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Sliding modes: from asymptoticity, to finite time and fixed time

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Abstract This paper proposes a new fixed-time sliding mode (FSM) control, where the settling time for reaching the system origin is bounded to a constant independent of the initial condition; this is in contrast to the initial condition-dependent constants used in the traditional linear sliding mode (LSM) and terminal sliding mode (TSM) controls. First, a new sliding mode control with a single power term is discussed, where the power term can have any nonnegative value. Except for the traditional LSM and TSM controls, a new sliding mode control called power sliding mode (PSM) is proposed, whose power term is larger than 1. Then, a new FSM control with two power terms is investigated, whose design is based on the combination of TSM and PSM. In particular, the two power terms on the plane in the first quadrant are carefully discussed, and a detailed classification is provided. Here, the first quadrant can be classified into six categories, including LSM, generalized LSM, TSM, fast TSM (FTSM), PSM, and FSM. Furthermore, the analytical settling time is calculated, and three different estimation bounds of the settling time are given for reaching the origin under any initial condition. It is also interesting to derive the lowest bound for the settling time. Finally, FSM control design for general nonlinear dynamical systems with the relative degree from the control input to the output is also discussed.

Keywords sliding mode control, fixed-time sliding mode, linear sliding mode, terminal sliding mode, power sliding mode, nonlinear systems

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1 Introduction

Sliding mode control has received significant attention recently [1–10] due to its robustness against parameter variations and uncertain disturbances. The key to sliding mode control lies in its two-step design process: sliding mode controller and surface. The controller is designed to drive the system to reach and then remain at the intersection of a set of prescribed switching manifolds. Then, the system states remaining on the prescribed surface can finally reach the origin. Asymptotic stable linear switching hyper-planes are commonly selected as the sliding mode surfaces.

However, in order to get higher performance, finite-time convergence is considered [5]. For reaching finite-time convergence, the terminal sliding mode (TSM) control is designed by [11], where nonlinear switching manifolds are designed for reaching the origin within a finite time [12]. Compared with the linear sliding mode (LSM) control based on linear switching hyperplanes, TSM cannot have the same convergence rate when the system state is far away from the origin. Hence, the fast TSM (FTSM) control is developed to combine the advantages of LSM and TSM controls [13]. The continuous-time sliding mode control is designed [14] to overcome the problem that some states in TSM may not be real. Modified super-twisting and integral sliding mode controls are discussed by [15, 16]. However, two additional challenges persist with TSM: singularity and chattering behavior. Nonsingular TSM control is proposed by [17, 18] to address the issue of singularity in TSM. The common chattering behaviors are addressed by [19–21], and practical implementation issues are addressed by [22–24].

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In the conventional sliding mode control design, the guaranteed convergence time depends on the initial state, which is unbounded and can be sufficiently large because the initial state is very large. Recently, a class of second-order fixed-time sliding mode (FSM) controller [25] was proposed where the convergence time was upper bounded by a finite value regardless of the initial conditions. It should be pointed out that the proposed FSM controller applies only to a class of specific second-order dynamical systems and that the upper bound estimate is conservative. The design of a new sliding mode surface called the FSM surface is presented in this paper using fixed-time stability in [26–30], where the convergence time is independent of the initial conditions and is uniformly bounded with respect to the initial conditions. Thus, irrespective of the initial state, one can design the FSM such that the controller can first drive the system into the FSM surface and remain on it, reaching the origin within a fixed time.

The main contributions of this paper can be summarized as follows. First, a unified framework for sliding mode control with two power terms is proposed. These two power terms on the plane in the first quadrant are discussed, and a detailed classification is presented, where the first quadrant can be classified into six categories, including three widely studied sliding mode controls (SMCs) (LSM, TSM, and FTSM), one generalized LSM, and two newly proposed SMCs in this paper, namely FSM control and power sliding mode (PSM) control. Second, the analytical settling time is calculated for FSM control, and three different estimation bounds of the settling time for reaching the origin are established for any initial condition. The lowest bound for the settling time is derived, which is essential for future research on fixed-time stability. Third, a new recursive design of FSM control is proposed in detail for a class of higher-order nonlinear systems, which is transferred from a general nonlinear dynamical system according to the relative degree.

The rest of this paper is organized as follows. Some preliminary descriptions of the sliding mode control and new concepts for fixed-time stability are discussed in Section 2. In Section 3, a new sliding mode control, PSM, with a single power term is discussed. In Section 4, a new sliding mode control with two power terms is investigated. In particular, the first quadrant on the plane can be classified into six categories, including LSM, generalized LSM, TSM, FTSM, PSM, and FSM. In addition, FSM control design for nonlinear dynamical systems is studied in Section 5. Section 6 provides a numerical example to verify the effectiveness of FSM control design proposed in this paper. Finally, Section 7 concludes this paper.

2 Preliminaries

In this section, some literature reviews for SMC and new concepts for fixed-time stability are first introduced, which are very important for presenting the main results of this paper.

Sliding mode control has been widely discussed in the literature [12], which can be designed through a two-step process. First, the sliding mode surface is designed on which the control objective is solved. Then, sliding mode control is applied to drive the states of the system into the designed sliding mode surface. Thus, the states of the system are first forced to drive on the sliding mode surface and then the control objective can finally be solved on this surface.

Suppose $z \in \mathbb{R}$ is the system state. For simplicity, the LSM surface is briefly introduced as follows:

$$s(t) = \dot{z} + \alpha z, \tag{1}$$

where $\alpha > 0$ is a constant.

In order to reach finite-time convergence, the TSM surface is designed by [11]

$$s(t) = \dot{z} + \beta z^{q/p}, \tag{2}$$

where $\beta > 0$ is a constant, and p and q are positive odd integers satisfying $p > q$. Note that for $z < 0$, the fractional power p/q may cause that $x^{q/p} \notin \mathbb{R}$ if p is even. Thus, p is simply assumed to be an odd number.

By combining the above LSM and TSM control advantages, the following FTSM surface is designed to reach a faster sliding mode control objective [13]:

$$s(t) = \dot{z} + \alpha z + \beta z^{q/p}, \tag{3}$$

where $\alpha > 0$ and $\beta > 0$ are constants, and p and q are positive odd integers satisfying $p > q$.

Let

$$y^{[\gamma]} = \mathrm{sign}(y)|y|^{\gamma}, \tag{4}$$

where $y \in \mathbb{R}$ and $\mathrm{sign}(y)$ is the sign function.

To avoid the above problem that z or $\dot{z}$ may not be real, the following modified TSM and FTSM surfaces are designed [14] by

$$s(t) = \dot{z} + \beta z^{[\eta]}, \tag{5}$$

and

$$s(t) = \dot{z} + \alpha z + \beta z^{[\eta]}, \tag{6}$$

respectively, where $z \in \mathbb{R}$, α, $\beta > 0$, and $0 < \eta < 1$.

In the modified TSM in (5) and FTSM in (6), the equilibrium point $z = 0$ is globally finite-time stable, which indicates that for any given initial condition z_0, the system state can reach the origin within finite time,

$$T(z_0) = \frac{|z_0|^{1-\eta}}{\beta(1-\eta)}, \tag{7}$$

and

$$T(z_0) = \frac{1}{\alpha(1-\eta)} \ln \frac{\alpha|z_0|^{1-\eta} + \beta}{\beta}, \tag{8}$$

respectively, and stay at $z = 0$ for all $t > T$.

For a general nonlinear system,

$$\dot{x}(t) = f(x(t)) + u(t), \quad x(0) = x_0, \tag{9}$$

where $x = (x_1, x_2, \ldots, x_N)^{\mathrm{T}} \in \mathbb{R}^n$ is the state, $f(\cdot)$ is a nonlinear function, u is the control input, and x_0 is the initial state. The following definitions are needed to present the main results.

Definition 1 ([31])**.** The origin of system (9) is said to be globally finite-time stable if it is globally asymptotically stable and any solution $x(t, x_0)$ of (9) reaches the equilibria at some finite time moment, i.e., $x(t, x_0) = 0, \forall t \geqslant T(x_0)$, where $T : \mathbb{R}^N \to \mathbb{R}_+ \cup \{0\}$ is called the settling-time function.

For simplicity, the modified TSM in (5) and FTSM in (6) are discussed instead of TSM in (2) and FTSM in (3), whose results can similarly be obtained. In order to solve the singularity problem, a nonsingular TSM surface is also proposed by [17].

Note that for the above TSM in (5) and FTSM in (6), the time for reaching the origin of the system depends on the initial conditions. For example, on the TSM surface in (5) ($s = \dot{z} + \beta z^{[\eta]} = 0$), the settling time for reaching the origin within finite time is $\frac{|z_0|^{1-\eta}}{\beta(1-\eta)}$ in (7), which depends on the initial condition z_0 and can be sufficiently large as $z_0 \to \infty$. In order to design a new fixed-time sliding mode control, the following definition of fixed-time stability is revisited.

Definition 2 ([26])**.** The origin of system (9) is said to be fixed-time stable if it is globally finite-time stable and the settling-time function $T(x_0)$ for reaching the origin is bounded, i.e., $\exists\, T_{\max} > 0 : T(x_0) \leqslant T_{\max}, \forall x_0 \in \mathbb{R}^N$.

In this paper, by using fixed-time stability in [26], the following new sliding mode surface called the FSM surface is designed:

$$s(t) = \dot{z} + \alpha z^{[\xi]} + \beta z^{[\eta]}, \tag{10}$$

where $\xi > 1 > \eta > 0$, $\alpha > 0$, and $\beta > 0$.

Then, on the FSM surface $s = \dot{z} + \alpha z^{[\xi]} + \beta z^{[\eta]} = 0$, one has $(|z|)' + \alpha|z|^{\xi} + \beta|z|^{\eta} = 0$. By invoking the following Lemma 1 in which the continuous positive definite function $V(z) = |z|$ is considered, one can obtain that the settling time $T(z_0)$ for reaching the origin within finite time satisfies

$$T(z_0) \leqslant \frac{1}{\alpha(\xi-1)} + \frac{1}{\beta(1-\eta)}, \tag{11}$$

which is independent of the initial condition z_0.

Lemma 1 ([26]). Consider the following system of differential equation:

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0,$$

where $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n, f(x) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $\mathbb{R}^n$, and $f(0) = 0$. Suppose there exists a continuous positive definite function $V(x) : \mathbb{R}^n \to \mathbb{R}$ such that for real numbers $a > 0$, $b > 0$, $p > 1$, and $q \in (0, 1)$,

$$\dot{V}(x) + a(V(x))^p + b(V(x))^q \leqslant 0, \quad x \in \mathbb{R}^N \backslash \{0\}. \tag{12}$$

Then, the origin is a globally fixed-time stable equilibrium and the settling time T satisfies

$$T(x_0) \leqslant \frac{1}{a(p-1)} + \frac{1}{b(1-q)}. \tag{13}$$

3 New sliding mode control with one power term

In this section, a unified framework for SMC with one power term as in (1) and (5) is analyzed, based on which the above new sliding mode called TSM in (10) is proposed.

First, a general sliding mode control surface is given as follows:

$$s(t) = \dot{z} + \beta z^{[\eta]}, \tag{14}$$

where η is a constant. The above sliding mode surface can include the LSM in (1) with $\eta = 1$ and TSM in (2) with $0 < \eta < 1$ as special cases. Next, some simple analysis for the general sliding mode control surface is proposed.

Theorem 1. On the general sliding mode surface $s = 0$ in (14), the state z can be stable. In addition, the following properties are satisfied:

(1) If $\eta = 1$, the state z is globally exponentially stable, and $|z(t)| = |z_0(t)| \exp^{-\beta t}$;

(2) If $0 \leqslant \eta < 1$, the state z is globally finite-time stable, and $|z(t)| = (|z_0|^{1-\eta} - (1-\eta)\beta t)^{\frac{1}{1-\eta}}$ for $t \leqslant T(z_0)$, where the settling time $T(z_0)$ for reaching origin is given in (7) where $T(z_0) = \frac{|z_0|^{1-\eta}}{\beta(1-\eta)}$;

(3) If $\eta > 1$, the state z is globally stable with power $-\frac{1}{\eta-1}$, and $|z(t)| = (\frac{1}{\frac{1}{|z_0|^{\eta-1}} + (\eta-1)\beta t})^{\frac{1}{\eta-1}}$.

Proof. Suppose that the initial time $t_0 = 0$. $|z(t)|$ at $z = 0$ is not differentiable. One can disregard this trivial condition since the goal for reaching the origin $z = 0$ has already been achieved. On the general sliding mode surface $s = 0$ in (14), one has $\dot{z} = -\beta z^{[\eta]}$, which yields that

$$\mathrm{d}(|z|)/\mathrm{d}t = -\beta |z|^\eta. \tag{15}$$

The analysis can be decomposed into several cases.

Case 1: $\eta = 1$. By integrating both sides of (15), one has

$$|z(t)| = |z_0(t)| \exp^{-\beta t}. \tag{16}$$

Case 2: $0 \leqslant \eta < 1$. Similarly, one obtains

$$\mathrm{d}(|z(t)|^{1-\eta}) = -(1-\eta)\beta \mathrm{d}t.$$

Then, one finally has

$$|z(t)|^{1-\eta} - |z_0|^{1-\eta} = -(1-\eta)\beta t, \tag{17}$$

which yields that

$$|z(t)| = (|z_0|^{1-\eta} - (1-\eta)\beta t)^{\frac{1}{1-\eta}}. \tag{18}$$

After time $T(z_0) = \frac{|z_0|^{1-\eta}}{\beta(1-\eta)}$, $z(t)$ reaches the origin and will stay at the origin for all $t > T(z_0)$.

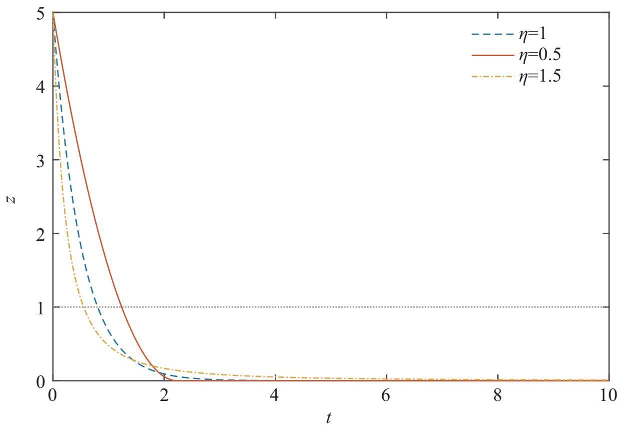

Figure 1 (Color online) State $z(t)$ on the sliding mode surface where $\eta = 1, 0.5$, and 1.5.

Case 3: $\eta > 1$. According to (17), one simply has

$$\frac{1}{|z(t)|^{\eta-1}} = \frac{1}{|z_0|^{\eta-1}} + (\eta - 1)\beta t. \tag{19}$$

Then, one finally has

$$|z(t)| = \left(\frac{1}{\frac{1}{|z_0|^{\eta-1}} + (\eta - 1)\beta t} \right)^{\frac{1}{\eta-1}}. \tag{20}$$

Note that as $t \to \infty$, $|z(t)| \sim (\frac{1}{\beta(\eta-1)})^{\frac{1}{\eta-1}} t^{-\frac{1}{\eta-1}} \to 0$.

Next, an illustrative example is given to show the convergence of the above results where η is chosen with different parameters. To simplify the description, set $\beta = 1$ and $\eta = 1, 0.5, 1.5$ for the corresponding three cases in Theorem 1. The initial condition is $z_0 = 5$. On the sliding mode surface $s = 0$, the state of z for different parameters η is illustrated in Figure 1.

One can observe that for Case 2 with $\eta = 0.5$, the convergence to $z = 1$ from $z_0 = 5$ is achieved in the longest time, while the convergence to $z = 0$ from $z = 5$ takes the shortest time. This implies that when $z > 1$, Case 2 exhibits a minimum decay rate, whereas when $0 < z < 1$, it demonstrates a maximum decay rate. Similar observations can be summarized for Case 3. Generally, it can be shown that if $z > 1$, Case 3 with $\eta > 1$ decreases faster than the other two cases. However, if $z < 1$, Case 2 with $0 \leqslant \eta < 1$ decreases in the fastest way among these three cases. The most important factor is that only in Case 2 with $0 \leqslant \eta < 1$, finite-time stability can be reached according to Theorem 1. This motivates a lot of interesting studies in the recent literature, based on which TSM control is proposed [11, 17].

Next, several definitions for some sliding mode control surfaces in (14) are given.

Definition 3. For a constant $\beta > 0$,

(1) If $\eta = 1$, the sliding mode surface in (14) is called LSM surface;

(2) If $0 \leqslant \eta < 1$, the sliding mode surface in (14) is called TSM surface;

(3) If $\eta > 1$, the sliding mode surface in (14) is called PSM surface.

Here, the first two kinds of sliding mode control (LSM and TSM) have been widely discussed recently. In this paper, a new sliding mode surface called the PSM surface is proposed, which is also be very helpful for the FSM control discussed later.

4 Fixed-time sliding mode control

In this section, a unified framework for SMC with two power terms as in (6) is analyzed, based on which the above new sliding mode called the FSM in (10) is proposed.

First, a general sliding mode control surface with two power terms is given as follows:

$$s(t) = \dot{z} + \alpha z^{[\xi]} + \beta z^{[\eta]}, \tag{21}$$

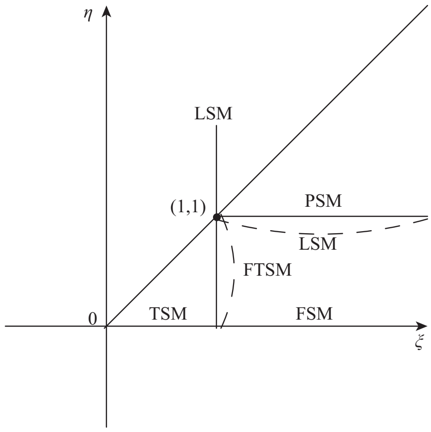

Figure 2 Parametric region for the classification of sliding mode surface.

where $\xi \geqslant \eta \geqslant 0$ for generality, $\alpha > 0$, and $\beta > 0$.

Since on the sliding mode surface $s(t) = 0$ of (21), one has

$$\mathrm{d}(|z|)/\mathrm{d}t = -\alpha|z|^{\xi} - \beta|z|^{\eta}. \tag{22}$$

From the analysis in Theorem 1 and Definition 3, $\eta = 1$ in the sliding mode control with one power term (14) can be considered as a bifurcation parameter. Then, for the sliding mode control with two power terms (22), there are totally 6 cases listed in the following.

Case 1: $\xi = \eta = 1$. The sliding mode surface in (21) is the LSM surface.

Case 2: $1 > \xi \geqslant \eta \geqslant 0$. The sliding mode surface in (21) is the TSM surface.

Case 3: $\xi \geqslant \eta > 1$. The sliding mode surface in (21) is the PSM surface.

The above three cases are directly satisfied since the same properties are kept when the two parameters are in the same region discussed in Theorem 1.

Case 4: $\xi > \eta = 1$. The sliding mode surface in (21) is a generalized LSM surface.

Even though the term $-\alpha|z|^{\xi}$ can induce a faster decrease than the other term $-\beta|z|^{\eta}$ if $z > 1$, as time tends to infinity, the term $-\beta|z|^{\eta}$ dominates the derivative on the right side of (22). Thus, such kind of sliding mode surface can be taken as a generalization of LSM surface, termed as generalized LSM surface.

Case 5: $\xi = 1 > \eta \geqslant 0$. The sliding mode surface in (21) is the FTSM surface. The settling time for reaching the origin is given by (8).

For the case of $\xi = 1$ and $1 > \eta \geqslant 0$, FTSM control (6) is discussed in [13], which can achieve terminal sliding mode control objective with a faster settling time compared with the conventional terminal sliding mode control in (5) under the same condition. This is intuitive since an additional linear term $-\alpha z^{[1]}$ is applied, which can induce a faster decrease when $z > 1$.

Case 6: $\xi > 1 > \eta \geqslant 0$. The sliding mode surface in (21) is the newly defined FSM surface.

Next, a definition as well as a detailed analysis of FSM is given.

Definition 4. The sliding mode surface in (21) is called the FSM surface if z is fixed-time stable.

Theorem 2. The sliding mode surface in (21) with $\xi > 1 > \eta \geqslant 0$ is the FSM surface and the settling time $T(z_0)$ for reaching the origin is bounded by

$$T(z_0) \leqslant \frac{1}{\alpha(\xi - 1)} + \frac{1}{\beta(1 - \eta)} = T_1.$$

Proof. The proof can be directly completed by using Lemma 1 and Definition 4.

Note that under Case 6 with $\xi > 1 > \eta \geqslant 0$, finite-time stable can be reached since $-\beta|z|^{\eta}$ dominates if $z < 1$. However, if $z > 1$, by adding the term $-\beta|z|^{\xi}$, a faster convergence can be reached. Thus, this new FSM control is different from the FTSM control. The key factor is that the settling time for reaching the origin in FSM is bounded by a constant independent of the initial condition as given T_1 in Theorem 2 compared with initial-dependent one (8) in FTSM, where the settling time can be sufficiently large if z_0 is very large.

The parametric region for ξ and η discussed in six cases is illustrated in Figure 2.

Under $\eta \in [0,1)$, finite-time stable for the state $z(t)$ in (21) can be reached as discussed in Cases 2, 5, and 6, which correspond to $\xi < 1$ for TSM, $\xi = 1$ for FTSM, and $\xi > 1$ for FSM, respectively.

If η=1, the sliding mode surface in (21) is LSM surface if $\xi \geqslant 1$ as discussed in Cases 1 and 4. The other case is that if $\xi \geqslant \eta > 1$, the new PSM surface is proposed for (21).

Next, some analysis is given to have a better bound for FSM in Case 6.

Theorem 3. The sliding mode surface in (21) with $\xi > 1 > \eta \geqslant 0$ is the FSM surface and the settling time $T(z_0)$ for reaching the origin is bounded by

$$T(z_0) \leqslant \frac{1}{|z^*|^{\xi-1}\alpha(\xi-1)} + \frac{|z^*|^{1-\eta}}{\beta(1-\eta)} = T_2, \tag{23}$$

where z^* is an arbitrarily chosen constant. In addition, T can obtain its minimal value when $|z^*|^{\xi-\eta} = \frac{\beta}{\alpha}$.

Proof. Suppose the initial state is z_0. One can calculate the time t_1 from z_0 to z^* along the trajectory of (22). One has that $\mathrm{d}(|z|)/\mathrm{d}t < -\alpha|z|^{\xi}$ if $z \neq 0$. Similar to the analysis (19) in Theorem 1, one can obtain that $1/|z^*|^{\xi-1} > 1/|z_0|^{\xi-1} + (\xi-1)\alpha t > (\xi-1)\alpha t$. Hence,

$$t_1 \leqslant \frac{1}{|z^*|^{\xi}\alpha(\xi-1)}. \tag{24}$$

Next, one can calculate the time t_2 from z^* to the origin $z = 0$. Similarly, one has $\mathrm{d}(|z|)/\mathrm{d}t < -\beta|z|^{\eta}$ if $z \neq 0$. Then, it follows from (18) in Theorem 1 that

$$t_2 \leqslant \frac{|z^*|^{1-\eta}}{\beta(1-\eta)}. \tag{25}$$

Thus, the settling time for reaching the origin satisfies

$$T(z_0) = t_1 + t_2 \leqslant \frac{1}{|z^*|^{\xi-1}\alpha(\xi-1)} + \frac{|z^*|^{1-\eta}}{\beta(1-\eta)}, \tag{26}$$

which obtains its minimal value when $|z^*|^{\xi+1-\eta} = \frac{\beta\xi}{\alpha(\xi-1)}$ to reach the origin.

It is quite easy to see that the result in Theorem 2 is a special case of that in Theorem 3 if $z^* = 1$. By carefully choosing an appropriate z^* which depends on ξ, η, α, and β, one can get a better lower bound for the settling time. It is important to note that a lower bound generally leads to less conservative parameter selection.

In order to get a lower bound, a detailed analysis is given.

Theorem 4. The sliding mode surface in (21) with $\xi > 1 > \eta \geqslant 0$ is the FSM surface and the settling time $T(z_0)$ for reaching the origin is given by

$$T(z_0) = \frac{|z_0|^{1-\eta}}{\beta(1-\eta)} F\left(1, \frac{1-\eta}{\xi-\eta}; 1 + \frac{1-\eta}{\xi-\eta}; -\alpha\beta^{-1}|z_0|^{\xi-\eta}\right), \tag{27}$$

where $F(\cdot)$ is the hypergeomnetric function defined in [32]:

$$F(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t$$

with $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$ being the Beta function. In addition, $T(z_0)$ is monotonically increasing with respect to $|z_0|$ and is bounded by T_1 in Theorem 2 or T_2 in Theorem 3. $T(z_0)$ reaches the maximum value $T_{\max} = \int_0^{+\infty} \frac{1}{\alpha x^{\xi}+\beta x^{\eta}}\mathrm{d}x$ as $|z_0| \to +\infty$.

Proof. Consider from the sliding mode surface in (21) that

$$T(z_0) = \int_0^{|z_0|} \frac{1}{\alpha x^{\xi} + \beta x^{\eta}}\mathrm{d}x. \tag{28}$$

According to (28), $T(z_0)$ is monotonically increasing with respect to $|z_0|$. From Theorems 2 and 3, one knows that the settling time $T(z_0)$ for reaching the origin is bounded by T_1 in Theorem 2 or T_2 in Theorem 3. Obviously, $T(z_0)$ converges and reaches the maximum value $T_{\max} = \int_0^{+\infty} \frac{1}{\alpha x^{\xi}+\beta x^{\eta}}\mathrm{d}x$.

Let $x = |z_0|s$; then $\mathrm{d}x = |z_0|\mathrm{d}s$. It follows that

$$\begin{aligned} T(z_0) &= \int_0^1 \frac{|z_0|}{\alpha|z_0|^\xi s^\xi + \beta|z_0|^\eta s^\eta}\mathrm{d}s \\ &= \frac{|z_0|}{\beta|z_0|^\eta}\int_0^1 \frac{1}{s^\eta}\frac{1}{\alpha\beta^{-1}|z_0|^{\xi-\eta}s^{\xi-\eta}+1}\mathrm{d}s \\ &= \frac{|z_0|}{(1-\eta)\beta|z_0|^\eta}\int_0^1 \frac{1}{\alpha\beta^{-1}|z_0|^{\xi-\eta}s^{\xi-\eta}+1}\mathrm{d}s^{1-\eta}. \end{aligned} \tag{29}$$

Let $t = s^{\xi-\eta}$; then

$$\begin{aligned} T(z_0) &= \frac{|z_0|}{(1-\eta)\beta|z_0|^\eta}\int_0^1 \frac{1}{1+\alpha\beta^{-1}|z_0|^{\xi-\eta}t}\mathrm{d}t^{\frac{1-\eta}{\xi-\eta}} \\ &= \frac{|z_0|}{(1-\eta)\beta|z_0|^\eta}\int_0^1 \frac{\frac{1-\eta}{\xi-\eta}t^{\frac{1-\eta}{\xi-\eta}-1}}{1+\alpha\beta^{-1}|z_0|^{\xi-\eta}t}\mathrm{d}t \\ &= \frac{|z_0|}{(\xi-\eta)\beta|z_0|^\eta}\int_0^1 t^{\frac{1-\eta}{\xi-\eta}-1}\frac{1}{1+\alpha\beta^{-1}|z_0|^{\xi-\eta}t}\mathrm{d}t. \end{aligned} \tag{30}$$

Now, consider the integral representation formula [32, 33]:

$$F(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\mathrm{d}t, \tag{31}$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$ is the Beta function and $F(\cdot)$ is the hypergeomnetric function defined in [32].

Choose parameters as follows: $a = 1$, $b = \frac{1-\eta}{\xi-\eta}$, $c = a + b$, and $z = -\alpha\beta^{-1}|z_0|^{\xi-\eta}$. Then it follows by (31) that

$$\begin{aligned} &F\left(1, \frac{1-\eta}{\xi-\eta}; 1+\frac{1-\eta}{\xi-\eta}; -\alpha\beta^{-1}|z_0|^{\xi-\eta}\right)B\left(\frac{1-\eta}{\xi-\eta}, 1\right) \\ &= \int_0^1 t^{\frac{1-\eta}{\xi-\eta}-1}\frac{1}{1+\alpha\beta^{-1}|z_0|^{\xi-\eta}t}\mathrm{d}t. \end{aligned} \tag{32}$$

Furthermore, it is easy to calculate that

$$B\left(\frac{1-\eta}{\xi-\eta}, 1\right) = \int_0^1 t^{\frac{1-\eta}{\xi-\eta}-1}\mathrm{d}t = \frac{\xi-\eta}{1-\eta}. \tag{33}$$

Combining (31) and (32), and noticing (33), one can conclude that

$$T(z_0) = \frac{|z_0|^{1-\eta}}{\beta(1-\eta)}F\left(1, \frac{1-\eta}{\xi-\eta}; 1+\frac{1-\eta}{\xi-\eta}; -\alpha\beta^{-1}|z_0|^{\xi-\eta}\right). \tag{34}$$

The analytical settling time $T(z_0)$ in Theorem 4 is given by a hypergeometric function and it is not intuitive to see that the righthand side of (27) is bounded under any initial condition z_0. However, under some special cases, the analytical settling time $T(z_0)$ can be calculated. For example, if $\xi - \eta = 2(1-\eta)$ or $\xi + \eta = 2$, then the settling time $T(z_0)$ can be simplified by

$$\begin{aligned} T(z_0) &= \frac{|z_0|}{(1-\eta)\beta|z_0|^\eta}\int_0^1 \frac{1}{1+\alpha\beta^{-1}|z_0|^{\xi-\eta}t}\mathrm{d}t^{\frac{1-\eta}{\xi-\eta}} \\ &= \frac{1}{(1-\eta)\sqrt{\alpha\beta}}\arctan\frac{\sqrt{\alpha}|z_0|^{1-\eta}}{\sqrt{\beta}}. \end{aligned}$$

As $|z_0| \to +\infty$, $T_{\max} = \frac{\pi}{(\xi-\eta)\sqrt{\alpha\beta}}$.

Next, the analytical function of $T_{\max}$ will be given, which is independent of the initial condition z_0.

Corollary 1. The sliding mode surface in (21) with $\xi > 1 > \eta \geqslant 0$ is the FSM surface and the settling time $T_{\max} = \lim_{|z_0|\to+\infty} T(z_0)$ under any initial condition z_0 for reaching the origin is given by

$$T_{\max} = \lim_{|z_0|\to+\infty} T(z_0) = \frac{1}{(\xi-\eta)\alpha^{\frac{1-\eta}{\xi-\eta}}\beta^{\frac{\xi-1}{\xi-\eta}}}\frac{\pi}{\sin(\frac{1-\eta}{\xi-\eta}\pi)}. \tag{35}$$

Proof. From (28), one has

$$T(z_0) = \int_0^{|z_0|}\frac{1}{\alpha x^{\xi}+\beta x^{\eta}}\mathrm{d}x = \frac{1}{\beta}\int_0^{|z_0|}\frac{1}{x^{\eta}}\frac{1}{\alpha\beta^{-1}x^{\xi-\eta}+1}\mathrm{d}x. \tag{36}$$

Let $t = \frac{\alpha}{\beta}x^{\xi-\eta}$; then

$$\begin{aligned} T(z_0) &= \frac{1}{\beta(1-\eta)}\int_0^{\frac{\alpha}{\beta}|z_0|^{\xi-\eta}}\frac{1}{1+t}\mathrm{d}\left(\frac{\beta t}{\alpha}\right)^{\frac{1-\eta}{\xi-\eta}} \\ &= \frac{1}{(\xi-\eta)\alpha^{\frac{1-\eta}{\xi-\eta}}\beta^{\frac{\xi-1}{\xi-\eta}}}\int_0^{\frac{\alpha}{\beta}|z_0|^{\xi-\eta}}\frac{t^{\frac{1-\eta}{\xi-\eta}-1}}{1+t}\mathrm{d}t. \end{aligned} \tag{37}$$

Since $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \int_0^{+\infty}\frac{t^{y-1}}{(1+t)^{x+y}}\mathrm{d}t$ and $\Gamma(s) = \int_0^{+\infty} t^{s-1}\mathrm{e}^{-t}\mathrm{d}t$, one has

$$\begin{aligned} T(\infty) &= \frac{1}{(\xi-\eta)\alpha^{\frac{1-\eta}{\xi-\eta}}\beta^{\frac{\xi-1}{\xi-\eta}}}\int_0^{+\infty}\frac{t^{\frac{1-\eta}{\xi-\eta}-1}}{1+t}\mathrm{d}t \\ &= \frac{1}{(\xi-\eta)\alpha^{\frac{1-\eta}{\xi-\eta}}\beta^{\frac{\xi-1}{\xi-\eta}}}B\left(\frac{\xi-1}{\xi-\eta},\frac{1-\eta}{\xi-\eta}\right) \\ &= \frac{1}{(\xi-\eta)\alpha^{\frac{1-\eta}{\xi-\eta}}\beta^{\frac{\xi-1}{\xi-\eta}}}\Gamma\left(\frac{\xi-1}{\xi-\eta}\right)\Gamma\left(\frac{1-\eta}{\xi-\eta}\right) \\ &= \frac{1}{(\xi-\eta)\alpha^{\frac{1-\eta}{\xi-\eta}}\beta^{\frac{\xi-1}{\xi-\eta}}}\frac{\pi}{\sin(\frac{1-\eta}{\xi-\eta}\pi)}. \end{aligned} \tag{38}$$

Remark 1. In Theorems 2 and 3, and Corollary 1, three different estimation bounds of the settling time $T(z_0)$ in Theorem 4 for reaching the origin in the sliding mode surface (21) are given. It is easy to see that $T(z_0) \leqslant T_{\max} \leqslant T_2 \leqslant T_1$, where T_1, T_2, and $T_{\max}$ are all independent of the initial condition z_0. This indicates the fixed-time stability of the system. Notice that $T_{\max} = \lim_{|z_0|\to+\infty} T(z_0)$. Thus, the lowest bound $T_{\max}$ for fixed-time stability is analytically given in Corollary 1 under any initial condition z_0.

5 Fixed-time sliding mode control design for nonlinear dynamical systems

In this section, the new FSM control for a general nonlinear dynamical system is discussed. Consider the following general nonlinear system:

$$\begin{aligned} &\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = x_0, \\ &y = h(x), \end{aligned} \tag{39}$$

where $x = (x_1, x_2, \ldots, x_N)^{\mathrm{T}} \in \mathbb{R}^n$ is the state, $f(\cdot)$ and $h(\cdot)$ are nonlinear functions, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the output state, and x_0 is the initial state.

Definition 5 ([34])**.** The nonlinear system (39) is said to have relative degree ρ, $1 \leqslant \rho \leqslant n$, in a region $D_0 \subset \mathbb{R}^n$, if

$$\begin{aligned} &\mathcal{L}g\mathcal{L}_f^k h(x) = 0, \ k < \rho - 1, \\ &\mathcal{L}g\mathcal{L}_f^{\rho-1} h(x) \neq 0, \end{aligned} \tag{40}$$

for all $x \in D_0$.

Here, we mainly focus on the FSM control design. Hence, the above single-input single-output (SISO) system (39) is assumed to have relative degree n and D_0 in Definition 5 is set as $\mathbb{R}^n$.

In order to transform the system (39) into a standard one, the Lie derivative is introduced. In particular, the Lie derivative of output function h with respect to function f is defined as the directional derivative $\mathcal{L}_f h = (\nabla h) f$, where $\nabla h = \partial h/\partial x$ is the gradient of h. The higher-order Lie derivative can be recursively defined as $\mathcal{L}_f^i h = \nabla(\mathcal{L}_f^{i-1} h) f$ for $i = 1, 2, \ldots, n$. In particular, $\mathcal{L}_f^0 h = h$.

Let $y = (h(x), \mathcal{L}_f h(x), \ldots, \mathcal{L}_f^{n-1} h(x))^{\mathrm{T}} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$. Then, one can transform the system (39) into

$$\begin{aligned} &\dot{y}_i(t) = y_{i+1},\ i = 1, 2, \ldots, n-1, \\ &\dot{y}_n(t) = a(y) + b(y)u, \end{aligned}$$

where $a(y) = \mathcal{L}_f^n h(x)$ and $b(y) = \mathcal{L} g \mathcal{L}_f^{n-1} h(x)$. The above system can be rewritten into a matrix form:

$$\dot{y}(t) = \hat{f}(y(t)) + \hat{g}(y(t))u(t), \tag{41}$$

where $\hat{f}(y(t)) = (y_2, y_3, \ldots, y_{n-1}, a(y))^{\mathrm{T}}$ and $\hat{g}(y(t)) = (0, \ldots, 0, b(y))^{\mathrm{T}}$.

Then, the following new recursive procedure for FSM control of a higher-order system (41) is designed:

$$\begin{aligned} &s_1(t) = \dot{s}_0 + \alpha_0 s_0^{[\xi_0]} + \beta_0 s_0^{[\eta_0]}, \\ &s_2(t) = \dot{s}_1 + \alpha_1 s_1^{[\xi_1]} + \beta_1 s_1^{[\eta_1]}, \\ &\vdots \\ &s_{n-1}(t) = \dot{s}_{n-2} + \alpha_{n-2} s_{n-2}^{[\xi_{n-2}]} + \beta_{n-2} s_{n-2}^{[\eta_{n-2}]}, \end{aligned} \tag{42}$$

where $s_0 = y_1$, $\alpha_i > 0$, $\beta_i > 0$, $\xi_i > 1$, $1 > \eta_i \geqslant 0$, $i = 0, 2, \ldots, n-2$. By designing the control input u, $s_{n-1} = 0$ is first reached in fixed time, then $s_{n-2} = 0$ can be reached in fixed time, and so will $s_{n-3}, \ldots, s_0$. Thus, fixed-time stability for the system (39) can be solved. According to Theorem 2, the settling time for reaching the origin can be bounded by

$$T(y(0)) \leqslant \sum_{i=0}^{n-2} \frac{1}{\alpha_i(\xi_i - 1)} + \frac{1}{\beta_i(1 - \eta_i)},$$

which is independent of the initial condition. Thus, if $s_i \to 0$ sequentially from $i = n-2$ to $i = 0$, then fixed-time stability for the system (39) can be reached.

Next, a theorem is given for the FSM control of a higher-order system (39).

Lemma 2. For the general nonlinear dynamical system (39), if the control input u is designed by

$$u(t) = u_{eq}(t) + \hat{u}(t), \tag{43}$$

where

$$\begin{aligned} &u_{eq}(t) = -b^{-1}(y)\left(a(y) + \sum_{k=0}^{n-2}\left(\alpha_k \mathcal{L}_{\hat{f}+\hat{g}u}^{n-1-k} s_k^{[\xi_k]} + \beta_k \mathcal{L}_{\hat{f}+\hat{g}u}^{n-1-k} s_k^{[\eta_k]}\right)\right), \\ &\hat{u}(t) = -b^{-1}(y)(\alpha s_{n-1}^{[\xi]} + \beta s_{n-1}^{[\eta]}), \end{aligned}$$

with $\alpha, \beta > 0$ and $\xi > 1 > \eta \geqslant 0$, then system (39) can reach the FSM $s_{n-1} = 0$ within fixed time.

Proof. Taking the derivative of s_{n-1}, one has

$$\dot{s}_{n-1}(t) = \ddot{s}_{n-2} + \alpha_{n-2} \mathcal{L}_{\hat{f}+\hat{g}u} s_{n-2}^{[\xi_{n-2}]} + \beta_{n-2} \mathcal{L}_{\hat{f}+\hat{g}u} s_{n-2}^{[\eta_{n-2}]}. \tag{44}$$

Since

$$\dot{s}_{n-2}(t) = \ddot{s}_{n-3} + \alpha_{n-3} \mathcal{L}_{\hat{f}+\hat{g}u} s_{n-3}^{[\xi_{n-3}]} + \beta_{n-3} \mathcal{L}_{\hat{f}+\hat{g}u} s_{n-3}^{[\eta_{n-3}]}, \tag{45}$$

one furthermore has

$$\begin{aligned}\dot{s}_{n-1}(t) = \dddot{s}_{n-3} + \alpha_{n-3}\mathcal{L}^2_{\hat{f}+\hat{g}u}s^{[\xi_{n-3}]}_{n-3} + \beta_{n-3}\mathcal{L}^2_{\hat{f}+\hat{g}u}s^{[\eta_{n-3}]}_{n-3} \\ +\alpha_{n-2}\mathcal{L}_{\hat{f}+\hat{g}u}s^{[\xi_{n-2}]}_{n-2} + \beta_{n-2}\mathcal{L}_{\hat{f}+\hat{g}u}s^{[\eta_{n-2}]}_{n-2}.\end{aligned} \tag{46}$$

The above process can be done recursively. Then, one finally has

$$\begin{aligned}\dot{s}_{n-1}(t) &= s_0^{(n)} + \sum_{k=0}^{n-2}\left(\alpha_k\mathcal{L}^{n-1-k}_{\hat{f}+\hat{g}u}s_k^{[\xi_k]} + \beta_k\mathcal{L}^{n-1-k}_{\hat{f}+\hat{g}u}s_k^{[\eta_k]}\right) \\ &= a(y) + b(y)u + \sum_{k=0}^{n-2}\left(\alpha_k\mathcal{L}^{n-1-k}_{\hat{f}+\hat{g}u}s_k^{[\xi_k]} + \beta_k\mathcal{L}^{n-1-k}_{\hat{f}+\hat{g}u}s_k^{[\eta_k]}\right).\end{aligned} \tag{47}$$

Substituting the designed control input (43) into (47) yields

$$\dot{s}_{n-1}(t) = -\alpha s^{[\xi]}_{n-1} - \beta s^{[\eta]}_{n-1}. \tag{48}$$

From the discussions in Theorems 2 and 3, one knows that system (39) can reach the FSM $s_{n-1} = 0$ within fixed time, where the settling time is bounded by $\frac{1}{|s^*|^{\xi-1}\alpha(\xi-1)} + \frac{|s^*|^{1-\eta}}{\beta(1-\eta)}$ with s^* being an arbitrarily chosen value.

To solve the fixed-time stability for the states in system (39), one should show that two terms $\alpha_k\mathcal{L}^{n-1-k}_{\hat{f}+\hat{g}u}s_k^{[\xi_k]}$ and $\beta_k\mathcal{L}^{n-1-k}_{\hat{f}+\hat{g}u}s_k^{[\eta_k]}$ are independent of the control input u and u is bounded, $k = 0, 1, \ldots, n-2$. Next, a theorem is provided to show this result.

Theorem 5. For the general nonlinear dynamical system (39), if the control input u is designed by (43) under the condition that

$$\eta_k > \frac{n-k-1}{n-k}, \tag{49}$$

and $s_k \to 0$ sequentially from $k = n-2$ to $k = 0$, then system (39) will first reach the FSM $s_k = 0$ and then reach the origin within fixed time as well for $k = 0, 1, \ldots, n-2$.

Proof. From the discussions in Lemma 2, system (39) will first reach the FSM $s_k = 0$, $k = 0, 1, \ldots, n-2$. Next, one aims to prove that the two terms $\mathcal{L}^{n-1-k}_{\hat{f}+\hat{g}u}s_k^{[\xi_k]}$ and $\mathcal{L}^{n-1-k}_{\hat{f}+\hat{g}u}s_k^{[\eta_k]}$ are independent of the control input u and u is bounded, $k = 0, 1, \ldots, n-2$.

First, one proves that $\mathcal{L}^{n-1-k}_{\hat{f}+\hat{g}u}s_k^{[\xi_k]} = F_k(y)$ and $\mathcal{L}^{n-1-k}_{\hat{f}+\hat{g}u}s_k^{[\eta_k]} = G_k(y)$, where $F_k(y)$ and $G_k(y)$ are continuous functions. Let $k = 0$; it is easy to see that $\mathcal{L}^{n-1}_{\hat{f}+\hat{g}u}s_0^{[\xi_0]} = \mathcal{L}^{n-1}_{\hat{f}+\hat{g}u}y_1^{[\xi_0]} = F_0(y)$ and $\mathcal{L}^{n-1}_{\hat{f}+\hat{g}u}s_0^{[\eta_0]} = \mathcal{L}^{n-1}_{\hat{f}+\hat{g}u}y_1^{[\eta_0]} = G_0(y)$. If $k = 1$, then $\mathcal{L}^{n-2}_{\hat{f}+\hat{g}u}s_1^{[\xi_1]} = \mathcal{L}^{n-2}_{\hat{f}+\hat{g}u}(\dot{y}_1 + \alpha_0 y_1^{[\xi_1]} + \beta_0 y_1^{[\eta_0]})^{[\xi_0]} = F_1(y)$ and so is that $\mathcal{L}^{n-2}_{\hat{f}+\hat{g}u}s_1^{[\eta_1]} = G_1(y)$.

Assume that $k = k_0$, $\mathcal{L}^{n-1-k_0}_{\hat{f}+\hat{g}u}s_{k_0}^{[\xi_{k_0}]} = F_{k_0}(y)$ and $\mathcal{L}^{n-1-k_0}_{\hat{f}+\hat{g}u}s_{k_0}^{[\eta_{k_0}]} = G_{k_0}(y)$. Then, one will prove that $k = k_0 + 1$ is also satisfied. For $k = k_0 + 1$ and based on (42),

$$\begin{aligned}\mathcal{L}^{n-1-(k_0+1)}_{\hat{f}+\hat{g}u}s_{k_0+1}^{[\xi_{k_0+1}]} &= \mathcal{L}^{n-1-(k_0+1)}_{\hat{f}+\hat{g}u}\left(\dot{s}_{k_0} + \alpha_{n-2}s_{k_0}^{[\xi_{k_0}]} + \beta_{k_0}s_{k_0}^{[\eta_{k_0}]}\right)^{[\xi_{k_0+1}]} \\ &= F_{k_0+1}(y).\end{aligned} \tag{50}$$

Similarly, one can obtain that $\mathcal{L}^{n-1-(k_0+1)}_{\hat{f}+\hat{g}u}s_{k_0+1}^{[\eta_{k_0+1}]} = G_{k_0+1}(y)$.

Next, one aims to prove that the control input u is bounded using similar analog in [13]. From the rule for the higher-order derivative of a composite function, one has that for a function $H(s)$:

$$\frac{\mathrm{d}^r}{\mathrm{d}t^r}H(s) = \sum \frac{r!}{i_1!i_2!\cdots i_l!}\frac{\partial^m H}{\partial s^m}\left(\frac{\dot{s}}{1!}\right)^{i_1}\left(\frac{\ddot{s}}{2!}\right)^{i_2}\cdots\left(\frac{s^{(l)}}{l!}\right)^{i_l}, \tag{51}$$

where the nonnegative integers satisfy $i_1 + 2i_2 + \cdots + li_l = r$ and $m = i_1 + i_2 + \cdots + i_l$.

On the fixed-time sliding mode surface $s_{k+1} = 0$, one has

$$\dot{s}_k = -\alpha_k s_k^{[\xi_k]} - \beta_k s_k^{[\eta_k]}. \tag{52}$$

Then, one knows that $\dot{s}_k = \mathcal{O}(s_k^{\eta_k})$ as $s_k \to 0$, where $\mathcal{O}(s_k^{\eta_k})$ has the same order infinitesimal as $s_k^{\eta_k}$. It follows that $s_k^{(d)} = \dot{s}_k^{(d-1)} = \mathcal{O}(s^{d\eta_k-(d-1)})$. From (41) and (51), one has

$$\begin{aligned}
\mathcal{L}_{\hat{f}+\hat{g}u}^{n-1-k} s_k^{[\eta_k]} &= \frac{\mathrm{d}^{n-1-k}}{\mathrm{d}t^{n-1-k}} s_k^{[\eta_k]} \\
&= \sum \mathcal{O}(s_k^{\eta_k-m})\mathcal{O}(s_k^{\eta_k})^{i_1}\mathcal{O}(s_k^{2\eta_k-1})^{i_2}\cdots\mathcal{O}(s_k^{l\eta_k-(l-1)})^{i_l} \\
&= \sum \mathcal{O}(s_k^{\eta_k-m})\mathcal{O}(s_k^{\eta_k(i_1+2i_2+\cdots+li_l)-i_2-2i_3-\cdots-(l-1)i_l}) \\
&= \sum \mathcal{O}(s_k^{\eta_k-m})\mathcal{O}(s_k^{\eta_k(n-1-k)+m-(n-1-k)}) \\
&= \mathcal{O}(s_k^{\eta_k(n-k)-(n-1-k)}).
\end{aligned} \tag{53}$$

Similarly, one obtains

$$\begin{aligned}
\mathcal{L}_{\hat{f}+\hat{g}u}^{n-1-k} s_k^{[\xi_k]} &= \sum \mathcal{O}(s_k^{\xi_k-m})\mathcal{O}(s_k^{\eta_k(n-1-k)+m-(n-1-k)}) \\
&= \mathcal{O}(s_k^{\eta_k(n-k-1)-(n-k-1)+\xi_k}).
\end{aligned} \tag{54}$$

If $\eta_k > \frac{n-k-1}{n-k}$ in condition (49), then $\eta_k(n-k)-(n-1-k) > 0$ and $\eta_k(n-k-1)-(n-k-1)+\xi_k > 0$. Thus, the control input u in (43) is bounded.

From Lemma 2, one can first design the control input in (43) to reach FSM $s_{n-1} = 0$ within the fixed time. If $s_k \to 0$ sequentially from $k = n-2$ to $k = 0$ under the condition (49), system (14) state will first reach the FSM surface (42) within fixed time and then reach the origin in fixed time according to Theorem 5.

6 Simulation example

In this section, one numerical simulation is provided to verify the effectiveness of the theoretical analysis.

Consider the following 3rd-order nonlinear dynamics:

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u = \begin{pmatrix} 0 \\ x_1 + \frac{x_2^3}{3} \\ x_1 - x_2 \end{pmatrix} + \begin{pmatrix} \mathrm{e}^{x_2} \\ \mathrm{e}^{x_2} \\ 0 \end{pmatrix} u, \\
y &= h(x) = x_3,
\end{aligned} \tag{55}$$

where $x = (x_1, x_2, x_3)^{\mathrm{T}}$. Let $z = (z_1, z_2, z_3)^{\mathrm{T}} = (h(x), L_f h(x), L_f^2 h(x))^{\mathrm{T}}$. With simple calculation, one has that $L_g h(x) = 0$, $L_g L_f h(x) = 0$, $b(z) = L_g L_f^2 h(x) = -(1 + x_2^2)\mathrm{e}^{x_2}$, $z_2 = L_f h(x) = x_1 - x_2$, $z_3 = L_f^2 h(x) = -x_1 - \frac{x_2^3}{3}$, and $a(z) = L_f^3 h(x) = -x_2^2(x_1 + \frac{x_2^3}{3})$. The dynamics of z can be written as

$$\begin{cases}
\dot{z}_1 = z_2, \\
\dot{z}_2 = z_3, \\
\dot{z}_3 = a(z) + b(z)u.
\end{cases} \tag{56}$$

Let $\hat{f}(z) = (z_2, z_3, a(z))^{\mathrm{T}}$ and $\hat{g}(z) = (0, 0, b(z))^{\mathrm{T}}$.

Choose $\alpha = 1$, $\beta = 1$, $\xi = \frac{5}{3}$, $\eta = \frac{1}{3}$, $\alpha_k = 1$, $\beta_k = 1$, $\xi_k = \frac{5}{3}$, and $\eta_k = \frac{2-k}{3-k} + \frac{1}{4}$, $k = 0, 1$.

For simplicity, let $w(z, u) = \hat{f}(z) + \hat{g}(z)u(t)$. Notice that $s_0 = z_1$ and $s_1 = z_2 + \alpha_0 z_1^{[\xi_0]} + \beta_0 z_1^{[\eta_0]}$. In the control design (31), critical terms should be clarified:

$$L_w^2 s_0^{[\frac{5}{3}]} = L_w\left(\frac{5}{3}|z_1|^{\frac{2}{3}} z_2\right) = \frac{10}{9} z_1^{[\frac{1}{3}]} z_2^2 + \frac{5}{3}|z_1|^{\frac{2}{3}} z_3,$$

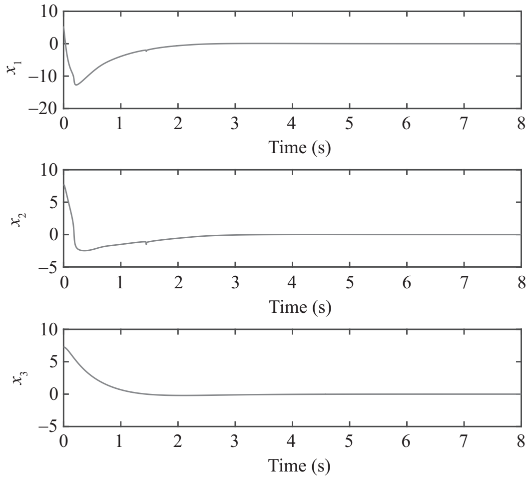

Figure 3 Trajectories of x_i.

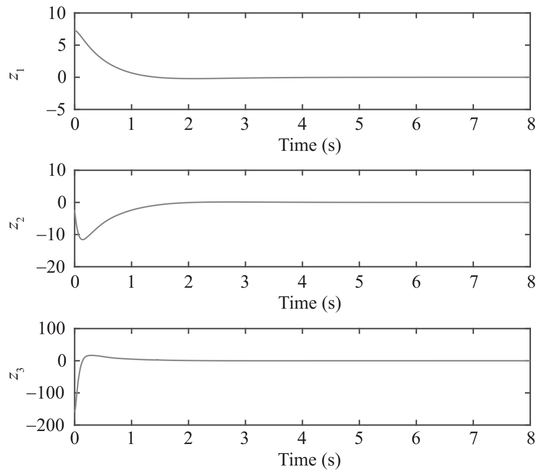

Figure 4 Trajectories of z_i.

$$\begin{aligned} L_w^2 s_0^{[\frac{11}{12}]} &= L_w\left(\frac{11}{12}|z_1|^{-\frac{1}{12}} z_2\right) = -\frac{11}{144} z_1^{[-\frac{13}{12}]} z_2^2 + \frac{11}{12}|z_1|^{-\frac{1}{12}} z_3, \\ L_w s_1^{[\frac{5}{3}]} &= L_w(z_2+\alpha_0 z_1^{[\frac{5}{3}]}+\beta_0 z_1^{[\frac{11}{12}]})^{[\frac{5}{3}]} = \frac{5}{3}|s_1|^{\frac{2}{3}}\left(\frac{5}{3}\alpha_0|z_1|^{\frac{2}{3}} z_2 + \frac{11}{12}\beta_0|z_1|^{-\frac{1}{12}} z_2 + z_3\right), \\ L_w s_1^{[\frac{3}{4}]} &= L_w(z_2+\alpha_0 z_1^{[\frac{5}{3}]}+\beta_0 z_1^{[\frac{11}{12}]})^{[\frac{3}{4}]} = \frac{3}{4}|s_1|^{-\frac{1}{4}}\left(\frac{5}{3}\alpha_0|z_1|^{\frac{2}{3}} z_2 + \frac{11}{12}\beta_0|z_1|^{-\frac{1}{12}} z_2 + z_3\right). \end{aligned}$$

The trajectories of x and z are depicted in Figures 3 and 4, respectively. One can observe that x reaches zero in fixed time.

7 Conclusion

In this paper, we have presented the novel FSM control as a valuable approach for real-world applications, offering the significant advantage that the settling time for reaching the system origin is bounded to a constant independent of the initial condition. First, a new sliding mode control with one power term was introduced, allowing for the power term to take any nonnegative value. Additionally, a new sliding mode control called PSM has been proposed, whose power term is larger than 1, complementing the traditional LSM and TSM controls. Subsequently, a new sliding mode control with two power terms was discussed, where the new FSM control was first designed. In particular, the two power terms on the plane in the first quadrant were discussed, and a comprehensive classification of the first quadrant into six categories was provided, including LSM, TSM, FTSM, and PSM. Furthermore, the analytical settling time was calculated, and three different estimation bounds of the settling time for reaching the origin were established for any initial condition. Particularly intriguing is the derivation of the lowest bound for the settling time, as a lower bound offers the potential for a less conservative control parameter selection, which is a critical aspect for future research on fixed-time stability. Finally, the fixed-time sliding mode control design for general nonlinear dynamical systems with the relative degree from the control input to the output was also discussed, which highlights the significance and applicability of the proposed FSM control.

Recently, sliding mode control has gained significant attention due to its superior performance and robustness against parameter variations and disturbances. The newly proposed FSM control exhibits more advantages in scenarios where the settling time is independent of the initial condition. Future work will focus on exploring practical implementations of FSM control. In addition, more complicated FSM controls will be investigated, such as FSM in multi-input multi-output (MIMO) systems, FSM in systems with disturbances, FSM in manipulators, and other intricate cases. Thus, these studies on FSM will certainly deserve future investigations. These would benefit the theoretical studies on sliding mode control and its real-world applications.

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