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Tracking performance limitations of MIMO discrete-time networked control systems with multiple constraints

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Appendix A Preliminaries and problem formulation

In this section, we first present symbols used in our calculations. All transfer function matrices that are proper, stable, and rational are represented as RH_∞. For any matrix A, vector u, and complex number z, the complex conjugate is denoted by A^H, u^H and $\bar{z}$, respectively. Furthermore, the transposes of A and u are represented by A^T and u^T, respectively. Open and closed unit discs are represented by $D := \{z \in C : |z| < 1\}$ and $\bar{D} := \{z \in C : |z| \leqslant 1\}$, respectively, and the complement of the closed unit disc is denoted by $\bar{D}^c := \{z \in C : |z| > 1\}$. We define $\partial D := \{z \in C : |z| = 1\}$ as a unit circle. The notations $\|\cdot\|_F$ and $\|\cdot\|_2$ represent the Frobenius norm and the Euclidean vector norm, respectively. In particular, we have $\|G\|_F^2 := tr(G^H G)$. The Hilbert space $\mathcal{L}_2$ is defined as

$$\mathcal{L}_2 := \left\{ F(z) : F(z) \text{ is measurable in } \partial D,\ \|F(z)\|_2 := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| F(e^{j\theta}) \right\|_F^2 d\theta \right)^{1/2} < \infty \right\}.$$

Furthermore, an inner product defined in $\mathcal{L}_2$ is represented by

$$\langle F, G \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} tr(F^H(e^{j\theta}) G^H(e^{j\theta})) d\theta.$$

$H_2(D)$ and $H_2^\perp(D)$ are subspaces of $\mathcal{L}_2$ and contain analytic functions in D and $\bar{D}^c$, defined as follows:

$$H_2(D) := \left\{ F(z) : F(z) \text{ analytic in } \overline{D}^c,\ \|F(z)\|_2 := \left(\sup_{r>1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| F(re^{j\theta}) \right\|_F^2 d\theta \right)^{1/2} < \infty \right\},$$

and

$$H_2^\perp(D) := \left\{ F(z) : F(z) \text{ analytic in } D,\ \|F(z)\|_2 := \left(\sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| F(re^{j\theta}) \right\|_F^2 d\theta \right)^{1/2} < \infty \right\}.$$

For any $F \in H_2$ and $G \in H_2^\perp$, we have $\langle F, G \rangle = 0$. If the transfer function matrix $G(z)$ is right-invertible and $F(z) \in RH_\infty$, the coprime factorization of $(1-\alpha)G(z)F(z)$ can be given by

$$(1-\alpha)G(z)F(z) = N(z)M^{-1}(z) = \tilde{M}^{-1}(z)\tilde{N}(z), \tag{A1}$$

where $N(z), M(z), \tilde{M}(z), \tilde{N}(z) \in RH_\infty$, and satisfy the double Bezout equation

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -z^{-\tau}\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ z^{-\tau}N & X \end{bmatrix} = I, \tag{A2}$$

with $X, Y, \tilde{X}, \tilde{Y} \in RH_\infty$. The all-pass factorization of $N(z)$ and $M(z)$ be given by $N(z) = L(z)N_m(z)$, and $M(z) = B(z)M_m(z)$, where $N_m(z)$ and $M_m(z)$ are the associated minimum phase part, respectively. $L(z)$ represent all-pass factor which can be constructed as

$$L(z) = \prod_{i=1}^{N_z} L_i(z),\ L_i(z) = \frac{1-\bar{s}_i}{1-s_i} \frac{z-s_i}{1-\bar{s}_i z} \eta_i \eta_i^H + U_i U_i^H, \tag{A3}$$

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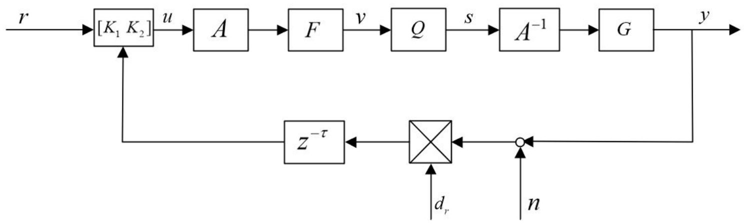

Figure A1 Networked control systems with multiple constraints.

where η_i is the direction vector of the nonminimum phase zero z_i, which satisfies the relation $\eta_i\eta_i^H + U_iU_i^H = I$ with the matrix U_i. In the same way, $\tilde{N}(z) = \tilde{N}_m(z)\tilde{L}(z)$, here the factorization of $\tilde{L}(z)$ can still be of the form (2).
Similarly, $\tilde{M}(z)$ can be factorized as $\tilde{M}(z) = \tilde{M}_m(z)\tilde{B}(z)$, where $\tilde{M}_m(z) \in RH_\infty$, and

$$\tilde{B}(z) = \prod_{i=1}^{N_p} \tilde{B}_i(z),\ \tilde{B}_i(z) = \frac{z-p_i}{1-\overline{p}_i z}\tilde{\omega}_i\tilde{\omega}_i^H + \tilde{W}_i\tilde{W}_i^H,$$

$\tilde{\omega}_i$ is the direction vector corresponding to the unstable pole p_i, where $\tilde{\omega}_i\tilde{\omega}_i^H + \tilde{W}_i\tilde{W}_i^H = I$. For real diagonal matrix W, $\tilde{M}(z)W$ can be factorized as

$$\tilde{M}(z)W = M_m(z)B(z), \tag{A4}$$

where $M_m(z) \in RH_\infty$, and

$$B(z) = \prod_{i=1}^{N_p} B_i(z),\ B_i(z) = \frac{z-p_i}{1-\overline{p}_i z}\omega_i\omega_i^H + W_iW_i^H, \tag{A5}$$

where ω_i is the unitary vector, and $\omega_i = \frac{W^{-1}\tilde{\omega}_i}{\|W^{-1}\tilde{\omega}_i\|}$, W_i is the matrix that satisfies $\omega_i\omega_i^H + W_iW_i^H = I$. In addition, define $\cos\angle(u,v) := \frac{|u^Hv|}{\|u\|\cdot\|v\|}$, so $\angle(u,v)$ is a directional angle between unit vector u and unit vector v.

Then, by invoking the Youla parametrization given in [1], in order to stabilize $G(z)$, each two-degree-of-freedom (TDOF) compensator $[K_1K_2]$ can be

$$\mathcal{K} := \left\{K : K = [K_1\ \ K_2] = (\tilde{X} - z^{-\tau}R\tilde{N})^{-1}[\,Q\ \ \tilde{Y} - R\tilde{M}\,]\right\} \tag{A6}$$

where $Q, R \in RH_\infty$.

In this letter, we study the NCSs as shown in Fig. 1, where $G(z)$ represents the controlled plant. $A(z)$ and $A^{-1}(z)$ stand for encoder and decoder, respectively. Bandwidth can be modeled by a low-pass Butterworth filters of order 1 with transfer function matrix $F(z)$. $Q(z)$ in Fig. 1 is used to model the uniform quantizer, which means $s = v + q$, where q is the quantization error and represented by $q(k) = \begin{bmatrix} q_1(k) & \cdots & q_m(k)\end{bmatrix}^{\mathrm{T}}$, the quantization error in each channel is independent and evenly distributed in the interval $\begin{bmatrix} -\Delta_i/2 & \Delta_i/2 \end{bmatrix}$, and we define $\Lambda = \frac{\sqrt{3}}{6}diag(\Delta_1,\cdots,\Delta_m)$. τ is network-induced delay, d_r indicates packet dropouts, be described as:

$$d_r(k) = \begin{cases} 0, & \text{when packet dropouts occurs at time } k, \\ 1, & \text{when no packet dropouts occurs at time } k. \end{cases} \tag{A7}$$

The probability of packet dropouts is $p\{d_r(k) = 1\} = 1 - \alpha$, and $p\{d_r(k) = 0\} = \alpha$, and the signals r, n, q, u and y represent the reference input, AWGN, quantization error, controller output ,and system output, respectively, the Z-transformed signals are $\tilde{r}$, $\tilde{n}$, $\tilde{q}$, $\tilde{u}$ and $\tilde{y}$, respectively. We assume that these signals are independent of each other.

Appendix B Proof

For the NCSs as shown in Fig. 1, we can get

$$\tilde{u} = K_1\tilde{r} + K_2d_rz^{-\tau}(\tilde{n} + \tilde{y}), \tag{B1}$$
$$\tilde{y} = GA^{-1}(\tilde{q} + FA\tilde{u}) = GA^{-1}\tilde{q} + GF\tilde{u}. \tag{B2}$$

Thus $\tilde{y} = T_1\tilde{r} + T_2\tilde{q} + T_3\tilde{n}$, $\tilde{e} = \tilde{r} - \tilde{y} = (I - T_1)\tilde{r} - T_2\tilde{q} - T_3\tilde{n}$, where

$$\begin{aligned} T_1 &= (I - z^{-\tau}d_rGFK_2)^{-1}GFK_1, \\ T_2 &= (I - z^{-\tau}d_rGFK_2)^{-1}GA^{-1}, \\ T_3 &= (I - z^{-\tau}d_rGFK_2)^{-1}z^{-\tau}d_rGFK_2. \end{aligned}$$

Form (A1), (A2), and (A6), we can obtain:

$$\begin{aligned} T_1 &= (I - z^{-\tau}d_rGFK_2)^{-1}GFK_1 \\ &= GF(I - z^{-\tau}K_2d_rGF)^{-1}K_1 \end{aligned}$$

$$\begin{aligned} &= \frac{1}{1-\alpha} NM^{-1}[I - z^{-\tau}(\tilde{X} - z^{-\tau}R\tilde{N})^{-1}(\tilde{Y} - R\tilde{M})\tilde{M}^{-1}\tilde{N}]^{-1}(\tilde{X} - z^{-\tau}R\tilde{N})^{-1}Q \\ &= \frac{1}{1-\alpha} NM^{-1}[(\tilde{X} - z^{-\tau}R\tilde{N})^{-1}M^{-1}]^{-1}(\tilde{X} - z^{-\tau}R\tilde{N})^{-1}Q \\ &= \frac{1}{1-\alpha} NQ \end{aligned} \tag{B3}$$

We can calculate T_2 and T_3 in a manner similar to the calculation of T_1. It then follows that

$$T_2 = \frac{1}{1-\alpha} N(\tilde{X} - z^{-\tau}R\tilde{N})F^{-1}A^{-1}, \tag{B4}$$

$$T_3 = z^{-\tau} N(\tilde{Y} - R\tilde{M}). \tag{B5}$$

Then the optlmal tracking performance can be further expressed as

$$\begin{aligned} J^* &= \inf_{K\in\mathcal{K}} (1-\varepsilon)E\left\{\|(I-T_1)\tilde{r} - T_2\tilde{q} - T_3\tilde{n}\|_2^2\right\} + \inf_{K\in\mathcal{K}} \varepsilon E\left\{\|T_1\tilde{r} + T_2\tilde{q} + T_3\tilde{n}\|_2^2 - \Gamma\right\} \\ &= \inf_{Q\in RH_\infty} \left\| \begin{bmatrix} \sqrt{1-\varepsilon}(I - T_1) \\ \sqrt{\varepsilon}T_1 \end{bmatrix} U \right\|_2^2 + \inf_{R\in RH_\infty} \|T_2\Lambda\|_2^2 + \inf_{R\in RH_\infty} \|T_3W\|_2^2 - \varepsilon\Gamma \end{aligned} \tag{B6}$$

Form (B3), (B4), (5B), and (B6), we can obtain

$$\begin{aligned} J^* = &\inf_{Q\in RH_\infty} \left\| \begin{bmatrix} \sqrt{1-\varepsilon}(I - \frac{1}{1-\alpha}NQ) \\ \sqrt{\varepsilon}\frac{1}{1-\alpha}NQ \end{bmatrix} U \right\|_2^2 + \inf_{R\in RH_\infty} \left\| \frac{1}{1-\alpha} N(\tilde{X} - z^{-\tau}R\tilde{N})F^{-1}A^{-1}\Lambda \right\|_2^2 \\ &+ \inf_{R\in RH_\infty} \left\| z^{-\tau}N(\tilde{Y} - R\tilde{M})W \right\|_2^2 - \varepsilon\Gamma, \end{aligned}$$

where $U = diag(\alpha_1, \cdots, \alpha_m)$, $W = diag(\gamma_1, \cdots, \gamma_m)$.
The problem studied in this letter can be expressed as $J^* = \inf_{K\in\mathcal{K}} J(Q, R, \varepsilon) \geqslant J_1^* + J_2^* + J_3^* - \varepsilon\Gamma$, where

$$J_1^* = \inf_{Q\in RH_\infty} \left\| \begin{bmatrix} \sqrt{1-\varepsilon}(I - \frac{1}{1-\alpha}NQ) \\ \sqrt{\varepsilon}\frac{1}{1-\alpha}NQ \end{bmatrix} U \right\|_2^2, \tag{B7}$$

$$J_2^* = \inf_{R\in RH_\infty} \left\| \frac{1}{1-\alpha} N(\tilde{X} - z^{-\tau}R\tilde{N})F^{-1}A^{-1}\Lambda \right\|_2^2, \tag{B8}$$

$$J_3^* = \inf_{R\in RH_\infty} \left\| z^{-\tau}N(\tilde{Y} - R\tilde{M})W \right\|_2^2. \tag{B9}$$

We first compute J_1^*, which is given by the all-pass factorization

$$J_1^* = \inf_{Q\in RH_\infty} \left\| \begin{bmatrix} \sqrt{1-\varepsilon}(I - \frac{1}{1-\alpha}LN_mQ) \\ \sqrt{\varepsilon}\frac{1}{1-\alpha}LN_mQ \end{bmatrix} U \right\|_2^2,$$

and since L is the all-pass factor, we can obtain:

$$J_1^* = \inf_{Q\in RH_\infty} \left\| \begin{bmatrix} \sqrt{1-\varepsilon}(L^{-1} - \frac{1}{1-\alpha}N_mQ) \\ \sqrt{\varepsilon}\frac{1}{1-\alpha}N_mQ \end{bmatrix} U \right\|_2^2, \tag{B10}$$

we define

$$\psi = \prod_{i=1}^{N_z} g(s_i),\ g(s_i) = \frac{\overline{s}_i(s_i - 1)}{1 - \overline{s}_i}\eta_i\eta_i^H + U_iU_i^H. \tag{B11}$$

From (B10) and (B11), it follows that

$$J_1^* = \inf_{Q\in RH_\infty} \left\| \begin{bmatrix} \sqrt{1-\varepsilon}(L^{-1} - \psi + \psi - \frac{1}{1-\alpha}N_mQ) \\ \sqrt{\varepsilon}\frac{1}{1-\alpha}N_mQ \end{bmatrix} U \right\|_2^2,$$

conspicuously,

$$\begin{bmatrix} \sqrt{1-\varepsilon}(L^{-1} - \psi) \\ 0 \end{bmatrix} \in H_2^\perp,\ \begin{bmatrix} \sqrt{1-\varepsilon}(\psi - \frac{1}{1-\alpha}N_mQ) \\ \sqrt{\varepsilon}\frac{1}{1-\alpha}N_mQ \end{bmatrix} \in H_2.$$

Furthermore, we can obtain

$$J_1^* = \left\| \begin{bmatrix} \sqrt{1-\varepsilon}(L^{-1} - \psi) \\ 0 \end{bmatrix} U \right\|_2^2 + \inf_{Q\in RH_\infty} \left\| \begin{bmatrix} \sqrt{1-\varepsilon}(\psi - \frac{1}{1-\alpha}N_mQ) \\ \sqrt{\varepsilon}\frac{1}{1-\alpha}N_mQ \end{bmatrix} U \right\|_2^2.$$

To facilitate the calculation of J_1^*, we define

$$J_{11}^* = \left\| \begin{bmatrix} \sqrt{1-\varepsilon}(L^{-1}-\psi) \\ 0 \end{bmatrix} U \right\|_2^2, \; J_{12}^* = \inf_{Q\in RH_\infty} \left\| \begin{bmatrix} \sqrt{1-\varepsilon}(\psi - \frac{1}{1-\alpha}N_m Q) \\ \sqrt{\varepsilon}\frac{1}{1-\alpha}N_m Q \end{bmatrix} U \right\|_2^2,$$

so we have $J_1^* = J_{11}^* + J_{12}^*$. By a direct calculation, we can obtain

$$\begin{aligned} J_{11}^* &= \left\| \left[\sqrt{1-\varepsilon}(L^{-1}-\psi)\right] U \right\|_2^2 \\ &= (1-\varepsilon)\sum_{i=1}^{N_z}\prod_{h=1}^{i}|g(s_{h-1})|^2 \left\| \left[L_i^{-1} - g(s_i)\right] U \right\|_2^2 \\ &= (1-\varepsilon)\sum_{i=1}^{N_z}\prod_{h=1}^{i}|g(s_{h-1})|^2 \left\| \frac{1-s_i}{1-\overline{s}_i}\frac{1-|s_i|^2}{z-s_i}\eta_i\eta_i^H U \right\|_2^2 \\ &= (1-\varepsilon)\sum_{i,j=1}^{N_z}\frac{(1-s_i)(1-s_j)(1-|s_i|^2)(1-|s_j|^2)}{(1-\overline{s}_i)(1-\overline{s}_j)}\left\langle \frac{1}{z-s_i}, \frac{1}{z-s_j}\right\rangle \prod_{h=1}^{i}|g(s_{h-1})|^2\left|\eta_i^H U\right|^2, \end{aligned}$$

and using Cauchy's integral theorem, we can obtain

$$\left\langle \frac{1}{z-s_i}, \frac{1}{z-s_j}\right\rangle = -\frac{1}{2\pi j}\int_{\partial D}\frac{dz}{(1-\overline{s}_i z)(z-s_j)} = \frac{1}{\overline{s}_i s_j - 1}.$$

Thus we have

$$J_{11}^* = (1-\varepsilon)\sum_{i,j=1}^{N_z}\frac{(1-s_i)(1-s_j)(1-|s_i|^2)(1-|s_j|^2)}{(1-\overline{s}_i)(1-\overline{s}_j)(\overline{s}_i s_j - 1)}\prod_{h=1}^{i}|g(s_{h-1})|^2\sum_{l=1}^{m}\alpha_l^2\cos^2\angle(\eta_i, e_l).$$

where e_l represents the unit vector with its $l-$th element being 1. Next, we calculate J_{12}^*.

$$J_{12}^* = \inf_{Q\in RH_\infty}\left\| \left(\begin{bmatrix} \sqrt{1-\varepsilon}\psi \\ 0 \end{bmatrix} + \begin{bmatrix} -\sqrt{1-\varepsilon} \\ \sqrt{\varepsilon} \end{bmatrix}\frac{1}{1-\alpha}N_m Q \right) U \right\|_2^2,$$

we introduce an inner-outer factorization

$$\begin{bmatrix} -\sqrt{1-\varepsilon} \\ \sqrt{\varepsilon} \end{bmatrix}\frac{1}{1-\alpha}N_m = \Delta_i\Delta_o,$$

where $\Delta_o \in RH_\infty$ and $\Delta_i \in RH_\infty$ represent an outer matrix and an inner matrix, respectively. Furthermore, we define

$$\Omega(z) = \begin{bmatrix} \Delta_i^T(-z) \\ I - \Delta_i(z)\Delta_i^T(-z) \end{bmatrix},$$

thus, $\Omega^H(e^{jw})\Omega(e^{-jw}) = I$, we then have

$$\begin{aligned} J_{12}^* &= \inf_{Q\in RH_\infty}\left\| \Omega\left(\begin{bmatrix} \sqrt{1-\varepsilon}\psi \\ 0 \end{bmatrix} + \begin{bmatrix} -\sqrt{1-\varepsilon} \\ \sqrt{\varepsilon} \end{bmatrix}\frac{1}{1-\alpha}N_m Q \right) U \right\|_2^2 \\ &= \inf_{Q\in RH_\infty}\left\| \left(\Delta_i^T\begin{bmatrix} \sqrt{1-\varepsilon}\psi \\ 0 \end{bmatrix} + \Delta_o Q + (I-\Delta_i\Delta_i^T)\begin{bmatrix} \sqrt{1-\varepsilon}\psi \\ 0 \end{bmatrix} \right) U \right\|_2^2 \\ &= \left\| (I-\Delta_i\Delta_i^T)\begin{bmatrix} \sqrt{1-\varepsilon}\psi \\ 0 \end{bmatrix} U \right\|_2^2 + \inf_{Q\in RH_\infty}\left\| \left(\Delta_i^T\begin{bmatrix} \sqrt{1-\varepsilon}\psi \\ 0 \end{bmatrix} + \Delta_o Q \right) U \right\|_2^2, \end{aligned}$$

it follows that $\inf_{Q\in RH_\infty}\left\| \left(\Delta_i^T\begin{bmatrix} \sqrt{1-\varepsilon}\psi \\ 0 \end{bmatrix} + \Delta_o Q \right) U \right\|_2^2$ can be made arbitrarily small by choosing appropriate $Q \in RH_\infty$. By a direct calculation, we can obtain

$$J_{12}^* = \varepsilon(1-\varepsilon)\sum_{i=1}^{N_z}|s_i|^2\sum_{l=1}^{m}\alpha_l^2\cos(\angle\eta_i, e_l).$$

Next, we calculate J_2^*.

$$J_2^* = \inf_{R\in RH_\infty}\frac{1}{(1-\alpha)^2}\left\| N(\tilde{X} - z^{-\tau}R\tilde{N})F^{-1}A^{-1}\Lambda \right\|_2^2$$

$$= \inf_{R\in RH_\infty} \frac{1}{(1-\alpha)^2} \left\| N_m \tilde{X} F^{-1} A^{-1} \Lambda - z^{-\tau} N_m R \tilde{N} F^{-1} A^{-1} \Lambda \right\|_2^2 .$$

Next, $\tilde{N} F^{-1} A^{-1} \Lambda$ is decomposed as $\tilde{N} F^{-1} A^{-1} \Lambda = \tilde{N}_m(z)\Xi(z)$, $\Xi(z)$ ia an allpass factor formed as $\Xi(z) = \prod_{i=1}^{N_z} \Xi_i(z)$, $\Xi_i(z) = \frac{1-\bar{s}_i}{1-s_i}\frac{z-s_i}{1-\bar{s}_i z}\varsigma_i\varsigma_i^H + \upsilon_i\upsilon_i^H$, where $\varsigma_i\varsigma_i^H + \upsilon_i\upsilon_i^H = I$ and $\tilde{N}_m$ is the minimum phase part.

$$\begin{aligned}
J_2^* &= \inf_{R\in RH_\infty} \frac{1}{(1-\alpha)^2} \left\| N_m \tilde{X} F^{-1} A^{-1} \Lambda - z^{-\tau} N_m R \tilde{N}_m \Xi \right\|_2^2 \\
&= \inf_{R\in RH_\infty} \frac{1}{(1-\alpha)^2} \left\| z^{\tau} N_m \tilde{X} F^{-1} A^{-1} \Lambda \Xi^{-1} - N_m R \tilde{N}_m \right\|_2^2 \\
&= \inf_{R\in RH_\infty} \frac{1}{(1-\alpha)^2} \left\| \sum_{i=1}^{N_z} (s_i)^{\tau} N_m(s_i) \tilde{X}(s_i) F^{-1}(s_i) A^{-1}(s_i) \Lambda H_i \Xi_i^{-1} G_i + R_1 - N_m R \tilde{N}_m \right\|_2^2 \\
&= \inf_{R\in RH_\infty} \frac{1}{(1-\alpha)^2} \left\| \sum_{i=1}^{N_z} (s_i)^{\tau} N_m(s_i) \tilde{X}(s_i) F^{-1}(s_i) A^{-1}(s_i) \Lambda H_i (\Xi_i^{-1} - \Xi_i^{-1}(\infty)) G_i + R_2 - N_m R \tilde{N}_m \right\|_2^2 ,
\end{aligned}$$

where $H_i = \left(\prod_{h=i+1}^{N_z} \Xi_h^{-H}(s_i)\right)^H$, $G_i = \left(\prod_{h=1}^{i-1} \Xi_h^{-H}(s_i)\right)^H$, and $R_1, R_2 \in RH_\infty$, $R_1 = R_2 - \sum_{i=1}^{N_z} (s_i)^{\tau} N_m(s_i) \tilde{X}(s_i) F^{-1}(s_i) A^{-1}(s_i) \Lambda$ $H_i \Xi_i^{-1}(\infty) G_i$. Since $\sum_{i=1}^{N_z} (s_i)^{\tau} N_m(s_i) \tilde{X}(s_i) F^{-1}(s_i) A^{-1}(s_i) \Lambda H_i (\Xi_i^{-1} - \Xi_i^{-1}(\infty)) G_i \in H_2^{\perp}$, In particular, $R_2 - N_m R \tilde{N}_m \in H_2$ can be rendered valid by choosing the appropriate controller parameter $R \in RH_\infty$, then we have

$$\begin{aligned}
J_2^* &= \frac{1}{(1-\alpha)^2} \left\| \sum_{i=1}^{N_z} (s_i)^{\tau} N_m(s_i) \tilde{X}(s_i) F^{-1}(s_i) A^{-1}(s_i) \Lambda H_i (\Xi_i^{-1} - \Xi_i^{-1}(\infty)) G_i \right\|_2^2 \\
&\quad + \inf_{R\in RH_\infty} \frac{1}{(1-\alpha)^2} \left\| R_2 - N_m R \tilde{N}_m \right\|_2^2 ,
\end{aligned}$$

and $\inf_{R\in RH_\infty} \frac{1}{(1-\alpha)^2} \left\| R_2 - N_m R \tilde{N}_m \right\|_2^2 = 0$ can be rendered valid by choosing a suitable controller parameter $R \in RH_\infty$, thus

$$\begin{aligned}
J_2^* &= \frac{1}{(1-\alpha)^2} \left\| \sum_{i=1}^{N_z} (s_i)^{\tau} N_m(s_i) \tilde{X}(s_i) F^{-1}(s_i) A^{-1}(s_i) \Lambda H_i (\Xi_i^{-1} - \Xi_i^{-1}(\infty)) G_i \right\|_2^2 \\
&= \frac{1}{(1-\alpha)^2} \left\| \sum_{i=1}^{N_z} \frac{(1-s_i)}{(1-\bar{s}_i)} \frac{(1-|s_i|^2)}{(z-s_i)} (s_i)^{\tau} N_m(s_i) \tilde{X}(s_i) F^{-1}(s_i) A^{-1}(s_i) \Lambda H_i \varsigma_i \varsigma_i^H G_i \right\|_2^2 \\
&= \frac{1}{(1-\alpha)^2} \sum_{i,j=1}^{N_z} \frac{(s_i-1)(1-s_j)(|s_i|^2-1)(1-|s_j|^2)}{(\bar{s}_i-1)(\bar{s}_j-1)(\bar{s}_i s_j - 1)} \\
&\quad \times \varsigma_i^H (s_i)^{\tau} H_i^H \Lambda A^{-H}(s_i) F^{-H}(s_i) \tilde{X}^H(s_i) N_m^H(s_i) (s_j)^{\tau} N_m(s_j) \tilde{X}(s_j) F^{-1}(s_j) A^{-1}(s_j) \Lambda H_j \varsigma_j \varsigma_j^H G_j G_i^H \varsigma_i .
\end{aligned}$$

From the double Bezout equation $M\tilde{X} - z^{-\tau} N \tilde{Y} = I$, and $N(s_i) = 0$, thus we have $\tilde{X}(s_i) = M^{-1}(s_i)$, so J_2^* can be expressed as

$$\begin{aligned}
J_2^* &= \frac{1}{(1-\alpha)^2} \sum_{i,j=1}^{N_z} \frac{(s_i-1)(1-s_j)(|s_i|^2-1)(1-|s_j|^2)}{(\bar{s}_i-1)(\bar{s}_j-1)(\bar{s}_i s_j - 1)} \\
&\quad \times \varsigma_i^H (s_i)^{\tau} H_i^H \Lambda A^{-H}(s_i) F^{-H}(s_i) M^{-H}(s_i) N_m^H(s_i) (s_j)^{\tau} N_m(s_j) M^{-1}(s_j) F^{-1}(s_j) A^{-1}(s_j) \Lambda H_j \varsigma_j \varsigma_j^H G_j G_i^H \varsigma_i .
\end{aligned}$$

This completes the calculation of J_2^*. Next, we calculate J_3^*. From the double Bezout equation: $z^{-\tau} N \tilde{Y} W B^{-1} = -W B^{-1} + R_3$, where $R_3 \in RH_\infty$, thus we can obtain:

$$\begin{aligned}
J_3^* &= \inf_{R\in RH_\infty} \left\| -W B^{-1} + R_3 - z^{-\tau} N R M_m \right\|_2^2 \\
&= \inf_{R\in RH_\infty} \left\| W - (R_3 - z^{-\tau} N R M_m) B \right\|_2^2 \\
&= \inf_{R\in RH_\infty} \left\{ \left\| W (B_{N_p}^{-1} - B_{N_p}^{-1}(\infty)) \right\|_2^2 + \left\| W B_{N_p}^{-1}(\infty) (B_{N_p-1}^{-1} - B_{N_p-1}^{-1}(\infty)) \right\|_2^2 \right. \\
&\quad \left. + \cdots + \left\| W B_{N_p}^{-1}(\infty) \cdots B_2^{-1}(\infty) (B_1^{-1} - B_1^{-1}(\infty)) \right\|_2^2 + \left\| W\theta - R_3 + z^{-\tau} N R M_m \right\|_2^2 \right\} ,
\end{aligned}$$

where $\theta = \prod_{l=1}^{N_p} B_{N_p+1-l}^{-1}(\infty) = \prod_{l=1}^{N_p} (I - (1+\overline{p}_{N_p+1-l}) \omega_{N_p+1-l} \omega_{N_p+1-l}^H)$.

Notably, $(B_j^{-1} - B_j^{-1}(\infty)) \in H_2^{\perp} (j = 1, \cdots, N_p)$, and an appropriate controller parameter R can be found to ensure that $W\theta - R_3(z) + z^{-\tau} N R M_m \in H_2$ is satisfied. Hence, we define $J_3^* = J_{31}^* + J_{32}^*$, where

$$J_{31}^* = \left\| W (B_{N_p}^{-1} - B_{N_p}^{-1}(\infty)) \right\|_2^2 + \left\| W B_{N_p}^{-1}(\infty) (B_{N_p-1}^{-1} - B_{N_p-1}^{-1}(\infty)) \right\|_2^2$$

$$
\begin{aligned}
&+\cdots+\left\|WB_{N_p}^{-1}(\infty)\cdots B_2^{-1}(\infty)(B_1^{-1}-B_1^{-1}(\infty))\right\|_2^2,\\
J_{32}^* &= \inf_{R\in RH_\infty}\left\|W\theta-R_3+z^{-\tau}NRM_m\right\|_2^2.
\end{aligned}
$$

By a direct calculation, we can obtain

$$
\begin{aligned}
J_{31}^* &= (\left|p_{N_p}\right|^2-1)\left\|W\omega_{N_p}\right\|_2^2+(\left|p_{N_p-1}\right|^2-1)\left\|W(I-(1+\overline{p}_{N_p})\omega_{N_p}\omega_{N_p}^H)\omega_{N_p-1}\right\|_2^2\\
&\quad+\cdots+(|p_1|^2-1)\left\|W\prod_{k=1}^{N_p-1}(I-(1+\overline{p}_{N_p+1-k})\omega_{N_p+1-k}\omega_{N_p+1-k}^H)\omega_1\right\|_2^2\\
&=\sum_{i=1}^{N_p}(\left|p_{N_p+1-i}\right|^2-1)\left\|W\zeta_{N_p+1-i}\right\|_2^2,
\end{aligned}
$$

where $\zeta_{N_p+1-i}=\begin{cases}\omega_{N_p} & i=1,\\ (\prod_{l=1}^{i-1}(I-(1+\overline{p}_{N_p+1-l})\omega_{N_p+1-l}\omega_{N_p+1-l}^H))\omega_{N_p+1-i} & i=2,\cdots,N_p.\end{cases}$

Next, we calculate J_{32}^*. Because $z^{-\tau}$ is the all-pass factor, we have

$$
\begin{aligned}
J_{32}^* &= \inf_{R\in RH_\infty}\left\|z^\tau L^{-1}(W\theta-R_3)+N_mRM_m\right\|_2^2\\
&= \inf_{R\in RH_\infty}\left\|\sum_{i=1}^{N_z}(s_i)^\tau E_iL_i^{-1}T_i(W\theta-R_3(s_i))+R_4+N_mRM_m\right\|_2^2\\
&= \inf_{R\in RH_\infty}\left\|\sum_{i=1}^{N_z}(s_i)^\tau E_i(L_i^{-1}-L_i^{-1}(\infty))T_i(W\theta-R_3(s_i))+R_5+N_mRM_m\right\|_2^2,
\end{aligned}
$$

where $R_4,\ R_5\in RH_\infty$, and $E_i=\left(\prod_{h=i+1}^{N_z}L_h^{-H}(s_i)\right)^H$, $T_i=\left(\prod_{h=1}^{i-1}L_h^{-H}(s_i)\right)^H$.

The appropriate $R\in RH_\infty$ can be found to satisfy $R_5+N_mRM_m\in H_2$. Note that $(s_i)^\tau E_i(L_i^{-1}-L_i^{-1}(\infty))T_i(W\theta-R_3(s_i))\in H_2^\perp$, thus we have

$$
J_{32}^*=\left\|\sum_{i=1}^{N_z}(s_i)^\tau E_i(L_i^{-1}-L_i^{-1}(\infty))T_i(W\theta-R_3(s_i))\right\|_2^2+\inf_{R\in RH_\infty}\|R_5+N_mRM_m\|_2^2.
$$

Note that an appropriate R can be chosen such that $\inf_{R\in RH_\infty}\|R_5+N_mRM_m\|_2^2=0$, then we have

$$
J_{32}^*=\left\|\sum_{i=1}^{N_z}(s_i)^\tau E_i(L_i^{-1}-L_i^{-1}(\infty))T_i(W\theta-R_3(s_i))\right\|_2^2,
$$

we can calculate J_{32}^* in a manner similar to the calculation of J_2^*. It then follows that

$$
\begin{aligned}
J_{32}^* &= \sum_{i,j=1}^{N_z}\frac{(1-s_i)(1-s_j)(1-|s_i|^2)(1-|s_j|^2)}{(1-\overline{s}_i)(1-\overline{s}_j)(\overline{s}_is_j-1)}\\
&\quad\times\eta_i^H(s_i)^\tau E_i^H(s_j)^\tau E_j\eta_j\eta_j^HT_jW(\theta-B^{-1}(s_j))(\theta-B^{-1}(s_i))^HWT_i^H\eta_i.
\end{aligned}
$$

This completes the proof.

Appendix C Illustrative example

This section discusses an illustrative example, to apply leader-follower systems [2], to verify the accuracy of the conclusions derived in Theorems 1. For a given MIMO plant with the following transfer function matrix

$$
G(z)=\begin{pmatrix}\frac{1}{z+0.2} & 0\\ 0 & \frac{z-k}{(z+0.2)(z-3)}\end{pmatrix},
$$

It is obvious that $G(z)$ has a nonminimum phase zero $z=k$ ($|k|>1$), the input zero direction and output zero direction of the NMP zero are both $\eta=(0,1)^T$, and an unstable pole $p=3$, it is polar direction is $\omega=(0,1)^T$. In addition, assume

$$
A(z)=\begin{pmatrix}\frac{1}{z+0.3} & 0\\ 0 & \frac{1}{z+0.2}\end{pmatrix},F(z)=\begin{pmatrix}\frac{\mu}{z+\mu} & 0\\ 0 & \frac{\mu}{z+\mu}\end{pmatrix}.
$$

Assume $U=diag(1,2)$, $W=I$, $\varepsilon=\frac{1}{2}$, $\Gamma=2$, $\tau=0.6$, $\mu=0.5$, $q=\frac{1}{3}$, and select

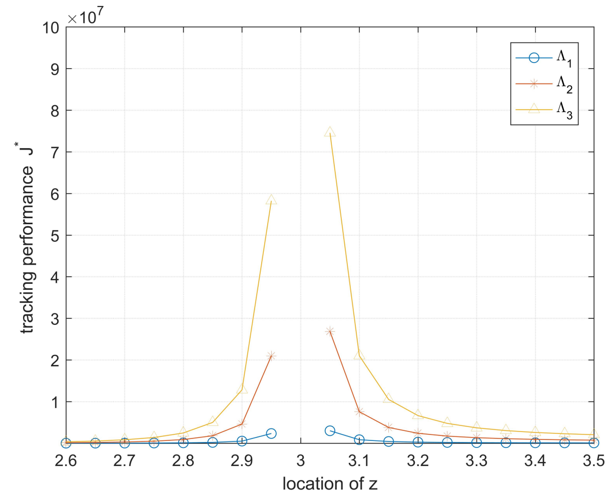

Figure C1 Tracking performance J^* under the influence of different quantization interval $\Lambda \in \{\Lambda_1, \Lambda_2, \Lambda_3\}$ and $k \in (1, +\infty)$.

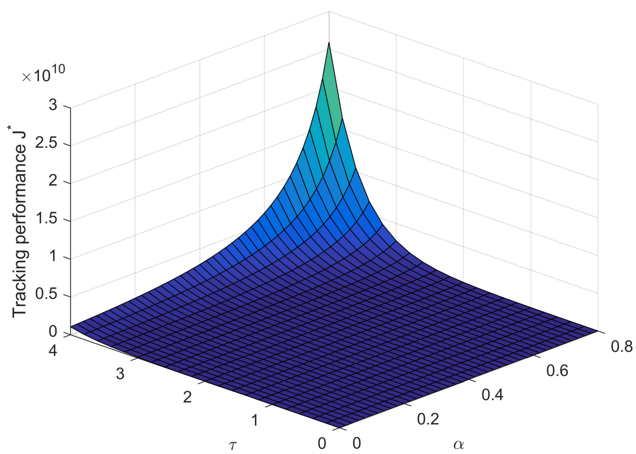

Figure C2 Tracking performance J^* under packet-dropouts $\alpha \in [0, 0.8]$ and time delay $\tau \in [0, 4]$.

$$\Lambda_1 = I,\ \Lambda_2 = \begin{pmatrix} 2 & \\ & 3 \end{pmatrix},\ \Lambda_3 = \begin{pmatrix} 4 & \\ & 5 \end{pmatrix}.$$

We mainly discuss the influence of quantization error on the tracking performance in Fig. C1 and the influence of packet dropouts and time delay on the tracking performance in Fig. C2.

In Fig. C1, it is evident that quantization error affects the tracking performance of NCSs, and the power spectral density(PSD) of the quantization error is negatively correlated with the tracking performance of the systems. In other words, the larger the PSD of the quantization error, the worse the tracking performance of the systems. We also noted that the tracking performance of the systems deteriorates dramatically when the zeros and poles of the plant are sufficiently close. In the actual design of NCSs, such proximity should be avoided. A similar conclusion can be drawn from Fig. C2, and the packet dropouts and time delay in communication channels will significantly affect the optimal tracking performance of the systems. The packet dropouts and time delay are negatively correlated with the tracking performance of the systems; the larger the packet dropouts probability and time delay, the worse the tracking performance.

In [3], the authors present detailed applications of theoretical results to a tracking problem, in which an inverted pendulum system mounted on a motor-driven cart is considered. We assumed that the pendulum moved only in the vertical plane.

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