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Predefined-time stabilization for nonlinear stochastic Itô systems

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Abstract In this paper, a control scheme for stochastic predefined-time stabilization of continuous-time stochastic Itô systems is proposed. Compared with stochastic finite-time or fixed-time stabilization, the proposed control scheme for stochastic predefined-time stabilization allows the upper bound of the mean value of the settling-time function to lie below an arbitrarily given positive value. Some Lyapunov-type results for predefined-time stabilization of general stochastic Itô systems are presented. Moreover, a state feedback control scheme is designed for a class of stochastic nonlinear systems in strict-feedback form. Two simulation examples are provided to show the usefulness of the proposed stochastic predefined-time stabilization.

Keywords stochastic predefined-time stabilization, nonlinear systems, stochastic systems, settling-time function

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1 Introduction

The study of the convergence time of dynamic systems has not only practical importance but also theoretical value. To satisfy some engineering requirements, we need to control the convergence speed to a faster or slower rate. Different from Lyapunov stability, which involves studying a system's asymptotic behavior in an infinite time horizon, finite-time or fixed-time stability investigates the transient response of a system state in a finite-time horizon. Finite-time control has important applications in robot manipulators [1]. This has given rise to extensive investigations on adaptive finite-time tracking control [2], fixed-time stabilization [3], and global fast finite-time stabilization of high-order nonlinear systems [4] in recent years. However, in many practical applications, it is expected that the state trajectory of a controlled system can converge to the equilibrium point at any admissible time by appropriately adjusting control parameters [5, 6]. A new concept called predefined-time stability was proposed in [7], which can achieve some consequences that cannot be provided by the traditional finite-time control schemes, such as the arbitrarily adjustable upper bound of convergence time independent of the initial value. Recently, great progress has been achieved for predefined-time control of deterministic nonlinear systems [8–11].

Because stochastic systems have wide applications, stochastic analysis and synthesis have been popular research areas over the past few decades [12–25]. Hence, it has become very valuable to study the finite-time convergence behavior of stochastic systems and to generalize the abovementioned studies to stochastic systems. Various researchers have recently conducted extensive studies on stochastic finite-time stability/stabilization [14,15,20–23] and stochastic fixed-time stability/stabilization [19,24,26]. Stochastic finite-time stability was first strictly analyzed in [21], the results of which highly likely prove that the stochastic settling-time function $T(x_0)$ is finite and that the system is stable in probability. Additionally, if we have $\sup_{x_0\in\mathcal{R}^n\setminus\{0\}}\mathcal{E}T(x_0)\leqslant T_{\max}$ for some $T_{\max}>0$, then the stochastic uncontrolled system would be what is called fixed-time stable. We expect that arbitrarily adjusting the upper bound of the mathematical expectation of the settling-time function by a control input would allow us to obtain a fast

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convergence speed. Therefore, it is necessary to generalize the predefined-time concept of deterministic systems to stochastic systems. However, to the best of our knowledge, no study has been conducted on stochastic predetermined-time stabilization based on stochastic settling-time function. What only exists in the literature is a recent work [18] on stochastic nonlinear prescribed-time stabilization in the mean square sense.

This paper investigates the problem of stochastic predefined-time stabilization of nonlinear stochastic Itô systems. The main contributions of this study are highlighted as follows:

(1) Stochastic predefined-time stabilization is introduced and a stochastic predefined-time stabilization theorem is obtained for general nonlinear stochastic Itô systems; see Theorem 1. Because predefined-time stabilization is stronger than finite-time stabilization and fixed-time stabilization, Theorem 1 is applicable to stochastic finite-time/fixed-time stabilization of [19, 21, 24], which can also be viewed as an extension of [8, 27] to stochastic systems. Theorem 1 is a Lyapunov-type theorem similar to the results of [21, 23]. Theorem 1 yields some useful corollaries that can be conveniently used in practice.

(2) The high-order nonlinear stochastic system is an important class of stochastic systems that can be used to describe many phenomena arising from mechanical systems, and its feedback stabilization has been investigated in [12, 19, 20]. Based on the results in Section 2, this paper also studied the predefined-time stabilization of high-order nonlinear stochastic systems and came up with a practical controller design algorithm. Through the addition of a power integrator technique together with a corollary given in Section 2, a state feedback control scheme is given for a class of high-order stochastic nonlinear systems in strict-feedback form, which guarantees that the closed-loop system is stochastically predefined-time stabilizable. In addition, the proposed stochastic predefined-time stabilization scheme can improve existing results [14, 15, 19].

The rest of this paper is organized as follows: Section 2 makes some preliminaries by introducing some new definitions, theorems, and corollaries. Lyapunov-type theorems about the stochastic predetermined-time stabilization are obtained. Section 3 presents the controller design procedure for high-order nonlinear stochastic systems and makes some comparisons with existing studies. Section 4 provides two simulation examples to show the effectiveness of our main results. Section 5 concludes this paper with some remarks.

2 Stochastic predefined-time stabilization

In this section, we will introduce a concept called "stochastic predefined-time control" and some useful criteria. Before that, we first introduce the following notations to be used: $\mathcal{R}^n$ denotes the n-dimensional real Euclidean vector space. $\mathcal{R}_+ := [0, \infty)$. D' is the transpose of a matrix or vector D. $\mathcal{C}^2$ stands for the set of real-valued twice continuously differentiable functions. $\mathrm{sign}(x) := 1$ for $x > 0$, -1 for $x < 0$, and 0 for $x = 0$. For any $b \geqslant 0$, $a \in \mathcal{R}$, the function $\lceil a \rfloor^b$ is defined as $\lceil a \rfloor^b = \mathrm{sign}(a)|a|^b$. $\mathcal{E}$ means the mathematical expectation operator. $\mathcal{P}(A)$ represents the probability of event A. $I_A(x)$ is the indicator function, i.e., $I_A(x) = 1$ for $x \in A$, otherwise, $I_A(x) = 0$. $\mathcal{K}^1$-functions: a scalar continuous function f defined from $\mathcal{R}_+$ to $[0, 1)$ is said to be a $\mathcal{K}^1$-function if it is strictly increasing, $f(0) = 0$, and $f(x) \to 1$ as $x \to \infty$.

We will consider the following continuous-time stochastic Itô system:

$$\begin{aligned} \mathrm{d}x(t) &= f(x(t), u(t))\,\mathrm{d}t + g(x(t), u(t))\,\mathrm{d}w(t), \\ x(0) &= x_0 \in \mathcal{R}^n \setminus \{0\}, \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ represents the system state. $u \in \mathcal{R}^{n_u}$ stands for the control input. $w(t)$ is a standard one-dimensional Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, \mathcal{P})$. The admissible control set $\mathcal{U}$ consists of all $\mathcal{F}_t$-adaptive control processes $u(x(t))$, $u(0) = 0$. In the considered system, we assume that $f : \mathcal{R}^n \times \mathcal{R}^{n_u} \to \mathcal{R}^n$ and $g : \mathcal{R}^n \times \mathcal{R}^{n_u} \to \mathcal{R}^n$ are continuous in x and u satisfying $f(0, 0) = 0$ and $g(0, 0) = 0$. The purpose of this section is to find an admissible control law $u^*(x) \in \mathcal{U}$ to stabilize the stochastic system (1) before a predefined time, i.e., achieve the stochastic predefined-time stabilization of system (1).

Remark 1. When studying finite-time stable systems, we are interested in having a unique solution in forward time [28], which means that, for any non-zero initial condition x_0, $x(t)$ is unique before reaching 0. For stochastic finite-time stable systems, the concept is extended to the solution in the weak sense [23]. Based on the assumption of $f(0, 0) = 0$ and $g(0, 0) = 0$, the origin is an equilibrium point of (1). The following lemma gives an existence result of a solution to system (1).

Lemma 1. Ref. [23] supposed that there exists a nonnegative radially unbounded $\mathcal{C}^2$-function $V(x)$ for system (1). If $\mathcal{L}V(x) \leqslant 0$, then system (1) has a regular continuous solution for any initial value.

Remark 2. The regular solution means that there is no finite explosion time with probability 1.

Now, we first introduce the following two definitions.

Definition 1. System (1) is said to be stochastically finite-time stabilizable or finite-time stabilizable in probability, if there exists a state feedback control $u(t) = u^*(x(t)) \in \mathcal{U}$, such that the closed-loop system

$$\begin{aligned} &\mathrm{d}x(t) = f(x(t), u^*(x(t)))\,\mathrm{d}t + g(x(t), u^*(x(t)))\,\mathrm{d}w(t), \\ &x(0) = x_0 \in \mathcal{R}^n \setminus \{0\}, \end{aligned} \tag{2}$$

is stochastically finite-time stable, i.e.,

- Finite-time attractiveness: for any non-zero initial value x_0, there exists the settling-time function $T(x_0) := \inf\{t : x(t, u^*, x_0) = 0\}$, such that

$$\mathcal{P}(T(x_0) < \infty) = 1,$$

where $x(t, u^*, x_0)$ is the solution of (2). Moreover, for any $t \geqslant 0$, $x(t + T(x_0), u^*, x_0) \equiv 0$ almost surely (a.s.);

- Stability in probability: for any pair (α, β), $\alpha \in (0, 1)$, $\beta > 0$, there exists a $\delta(\alpha, \beta) > 0$, such that for all $\|x_0\| < \delta(\alpha, \beta)$, we have

$$\mathcal{P}(\|x(t, u^*(x(t)), x_0)\| < \beta, \forall t > 0) \geqslant 1 - \alpha.$$

Moreover, if there exists a positive constant $T_{\max}$ such that

$$\sup_{x_0 \in \mathcal{R}^n \setminus \{0\}} \mathcal{E}T(x_0) \leqslant T_{\max},$$

then system (2) is said to be stochastically fixed-time stable, and system (1) is said to be stochastically fixed-time stabilizable.

Remark 3. In recent literature, new and improved definitions called predefined-time stable and stabilizable were proposed for deterministic systems based on fixed-time stability [7, 8]. This paper aims to extend these concepts and related results to stochastic system (1).

Definition 2. System (1) is said to be stochastically predefined-time stabilizable if it is not only stochastically fixed-time stabilizable but also for any $\varsigma > 0$, there exists a control $u(t) = u^*(x(t)) \in \mathcal{U}$, such that

$$\sup_{x_0 \in \mathcal{R}^n \setminus \{0\}} \mathcal{E}T(x_0) \leqslant \varsigma.$$

Remark 4. Obviously, a main difference between stochastic predefined-time stabilization and stochastic fixed-time stabilization exists in that whether the upper bound $\sup_{x_0 \in \mathcal{R}^n \setminus \{0\}} \mathcal{E}T(x_0)$ can be arbitrarily adjusted by selecting a suitable admissible control $u^*(x(t)) \in \mathcal{U}$.

Remark 5. Recently, another newly proposed definition called prescribed-time mean-square stability was introduced [18], which is different from Definition 2. Definition 2 is based on a stochastic settling-time function. When the system (1) degenerates into a deterministic system, Definition 2 is consistent with the corresponding definition of deterministic systems.

Remark 6. From Definitions 1 and 2, for the system (1) with non-zero initial value, there must be $x(t + T(x_0), u^*, x_0) \equiv 0$ with $t \geqslant 0$ and $T(x_0)$ being the stochastic settling-time function. Therefore, the origin is an absorbing state. Moreover, without loss of generality, when discussing the predefined-time stabilization or finite-time stabilization problem, a non-zero initial value is usually assumed [8, 22].

The following theorem is a Lyapunov-type theorem about stochastic predefined-time stabilization.

Theorem 1. For any $\alpha > 0$, there exists a control input $u^*(x(t)) \in \mathcal{U}$, driving the state of system (1) to satisfy

$$\mathcal{L}_{u=u^*} V(x) \leqslant -\frac{1}{\alpha}\beta(V(x)),\ x \in \mathcal{R}^n \setminus \{0\}, \tag{3}$$

where $\beta : \mathcal{R}_+ \mapsto \mathcal{R}_+$ with $\dot{\beta}(\cdot) \geqslant 0$, $\beta(s) > 0$ for any $s > 0$, and $\int_0^\infty \frac{1}{\beta(s)}\,\mathrm{d}s \leqslant 1$. $V : \mathcal{R}^n \mapsto \mathcal{R}_+$ is a $\mathcal{C}^2$-positive definite and radially unbounded function, and $\mathcal{L}_{u=u^*}$ represents the infinitesimal generator of system (2). Then system (1) is stochastically predefined-time stabilizable, and $\sup_{x_0 \in \mathcal{R}^n \setminus \{0\}} \mathcal{E}T(x_0) \leqslant \alpha$.

Proof. The proof is given in Appendix A.

Remark 7. In inequality (3) of Theorem 1, the zero point is excluded due to the following reasons: (i) Because the zero point is assumed to be an equilibrium point, $\mathcal{E}T(0) = 0 < \zeta$ for any $\zeta > 0$. Hence, in the study on finite-time stability and stabilization, $x = 0$ is a trivial case and can be excluded. This is why we assume $x_0 \neq 0$ in system (1) and Definition 2. (ii) The removal of $x = 0$ is also to avoid the division by zero when estimating $\mathcal{L}_{u=u^*}V(x)$.

Generally speaking, a stochastic fixed-time stabilizing controller is not necessarily a stochastic predefined-time stabilizing controller; see the following remark.

Remark 8. Consider a scalar system:

$$\mathrm{d}x = u\mathrm{d}t + x\mathrm{d}w,\ x(0) = x_0 \neq 0. \tag{4}$$

By selecting the control input

$$u = u^* = -\frac{1}{2}x - x^a - x^b, \tag{5}$$

with $a \in (0, 1)$ and $b > 1$, the feedback system of (4) becomes

$$\mathrm{d}x = \left(-\frac{1}{2}x - x^a - x^b\right)\mathrm{d}t + x\,\mathrm{d}w,\ x(0) = x_0 \neq 0. \tag{6}$$

For system (6) and the Lyapunov function $V(x) = x^2$, it is easy to compute

$$\mathcal{L}_{u=u^*}V(x) = -2x^{a+1} - 2x^{b+1},$$

where $\mathcal{L}_{u=u^*}$ is the infinitesimal generator of system (6). According to Corollary 3.4 of [24], we have

$$\sup_{x_0 \neq 0} \mathcal{E}T(x_0) \leqslant S(a, b) := \frac{1}{1-a} + \frac{1}{b-1}.$$

Furthermore, $\frac{\partial S}{\partial a} > 0$ for $a \in (0, 1)$ and $\frac{\partial S}{\partial b} < 0$ for $b > 1$. Therefore, we can find that the minimum upper limit of the mean value of the settling-time function is $\inf_{a\in(0,1),b>1} S(a, b) = 1$. Hence, system (4) is stochastically fixed-time stabilizable.

In addition, for any $\alpha > 0$, if we choose the control input as

$$u = u^*(x) = -\frac{\sqrt{\pi}}{2\alpha}\mathrm{sign}(x)\mathrm{e}^{x^2} - \frac{1}{2}x, \tag{7}$$

and a $\mathcal{C}^2$-positive definite and radially unbounded function $V(x) = x^2$, then by Itô formula, we have

$$\begin{aligned}\mathcal{L}_{u=u^*}V(x) &= 2xu^* + x^2 = x\left(-\frac{\sqrt{\pi}}{\alpha}\mathrm{sign}(x)\mathrm{e}^{x^2}\right)\\ &= -\frac{\sqrt{\pi}}{\alpha}|x|\mathrm{e}^{x^2} = -\frac{\sqrt{\pi}}{\alpha}V(x)^{1/2}\mathrm{e}^{V(x)}.\end{aligned}$$

Let $\beta(s) = \sqrt{\pi}s^{1/2}\mathrm{e}^s$; then $\beta(s)$ satisfies the conditions of Theorem 1. By Theorem 1, system (4) is also stochastically predefined-time stabilizable, and Eq. (7) is a stochastic predefined-time stabilizing controller.

Remark 9. System (4) is not only stochastically fixed-time stabilizable but also stochastically predefined-time stabilizable with (5) a fixed-time stabilizing controller and (7) a predefined-time stabilizing controller. Fixed-time stabilizing controller (5) cannot make the upper bound $\sup_{x_0\in\mathcal{R}\setminus\{0\}} \mathcal{E}T(x_0)$ arbitrarily small. On this point, we refer the reader to [8, 11]. It is easy to construct a counterexample that is stochastically fixed-time stabilizable but is not stochastically predefined-time stabilizable. It can be found that, although there are some errors in the mathematical derivations in [24], Corollary 3.4 of [24] is correct.

Corollary 1. For any $\alpha > 0$, there exists a control input $u^*(x(t)) \in \mathcal{U}$, driving the state of system (1) to satisfy

$$\mathcal{L}_{u=u^*}V(x) \leqslant -\frac{\tilde{\beta}(V(x))^p}{(1-p)\alpha\dot{\tilde{\beta}}(V(x))},\ x \in \mathcal{R}^n \setminus \{0\}, \tag{8}$$

where $\tilde{\beta}(\cdot)$ is a function belonging to $\mathcal{K}^1$ with $\ddot{\tilde{\beta}}(\cdot) \leqslant 0$, $0 \leqslant p < 1$. $V : \mathcal{R}^n \mapsto \mathcal{R}_+$ is a $\mathcal{C}^2$-positive definite and radially unbounded function. Then system (1) is stochastically predefined-time stabilizable, and for any $x_0 \in \mathcal{R}^n \setminus \{0\}$, the settling-time function satisfies $\mathcal{E}T(x_0) \leqslant \alpha$.

Proof. Corollary 1 can be directly obtained by Theorem 1 via setting

$$\beta(s) = \frac{\tilde{\beta}(s)^p}{(1-p)\dot{\tilde{\beta}}(s)}.$$

We can also give another Lyapunov-type theorem inspired by [27].

Corollary 2. For any $\alpha > 0$, there exists a control input $u^*(x(t)) \in \mathcal{U}$, driving the state of system (1) to satisfy

$$\mathcal{L}_{u=u^*} W(x) \leqslant -\frac{1}{\alpha},\ x \in \mathcal{R}^n \setminus \{0\}, \tag{9}$$

where $W : \mathcal{R}^n \mapsto \mathcal{R}_+$ is a $\mathcal{C}^2$-positive definite function with $0 \leqslant W(s) < 1$, $\lim_{s\to\infty} W(s) = 1$. Then system (1) is stochastically predefined-time stabilizable, and for any $x_0 \in \mathcal{R}^n \setminus \{0\}$, the settling-time function satisfies $\mathcal{E}T(x_0) \leqslant \alpha$.

Proof. Set $V(x) = -\ln(1 - W(x))$. By (9), it is easy to obtain that

$$\mathcal{L}_{u=u^*} W(x) = \mathrm{e}^{-V(x)} \mathcal{L}_{u=u^*} V(x) \leqslant -\frac{1}{\alpha}. \tag{10}$$

This gives

$$\mathcal{L}_{u=u^*} V(x) \leqslant -\frac{1}{\alpha}\mathrm{e}^{V(x)}.$$

By setting $\beta(s) = \mathrm{e}^s$, and using Theorem 1, we have the desired results immediately.

Corollary 3. Assume that there exist a $\mathcal{C}^2$-positive definite and radially unbounded function V, positive constants $a > 0$, $0 < b_1 < 1$, and $b_2 > 1$ satisfying $a - ab_1 - 1 > 0$. If for any $\alpha > 0$, there exists a control input $u^*(x(t)) \in \mathcal{U}$, such that

$$\mathcal{L}_{u=u^*} V(x) \leqslant -\frac{a - ab_1}{\alpha(a - ab_1 - 1)(b_2 - 1)} V(x)^{b_2} - \frac{a}{\alpha} V(x)^{b_1}, \tag{11}$$

for $x \in \mathcal{R}^n \setminus \{0\}$, then system (1) is stochastically predefined-time stabilizable, i.e., $\sup_{\forall x_0 \in \mathcal{R}^n\setminus\{0\}} \mathcal{E}T(x_0) \leqslant \alpha$.

Proof. In Theorem 1, let

$$\beta(V(x)) = aV(x)^{b_1} + \frac{a - ab_1}{(a - ab_1 - 1)(b_2 - 1)} V(x)^{b_2}.$$

Obviously, $\beta(s) \geqslant 0$ for $s \in \mathcal{R}_+$, and $\dot{\beta}(\cdot) \geqslant 0$. Moreover,

$$\begin{aligned}\int_0^\infty \frac{1}{\beta(s)}\mathrm{d}s &= \int_0^1 \frac{1}{\beta(s)}\mathrm{d}s + \int_1^\infty \frac{1}{\beta(s)}\mathrm{d}s \leqslant \int_0^1 \frac{1}{as^{b_1}}\mathrm{d}s + \int_1^\infty \frac{(a - ab_1 - 1)(b_2 - 1)}{(a - ab_1)s^{b_2}}\mathrm{d}s \\ &= \frac{1}{(1 - b_1)a} + \frac{a - ab_1 - 1}{a - ab_1} = 1.\end{aligned}$$

Therefore, Eq. (11) implies that conditions of Theorem 1 are satisfied. This corollary is proven.

3 Controller design in strict-feedback form

3.1 Main results

In this subsection, combining Corollary 3 and the backstepping method, we give a stochastic predefined-time stabilizing controller design scheme for a class of high-order stochastic nonlinear systems. Consider the following high-order stochastic nonlinear system described by

$$\begin{cases} \mathrm{d}x_i = (h_i \lceil x_{i+1} \rfloor^{q_i} + f_i(\bar{x}_i))\mathrm{d}t + g_i(\bar{x}_i)\mathrm{d}w, \\ \mathrm{d}x_n = (h_n \lceil u \rfloor^{q_n} + f_n(x))\mathrm{d}t + g_n(x)\mathrm{d}w, \end{cases} \tag{12}$$

where $i = 1, 2, \ldots, n-1$, $x = [x_1, \ldots, x_n]' \in \mathcal{R}^n$ stands for the system state, and $u \in \mathcal{R}$ is the control input. $\bar{x}_i$ is defined as $\bar{x}_i = [x_1, \ldots, x_i]'$. $h_1, \ldots, h_n$ are unknown virtual control coefficients. The drift terms $f_i(x)$ and diffusion terms $g_i(x)$, $i = 1, 2, \ldots, n$, are continuous functions with $f_i(0) = 0$ and $g_i(0) = 0$. For any i, $1 < q_i \in \mathcal{R}_+$ is called the high-order power of system (12).

The following assumption allows the functions $f_i(\bar{x}_i)$ and $g_i(\bar{x}_i)$ to have high-order and low-order nonlinear growth rates.

Assumption 1. For any $i = 1, 2, \ldots, n$, the drift terms $f_i(\bar{x}_i)$ and the diffusion terms $g_i(\bar{x}_i)$ satisfy the following conditions:

$$|f_i(\bar{x}_i)| \leqslant \phi_i(\bar{x}_i) \sum_{j=1}^{i} |x_j|^{\varpi_{ij} + \frac{r_i+\kappa}{r_j}},$$

$$|g_i(\bar{x}_i)| \leqslant \varphi_i(\bar{x}_i) \sum_{j=1}^{i} |x_j|^{\rho_{ij} + \frac{2r_i+\kappa}{2r_j}},$$

where $\varpi_{ij} \geqslant 0$, $\rho_{ij} \geqslant 0$, $\phi_i(\bar{x}_i)$, and $\varphi_i(\bar{x}_i)$ are nonnegative smooth functions with $\phi_i(0) = 0$ and $\varphi_i(0) = 0$, $\kappa \in (-\frac{1}{1+\sum_{s=1}^{n-1} q_1 \cdots q_s}, 0)$. r_{i+1} is recursively defined as $r_{i+1} = \frac{r_i+\kappa}{q_i}$ with $r_1 = 1$.

Remark 10. Assumption 1 is a common nonlinear growth rate [4,14]. The powers in growth condition of $f_i(\bar{x}_i)$ and $g_i(\bar{x}_i)$ are defined as $\varpi_{ij} + \frac{r_i+\kappa}{r_j} \in \mathcal{R}_+$ and $\rho_{ij} + \frac{2r_i+\kappa}{2r_j} \in \mathcal{R}_+$, respectively. $\varpi_{ij} + \frac{r_i+\kappa}{r_j}$ and $\rho_{ij} + \frac{2r_i+\kappa}{2r_j}$ belonging to an interval $(0, \infty)$ allow $f_i(\bar{x}_i)$ and $g_i(\bar{x}_i)$ to have both high-order and low-order nonlinear growth rates. Assumption 1 is very general than the previous assumptions. For example, when $\varpi_{ij} = \rho_{ij} = 0$, Assumption 1 degenerates to Assumption 1 in [13]. If $\phi_i(\bar{x}_i)$ and $\varphi_i(\bar{x}_i)$ are further specialized as constants, Assumption 1 becomes the low-order growth rate used in [20]. When $\varpi_{ij} = 1 - \frac{r_i+\kappa}{r_j}$ and $\rho_{ij} = 1 - \frac{2r_i+\kappa}{2r_j}$, Assumption 1 comes down to the linear-like growth rate of [17].

Assumption 2. For any $i = 1, 2, \ldots, n$, there exist constants $\overline{h}_i > 0$ and $\underline{h}_i > 0$ such that $\underline{h}_i \leqslant h_i \leqslant \overline{h}_i$.

Lemma 2 ([20])**.** For any $a \in \mathcal{R}$ and $b \geqslant 2$, we have that the function $f(a) := \lceil a \rfloor^b \in \mathcal{C}^2$, $\frac{\partial f(a)}{\partial a} = b|a|^{b-1}$, and $\frac{\partial^2 f(a)}{\partial a^2} = b(b-1)\lceil a \rfloor^{b-2}$.

Lemma 3 ([29])**.** Supposing $0 < p < 1$ and $q > 1$, then for any $x, y \in \mathcal{R}$, we have

$$|\lceil x \rfloor^{pq} - \lceil y \rfloor^{pq}| \leqslant 2^{1-p} |\lceil x \rfloor^q - \lceil y \rfloor^q|^p.$$

Lemma 4 ([30])**.** For any positive real numbers p, q, and any real-valued function $f(x, y) > 0$, the following relationship holds:

$$|x|^p |y|^q \leqslant \frac{p}{p+q} f(x, y) |x|^{p+q} + \frac{q}{p+q} f^{-\frac{p}{q}}(x, y) |y|^{p+q}.$$

Lemma 5 ([30])**.** For any $x, y \in \mathcal{R}$, $a \geqslant 1$, we have $(|x| + |y|)^{\frac{1}{a}} \leqslant |x|^{\frac{1}{a}} + |y|^{\frac{1}{a}} \leqslant 2^{\frac{a-1}{a}} (|x| + |y|)^{\frac{1}{a}}$.

Lemma 6 ([30])**.** $(\sum_{i=1}^{j} a_i)^b \leqslant \max\{j^{b-1}, 1\} (\sum_{i=1}^{j} a_i^b)$ always holds for any positive numbers $a_1, a_2, \ldots$, a_j, and b.

Now, we give our main theorem in this section.

Theorem 2. If the high-order stochastic nonlinear system (12) satisfies Assumptions 1 and 2, then there exists a control input $u = u^*$ such that system (12) is stochastically predefined-time stabilizable.

Proof. The proof is given in Appendix A.

3.2 Related studies and comparisons

In this subsection, we make some comparisons with related existing studies.

Based on Corollary 3 and Theorem 2, this paper gives a stochastic predefined-time stabilizing controller design scheme for strict feedback nonlinear stochastic system (12). Although there have been many articles to study the finite-time or fixed-time stabilization or tracking for strict feedback nonlinear stochastic systems [4,12,13,19], there seems no work considering the stochastic predefined-time stabilizing control problem for this class of systems. A new stabilization theory needs to be established to realize the requirements of stochastic predefined-time stabilizing control, which brings a great challenge.

In Section 2, different constructions of the function β produce different criteria (Corollaries 1–3) of stochastic predefined-time stabilization. Among these criteria, Corollaries 1 and 2 can be regarded as the stochastic version of deterministic systems presented in [8, 27], respectively. Corollaries 1 and 2 are not able to use adding power integrator technique [17, 20, 30] to deal with high-order stochastic nonlinear problems, so this paper adopts Corollary 3 in the proof of Theorem 2.

Theorem 1 presents a Lyapunov-type result about stochastic predefined-time stabilization. When $\int_0^t \frac{1}{\beta(s)}\,\mathrm{d}s < \infty$ with $t \geqslant 0$ replaces $\int_0^\infty \frac{1}{\beta(s)}\,\mathrm{d}s \leqslant 1$ in Theorem 1, then system (1) is stochastically finite-time stabilizable [21]. In addition, if the condition $\int_0^\infty \frac{1}{\beta(s)}\,\mathrm{d}s \leqslant 1$ is changed as $\int_0^\infty \frac{1}{\beta(s)}\,\mathrm{d}s < \infty$, then system (1) is stochastically fixed-time stabilizable; see Remark 9 of [23].

In [8, 27, 28, 31], some Lyapunov-type theorems for finite-/predefined-time stabilization of deterministic systems were given, and some finite-/predefined-time controllers for strict-feedback systems were studied in [10, 11, 32, 33]. The work in this paper is fundamentally different from those studies of deterministic systems both in definitions and theorem derivations. Compared with the latest research on stochastic predefined-time control, the results of this paper also have certain innovations. Refs. [34] focused on the practically predefined-time adaptive control for nonlinear stochastic systems with actuator dead zone, where the designed controller cannot address high-order systems. In addition, Lemma 2.1 in [34] can be regarded as a special case of Theorem 1.

4 Simulation examples

In this section, we present two examples to illustrate the validity of our main results.

In this example, we consider the following one-dimensional system:

$$\mathrm{d}x = (x^{\frac{5}{3}} + u)\,\mathrm{d}t + 12x\,\mathrm{d}w. \tag{13}$$

For system (13), the stochastic fixed-time design scheme of [19] cannot solve its predefined-time stabilization. Based on [8, 27], a deterministic predefined-time control can be designed as

$$u = u^* = -x^{\frac{5}{3}} - \frac{\pi}{4qT_c}(x^{1+2q} + x^{1-2q}), \tag{14}$$

where $0 < q < 0.5$ and $T_c > 0$ determine the convergence time of system (13). Setting $T_c = 4, 2, 0.5$, then the corresponding state trajectories are shown in Figure 1. In fact, when there are stochastic noises in systems, the predefined-time control for deterministic systems is no longer applicable. Therefore, the state trajectories in Figure 1 will fluctuate significantly. Through the design method proposed in Corollary 3, the stochastic predefined-time control can be constructed as

$$u = u^* = -x^{\frac{5}{3}} - 36x - \frac{k_1}{2k_4}x^{\frac{1}{3}} - \frac{k_2}{2k_4}x^{k_3}. \tag{15}$$

Set $V(x) = x^2$. By Itô formula, $\mathcal{L}_{u=u^*}V = -\frac{k_1}{k_4}x^{\frac{4}{3}} - \frac{k_2}{2k_4}x^{k_3+1} = -\frac{k_1}{k_4}V^{\frac{2}{3}} - \frac{k_2}{k_4}V^{\frac{k_3+1}{2}}$. Therefore, k_1, k_2, and k_3 in stochastic predefined-time control (15) can be chosen as $k_1 > 3$, $k_2 = \frac{2k_1}{(k_1-3)(k_3-1)}$, and $k_3 > 1$. k_4 is a positive control parameter that can be adjusted arbitrarily. Then, the mean value of the first time to reach the equilibrium point must be less than k_4. Here we choose $x(0) = 1$, $k_1 = 4.1$, $k_3 = 3$. In order to compare the convergence speed, k_4 is selected as a variety of different values of 4, 2 and 0.5. The corresponding trajectories of the states for $k_4 = 4, 2, 0.5$ are all depicted in Figure 2.

Example 1. We have completed 25 random experiments for $k_4 = 4, 2, 0.5$, respectively. From Figure 2, we can find that smaller k_4 leads to faster convergence speed. It is obvious that the rapidity of the convergence time meets the requirement of the stochastic predefined-time stabilization proposed in this paper.

Example 2. A simulation example is given to illustrate how to apply Theorem 2. Consider the following nonlinear stochastic system:

$$\begin{cases} \mathrm{d}x_1 = (h_1\lceil x_2\rfloor^{\frac{5}{3}} - \lceil x_1\rfloor^{\frac{3}{4}})\mathrm{d}t + \sin(x_1)|x_1|\mathrm{d}w, \\ \mathrm{d}x_2 = h_2\lceil u\rfloor^{\frac{4}{3}}\,\mathrm{d}t. \end{cases} \tag{16}$$

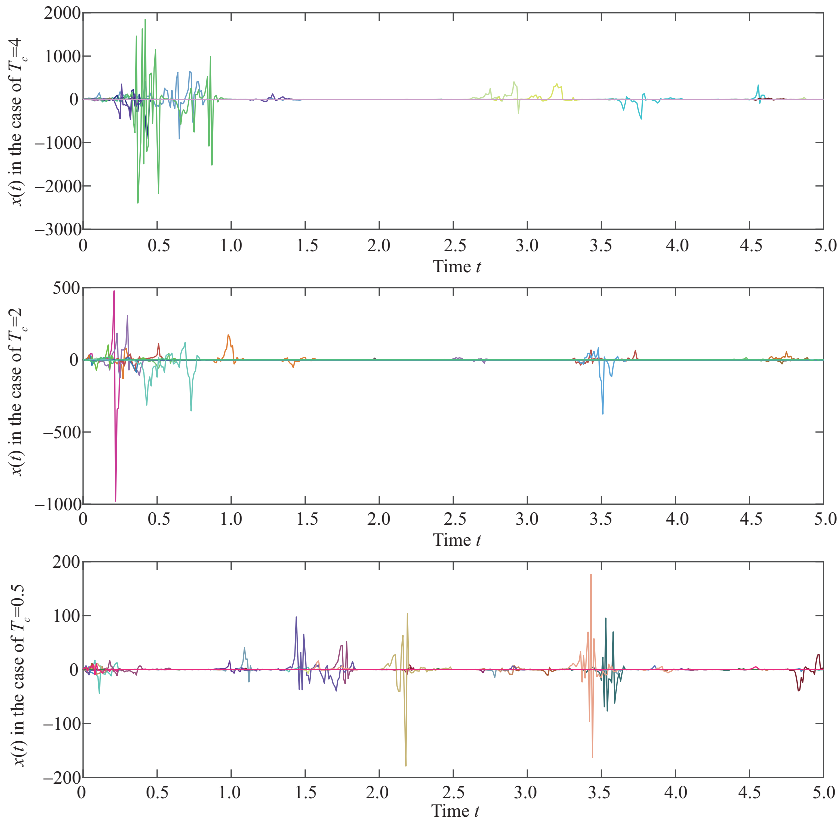

Figure 1 (Color online) The state $x(t)$ of the closed-loop system (13) with $u = u^*$ as in (14) and $T_c = 4, 2, 0.5$.

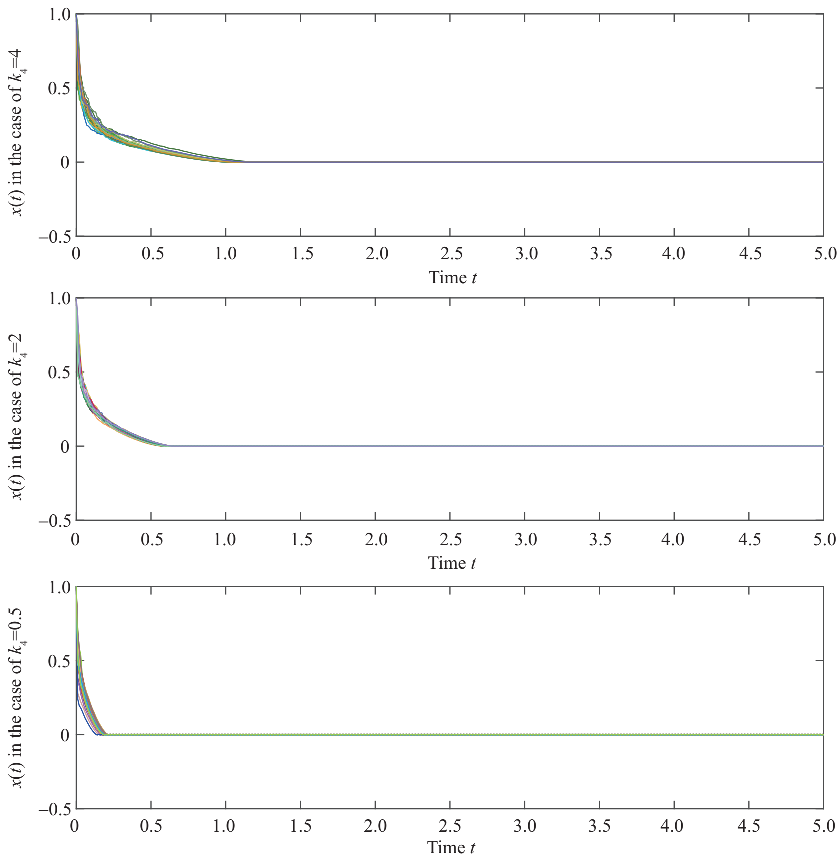

Figure 2 (Color online) The state $x(t)$ of the closed-loop system (13) with $u = u^*$ as in (15) and $k_4 = 4, 2, 0.5$.

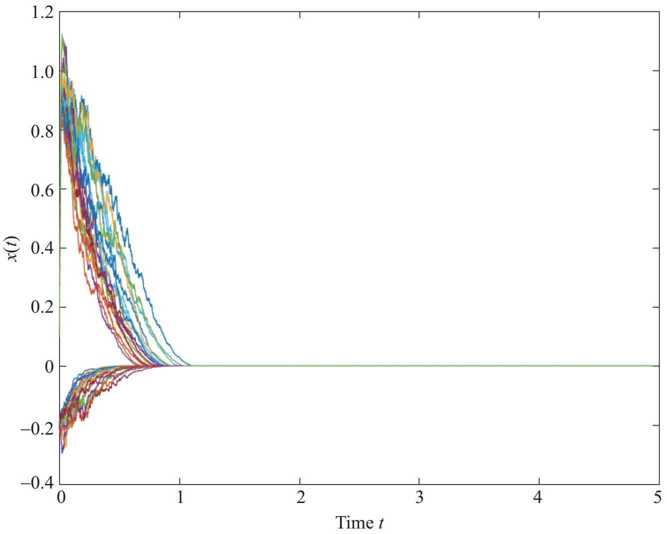

Figure 3 (Color online) The state $x(t)$ of the closed-loop system (16) with $u = u^*$ as in (17) and $k_4 = 1$.

Clearly, Assumptions 1 and 2 are satisfied with $\underline{h}_2 = 1$, $\overline{h}_2 = 2$, $\phi_1 = \varphi_1 = 1$, $\kappa = -\frac{1}{4}$, $\varpi_{11} = 0$, $\rho_{11} = \frac{1}{8}$. According to Theorem 2, we can select the control parameters as $k_1 = 65.6$, $k_2 = 494.6$, $k_3 = 3.1$, $k_4 = 3$, and the control input can be designed as

$$u = u^* = -(10.9 + 3.5\beta_1(x_1) + 82.4|\xi_2|^{0.75})\lfloor\xi_2\rceil^{\frac{1}{8}}, \tag{17}$$

with $\beta_1(x_1) = (47.1 + 165|\xi_1|^{3.1})^{0.6}$. Figure 3 shows that the designed controller renders the origin of system (16) stochastically predefined-time stabilizable with $\sup_{x_0 \in \mathcal{R}^n \setminus \{0\}} \mathcal{E}T(x_0) \leqslant k_4$.

5 Conclusion

In our study, we obtained several Lyapunov-type results about stochastic predefined-time stabilization of stochastic nonlinear systems, which extend the results of finite-time/fixed-time stabilization. Furthermore, in this paper, we present feasible conditions and a constructive solution to stochastic predefined-time stabilization of a class of stochastic nonlinear systems in strict-feedback form. Examples are provided to demonstrate the validity of the obtained results.

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Appendix A

Proof of Theorem 1. Because the origin is an equilibrium point and $u^*(0)=0$, $\mathcal{L}_{u=u^*}V(x)=0$ for $x=0$, this together with (3) yields that $\mathcal{L}_{u=u^*}V(x)\leqslant 0$ for $x\in\mathcal{R}^n$, i.e., the inequality (7) of [23] holds. In addition, under the conditions of this theorem, the inequality (8) of [23] always holds. Therefore, by Theorem 1 of [23], system (1) is stochastically finite-time stabilizable. Hence, in order to prove stochastic predefined-time stabilization of system (1), we only need to show that for any $\alpha>0$, we can always find an admissible control $u^*(x)\in\mathcal{U}$, such that the following holds:

$$\sup_{x_0\in\mathcal{R}^n\setminus\{0\}}\mathcal{E}T(x_0)\leqslant\alpha.$$

Obviously, if $x_0=0$, then $T(x_0)=0$. Therefore, we only need to consider the case of non-zero initial state. From condition (3) and Lemma 1, it leads to that for each $x_0\neq 0$, there exists a regular continuous solution $x(t,u^*,x_0)$ to (1). Therefore, there exists a positive integer $k\in\{1,2,\ldots\}$ such that $\frac{1}{k}<\|x_0\|<k$. Similar to [23], define three sequences as follows:

$$\tau_k=\inf\left\{t\geqslant 0:\|x(t,u^*(t),x_0)\|\notin\left(\frac{1}{k},k\right)\right\},$$
$$\tau_{1k}=\inf\left\{t\geqslant 0:\|x(t,u^*(t),x_0)\|\in\left[0,\frac{1}{k}\right]\right\},$$
$$\tau_{2k}=\inf\left\{t\geqslant 0:\|x(t,u^*(t),x_0)\|\in[k,\infty)\right\}.$$

Because $x(t,u^*(x(t)),x_0)$ is a regular continuous solution and $u^*(x(t))$ is $\mathcal{F}_t$-adapted, so τ_k, τ_{1k}, and τ_{2k} are $\mathcal{F}_{\tau_k}$-, $\mathcal{F}_{\tau_{1k}}$- and $\mathcal{F}_{\tau_{2k}}$-measurable, respectively, that is, they are all stopping times. For convenience, in the sequel, we define the solution $x(t,u^*(x(t)),x_0)$ as $x(t)$ for short. We introduce a new Lyapunov function $P(x)=\int_0^{V(x)}\frac{\alpha}{\beta(s)}\,\mathrm{d}s$. By Itô formula, we get that

$$\mathrm{d}P(x(t))=\mathcal{L}_{u=u^*}P(x(t))\,\mathrm{d}t+\frac{\alpha}{\beta(V(x(t)))}\frac{\partial V'}{\partial x}g(x(t),u^*(x(t)))\,\mathrm{d}w(t),$$

where $\mathcal{L}_{u=u^*}P(x)$ can be computed as

$$\mathcal{L}_{u=u^*}P(x)=\frac{\alpha\mathcal{L}_{u=u^*}V(x)}{\beta(V(x))}-\frac{\alpha\dot{\beta}(V)}{2\beta^2(V)}\left(g'(x,u^*)\frac{\partial V}{\partial x}\right)\left(\frac{\partial V'}{\partial x}g(x,u^*)\right).\tag{A1}$$

Then, similar to Theorem 3.1 in [21], it follows that

$$\mathcal{E}P(x(t\wedge\tau_k))-P(x_0)=\mathcal{E}\int_0^{t\wedge\tau_k}\mathcal{L}_{u=u^*}P(x(v))\mathrm{d}v. \tag{A2}$$

From (3) and (A1), we have $\mathcal{L}_{u=u^*}P(x)\leqslant -1$ for any $x\neq 0$. Therefore, $\mathcal{E}P(x(t\wedge\tau_k))-P(x_0)\leqslant -\mathcal{E}(t\wedge\tau_k)$ and $\mathcal{E}(t\wedge\tau_k)\leqslant P(x_0)$ hold. Because $\{t\wedge\tau_k\}_k$ is an increasing sequence of nonnegative random variables, by the monotonic convergence theorem,

$$\lim_{k\to\infty}\mathcal{E}(t\wedge\tau_k)=\mathcal{E}\left(\lim_{k\to\infty}t\wedge\tau_k\right)=\mathcal{E}(t\wedge\tau_\infty).$$

Note that

$$\tau_\infty=\tau_{1,\infty}\wedge\tau_{2,\infty}.$$

Since $x(t,u^*(x(t)),x_0)$ is a regular solution, $\tau_{2,\infty}=\infty$ a.s.. Therefore, $\mathcal{E}(t\wedge\tau_\infty)=\mathcal{E}(t\wedge\tau_{1,\infty})\leqslant P(x_0)\leqslant\alpha$. Due to the arbitrariness of x_0 and $\tau_{1,\infty}=T(x_0)$, it yields that

$$\sup_{x_0\in\mathcal{R}^n\setminus\{0\}}\mathcal{E}T(x_0)\leqslant\alpha.$$

From (3) and $V(x)\geqslant 0$, $V_t:=V(x(t))$ is a nonnegative continuous supermartingale with augmented filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. Through Doob's optional-sampling theorem for continuous nonnegative supermartingales[1)],

$$\mathcal{E}(V_{T(x_0)+t}|\mathcal{F}_{T(x_0)})\leqslant V_{T(x_0)}=0,\ \text{a.s.},\ \forall t\geqslant 0.$$

Taking the mathematical expectation on both sides of the above inequality, we have $\mathcal{E}V_{T(x_0)+t}\equiv 0$. Since V is positive definite, it follows that $x(T(x_0)+t)\equiv 0$, a.s.. The proof is completed.

Proof of Theorem 2. The proof is based on inductive arguments.

Step 1: Firstly, by adding power integrator technique, we choose the Lyapunov function for the first subsystem as

$$V_1(x_1)=\int_{x_1^*}^{x_1}\left\lceil \lceil s\rfloor^{\frac{r}{r_1}}-\lceil x_1^*\rfloor^{\frac{r}{r_1}}\right\rfloor^{\frac{4r-\kappa-r_1}{r}}\mathrm{d}s,$$

where $r\geqslant\max_{1\leqslant i\leqslant n}\{2r_i\}=2$ and $x_1^*=0$. By Itô's formula, Assumption 1, and Lemma 2, we can get that

$$\begin{aligned}
&\mathcal{L}_1V_1(x_1)\\
&=\frac{\partial V_1}{\partial x_1}(h_1\lceil x_2\rfloor^{q_1}+f_1(\bar{x}_1))+\frac{1}{2}\frac{\partial^2V_1}{\partial x_1^2}|g_1(\bar{x}_1)|^2\\
&=\lceil\xi_1\rfloor^{\frac{4r-\kappa-r_1}{r}}(h_1\lceil x_2\rfloor^{q_1}+f_1(\bar{x}_1))+\frac{4r-\kappa-r_1}{2r}|\xi_1|^{\frac{3r-\kappa-r_1}{r}}\frac{r}{r_1}|x_1|^{\frac{r}{r_1}-1}|g_1(\bar{x}_1)|^2\\
&\leqslant\lceil\xi_1\rfloor^{\frac{4r-\kappa-r_1}{r}}h_1(\lceil x_2\rfloor^{q_1}-\lceil x_2^*\rfloor^{q_1})+\lceil\xi_1\rfloor^{\frac{4r-\kappa-r_1}{r}}h_1\lceil x_2^*\rfloor^{q_1}+\frac{4r-\kappa-r_1}{2r}|\xi_1|^{\frac{3r-\kappa-r_1}{r}}\frac{r}{r_1}|x_1|^{\frac{r}{r_1}-1}\left(\bar{\varphi}_1(\bar{x}_1)|x_1|^{\frac{2r_1+\kappa}{2r_1}}\right)^2\\
&\quad+\lceil\xi_1\rfloor^{\frac{4r-\kappa-r_1}{r}}\bar{\phi}_1(x_1)|x_1|^{\frac{r_1+\kappa}{r_1}}\\
&\leqslant\lceil\xi_1\rfloor^{\frac{4r-\kappa-r_1}{r}}h_1(\lceil x_2\rfloor^{q_1}-\lceil x_2^*\rfloor^{q_1})+\lceil\xi_1\rfloor^{\frac{4r-\kappa-r_1}{r}}h_1\lceil x_2^*\rfloor^{q_1}+|\xi_1|^4\bar{\phi}_1(x_1)+\frac{4r-\kappa-r_1}{2r_1}|\xi_1|^4\bar{\varphi}_1^2(\bar{x}_1),
\end{aligned}$$

where $\xi_1=\lceil x_1\rfloor^{\frac{r}{r_1}}$, $\bar{\varphi}_1(x_1)\geqslant\varphi_1(x_1)|x|^{\rho_{11}}$, and $\bar{\phi}_1(x_1)\geqslant\phi_1(x_1)|x_1|^{\varpi_{11}}$ are smooth functions. $\mathcal{L}_1$ is the infinitesimal generator of the first subsystem. The definitions of $\bar{\varphi}_i(\bar{x}_i)$ and $\bar{\phi}_i(\bar{x}_i)$ used in the rest proofs are similar and will not be repeated. Designing $x_2^*=-\beta_1\lceil\xi_1\rfloor^{\frac{r_2}{r}}$ and $\beta_1>0$, it leads to

$$\lceil x_2^*\rfloor^{q_1}=\mathrm{sign}\left(-\beta_1\lceil\xi_1\rfloor^{\frac{r_2}{r}}\right)\left|\beta_1\lceil\xi_1\rfloor^{\frac{r_2}{r}}\right|^{q_1}=-\mathrm{sign}\left(\lceil\xi_1\rfloor^{\frac{r_2}{r}}\right)\left|\beta_1\lceil\xi_1\rfloor^{\frac{r_2}{r}}\right|^{q_1}=-\beta_1^{q_1}\lceil\xi_1\rfloor^{\frac{r_2q_1}{r}}.$$

Note that $\lceil\xi_1\rfloor^{\frac{4r-\kappa-r_1}{r}}\lceil\xi_1\rfloor^{\frac{r_2q_1}{r}}=|\xi_1|^4$. Set $\beta_1(x_1)=(\frac{4r-\kappa-r_1}{2r_1\underline{h}_1}\bar{\varphi}_1^2(x_1)+\frac{nk_1}{\underline{h}_1k_4}+\frac{k_2}{\underline{h}_1k_4}|\xi_1|^{k_3}+\frac{\bar{\phi}_1(x_1)}{\underline{h}_1})^{\frac{1}{q_1}}$ with k_1, k_2, k_3, and k_4 to be designed. Then

$$\mathcal{L}_1V_1(x_1)\leqslant\lceil\xi_1\rfloor^{\frac{4r-\kappa-r_1}{r}}h_1(\lceil x_2\rfloor^{q_1}-\lceil x_2^*\rfloor^{q_1})-\frac{nk_1}{k_4}|\xi_1|^4-\frac{k_2}{k_4}|\xi_1|^{4+k_3}.$$

Step 2-Inductive assumption: Suppose for $j\in\{3,4,\ldots,n\}$, there is a $\mathcal{C}^2$-positive definite radially unbounded function V_{j-1}, and a set of virtual controllers $x_1^*,x_2^*,\ldots,x_j^*$ defined by

$$\begin{aligned}
&x_1^*=0, &&\xi_1=\lceil x_1\rfloor^{\frac{r}{r_1}}-\lceil x_1^*\rfloor^{\frac{r}{r_1}};\\
&x_2^*=-\beta_1(\bar{x}_1)\lceil\xi_1\rfloor^{\frac{r_2}{r}}, &&\xi_2=\lceil x_2\rfloor^{\frac{r}{r_2}}-\lceil x_2^*\rfloor^{\frac{r}{r_2}};\\
&\vdots &&\vdots\\
&x_j^*=-\beta_{j-1}(\bar{x}_{j-1})\lceil\xi_{j-1}\rfloor^{\frac{r_j}{r}}, &&\xi_j=\lceil x_j\rfloor^{\frac{r}{r_j}}-\lceil x_j^*\rfloor^{\frac{r}{r_j}},
\end{aligned}$$

1) Khasminskii R Z. Stochastic Stability of Differential Equations. Alphen: Sijtjoff and Noordhoff, 1980.

with functions $\beta_i(\bar{x}_i) > 0$, $i \in \{1, 2, \dots, j-1\}$, such that

$$\begin{aligned}\mathcal{L}_{j-1}V_{j-1}(\bar{x}_{j-1}) \leqslant & \lceil \xi_{j-1} \rfloor^{\frac{4r-\kappa-r_{j-1}}{r}} h_{j-1}(\lceil x_j \rfloor^{q_{j-1}} - \lceil x_j^* \rfloor^{q_{j-1}}) \\ & -(n-j+2)\sum_{i=1}^{j-1}\frac{k_1}{k_4}|\xi_i|^4 - \sum_{i=1}^{j-1}\frac{k_2}{k_4}|\xi_i|^{4+k_3},\end{aligned} \tag{A3}$$

where $\mathcal{L}_{j-1}$ is the infinitesimal generator of the first $j-1$ subsystems.

Step 3: In the following, we shall show that Eq. (A3) still holds when $j-1$ is replaced by j with

$$V_j(\bar{x}_j) = V_{j-1}(\bar{x}_{j-1}) + W_j(\bar{x}_j), \tag{A4}$$

and

$$W_j(\bar{x}_j) = \int_{x_j^*}^{x_j} \left\lceil \lceil s \rfloor^{\frac{r}{r_j}} - \lceil x_j^* \rfloor^{\frac{r}{r_j}} \right\rfloor^{\frac{4r-\kappa-r_j}{r}} \mathrm{d}s. \tag{A5}$$

With the help of Proposition A1, it can be deduced from $V_j(\bar{x}_j)$ that

$$\begin{aligned}&\mathcal{L}_j V_j(\bar{x}_j) \\ &\leqslant \lceil \xi_{j-1} \rfloor^{\frac{4r-\kappa-r_{j-1}}{r}} h_{j-1}(\lceil x_j \rfloor^{q_{j-1}} - \lceil x_j^* \rfloor^{q_{j-1}}) - (n-j+2)\sum_{i=1}^{j-1}\frac{k_1}{k_4}|\xi_i|^4 - \sum_{i=1}^{j-1}\frac{k_2}{k_4}|\xi_i|^{4+k_3} \\ &\quad + \lceil \xi_j \rfloor^{\frac{4r-\kappa-r_j}{r}} h_j \lceil x_{j+1} \rfloor^{q_j} + \lceil \xi_j \rfloor^{\frac{4r-\kappa-r_j}{r}} f_j(\bar{x}_j) + \sum_{i=1}^{j-1}\frac{\partial W_j}{\partial x_i}(h_i \lceil x_{i+1} \rfloor^{q_i} + f_i(\bar{x}_i)) \\ &\quad + \frac{1}{2}\sum_{i,m=1}^{j}\left|\frac{\partial^2 W_j}{\partial x_i \partial x_m}\right| \left| g_i(\bar{x}_i)' g_m(\bar{x}_m) \right|.\end{aligned} \tag{A6}$$

Based on Propositions A2–A5, we estimate some terms in (A6) and obtain that

$$\begin{aligned}\mathcal{L}_j V_j(\bar{x}_j) \leqslant & (\alpha_{1,j} + \alpha_{2,j} + \alpha_{3,j} + \alpha_{4,j})|\xi_j|^4 - (n-j+1)\sum_{i=1}^{j-1}\frac{k_1}{k_4}|\xi_i|^4 \\ & - \sum_{i=1}^{j-1}\frac{k_2}{k_4}|\xi_i|^{4+k_3} + \lceil \xi_j \rfloor^{\frac{4r-\kappa-r_j}{r}} h_j \lceil x_{j+1}^* \rfloor^{q_j} \\ & + \lceil \xi_j \rfloor^{\frac{4r-\kappa-r_j}{r}} h_j(\lceil x_{j+1} \rfloor^{q_j} - \lceil x_{j+1}^* \rfloor^{q_j}).\end{aligned}$$

It is easy to see that the virtual controller

$$x_{j+1}^* = -\beta_j \lceil \xi_j \rfloor^{\frac{r_{j+1}}{r}},$$

and

$$\beta_j = \left(\frac{(n-j+1)k_1}{\underline{h}_j k_4} + \frac{\alpha_{1,j} + \alpha_{2,j} + \alpha_{3,j} + \alpha_{4,j}}{\underline{h}_j} + \frac{k_2}{k_4 \underline{h}_j}|\xi_j|^{k_3} \right)^{\frac{1}{q_j}}.$$

Then

$$\mathcal{L}_j V_j(\bar{x}_j) \leqslant -(n-j+1)\sum_{i=1}^{j}\frac{k_1}{k_4}|\xi_i|^4 - \sum_{i=1}^{j}\frac{k_2}{k_4}|\xi_i|^{4+k_3} + \lceil \xi_j \rfloor^{\frac{4r-\kappa-r_j}{r}} h_j(\lceil x_{j+1} \rfloor^{q_j} - \lceil x_{j+1}^* \rfloor^{q_j}). \tag{A7}$$

From Steps 1–3, we have proven that Eq. (A7) holds for any $j \in \{1, 2, \dots, n\}$. Hence, at the nth step, one concludes that

$$\begin{aligned}&\mathcal{L}_n V_n(\bar{x}_n) \\ &\leqslant -\sum_{i=1}^{n-1}\frac{k_1}{k_4}|\xi_i|^4 - \sum_{i=1}^{n-1}\frac{k_2}{k_4}|\xi_i|^{4+k_3} + \lceil \xi_n \rfloor^{\frac{4r-\kappa-r_n}{r}} h_n \lceil u \rfloor^{q_n} + (\alpha_{1,n} + \alpha_{2,n} + \alpha_{3,n} + \alpha_{4,n})|\xi_n|^4 \\ &\leqslant -\sum_{i=1}^{n}\frac{k_1}{k_4}|\xi_i|^4 - \sum_{i=1}^{n}\frac{k_2}{k_4}|\xi_i|^{4+k_3},\end{aligned}$$

with the Lyapunov function as

$$V_n(\bar{x}_n) = V_{n-1}(\bar{x}_{n-1}) + W_n(\bar{x}_n),$$

and

$$W_n(\bar{x}_n) = \int_{x_n^*}^{x_n} \left\lceil \lceil s \rfloor^{\frac{r}{r_j}} - \lceil x_n^* \rfloor^{\frac{r}{r_n}} \right\rfloor^{\frac{4r-\kappa-r_n}{r}} \mathrm{d}s.$$

Consequently, the control u can be designed as

$$u = -\beta_n \lceil \xi_n \rfloor^{\frac{r_{n+1}}{r}}$$

$$= -\left(\frac{k_1}{\bar{h}_n k_4} + \frac{k_2}{k_4 \bar{h}_n}|\xi_n|^{k_3} + \frac{\alpha_{1,n}+\alpha_{2,n}+\alpha_{3,n}+\alpha_{4,n}}{\bar{h}_n}\right)^{\frac{1}{q_n}} \lceil \xi_n \rfloor^{\frac{r_{n+1}}{r}}.$$

Based on the proof of Proposition A4, we can get that

$$\int_{x_j^*}^{x_j} \left\lceil \lceil s \rfloor^{\frac{r}{r_j}} - \lceil x_j^* \rfloor^{\frac{r}{r_j}} \right\rfloor^{\frac{4r-\kappa-r_j}{r}} \mathrm{d}s \leqslant 2^{1-\frac{r_j}{r}} |\xi_j|^{\frac{4r-\kappa}{r}},$$

and

$$V_n \leqslant 2\sum_{i=1}^{n} |\xi_i|^{\frac{4r-\kappa}{r}}.$$

Setting $k_3 \geqslant -\frac{\kappa}{r}$ and using Lemma 6 arrive at

$$-\sum_{i=1}^{n} |\xi_i|^{4+k_3} \leqslant -n^{\frac{-rk_3-\kappa}{4r-\kappa}} \left(\sum_{i=1}^{n} |\xi_i|^{\frac{4r-\kappa}{r}}\right)^{\frac{r(4+k_3)}{4r-\kappa}} \leqslant -n^{\frac{-rk_3-\kappa}{4r-\kappa}} \frac{1}{2}^{\frac{r(4+k_3)}{4r-\kappa}} V_n^{\frac{r(4+k_3)}{4r-\kappa}},$$

and

$$-\sum_{i=1}^{n} |\xi_i|^4 \leqslant -\frac{1}{2}^{\frac{4r}{4r-\kappa}} V_n^{\frac{4r}{4r-\kappa}}.$$

Accordingly, one has

$$\mathcal{L}_n V_n(\bar{x}_n) \leqslant -\frac{k_1}{k_4}\left(\frac{V_n}{2}\right)^{\frac{4r}{4r-\kappa}} - \frac{k_2}{k_4} n^{\frac{-rk_3-\kappa}{4r-\kappa}} \left(\frac{V_n}{2}\right)^{\frac{4r+rk_3}{4r-\kappa}}.$$

Let $k_3 > -\frac{\kappa}{r}$, $b_1 = \frac{4r}{4r-\kappa}$, $b_2 = \frac{4r+rk_3}{4r-\kappa}$, $a = 2^{-b1}k_1$, $k_1 > \frac{2^{b1}}{1-b_1}$, $k_2 = \frac{a-ab_1}{(a-ab_1-1)(b_2-1)} n^{\frac{rk_3+\kappa}{4r-\kappa}} 2^{b_2}$. The control parameter k_4 can be chosen as an arbitrary positive number. Then, considering Corollary 3, system (12) is stochastically predefined-time stabilizable and $\sup_{x_0 \in \mathcal{R}^n \setminus \{0\}} \mathcal{E}T(x_0) \leqslant k_4$.

The following proposition can be obtained from [19].

Proposition A1. The following statements accordingly hold when the positive definite and proper Lyapunov function V_j and W_j are defined by (A4) and (A5), respectively.

$$\begin{aligned}
&\frac{\partial W_j}{\partial x_j} = \lceil \xi_j \rfloor^{\frac{4r-\kappa-r_j}{r}},\\
&\frac{\partial W_j}{\partial x_i} = -\frac{4r-\kappa-r_j}{r}\int_{x_j^*}^{x_j} \left|\lceil s \rfloor^{\frac{r}{r_j}} - \lceil x_j^* \rfloor^{\frac{r}{r_j}}\right|^{\frac{3r-\kappa-r_j}{r}} \mathrm{d}s \frac{\partial \lceil x_j^* \rfloor^{\frac{r}{r_j}}}{\partial x_i},\\
&\frac{\partial^2 W_j}{\partial^2 x_j} = \frac{4r-\kappa-r_j}{r_j}|x_j|^{\frac{r-r_j}{r_j}} |\xi_j|^{\frac{3r-\kappa-r_j}{r}},\\
&\frac{\partial^2 W_j}{\partial x_i \partial x_j} = \frac{\partial^2 W_j}{\partial x_j \partial x_i} = -\frac{4r-\kappa-r_j}{r}|\xi_j|^{\frac{3r-\kappa-r_j}{r}} \frac{\partial \lceil x_j^* \rfloor^{\frac{r}{r_j}}}{\partial x_i},\\
&\frac{\partial^2 W_j}{\partial x_i \partial x_m} = \frac{(4r-\kappa-r_j)(3r-\kappa-r_j)}{r^2} \frac{\partial \lceil x_j^* \rfloor^{\frac{r}{r_j}}}{\partial x_i} \frac{\partial \lceil x_j^* \rfloor^{\frac{r}{r_j}}}{\partial x_m} \int_{x_j^*}^{x_j} \left\lceil \lceil s \rfloor^{\frac{r}{r_j}} - \lceil x_j^* \rfloor^{\frac{r}{r_j}} \right\rfloor^{\frac{2r-\kappa-r_j}{r}} \mathrm{d}s\\
&\qquad - \frac{4r-\kappa-r_j}{r} \frac{\partial^2 \lceil x_j^* \rfloor^{\frac{r}{r_j}}}{\partial x_i \partial x_m} \int_{x_j^*}^{x_j} \left|\lceil s \rfloor^{\frac{r}{r_j}} - \lceil x_j^* \rfloor^{\frac{r}{r_j}}\right|^{\frac{3r-\kappa-r_j}{r}} \mathrm{d}s,
\end{aligned}$$

where $i, m < j$.

Next, we need to prove some necessary propositions listed below in order to estimate some terms in (A6).

Proposition A2. There exists a nonnegative smooth function $\alpha_{1j}(\bar{x}_j)$ such that

$$\lceil \xi_{j-1} \rfloor^{\frac{4r-\kappa-r_{j-1}}{r}} h_{j-1}(\lceil x_j \rfloor^{q_{j-1}} - \lceil x_j^* \rfloor^{q_{j-1}}) \leqslant \frac{k_1}{4k_4}|\xi_{j-1}|^4 + \alpha_{1j}(\bar{x}_j)|\xi_j|^4.$$

Proposition A3. There exists a nonnegative smooth function $\alpha_{2j}(\bar{x}_j)$ such that

$$\lceil \xi_j \rfloor^{\frac{4r-\kappa-r_j}{r}} f_j(\bar{x}_j) \leqslant \frac{k_1}{4k_4}\sum_{i=1}^{j-1} |\xi_i|^4 + \alpha_{2j}(\bar{x}_j)|\xi_j|^4.$$

Proposition A4. There exists a nonnegative smooth function $\alpha_{3j}(\bar{x}_j)$ such that

$$\sum_{i=1}^{j-1} \frac{\partial W_j}{\partial x_i}(h_i \lceil x_{i+1} \rfloor^{q_i} + f_i(\bar{x}_i)) \leqslant \frac{k_1}{4k_4}\sum_{i=1}^{j-1} |\xi_i|^4 + \alpha_{3j}(\bar{x}_j)|\xi_j|^4.$$

Proposition A5. There exists a nonnegative smooth function $\alpha_{4j}(\bar{x}_j)$ such that

$$\frac{1}{2}\sum_{i,m=1}^{j}\left|\frac{\partial^2 W_j}{\partial x_i \partial x_m}\right|\left|g_i(\bar{x}_i)' g_m(\bar{x}_m)\right| \leqslant \frac{k_1}{4k_4}\sum_{i=1}^{j-1}|\xi_i|^4 + \alpha_{4,j}|\xi_j|^4.$$

Proof of Proposition A2. Note that

$$\begin{aligned}&\lceil\xi_{j-1}\rfloor^{\frac{4r-\kappa-r_{j-1}}{r}} h_{j-1}(\lceil x_j\rfloor^{q_{j-1}} - \lceil x_j^*\rfloor^{q_{j-1}})\\ &\leqslant |\xi_{j-1}|^{\frac{4r-\kappa-r_{j-1}}{r}}\bar{h}_{j-1}\left|\left(\lceil x_j\rfloor^{q_{j-1}\frac{r}{r_j}}\right)^{\frac{r_j}{r}} - \left(\lceil x_j^*\rfloor^{q_{j-1}\frac{r}{r_j}}\right)^{\frac{r_j}{r}}\right|.\end{aligned}$$

Based on Lemmas 3 and 4, and $q_{j-1}r_j = r_{j-1}+\kappa$, the above inequality leads to

$$\begin{aligned}&\lceil\xi_{j-1}\rfloor^{\frac{4r-\kappa-r_{j-1}}{r}}\bar{h}_{j-1}(\lceil x_j\rfloor^{q_{j-1}} - \lceil x_j^*\rfloor^{q_{j-1}})\\ &\leqslant |\xi_{j-1}|^{\frac{4r-\kappa-r_{j-1}}{r}}\bar{h}_{j-1}2^{1-\frac{r_{j-1}+\kappa}{r}}\left|\lceil x_j\rfloor^{\frac{r}{r_j}} - \lceil x_j^*\rfloor^{\frac{r}{r_j}}\right|^{\frac{r_{j-1}+\kappa}{r}}\\ &\leqslant \frac{k_1}{4k_4}|\xi_{j-1}|^4 + \bar{h}_{j-1}2^{1-\frac{r_{j-1}+\kappa}{r}}\frac{r_{j-1}+\kappa}{4r}\Phi_j^{-\frac{4r-\kappa-r_{j-1}}{r_{j-1}+\kappa}}|\xi_j|^4,\end{aligned}$$

where the real-valued function Φ_j can be selected as $\Phi_j = \frac{rk_1}{k_4(4r-\kappa-r_{j-1})\bar{h}_{j-1}2^{1-\frac{r_{j-1}+\kappa}{r}}}$. The proof is completed.

Proof of Proposition A3. In view of Assumption 1, we have

$$\lceil\xi_j\rfloor^{\frac{4r-\kappa-r_j}{r}} f_j(\bar{x}_j) \leqslant |\xi_j|^{\frac{4r-\kappa-r_j}{r}}\bar{\phi}_j(\bar{x}_j)\sum_{i=1}^{j}\left||\xi_i| + |\beta_{i-1}|^{\frac{r}{r_i}}|\xi_{i-1}|\right|^{\frac{r_j+\kappa}{r}},$$

with $\xi_0 = 0$. Note that $r_j+\kappa < r$. It is obtained from Lemma 5 that

$$\lceil\xi_j\rfloor^{\frac{4r-\kappa-r_j}{r}} f_j(\bar{x}_j) \leqslant |\xi_j|^{\frac{4r-\kappa-r_j}{r}}\bar{\phi}_j(\bar{x}_j)\left(\sum_{i=1}^{j-1}\left(1+|\beta_i|^{\frac{r_j+\kappa}{r_{i+1}}}\right)|\xi_i|^{\frac{r_j+\kappa}{r}} + |\xi_j|^{\frac{r_j+\kappa}{r}}\right).$$

From Lemma 4,

$$\lceil\xi_j\rfloor^{\frac{4r-\kappa-r_j}{r}} f_j(\bar{x}_j) \leqslant \frac{k_1}{4k_4}\sum_{i=1}^{j-1}|\xi_i|^4 + \alpha_{2j}(\bar{x}_j)|\xi_j|^4,$$

where α_{2j} is a nonnegative smooth function. The proof is completed.

Proof of Proposition A4. According to Lemma 5 and the proof of Proposition A3, we have

$$f_i(\bar{x}_i) \leqslant \bar{\phi}_i(\bar{x}_i)\left(\sum_{m=1}^{i-1}\left(1+|\beta_m|^{\frac{r_i+\kappa}{r_{m+1}}}\right)|\xi_m|^{\frac{r_i+\kappa}{r}} + |\xi_i|^{\frac{r_i+\kappa}{r}}\right),$$

and $\lceil x_{i+1}\rfloor^{q_i} \leqslant |\xi_{i+1}|^{\frac{r_i+\kappa}{r}} + |\beta_i|^{\frac{r_i+\kappa}{r_{i+1}}}|\xi_i|^{\frac{r_i+\kappa}{r}}$. Considering Proposition A1, we can get that

$$\begin{aligned}&\sum_{i=1}^{j-1}\frac{\partial W_j}{\partial x_i}(h_i\lceil x_{i+1}\rfloor^{q_i} + f_i(\bar{x}_i))\\ &\leqslant \sum_{i=1}^{j-1}\left|\frac{4r-\kappa-r_j}{r}\frac{\partial\lceil x_j^*\rfloor^{\frac{r}{r_j}}}{\partial x_i}\int_{x_j^*}^{x_j}\left|\lceil s\rfloor^{\frac{r}{r_j}} - \lceil x_j^*\rfloor^{\frac{r}{r_j}}\right|^{\frac{3r-\kappa-r_j}{r}}\mathrm{d}s\right|\\ &\cdot\left[\bar{\phi}_i(\bar{x}_i)\left(\sum_{m=1}^{i-1}(1+|\beta_m|^{\frac{r_i+\kappa}{r_{m+1}}})|\xi_m|^{\frac{r_i+\kappa}{r}} + |\xi_i|^{\frac{r_i+\kappa}{r}}\right) + |\xi_{i+1}|^{\frac{r_i+\kappa}{r}} + |\beta_i|^{\frac{r_i+\kappa}{r_{i+1}}}|\xi_i|^{\frac{r_i+\kappa}{r}}\right].\end{aligned} \tag{A8}$$

Using Lemma 3, one has

$$\int_{x_j^*}^{x_j}\left\lceil\lceil s\rfloor^{\frac{r}{r_j}} - \lceil x_j^*\rfloor^{\frac{r}{r_j}}\right\rfloor^{\frac{3r-\kappa-r_j}{r}}\mathrm{d}s \leqslant 2^{1-\frac{r_j}{r}}|\xi_j|^{\frac{3r-\kappa}{r}}. \tag{A9}$$

Next, by induction, we can estimate

$$\left|\frac{\partial\lceil x_j^*\rfloor^{\frac{r}{r_j}}}{\partial x_i}\right| \leqslant \Theta_{1ji}\sum_{l=1}^{j-1}|\xi_l|^{\frac{r-r_i}{r}},\ j > i \geqslant 1, \tag{A10}$$

where Θ_{1ji} is a nonnegative smooth function. According to (A8)–(A10), we have

$$\begin{aligned}
&\sum_{i=1}^{j-1}\frac{\partial W_j}{\partial x_i}(h_i\lceil x_{i+1}\rfloor^{q_i}+f_i(\bar{x}_i))\\
&\leqslant \sum_{i=1}^{j-1}\frac{4r-\kappa-r_j}{r}2^{1-\frac{r_j}{r}}|\xi_j|^{\frac{3r-\kappa}{r}}\Theta_{1ji}\sum_{l=1}^{j-1}|\xi_l|^{\frac{r-r_i}{r}}\Theta_{2i}\sum_{m=1}^{j}|\xi_m|^{\frac{r_i+\kappa}{r}}\\
&\leqslant \frac{k_1}{4k_4}\sum_{i=1}^{j-1}|\xi_i|^4+\alpha_{3j}(\bar{x}_j)|\xi_j|^4,
\end{aligned}$$

where α_{3j} is a nonnegative smooth function. Proposition A4 is proven.

Proof of Proposition A5. Note that

$$\begin{aligned}
&\frac{1}{2}\sum_{i,m=1}^{j}\left|\frac{\partial^2 W_j}{\partial x_i\partial x_m}\right|\left|g_i(\bar{x}_i)'g_m(\bar{x}_m)\right|\\
&=\frac{1}{2}\left|\frac{\partial^2 W_j}{\partial^2 x_j}\right|\left|g_j(\bar{x}_j)'g_j(\bar{x}_j)\right|+\sum_{i=1}^{j-1}\left|\frac{\partial^2 W_j}{\partial x_j\partial x_i}\right|\left|g_j(\bar{x}_j)'g_i(\bar{x}_i)\right|+\frac{1}{2}\sum_{i,m=1}^{j-1}\left|\frac{\partial^2 W_j}{\partial x_i\partial x_m}\right|\left|g_i(\bar{x}_i)'g_m(\bar{x}_m)\right|.
\end{aligned}$$

Firstly, based on Propositions A1, A3, and Lemma 6, we have

$$\begin{aligned}
&\frac{1}{2}\left|\frac{\partial^2 W_j}{\partial^2 x_j}\right|\left|g_j(\bar{x}_j)'g_j(\bar{x}_j)\right|\\
&\leqslant \frac{(4r-\kappa-r_j)j}{2r_j}\left(|\xi_j|^{\frac{r-r_j}{r}}+|\beta_{j-1}|^{\frac{r-r_j}{r_j}}|\xi_{j-1}|^{\frac{r-r_j}{r}}\right)\bar{\varphi}_j(\bar{x}_j)^2\\
&\quad\times|\xi_j|^{\frac{3r-\kappa-r_j}{r}}\left(\sum_{s=1}^{j-1}\left(1+|\beta_s|^{\frac{2r_j+\kappa}{r_{s+1}}}\right)|\xi_s|^{\frac{2r_j+\kappa}{r}}+|\xi_j|^{\frac{2r_j+\kappa}{r}}\right)\\
&\leqslant \Xi_{1j}(\zeta_{1j},\bar{x}_j)|\xi_j|^4+\zeta_{1j}\sum_{s=1}^{j-1}|\xi_s|^4,
\end{aligned}\tag{A11}$$

where $\Xi_{1j}(\zeta_{1j},\bar{x}_j)$ is a positive real function and ζ_{1j} can be an any positive real number.

Secondly, we will consider

$$\sum_{i=1}^{j-1}\left|\frac{\partial^2 W_j}{\partial x_j\partial x_i}\right|\left|g_j(\bar{x}_j)'g_i(\bar{x}_i)\right|\leqslant\sum_{i=1}^{j}\frac{4r-\kappa-r_j}{r}|\xi_j|^{\frac{3r-\kappa-r_j}{r}}\Theta_{1ji}\sum_{l=1}^{j-1}|\xi_l|^{\frac{r-r_i}{r}}\left|g_j(\bar{x}_j)'g_i(\bar{x}_i)\right|.$$

Note that

$$\left|g_j(\bar{x}_j)'g_i(\bar{x}_i)\right|\leqslant\bar{\varphi}_i(\bar{x}_i)\bar{\varphi}_j(\bar{x}_j)\Theta_{2ji}\sum_{s=1}^{j}|\xi_s|^{\frac{r_j+r_i+\kappa}{r}}$$

with Θ_{2ji} being a positive continuous function. Summarizing the above analysis leads to that

$$\begin{aligned}
&\sum_{i=1}^{j-1}\left|\frac{\partial^2 W_j}{\partial x_j\partial x_i}\right|\left|g_j(\bar{x}_j)'g_i(\bar{x}_i)\right|\\
&\leqslant\sum_{i=1}^{j}\frac{(4r-\kappa-r_j)j}{r}|\xi_j|^{\frac{3r-\kappa-r_j}{r}}\Theta_{1ji}\Theta_{2ji}\bar{\varphi}_i(\bar{x}_i)\bar{\varphi}_j(\bar{x}_j)\sum_{l=1}^{j-1}|\xi_l|^{\frac{r-r_i}{r}}\sum_{s=1}^{j}|\xi_s|^{\frac{r_j+r_i+\kappa}{r}}\\
&\leqslant\Xi_{2j}(\zeta_{2j},\bar{x}_j)|\xi_j|^4+\zeta_{2j}\sum_{l=1}^{j-1}|\xi_l|^4,
\end{aligned}\tag{A12}$$

where $\Xi_{2j}(\zeta_{2j},\bar{x}_j)$ is a positive real function and ζ_{2j} can be an any positive real number.

Finally, we have the following estimation:

$$\begin{aligned}
&\frac{1}{2}\sum_{i,m=1}^{j-1}\left|\frac{\partial^2 W_j}{\partial x_i\partial x_m}\right|\left|g_i(\bar{x}_i)'g_m(\bar{x}_m)\right|\\
&\leqslant\sum_{i,m=1}^{j-1}\left|\frac{(4r-\kappa-r_j)(3r-\kappa-r_j)}{r^2}\frac{\partial\lceil x_j^*\rfloor^{\frac{r}{r_j}}}{\partial x_i}\frac{\partial\lceil x_j^*\rfloor^{\frac{r}{r_j}}}{\partial x_m}\right.\\
&\quad\times\int_{x_j^*}^{x_j}\left\lceil\lceil s\rfloor^{\frac{r}{r_j}}-\lceil x_j^*\rfloor^{\frac{r}{r_j}}\right\rfloor^{\frac{2r-\kappa-r_j}{r}}\mathrm{d}s-\frac{4r-\kappa-r_j}{r}\frac{\partial^2\lceil x_j^*\rfloor^{\frac{r}{r_j}}}{\partial x_i\partial x_m}\\
&\quad\times\left.\int_{x_j^*}^{x_j}\left|\lceil s\rfloor^{\frac{r}{r_j}}-\lceil x_j^*\rfloor^{\frac{r}{r_j}}\right|^{\frac{3r-\kappa-r_j}{r}}\mathrm{d}s\right|\bar{\varphi}_i(\bar{x}_i)\bar{\varphi}_m(\bar{x}_m)\Theta_{2im}\sum_{s=1}^{j-1}|\xi_s|^{\frac{r_m+r_j+\kappa}{r}}.
\end{aligned}$$

According to (A9) and (A10), we have

$$\int_{x_j^*}^{x_j} \left\lceil \lceil s \rfloor^{\frac{r}{r_j}} - \lceil x_j^* \rfloor^{\frac{r}{r_j}} \right\rfloor^{\frac{2r-\kappa-r_j}{r}} \mathrm{d}s \leqslant 2^{1-\frac{r_j}{r}} |\xi_j|^{\frac{2r-\kappa}{r}}$$

and

$$\left|\frac{\partial \lceil x_j^* \rfloor^{\frac{r}{r_j}}}{\partial x_i}\right| \left|\frac{\partial \lceil x_j^* \rfloor^{\frac{r}{r_j}}}{\partial x_m}\right| \leqslant \Theta_{3jim} \sum_{l=1}^{j-1} |\xi_l|^{\frac{2r-r_i-r_m}{r}},$$

with Θ_{3jim} being a positive real function. Therefore, we can get that

$$\begin{aligned}
&\sum_{i,m=1}^{j-1} \frac{(4r-\kappa-r_j)(3r-\kappa-r_j)}{r^2} \frac{\partial \lceil x_j^* \rfloor^{\frac{r}{r_j}}}{\partial x_i} \frac{\partial \lceil x_j^* \rfloor^{\frac{r}{r_j}}}{\partial x_m} \\
&\quad \times \int_{x_j^*}^{x_j} \left\lceil \lceil s \rfloor^{\frac{r}{r_j}} - \lceil x_j^* \rfloor^{\frac{r}{r_j}} \right\rfloor^{\frac{2r-\kappa-r_j}{r}} \mathrm{d}s \bar{\varphi}_i(\bar{x}_i)\bar{\varphi}_m(\bar{x}_m)\Theta_{2im} \sum_{s=1}^{j-1} |\xi_s|^{\frac{r_m+r_i+\kappa}{r}} \\
&\leqslant \Xi_{3j}(\zeta_{3j}, \bar{x}_j)|\xi_j|^4 + \zeta_{3j} \sum_{l=1}^{j-1} |\xi_l|^4,
\end{aligned} \tag{A13}$$

where $\Xi_{3j}(\zeta_{3j}, \bar{x}_j)$ is a positive real function and ζ_{3j} can be an any positive real number. Now, we are in a position to consider

$$\sum_{i,m=1}^{j-1} \left| \frac{4r-\kappa-r_j}{r} \frac{\partial^2 \lceil x_j^* \rfloor^{\frac{r}{r_j}}}{\partial x_i \partial x_m} \int_{x_j^*}^{x_j} \left| \lceil s \rfloor^{\frac{r}{r_j}} - \lceil x_j^* \rfloor^{\frac{r}{r_j}} \right|^{\frac{3r-\kappa-r_j}{r}} \mathrm{d}s \right| \bar{\varphi}_i(\bar{x}_i)\bar{\varphi}_m(\bar{x}_m)\Theta_{3im} \sum_{s=1}^{j-1} |\xi_s|^{\frac{r_m+r_i+\kappa}{r}}.$$

Through some calculations, we can obtain $\frac{\partial^2 \lceil x_j^* \rfloor^{\frac{r}{r_j}}}{\partial x_i \partial x_m} \leqslant \sum_{s=1}^{j-1} \Theta_{4jims} |\xi_s|^{\frac{r-r_i-r_m}{r}}$ with

$$\Theta_{4jims} = \prod_{l=s+1}^{j-1} \left|\beta_l^{\frac{r}{r_{l+1}}}\right| \left|\frac{\partial^2 \beta_s^{\frac{r}{r_{s+1}}}}{\partial x_i \partial x_m}\right| |\xi_s|^{\frac{r_i+r_m}{r}} + I_{i=m} \prod_{l=s}^{j-1} \left|\beta_l^{\frac{r}{r_{l+1}}}\right|.$$

As a consequence,

$$\begin{aligned}
&\sum_{i,m=1}^{j-1} \left| \frac{4r-\kappa-r_j}{r} \frac{\partial^2 \lceil x_j^* \rfloor^{\frac{r}{r_j}}}{\partial x_i \partial x_m} \int_{x_j^*}^{x_j} \left\lceil \lceil s \rfloor^{\frac{r}{r_j}} - \lceil x_j^* \rfloor^{\frac{r}{r_j}} \right\rfloor^{\frac{3r-\kappa-r_j}{r}} \mathrm{d}s \right| \bar{\varphi}_i(\bar{x}_i)\bar{\varphi}_m(\bar{x}_m) \sum_{s=1}^{m} |x_s|^{\frac{2r_m+\kappa}{2r_s}} \sum_{s=1}^{i} |x_s|^{\frac{2r_i+\kappa}{2r_s}} \\
&\leqslant \sum_{i,m=1}^{j-1} \frac{4r-\kappa-r_j}{r} 2^{1-\frac{r_j}{r}} |\xi_j|^{\frac{3r-\kappa}{r}} \bar{\varphi}_i(\bar{x}_i)\bar{\varphi}_m(\bar{x}_m) \sum_{s=1}^{j-1} \Theta_{4jims} |\xi_s|^{\frac{r-r_i-r_m}{r}} \Theta_{3im} \sum_{s=1}^{j-1} |\xi_s|^{\frac{r_m+r_i+\kappa}{r}} \\
&\leqslant \Xi_{4j}(\zeta_{4j}, \bar{x}_j)|\xi_j|^4 + \zeta_{4j} \sum_{s=1}^{j-1} |\xi_s|^4,
\end{aligned} \tag{A14}$$

where $\Xi_{4j}(\zeta_{4j}, \bar{x}_j)$ is a positive real function and ζ_{4j} can be an any positive real number. Based on (A11)–(A14), we can find a suitable ζ_{ij} such that $\sum_{i=1}^{4} \zeta_{ij} = \frac{k_1}{4k_4}$, which completes the proof of Proposition A5.