corollary 1, and $\mathcal{S}_{\hat{\theta}_0}(d_0) \subset \mathcal{S}_\theta$, except that the adaptive estimator described by $\dot{\hat{\theta}} = \Gamma H(x)'B'Xx$ is re-initialized at each time sample $t = t_i$ by

$$\hat{\theta}(t_i) = \underset{\widehat{\theta} \in \mathcal{S}_\theta}{\operatorname{argmin}} \|\widehat{\theta}_{\mathrm{LS}}(t_i) - \widehat{\theta}\|, \quad i \geqslant 1.$$

Then, under the PE condition on $\Phi(t)$, the adaptive control law in (3) can be designed to not only asymptotically stabilize the closed-loop system for any given bounded initial condition, but also achieve the asymptotic linearization of the closed-loop system.

Proof. The asymptotic stabilization results are shown in Theorem 1 and Corollary 1. We will be focused on the proof of the asymptotic linearization. As discussed earlier, the estimation error of the adaptive parameter estimator in Theorem 1 admits the non-increasing property. Since the adaptive estimator re-initializes with the LS estimate or its closest approximate in $\mathcal{S}_\theta$ at each time sample $t = t_i$, the solution to $\dot{\hat{\theta}} = \Gamma H(x)'B'Xx$ remains in $\mathcal{S}_\theta$ over $[t_i,\ t_{i+1})$ for each $i \geqslant 1$, in light of the bounded convex assumption on $\mathcal{S}_\theta$ in Assumption 2, and Remark 2. The PE hypothesis on $\Phi(t)$ implies that the LS estimate converges to the true parameter vector, implying that $\tilde{\theta} \to 0$ and $\Delta_J \to 0$, as $t \to \infty$. It follows that the closed-loop system as described in (19) converges to $\dot{x} = A_F x = (A + BF)x$ asymptotically, by $\omega \to 0$, as $t \to \infty$. Hence, the asymptotic linearization holds for the closed-loop system.

For the nonlinear systems described in (16), the secondary LS estimator encounters the problem of nonlinear parameter vectors. Specifically, the closed-loop system (18) can be rewritten as

$$\dot{x} = A_F x + B\omega, \quad \omega = J(x, \theta)H(x)\theta - Fx + J(x, \theta)\tilde{\ell}(x, \hat{\theta}), \quad \tilde{\ell}(x, \hat{\theta}) = J(x, \hat{\theta})^{-1}Fx - H(x)\hat{\theta}. \tag{26}$$

Let $\tilde{\ell}_i(x,\hat{\theta})$ be the ith entry of $\tilde{\ell}(x,\hat{\theta})$ and $h_i(x)$ the ith row of $H(x)$. Signal ω is obtained as

$$\omega = \varphi(x,\hat{\theta}) + \Psi(x,\hat{\theta})\theta + \sum_{i=1}^{m} J_i(x)\theta h_i(x)\theta, \tag{27}$$

$$\varphi(x,\hat{\theta}) = J_0(x)\tilde{\ell}(x,\hat{\theta}) - Fx, \quad \Psi(x,\hat{\theta}) = J_0(x)H(x) + \sum_{i=1}^{m} J_i(x)\tilde{\ell}_i(x,\hat{\theta}). \tag{28}$$

The bilinear summation terms in (27) poses an issue for the secondary LS estimator. In the case when $J(x,\theta)$ is a nonlinear function of θ, the issue becomes more severe. One possible remedy replaces some θ by $\hat{\theta}$ for the LS estimation. This method is adopted in the simulation section. By taking this remedy and assuming the linear form for $J(x,\theta)$ as in Assumption 2, ω in (27) is modified to

$$\omega = \varphi(x,\hat{\theta}) + \left[\Psi(x,\hat{\theta}) + \sum_{i=1}^{m} J_i(x)h_i(x)\hat{\theta}\right]\theta. \tag{29}$$

The reconstruction filter and LS estimator developed earlier apply to the above ω. However, the convergence of the LS estimate to the true parameter vector remains unknown, even if the PE condition holds. The simulation section provides some insights to the convergence issue.

3 Asymptotic full actuation control

Recently, a series of studies by Duan [16–20, 34, 35] on the full actuation control has received great attention in the control community. While these results solve the linearized control problem for many different families of nonlinear systems, there lacks the full actuation control results for uncertain nonlinear systems in the sense of linearization. Since most of the nonlinear systems involve parameter uncertainties, the objectives of the full actuation control are difficult to achieve. This section will be focused on the study of the asymptotic full actuation control, employing the adaptive control algorithm similar to that in Section 2, aiming at achieving the objectives of full actuation control asymptotically.

The family of uncertain nonlinear systems under the study can be described by

$$\dot{x} = Ax + g(x) + H(x)\theta + B(x,\theta)u, \quad x(0) = x_0, \tag{30}$$

where $x, u \in \mathbb{R}^n$ are state and control input, respectively. Different from the series of studies by Duan, $B(x,\theta)$ of dimension $n \times n$ is also a function of θ. Note that $H(x)$ has dimension $n \times \eta$. The family of nonlinear systems described in (30) includes high order nonlinear systems. The following assumptions are made.

Assumption 4. (1) The origin of the state space is an equilibrium point of the autonomous system, i.e., $g(0) = 0 \in \mathbb{R}^n$ and $H(0) = 0 \in \mathbb{R}^{n\times\eta}$. (2) $\mathrm{rank}\{B(x,\theta)\} = n\ \forall x \in \mathbb{R}^n$. (3) $\{g(x), H(x)\}$ are continuous functions of $x \in \mathbb{R}^n$ and satisfy the local Lipschitz condition $\|g(x)\| \leqslant \ell_g\|x\|$ and $\overline{\sigma}[H(x)] \leqslant \ell_H\|x\|$ in small neighborhoods of the origin of $\mathbb{R}^n$ for some constants $\ell_g > 0$ and $\ell_H > 0$.

Assumption 5. Parameter vector $\theta \in \mathcal{S}_\theta$ that is a known bounded convex subset in $\mathbb{R}^\eta$, and $B(x,\theta) = B_0(x) + [\, B_1(x)\theta \ \cdots \ B_n(x)\theta \,]$. There holds $\det[B(x,\hat{\theta})] \neq 0$ for all $x \in \mathbb{R}^n$ and $\hat{\theta} \in \mathcal{S}_\theta$.

Assumption 6. Let $\Delta_B = I - B(x,\theta)B(x,\hat{\theta})^{-1}$ with $\hat{\theta} \in \mathcal{S}_\theta$. There holds $\overline{\sigma}(\Delta_B) \leqslant \delta$ over all $x \in \mathbb{R}^n$ and $\hat{\theta} \in \mathcal{S}_\theta$, and $\delta \in (0,\ 1)$ is known.

Assumptions 4–6 are similar to Assumptions 1–3, respectively, and will be assumed throughout the rest of this section. Recall that $\mathcal{S}_{\mathrm{RoA}}$ is the set of all initial condition $x_0 \in \mathbb{R}^n$ such that the solution trajectory $x(t) \to 0$ as $t \to \infty$. The control law for the fully actuated model described in (30) is given by

$$u = B(x,\hat{\theta})^{-1}[Fx - g(x) - H(x)\hat{\theta}] \tag{31}$$

where $\hat{\theta}$ is adaptively estimated in realtime. The above adaptive control law is similar to (3) in Section 2. If $\hat{\theta} = \theta$, the true parameter vector, then $\dot{x} = A_F x$ with $A_F = A + F$ representing the desired linear system matrix, thereby achieving the control objective. Upon substitution of u in (31) into (30) yields

$$\dot{x} = A_F x + H(x)\tilde{\theta} - \Delta_B[Fx - g(x) - H(x)\hat{\theta}], \quad \Delta_B = I - B(x,\theta)B(x,\hat{\theta})^{-1}. \tag{32}$$

Theorem 4. Let $\Gamma \in \mathbb{R}^{\eta\times\eta}$ be positive definite. Consider adaptive estimator $\dot{\hat{\theta}} = \Gamma H(x)'Xx$ with $\hat{\theta}(0) = \hat{\theta}_0 \in \mathcal{S}_\theta$, where $X > 0$ is the stabilizing solution to ARE,

$$A'X + XA - X(R^{-1} - \Pi)X + Q = 0, \quad R^{-1} - \Pi > 0, \tag{33}$$

satisfying $R > 0$, $\Pi > 0$, and $Q > 0$, which can all be designed. Then, $\hat{\theta} \in \mathcal{S}_\theta$. In addition, the adaptive controller can be designed so that the region of attraction covers any given bounded set in $\mathbb{R}^n$.

Proof. We note that the nonlinear system described in (30) can be written as

$$\dot{x} = Ax + B_\mathrm{o}\left[J_\mathrm{o}(x,\theta)u + g(x) + H(x)\theta\right], \quad x(0) = x_0,$$

by taking $B_\mathrm{o} = I$ and $J_\mathrm{o}(x,\theta) = B(x,\theta)$. The above has the same form as in (1) except that B is replaced by B_o and $J(x,\theta)$ by $J_\mathrm{o}(x,\theta)$. Hence, the proof of Theorem 1 applies, leading to the conclusion that a nonempty region of attraction exists. In addition, the proof for Corollary 1 can be applied to conclude that the adaptive control law can be designed such that the region of attraction can cover any bounded region in the state-space. The theorem is thus true.

Even though asymptotic stabilization can be ensured for the closed-loop system under the adaptive control law (31), there are no guarantees that $\hat{\theta}$ converges to the true parameter vector, due to the same reason as that in Section 2. Therefore, we need to employ secondary realtime estimator for θ.

Under adaptive control law (31), the closed-loop system is described by

$$\dot{x} = A_F x + \omega, \quad \omega = H(x)\theta - \Delta_B[Fx - g(x) - H(x)\hat{\theta}] - H(x)\hat{\theta}. \tag{34}$$

Note that ω in the above can be written as $\omega = \varphi(x,\hat{\theta}) + \Psi(x,\hat{\theta})\theta$ with

$$\varphi(x,\hat{\theta}) = g(x) + B_0(x)\ell(x,\hat{\theta}) - Fx, \quad \Psi(x,\hat{\theta}) = H(x) + \sum_{i=1}^{n} B_i(x)\ell_i(x,\hat{\theta}),$$

where $\ell_i(x,\hat{\theta})$ is the ith entry of $\ell(x,\hat{\theta}) = B(x,\hat{\theta})^{-1}[Fx - g(x) - H(x)\hat{\theta}]$. It can be verified that the relations in Figure 1 hold, by setting the receiver filter as

$$R(s) = (sI + D)^{-1}[(sI + D)F(sI - A_F)^{-1}B]^{-1}(sI + D)F, \quad D = D' > 0.$$

A realization of $R(s)$ can be obtained following the same steps as in Section 2. In fact, setting $B = I$ in the expression of (A_R, B_R, C_R, D_R) in (22) of Section 2 gives the realization of $R(s)$ in this section. The relations between $\{x(t), \omega(t)\}$ and $z(t)$ lead to

$$\dot{z} + Dz - \varphi(x,\hat{\theta}) = \Psi(x,\hat{\theta})\theta. \tag{35}$$

The above is identical to (24), although the physical meanings of various signals change. In any case, the LS algorithm from Section 2 applies. In fact, any convergent algorithms other than the LS can be employed to estimate the unknown parameter vector, including those in [36]. Hence, we skip the development of the estimation algorithm for θ in the secondary stage, based on realtime measurements of x and adaptive estimate $\hat{\theta}$. Let $\widehat{\theta}_{\mathrm{LS}}(t)$ be the corresponding LS estimate at time t. The following result is parallel to Theorem 3 of which the proof is skipped.

Theorem 5. Partition time into consecutive intervals $[t_i,\ t_{i+1}]$ with $L_i = t_{i+1} - t_i > 0$ and $t_0 = 0$. Under the same hypotheses as those in Theorem 4, except that the adaptive estimator described by $\dot{\hat{\theta}} = \Gamma H(x)'B'Xx$ is re-initialized at each time sample $t = t_i$ by

$$\hat{\theta}(t_i) = \operatorname*{argmin}_{\theta\in\mathcal{S}_\theta} \|\widehat{\theta}_{\mathrm{LS}}(t_i) - \widehat{\theta}\|, \quad i \geqslant 1.$$

Then, under the PE condition on $\Phi(t)$, the adaptive control law in (31) can be designed to not only asymptotically stabilize the closed-loop system for any given bounded initial condition, but also achieve the asymptotic linearization of the closed-loop system.

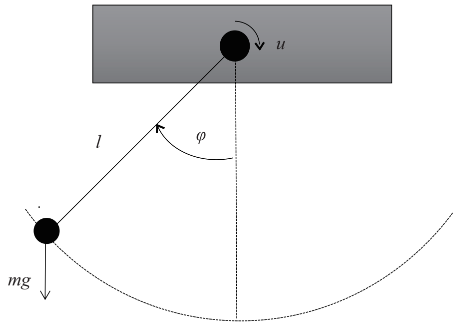

Figure 2 Schematic illustration of the pendulum system.

4 Simulation studies

This section is focused on the simulation studies. The first example comes from [23, 41].

Example 1. The pendulum system is shown in Figure 2. Its dynamics is described by

$$u(t) = ml\ddot{\varphi} + mg\sin(\varphi) + kl\dot{\varphi} \iff \ddot{\varphi} = \frac{1}{ml}u - \frac{g}{l}\sin(\varphi) - \frac{k}{m}\dot{\varphi}, \tag{36}$$

where u is the control input, and φ, m, l, g, and k represent the angle, mass, length, gravity, and friction coefficient, respectively. These parameters are specified as $m = 0.15$ kg, $l = 0.425$ m, $k = 0.165$, and $g = 9.8$ m/s^2. We take $\theta_1 = 1/l$ and $\theta_2 = k$ as uncertain parameters, assuming adjustable length, and thus both l and k are subject to tear-wear and aging.

By setting $x_1 = \varphi$ and $x_2 = \dot{\varphi}$ as the $n = 2$ state variables, the pendulum system in (36) can be converted into the same form as that in (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, \quad J(\theta) = \theta_1, \quad g(x) = 0, \quad H(x) = \begin{bmatrix} -mg\sin(x_1) & -x_2 \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}.$$

Clearly, Assumptions 1–3 are all satisfied. The control objective is for the output signal $\varphi(t)$ to track the reference signal $r(t) = \sin(\omega_0 t)$. The adaptive control law in Theorem 1 is applicable to asymptotic stabilization of the nonlinear system described in (36). Although it does not apply to solving the problem of asymptotic tracking, the tracking issue is solved for the pendulum control system in the presence of logarithmic quantization in [43].

Figure 3(a) shows the output response (in the dashed blue curve), compared with that of the backstepping control law (in the dot-dashed green curve) employed in [23] in the presence of the logarithmic quantification with quantization density equal to 0.1, taking $\omega_0 = 1$. The initial values are given by $l = 0.5$ m, 17.6% greater than its true value, and by $k = 0.15$, 9.1% smaller than its true value. Hence, $\theta_0 \in \mathcal{S}_\theta$. The simulation results on Figure 3(a) show that the proposed adaptive control law in this paper has considerably smaller transient tracking error, but a little larger settling time, compared with the backstepping method employed in [23]. Figure 3(b) shows the parameter estimation results for the adaptive estimator. As discussed in Section 2, the estimation errors of the adaptive estimator are not equal to zero, although the tracking error converges to zero asymptotically, entailing the second stage (LS) estimation.

The LS estimator in Subsection 2.2 is employed to estimate θ, and the results are shown in Figure 4 in solid lines, validating the convergence and achieving the asymptotic linearization. By Remark 4, other convergent non-LS algorithms can also be employed for the second stage estimation. We choose one of the three convergent algorithms from [36] (Theorem 2.1) for the comparison study. The estimation results using [36] are shown in dashed lines in Figure 4. It can be seen that both estimation methods can achieve convergence to the true parameter values, albeit much smaller convergence time for the LS estimator. It needs to be mentioned that the logarithmic quantization is removed in this comparative study in order to fulfill the PE condition in [36].

Example 2. Consider the velocity control for the vehicle model widely studied in [21, 22, 38, 39]. Assuming longitudinal motion, the vehicle dynamics are described by [21, 37]

$$\dot{p}(t) = v(t), \quad \dot{v}(t) = a(t), \quad \dot{a}(t) = b(v, a) + c(v)u(t), \tag{37}$$

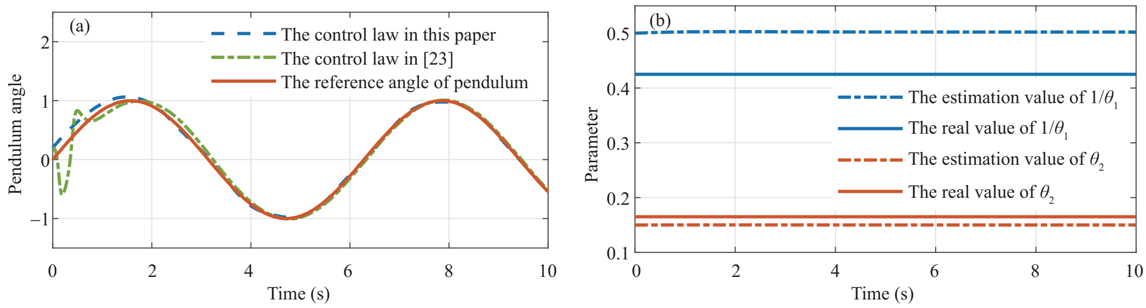

Figure 3 (Color online) Adaptive control and estimation results for Example 1, compared with [23]. (a) Angle response of the pendulum; (b) parameter estimation results for the adaptive estimator.

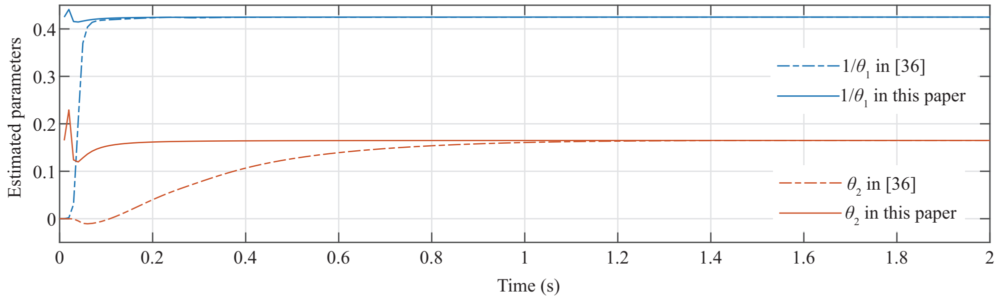

Figure 4 (Color online) The LS estimation results for Example 1, compared with the results using [36].

where $p(t)$ is the position, $v(t)$ is the velocity, $a(t)$ is the acceleration of the vehicle, and

$$b(v,a) = -\frac{1}{m\tau}\left[2k_d\tau v(t)a(t) + ma(t) + k_d v(t)^2 + d_m\right], \quad c(v) = \frac{1}{m\tau}.$$

We assume that the position, velocity, and acceleration of the vehicle are all measurable. Although position measurements can have large errors, it is not a serious issue. Because if the vehicle drives far away from other vehicles, the bias in position measurements does not matter; if the vehicle drives in a platoon, the local sensors such as camera, lidar, and microwave radar, can measure the relative distance to the neighboring vehicles quite accurately.

According to [21,37], the mechanical drag coefficient $d_m = 0$, if $v(t) = 0$ and d_m is equal to a constant, if $v(t) \neq 0$. Furthermore, the aerodynamic drag coefficient k_d remains a constant over time. Hence, we assume that both d_m and k_d are known, which can be experimentally determined. The only uncertain parameters are m and τ: change of m is due to the variation of vehicle load, and change of the engine time constant is due to tear-wear and aging with respect to time. However, we will assume that

$$|1 - \hat{m}\hat{\tau}/(m\tau)| \leqslant \delta < 1 \iff (1-\delta)m\tau \leqslant \hat{m}\hat{\tau} \leqslant (1+\delta)m\tau, \tag{38}$$

where $\hat{m}$ and $\hat{\tau}$ are estimates of m and τ, respectively. Considering that the load change of the vehicle is much smaller than the vehicle's mass, and τ changes slowly, the assumption in (38) is quite reasonable, implying that Assumption 3 holds.

For the longitudinal motion and velocity control, the reference position is specified as

$$r(t) = v_{\rm ref}(t - t_0) + p_0, \quad t \geqslant 0, \tag{39}$$

with p_0 and v_0 being the initial position and velocity at time t_0, and $v_{\rm ref}$ the reference velocity. The reference in (39) is common for roadway drive due to the speed limit. Without loss of generality, $p_0 = 0$ and $t_0 = 0$ are taken. As a result, the desired equilibrium point, i.e., the operating point of the nonlinear vehicle system described in (37) is given by

$$\begin{bmatrix} p_{\rm e}(t) \\ v_{\rm e} \\ a_{\rm e} \end{bmatrix} = \begin{bmatrix} r(t) \\ \dot{r}(t) \\ \ddot{r}(t) \end{bmatrix} = \begin{bmatrix} v_{\rm ref}t \\ v_{\rm ref} \\ 0 \end{bmatrix} \implies x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} := \begin{bmatrix} p(t) - v_{\rm ref}t \\ v(t) - v_{\rm ref} \\ a(t) \end{bmatrix}.$$

Table 1 Parameters used for the single vehicle

Vehicle mass	Aerodynamic drag coefficient	Engine time constant	Mechanical drag coefficient
$m = 2000$ kg	$k_d = 0.51$ kg/m	$\tau = 0.25$ s	$d_m = 4$ kg $\cdot$ m/s^2

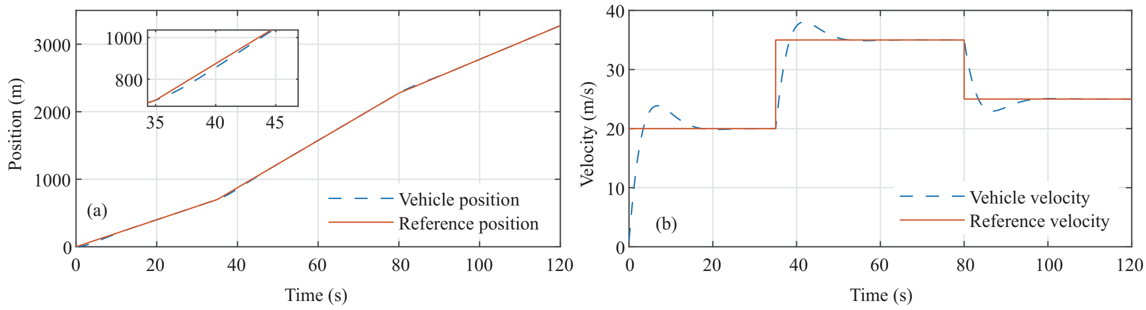

Figure 5 (Color online) The tracking results for a single vehicle in terms of (a) position and (b) velocity.

That is, we take the above $x(t)$ as the state vector. The equilibrium of the control input u is given by

$$u_{\rm e} = k_d v_{\rm e}^2 + d_m \quad \Longrightarrow \quad \dot{x}_3 = \frac{\delta u - [2k_d v_{\rm e} x_2 + (m + 2k_d v_{\rm e}\tau)x_3 + 2k_d \tau x_2 x_3 + k_d x_2^2]}{m\tau}, \tag{40}$$

where $\delta u = u - u_{\rm e}$ and argument of time t is skipped. Let $\theta_1 = \tau$ and $\theta_2 = m + 2k_d v_{\rm e}\tau$. Then, we obtain

$$\dot{x}(t) = Ax(t) + BJ(\theta)\left[u(t) - H\{x(t)\}\theta\right], \quad x(0) = x_0, \tag{41}$$

with various terms specified by $H(x) = x_3[\, 2k_d x_2 \quad 1\,]$,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2k_d v_{\rm e} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad J(\theta) = \frac{1}{\theta_1(\theta_2 - 2k_d v_{\rm e}\theta_1)}.$$

Note that if an estimate $\hat{\theta}$ is available, then $\hat{\tau} = \hat{\theta}_1$ and $\hat{m} = \hat{\theta}_2 - 2k_d v_{\rm e}\hat{\theta}_1$ are also known, which are required to satisfy (38), implying that Assumption 3 holds. It can be seen that Assumption 1 holds, but Assumption 2 fails, due to the nonlinear form of $J(\theta)$. Nevertheless, Theorem 2 applies, i.e., we can design an adaptive control law (17) to asymptotically stabilize the closed-loop system, even though J is a nonlinear function of θ. The nominal values of the vehicle parameters are given in Table 1 [21], of which the actual m and τ values have no more than 5% of variations in the simulation study. The position and velocity responses of the vehicle versus time (in seconds) are shown in Figure 5, using the adaptive control law in Theorem 2, assuming that the reference velocity changes from 20 to 35 m/s at $t = 35$ s, and then to 25 m/s at $t = 80$ s.

The asymptotic stabilization and thus the velocity tracking are clearly achieved with decent performance, but adaptive estimation for θ fails, which is skipped. The reason lies in the fact that $x_3(t) = a(t)$ is present at each entry of $H(x)$, which quickly settles to zero after each change of the reference velocity, providing very short time duration for the effective adaptive estimation, entailing the LS estimation. Figure 6 shows the simulation results for the LS estimation, making use of (29).

It needs to be mentioned that the convergence of the LS estimates to the true parameter vector does not hold in general for the general system described in (16). Even for the vehicle system in (37), the initial condition in the LS algorithm needs to be suitably chosen in order to achieve the convergence.

Example 3. Control of vehicle platoons. In this simulation study, there are five vehicles with the same dynamic model, driving in the platoon with mass and the corresponding aerodynamic drag coefficient of the five vehicles given by

$$m_1 = 2000 \text{ kg}, \quad m_2 = 1800 \text{ kg}, \quad m_3 = 2200 \text{ kg}, \quad m_4 = 1600 \text{ kg}, \quad m_5 = 2400 \text{ kg},$$

$$k_{d1} = 0.51 \text{ kg/m}, \quad k_{d2} = 0.46 \text{ kg/m}, \quad k_{d3} = 0.57 \text{ kg/m}, \quad k_{d4} = 0.43 \text{ kg/m}, \quad k_{d5} = 0.62 \text{ kg/m}.$$

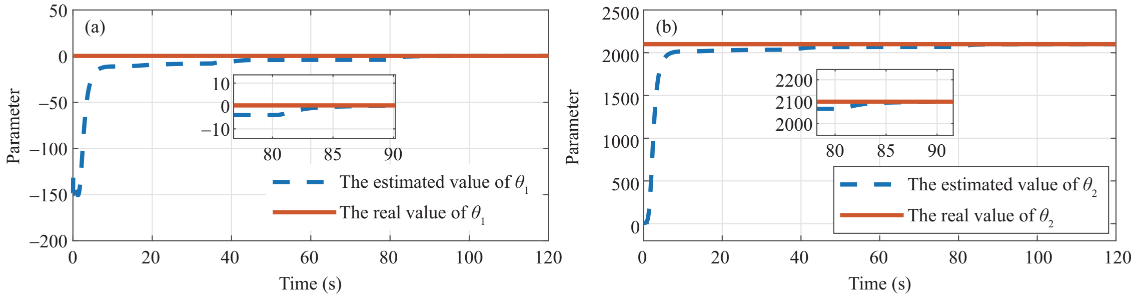

Figure 6 (Color online) The LS estimation results in Example 2. (a) θ_1; (b) θ_2.

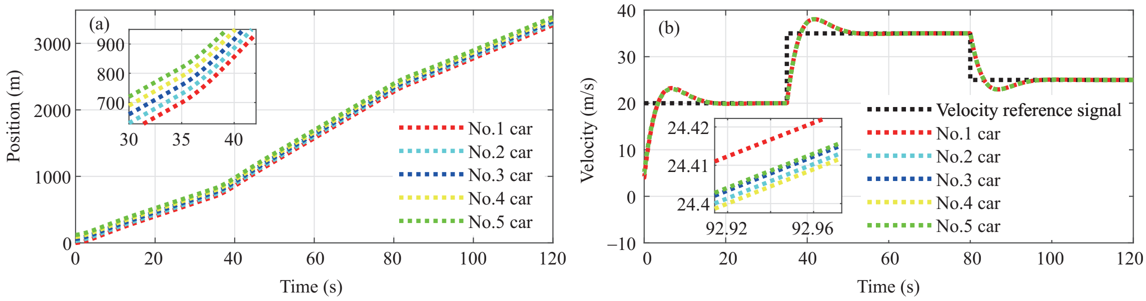

Figure 7 (Color online) The simulation results for multiple vehicles in Example 3. (a) The position responses; (b) the velocity responses.

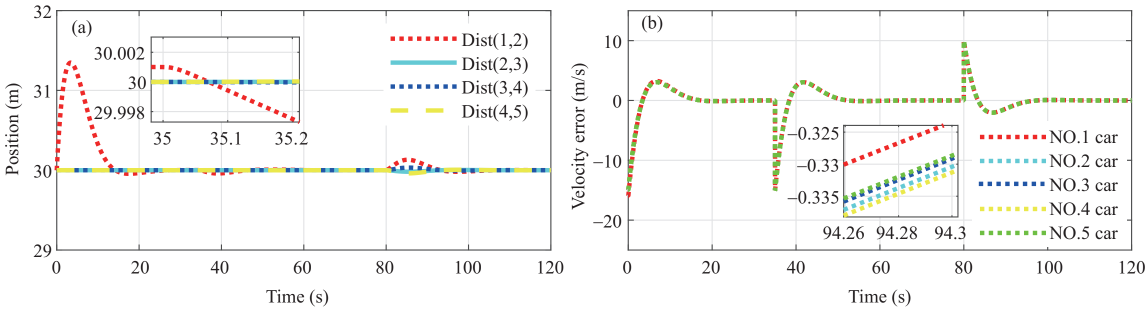

Figure 8 (Color online) The tracking results for multiple vehicles in Example 3. (a) The distance between the neighboring vehicles; (b) the velocity tracking errors for five vehicles.

Other parameters are the same as those in Table 1. Since the parameters can be estimated accurately by the LS algorithm, we assume that the nonlinear dynamics can be canceled completely as in [21, 22], and an $\mathcal{H}_\infty$ state feedback control law is employed for the linearized vehicle systems, in order to guarantee the string stability. Again, we assume that the reference velocity changes from 20 to 35 m/s at $t = 35$ s, then to 25 m/s at $t = 80$ s, and the initial distance between the neighboring vehicle is 30 m. The position and velocity responses of the five vehicles versus the time are shown in Figure 7. The distance between the neighboring vehicles and velocity tracking errors versus time are shown in Figure 8, illustrating no vehicle collisions between the neighboring vehicles. The optimistic results for neighboring distances in Figure 8(a) are owing to the fact that all vehicle systems are linearized, the closed-loop vehicle systems have similar time constants, and disturbances are absent in the simulation studies. The neighboring vehicle distances in practice can be very different and are worth future study.

5 Conclusion

In this paper, an adaptive control law is proposed for a family of nonlinear systems involving uncertain parameters, including control inputs. The proposed adaptive control law achieves the asymptotic stability for the closed-loop nonlinear system over any given bounded region without assuming the convergence of the estimates of the system parameters to their respective true values. A secondary estimator is developed to estimate the system parameters using the LS algorithm based on the error residual signal from the adaptive estimator. Under the persistent excitation condition, the LS estimates converge to the true system parameters, if the error residue signal is a linear function of the unknown parameter vector, thereby enabling the asymptotic linearization. Numerical examples are presented to illustrate the utility of the proposed adaptive control and estimation methods.

While the strengths and advantages of the proposed adaptive control method are stressed in several parts of the paper, the weaknesses and disadvantages are less discussed, which are briefly summarized as follows. First of all, the uncertain nonlinear system model assumes that the unknown parameter vector has the linear form in order for the adaptive estimator to be applicable. This can limit the use of the proposed adaptive control method, considering that parameter vectors may appear in the nonlinear form in practice, which is the true for the vehicle model in Example 2 of Section 4. In addition, the secondary LS estimation requires the PE condition. A conventional wisdom to ensure the PE condition is to inject a persistent exciting signal at the control input, which can have a detrimental effect on the control performance, unless it has a very small magnitude. Finally, the secondary LS estimation increases the complexity of the feedback controller. One may consider to use the recursive LS algorithm in the adaptive estimator without the secondary estimation, but then, the asymptotic stability of the closed-loop system cannot be ensured. The aforementioned issues are very important, representing significant challenges to control of uncertain nonlinear systems, which are worth further study in the future.

Acknowledgements This work was supported in part by National Natural Science Foundation of China (Grant No. 61873215) and Natural Science Foundation of Sichuan Province (Grant No. 2022NSFSC0470). The authors would like to thank the anonymous reviewers for their valuable and constructive comments.

References

1 Isidori A. Nonlinear Control Systems. Berlin: Springer, 1993
2 Enns D, Bugajski D, Hendrick R, et al. Dynamic inversion: an evolving methodology for flight control design. Int J Control, 1994, 59: 71–91
3 Chen W H. Nonlinear disturbance observer-enhanced dynamic inversion control of missiles. J Guidance Control Dyn, 2003, 26: 161–166
4 Lungu R, Lungu M, Efrim C. Adaptive control of DGMSCMG using dynamic inversion and neural networks. Adv Space Res, 2021, 68: 3478–3494
5 Lewis F, Jagannathan S, Yesildirak A. Neural Network Control of Robot Manipulators and Nonlinear Systems. Boca Raton: CRC Press, 1998
6 Ge S S, Hang C C, Lee T H. Stable Adaptive Neural Network Control. Berlin: Springer, 2013
7 Xie K, Chen C, Lewis F L, et al. Adaptive asymptotic neural network control of nonlinear systems with unknown actuator quantization. IEEE Trans Neural Netw Learn Syst, 2018, 29: 6303–6312
8 Esfandiari K, Abdollahi F, Talebi H A. Neural Network-based Adaptive Control of Uncertain Nonlinear Systems. Berlin: Springer, 2022
9 Ma H, Li H Y, Lu R Q, et al. Adaptive event-triggered control for a class of nonlinear systems with periodic disturbances. Sci China Inf Sci, 2020, 63: 150212
10 Hill D, Moylan P. The stability of nonlinear dissipative systems. IEEE Trans Automat Contr, 1976, 21: 708–711
11 Byrnes C I, Isidori A, Willems J C. Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. IEEE Trans Automat Contr, 1991, 36: 1228–1240
12 Kokotović P, Khalil H K, O'Reilly J. Singular Perturbation Methods in Control: Analysis and Design. Philadelphia: SIAM, 1986
13 Krstic M, Kanellakopoulos I, Kokotović P. Nonlinear and Adaptive Control Design. New York: John Wiley, 1993
14 Zouari F, Boubellouta A. Neural approximation-based adaptive control for pure-feedback fractional-order systems with output constraints and actuator nonlinearities. In: Advanced Synchronization Control and Bifurcation of Chaotic Fractional-Order Systems. Hershey: IGI Global, 2018
15 Zouari F, Boubellouta A. Adaptive neural control for unknown nonlinear time-delay fractional-order systems with input saturation. In: Advanced Synchronization Control and Bifurcation of Chaotic Fractional-order Systems. Hershey: IGI Global, 2018
16 Duan G R. High-order fully actuated system approaches: part I. Models and basic procedure. Int J Syst Sci, 2021, 52: 422–435
17 Duan G R. High-order fully actuated system approaches: part II. Generalized strict-feedback systems. Int J Syst Sci, 2021, 52: 437–454
18 Duan G R. High-order fully actuated system approaches: part III. Robust control and high-order backstepping. Int J Syst Sci, 2021, 52: 952–971
19 Duan G R. Discrete-time delay systems: part 1. Global fully actuated case. Sci China Inf Sci, 2022, 65: 182201
20 Duan G R. Discrete-time delay systems: part 2. Sub-fully actuated case. Sci China Inf Sci, 2022, 65: 192201
21 Ioannou P A, Chien C C. Autonomous intelligent cruise control. IEEE Trans Veh Technol, 1993, 42: 657–672
22 Naus G J L, Vugts R P A, Ploeg J, et al. String-stable CACC design and experimental validation: a frequency-domain approach. IEEE Trans Veh Technol, 2010, 59: 4268–4279
23 Zhou J, Wen C, Wang W, et al. Adaptive backstepping control of nonlinear uncertain systems with quantized states. IEEE Trans Automat Contr, 2019, 64: 4756–4763
24 Zhang Z, Wen C, Xing L, et al. Adaptive Output Feedback Control of Nonlinear Systems With Mismatched Uncertainties Under Input/Output Quantization. IEEE Trans Automat Contr, 2022, 67: 4801–4808
25 Hung C P. Integral variable structure control of nonlinear system using a CMAC neural network learning approach. IEEE Trans Syst Man Cybern B, 2004, 34: 702–709

26 Lee C Y, Lee J J. Adaptive control for uncertain nonlinear systems based on multiple neural networks. IEEE Trans Syst Man Cybern B, 2004, 34: 325–333

27 Merazka L, Zouari F, Boulkroune A. Fuzzy state-feedback control of uncertain nonlinear MIMO systems. In: Proceedings of the 6th International Conference on Systems and Control (ICSC), 2017. 103–108

28 Merazka L, Zouari F, Boulkroune A. High-gain observer-based adaptive fuzzy control for a class of multivariable nonlinear systems. In: Proceedings of the 6th International Conference on Systems and Control (ICSC), 2017. 96–102

29 Wang Y, Liu Y G. Adaptive output-feedback tracking for nonlinear systems with unknown control direction and generic inverse dynamics. Sci China Inf Sci, 2022, 65: 182204

30 Yu J P, Shi P, Chen X K, et al. Finite-time command filtered adaptive control for nonlinear systems via immersion and invariance. Sci China Inf Sci, 2021, 64: 192202

31 Hou Z, Xiong S. On model-free adaptive control and its stability analysis. IEEE Trans Automat Contr, 2019, 64: 4555–4569

32 Liang Y, Li Y X, Che W W, et al. Adaptive fuzzy asymptotic tracking for nonlinear systems with nonstrict-feedback structure. IEEE Trans Cybern, 2021, 51: 853–861

33 Zouari F, Saad K B, Benrejeb M. Adaptive backstepping control for a single-link flexible robot manipulator driven DC motor. In: Proceedings of International Conference on Control, Decision and Information Technologies (CoDIT), 2013. 864–871

34 Duan G R. High-order fully actuated system approaches: part IV. Adaptive control and high-order backstepping. Int J Syst Sci, 2021, 52: 972–989

35 Duan G R. High-order fully actuated system approaches: part V. Robust adaptive control. Int J Syst Sci, 2021, 52: 2129–2143

36 Na J, Mahyuddin M N, Herrmann G, et al. Robust adaptive finite-time parameter estimation and control for robotic systems. Int J Robust Nonlinear Control, 2015, 25: 3045–3071

37 Sheikholeslam S. Control of a class of interconnected nonlinear dynamical systems: the platoon problem. Dissertation for Ph.D. Degree. Berkeley: University of California, Berkeley, 1991

38 Zheng Y, Li S E, Li K, et al. Platooning of connected vehicles with undirected topologies: robustness analysis and distributed H_∞ controller synthesis. IEEE Trans Intell Transp Syst, 2018, 19: 1353–1364

39 Wang L Y, Syed A, Yin G G, et al. Control of vehicle platoons for highway safety and efficient utility: Consensus with communications and vehicle dynamics. J Syst Sci Complex, 2014, 27: 605–631

40 Zhou J, Zhang C, Wen C. Robust adaptive output control of uncertain nonlinear plants with unknown backlash nonlinearity. IEEE Trans Automat Contr, 2007, 52: 503–509

41 Khalil K. Nonlinear Systems. New York: Prentice-Hall, 2000

42 Marino R, Tomei P. Nonlinear Control Design: Geometric, Adaptive and Robust. New York: Prentice-Hall, 1995

43 Zhang M, Yan F, Zhu Y. Robust backstepping control for a class of nonlinear uncertain systems with input quantization. In: Proceedings of the 41st Chinese Control Conference (CCC), 2022. 2093–2098