

Detecting coherence with respect to general quantum measurements

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Abstract Quantum coherence is crucial in quantum resource theory. Previous studies have mainly focused on standard coherence based on a complete orthogonal reference. Standard coherence has recently been extended to general positive-operator-valued measure (POVM)-based coherence, including block coherence as a special case. Therefore, it is necessary to construct block and POVM-based coherence witnesses to detect them. In this study, we present witnesses for block and POVM-based coherence and obtain the necessary and sufficient conditions for constructing these witnesses. We also discuss possible realizations of some block and POVM-based coherence witnesses in experiments and present examples of measuring block coherence witnesses based on real experimental data. Furthermore, we present an application of block coherence witnesses in a quantum-parameter estimation task with a degenerate Hamiltonian and estimate the unknown parameter by measuring the block coherence witnesses when the input state is block coherent. Finally, we prove that the quantum Fisher information of any block-incoherent state equals zero, which coincides with the result obtained from measuring block coherence witnesses.

Keywords coherence witness, general quantum measurements, block coherence, POVM-based coherence, quantum coherence

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1 Introduction

Quantum coherence plays a significant role in quantum mechanics, with many applications in quantum optics, quantum information processing, quantum network, nanoscale thermodynamics, and biological systems [1–13]. Recently, researchers have recognized that coherence can be viewed as a type of quantum resource. Therefore, detecting coherence is crucial in quantum physics.

In [14], quantitative investigations of quantum coherence were launched, and several coherence measures for standard coherence were proposed with respect to von Neumann measurements. For standard coherence, a state δ in a d -dimensional Hilbert space \mathcal{H} is incoherent under a fixed reference basis $\{|i\rangle\}_{i=1}^d$, if and only if δ is diagonal under the reference basis [14], i.e.,

$$\delta = \sum_{i=1}^d p_i |i\rangle\langle i|, \quad (1)$$

with probabilities $\{p_i\}$. One can define the standard dephasing operation Δ as

$$\Delta(\rho) := \sum_{i=1}^d |i\rangle\langle i| \rho |i\rangle\langle i|. \quad (2)$$

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Thus, a state δ is incoherent under a chosen reference basis $\{|i\rangle\}_{i=1}^d$ if and only if the following condition holds:

$$\delta = \Delta(\delta), \tag{3}$$

where Δ is defined in (2).

In [15], Åberg proposed a framework that defined block coherence with respect to projective measurements. Standard coherence can be viewed as a special case of block coherence. Moreover, Bischof et al. [16] generalized block coherence to the coherence with respect to general quantum measurements, i.e., positive-operator-valued measure (POVM)-based coherence. Therefore, the resource theory of coherence was generalized from the standard coherence to the block coherence and even the POVM-based coherence. Notably, POVMs describe the most general type of quantum measurements, and they might be more advantageous than the projective measurements.

However, in contrast to standard coherence, only a few reported studies quantified block coherence and POVM-based coherence [16–19]. Moreover, methods to detect whether a state has nonzero block coherence and POVM-based coherence do not exist currently. For standard coherence, Ref. [13] first introduced the standard coherence witness W . Similar to entanglement witnesses, a standard coherence witness W is a Hermitian operator and $\text{Tr}(\delta W) \geq 0$ holds for all incoherent states δ . If $\text{Tr}(\rho W) < 0$ can be found for a state ρ , then the state ρ must be a standard coherent state. Coherence measures generally require full state information obtained via quantum state tomography, involving exponentially increasing measurements with the number of qubits. In contrast, coherence witnesses can be measured with considerably fewer measurements, without the need of quantum state tomography. Thus, it is necessary to construct the block and POVM-based coherence witnesses to detect them without quantum state tomography, particularly for experimentally unknown states.

In this study, we present witnesses for block and POVM-based coherence and obtain the necessary and sufficient conditions for them. Moreover, we discuss possible realizations of some block and POVM-based coherence witnesses in real experiments and present examples of detecting block coherence by measuring block coherence witnesses. Furthermore, we provide an application of block coherence witnesses in a quantum-parameter estimation task with a degenerate Hamiltonian and estimate the unknown parameter by measuring the block coherence witnesses if the input state is block-coherent. We also prove that the quantum Fisher information of any block-incoherent state is equal to zero, which agrees with the results from measuring block coherence witnesses.

2 Detecting block coherence based on block coherence witnesses

Before discussing our main results, we first review the definition of block-incoherent states. In [15–21], the block-incoherent state has been defined as follows.

Given a d -dimensional Hilbert space \mathcal{H} that has been divided into n ($n \leq d$) subspaces, the subspace projectors are $\mathbf{P} := \{P_s\}_{s=1}^n$ with $\sum_{s=1}^n P_s = \mathbb{1}$ (where $\mathbb{1}$ is the identity operator). A state $\tilde{\delta}$ is block-incoherent under the reference subspace projectors \mathbf{P} if and only if $\tilde{\delta}$ is block-diagonal under the reference \mathbf{P} , that is,

$$\tilde{\delta} = \sum_{s=1}^n P_s \tilde{\delta} P_s := \tilde{\Delta}(\tilde{\delta}), \tag{4}$$

where we define the modified dephasing operation as follows:

$$\tilde{\Delta}(\rho) := \sum_s P_s \rho P_s. \tag{5}$$

Similar to the witnesses for standard coherence [13, 22–24], we can construct block coherence witnesses as follows.

Theorem 1. (a) For any Hermitian operator A , we can construct a block coherence witness,

$$\tilde{W}_A = \tilde{\Delta}(A) - A. \tag{6}$$

(b) An arbitrary Hermitian operator \tilde{W} is a block coherence witness if and only if $\tilde{\Delta}(\tilde{W}) \geq 0$.

Proof. (a) We first prove that \widetilde{W}_A is a block coherence witness. Since A is a Hermitian operator, \widetilde{W}_A must also be Hermitian. Thus, for an arbitrary block-incoherent state $\widetilde{\delta} = \sum_s P_s \widetilde{\delta} P_s$, we obtain

$$\begin{aligned} \text{Tr}(\widetilde{\delta}\widetilde{W}_A) &= \text{Tr}[\widetilde{\delta}\widetilde{\Delta}(A)] - \text{Tr}[\widetilde{\delta}A] \\ &= \text{Tr}\left[\widetilde{\delta}\sum_s P_s A P_s\right] - \text{Tr}[\widetilde{\delta}A] \\ &= \text{Tr}\left[\sum_s P_s \widetilde{\delta} P_s A\right] - \text{Tr}[\widetilde{\delta}A] \\ &= 0, \end{aligned} \tag{7}$$

which implies that \widetilde{W}_A is a block coherence witness.

(b) It is important to note that a Hermitian operator W is a coherence witness for standard coherence if and only if $\Delta(W) \geq 0$ [13]. Similarly, we can prove that a Hermitian operator \widetilde{W} is a block coherence witness if and only if $\widetilde{\Delta}(\widetilde{W}) \geq 0$.

First, if $\widetilde{\Delta}(\widetilde{W}) \geq 0$ holds, for any block-incoherent state $\widetilde{\delta}$ we can obtain

$$\begin{aligned} \text{Tr}[\widetilde{\delta}\widetilde{W}] &= \text{Tr}[\widetilde{\Delta}(\widetilde{\delta})\widetilde{W}] \\ &= \text{Tr}[\widetilde{\Delta}(\widetilde{W})\widetilde{\delta}] \\ &\geq 0, \end{aligned} \tag{8}$$

that is, \widetilde{W} is a block coherence witness.

Conversely, we prove that if $\widetilde{\delta} \text{Tr}[\widetilde{\delta}\widetilde{W}] \geq 0$ holds for any block-incoherent state, then $\widetilde{\Delta}(\widetilde{W}) \geq 0$. For any quantum state ρ , we obtain

$$\begin{aligned} \text{Tr}[\rho\widetilde{\Delta}(\widetilde{W})] &= \text{Tr}[\widetilde{\Delta}(\rho)\widetilde{W}] \\ &= \text{Tr}[\widetilde{\delta}_\rho\widetilde{W}] \\ &\geq 0, \end{aligned} \tag{9}$$

where $\widetilde{\delta}_\rho := \widetilde{\Delta}(\rho)$ is a block-incoherent state. Thus, $\widetilde{\Delta}(\widetilde{W})$ is positive-semidefinite, that is, $\widetilde{\Delta}(\widetilde{W}) \geq 0$.

Therefore, \widetilde{W} is block coherence witness, if and only if $\widetilde{\Delta}(\widetilde{W}) \geq 0$.

Remark 1. Based on Theorem 1(a), we can construct a block coherence witness \widetilde{W}_σ using $A = \sigma$, where σ is any density matrix,

$$\widetilde{W}_\sigma = \widetilde{\Delta}(\sigma) - \sigma. \tag{10}$$

Moreover, if σ is a pure state $|\phi\rangle$, then we obtain

$$\widetilde{W}_\phi = \widetilde{\Delta}(|\phi\rangle\langle\phi|) - |\phi\rangle\langle\phi|, \tag{11}$$

and

$$\begin{aligned} \text{Tr}[\rho\widetilde{W}_\phi] &= \text{Tr}[\rho(\widetilde{\Delta}(|\phi\rangle\langle\phi|) - |\phi\rangle\langle\phi|)] \\ &= \langle\phi|\widetilde{\Delta}(\rho)|\phi\rangle - \langle\phi|\rho|\phi\rangle \\ &= F(\widetilde{\Delta}(\rho), |\phi\rangle) - F(\rho, |\phi\rangle), \end{aligned} \tag{12}$$

where $F(\rho, |\phi\rangle) := \langle\phi|\rho|\phi\rangle$ is the fidelity between the state ρ and the pure state $|\phi\rangle$. Therefore, the expected value of the block coherence witness \widetilde{W}_ϕ is related to these two fidelities.

3 Coherence witness with respect to general measurements

Recently, Bischof et al. [16–19] introduced POVM-based coherence that is defined as follows.

Let \mathbf{E} be a set of n positive operators $\mathbf{E} := \{E_i\}_{i=1}^n$ with $\sum_{i=1}^n E_i = \mathbb{1}$. The corresponding measurement operator of each E_i is defined as A_i , such that $E_i = A_i^\dagger A_i$ holds. Thus, a state $\bar{\delta}$ is incoherent with respect to the general measurement \mathbf{E} if and only if

$$E_i \bar{\delta} E_{i'} = 0, \quad \forall i \neq i'. \tag{13}$$

Moreover, this is equivalent to [16]

$$A_i \bar{\delta} A_{i'}^\dagger = 0, \forall i \neq i'. \tag{14}$$

Therefore, any POVM-based incoherent state $\bar{\delta}$ should satisfy

$$\bar{\delta} = \sum_i E_i \bar{\delta} E_i := \bar{\Delta}(\bar{\delta}), \tag{15}$$

where $\bar{\Delta}$ is defined as

$$\bar{\Delta}(\rho) := \sum_i E_i \rho E_i. \tag{16}$$

It is noteworthy that Eq. (15) can be easily proved from the definition of POVM-based incoherent state because for any POVM-based incoherent state $\bar{\delta}$, we can obtain that

$$\begin{aligned} \bar{\delta} &= \left(\sum_i E_i \right) \bar{\delta} \left(\sum_j E_j \right) \\ &= \sum_i E_i \bar{\delta} E_i + \sum_{i \neq j} E_i \bar{\delta} E_j \\ &= \sum_i E_i \bar{\delta} E_i, \end{aligned} \tag{17}$$

where we have used $E_i \bar{\delta} E_j = 0, \forall i \neq j$.

Theorem 2. (a) For any Hermitian operator A , we can construct a POVM-based coherence witness \bar{W}_A as follows:

$$\bar{W}_A = \bar{\Delta}(A) - A. \tag{18}$$

(b) An arbitrary Hermitian operator \bar{W} is a POVM-based coherence witness if and only if $\bar{\Delta}(\bar{W}) \geq 0$.

Proof. (a) We show that \bar{W}_A is a POVM-based coherence witness. Because A is a Hermitian operator, \bar{W}_A must be Hermitian. For any incoherent state $\bar{\delta}$ with respect to $\{E_i\}$, $\bar{\delta} = \sum_i E_i \bar{\delta} E_i$ holds; thus,

$$\begin{aligned} \text{Tr}(\bar{\delta} \bar{W}_A) &= \text{Tr}[\bar{\delta} \bar{\Delta}(A)] - \text{Tr}[\bar{\delta} A] \\ &= \text{Tr} \left[\bar{\delta} \sum_i E_i A E_i \right] - \text{Tr}[\bar{\delta} A] \\ &= \text{Tr} \left[\sum_i E_i \bar{\delta} E_i A \right] - \text{Tr}[\bar{\delta} A] \\ &= 0, \end{aligned} \tag{19}$$

implying that \bar{W}_A is a POVM-based coherence witness.

(b) First, if $\bar{\Delta}(\bar{W}) \geq 0$ holds, then for any POVM-based incoherent state $\bar{\delta}$ we obtain

$$\begin{aligned} \text{Tr}[\bar{\delta} \bar{W}] &= \text{Tr}[\bar{\Delta}(\bar{\delta}) \bar{W}] \\ &= \text{Tr}[\bar{\Delta}(\bar{W}) \bar{\delta}] \\ &\geq 0. \end{aligned} \tag{20}$$

Thus, \bar{W} is a POVM-based coherence witness.

Conversely, we prove that if $\text{Tr}[\bar{\delta} \bar{W}] \geq 0$, then $\bar{\Delta}(\bar{W}) \geq 0$. For any quantum state ρ , we obtain

$$\begin{aligned} \text{Tr}[\rho \bar{\Delta}(\bar{W})] &= \text{Tr}[\bar{\Delta}(\rho) \bar{W}] \\ &= \text{Tr}[\bar{\delta}_\rho \bar{W}] \\ &\geq 0, \end{aligned} \tag{21}$$

where $\bar{\delta}_\rho := \bar{\Delta}(\rho)$ is a POVM-based incoherent state. Thus, $\bar{\Delta}(\bar{W})$ is positive-semidefinite, that is, $\bar{\Delta}(\bar{W}) \geq 0$.

Therefore, we prove that \bar{W} is a POVM-based coherence witness if and only if $\bar{\Delta}(\bar{W}) \geq 0$.

Remark 2. Based on Theorem 2(a), we can also construct a POVM-based coherence witness \bar{W}_σ by choosing $A = \sigma$ where σ is an arbitrary density matrix,

$$\bar{W}_\sigma = \bar{\Delta}(\sigma) - \sigma. \tag{22}$$

Moreover, if σ is a pure state $|\phi\rangle$, we obtain that

$$\bar{W}_\phi = \bar{\Delta}(|\phi\rangle\langle\phi|) - |\phi\rangle\langle\phi|, \tag{23}$$

and

$$\begin{aligned} \text{Tr}[\rho\bar{W}_\phi] &= \text{Tr}[\rho(\bar{\Delta}(|\phi\rangle\langle\phi|) - |\phi\rangle\langle\phi|)] \\ &= \text{Tr}[\rho\bar{\Delta}(|\phi\rangle\langle\phi|)] - \text{Tr}[\rho|\phi\rangle\langle\phi|] \\ &= F(\bar{\Delta}(\rho), |\phi\rangle) - F(\rho, |\phi\rangle), \end{aligned} \tag{24}$$

where it demonstrates the relationship of the POVM-based coherence witness \bar{W}_ϕ and the fidelity between the state ρ (or $\bar{\Delta}(\rho)$) and the pure state $|\phi\rangle$.

4 Possible experimental realization for witnesses and examples

Many experiments have measured the fidelity $F = \langle\phi|\rho_{\text{exp}}|\phi\rangle$ between the experimental state ρ_{exp} and the target pure state $|\phi\rangle$ [25]. For bipartite and multipartite systems, fidelities can be measured by decomposing the operator $|\phi\rangle\langle\phi|$ as the sum of the tensor products of local observables. Therefore, we can measure witnesses (11) and (23) in the same manner.

In the following, we present examples of N -qubit W states $|W_N\rangle$ obtained from real experimental data, where

$$|W_N\rangle = (|0\cdots 001\rangle + |0\cdots 010\rangle + \cdots + |10\cdots 0\rangle)/\sqrt{N}. \tag{25}$$

We used the block coherence witness (11) to detect the block coherence of the W states. In [25], N -qubit W states ($4 \leq N \leq 8$) must be experimentally generated by trapped ions, with fidelities between the experimental states and perfect W states $F_4 = 0.846$, $F_5 = 0.759$, $F_6 = 0.788$, $F_7 = 0.763$, $F_8 = 0.722$ for the 4-, 5-, 6-, 7- and 8-ion W states, respectively. Moreover, the numerical values of the density matrices of the experimental states with $4 \leq N \leq 8$ are presented in [25]. It is important to note that the experimental states have local phases, and the local phases can be found by maximizing the fidelity $F = \langle\tilde{W}_N|\rho_{\text{exp}}|\tilde{W}_N\rangle$ where $|\tilde{W}_N\rangle$ is W states containing local phases, as shown in [25]. After choosing local unitary transformations based on the local phases, we can transform ρ_{exp} to ρ'_{exp} such that $F = \langle\tilde{W}_N|\rho_{\text{exp}}|\tilde{W}_N\rangle = \langle W_N|\rho'_{\text{exp}}|W_N\rangle$. We consider the following reference subspace projectors P with

$$\begin{aligned} P_0 &= |\phi^-\rangle\langle\phi^-|, \\ P_1 &= |\phi^+\rangle\langle\phi^+|, \\ P_2 &= |0\cdots 010\rangle\langle 0\cdots 010|, \\ &\cdots, \\ P_{N-1} &= |010\cdots 0\rangle\langle 010\cdots 0|, \\ P_N &= \mathbb{1} - \sum_{i=0}^{N-1} P_i, \end{aligned}$$

where $|\phi^\pm\rangle := (|00\cdots 01\rangle \pm |10\cdots 0\rangle)/\sqrt{2}$. Thus, our block coherence witness is given by

$$\begin{aligned} \tilde{W} &= \tilde{\Delta}(|W_N\rangle\langle W_N|) - |W_N\rangle\langle W_N| \\ &= \sum_{i=0}^N P_i|W_N\rangle\langle W_N|P_i - |W_N\rangle\langle W_N|. \end{aligned} \tag{26}$$

From Table 1, we can see that $-\text{Tr}(\rho'_{\text{exp}}\tilde{W})$ is always greater than zero, which means that all experimental states in [25] are block-coherent under the above reference subspace projectors P .

Table 1 Results of the block coherent witness (26) for 4–8 qubit W states^{a)}

$ W_N\rangle$	4	5	6	7	8
F_N	0.846	0.759	0.788	0.763	0.722
$\langle \tilde{\Delta}(W_N\rangle\langle W_N) \rangle$	0.321	0.207	0.173	0.141	0.119
$-\text{Tr}(\rho'_{\text{exp}} \tilde{W})$	0.525	0.552	0.615	0.622	0.603

a) The fidelity F_N is the result of the overlap $\langle W_N | \rho'_{\text{exp}} | W_N \rangle$ according to [25]. The values $\langle \tilde{\Delta}(|W_N\rangle\langle W_N|) \rangle$ and $-\text{Tr}(\rho'_{\text{exp}} \tilde{W})$ are obtained from the density matrices of the experimental states in [25]. It is observed that $-\text{Tr}(\rho'_{\text{exp}} \tilde{W})$ is always greater than zero.

5 Quantum parameter estimation task with degenerate Hamiltonians

In quantum metrology, one of the main tasks is to improve the accuracy of parameter estimation in a quantum channel, which is different from the standard quantum limits [26, 27]. Quantum coherence plays a fundamental role in quantum parameter estimation [28]. For unitary evolution with a degenerate Hamiltonian, we propose a simple application of quantum block coherence and find that the quantum Fisher information is strongly related to block coherence.

5.1 Quantum parameter estimation with block coherent states

We now consider a d -dimensional Hilbert space, assuming that H is a degenerate Hamiltonian,

$$H = \sum_{s=1}^n \sum_{g=1}^{k_s} E_s |s, g\rangle\langle s, g|, \quad (27)$$

where H has n different eigenvalues $\{E_s\}_{s=1}^n$ and each eigenvalue E_s has k_s degenerate eigenstates $\{|s, g\rangle\}_{g=1}^{k_s}$. Here, the index s is for the s -th different eigenvalue, and the index g denotes the g -th E_s (the total number of the same eigenvalue E_s is k_s). $|s, g\rangle$ is the eigenstate corresponding to the eigenvalue, that is, the g th E_s . According to the definition of block coherence, we can naturally choose the degenerate subspaces of H in (27) as reference subspaces. Therefore,

$$P_s = \sum_{g=1}^{k_s} |s, g\rangle\langle s, g| \quad (28)$$

is the s th subspace projector, where every pure state in this subspace is an eigenstate of H with an eigenvalue of E_s .

Proposition 1. With a degenerate Hamiltonian H in (27), we choose $\{P_s\}_{s=1}^n$ in (28) as the reference subspace projector P . The output state $\rho_{\text{out}} = U_\varphi \rho_{\text{in}} U_\varphi^\dagger := \rho_\varphi$ is used to estimate the unknown parameter φ in the black box in Figure 1, if and only if ρ_{in} and ρ_{out} have nonzero block coherence under the reference subspaces P .

Proof. For an arbitrary input state, ρ_{in} can be expressed in terms of the eigenstates of H (27), that is,

$$\rho_{\text{in}} = \sum_{s, s'} \sum_{g, g'} \rho_{(s, g), (s', g')} |s, g\rangle\langle s', g'|, \quad (29)$$

where $\rho_{(s, g), (s', g')} := \langle s, g | \rho_{\text{in}} | s', g' \rangle$ with $\sum_{s, g} \rho_{(s, g), (s, g)} = 1$. Thus, the corresponding output state $\rho_{\text{out}} = U_\varphi \rho_{\text{in}} U_\varphi^\dagger := \rho_\varphi$ can be expressed as follows:

$$\begin{aligned} \rho_{\text{out}} &= U_\varphi \rho_{\text{in}} U_\varphi^\dagger \\ &= \sum_{\substack{s, s' \\ s \neq s'}} \sum_{g, g'} \rho_{(s, g), (s', g')} e^{-i(E_s - E_{s'})\varphi} |s, g\rangle\langle s', g'| \\ &\quad + \sum_s \sum_{g, g'} \rho_{(s, g), (s, g')} |s, g\rangle\langle s, g'|. \end{aligned} \quad (30)$$

We can see that ρ_{out} depends on φ if and only if there exists a nonzero $\rho_{(s, g), (s', g')}$ with $s \neq s'$, that is, ρ_{out} and ρ_{in} have nonzero block coherence under the reference subspaces.

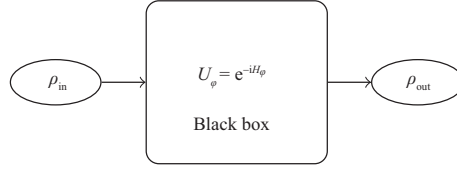


Figure 1 The black box implements a unitary evolution $U_\varphi = e^{-iH\varphi}$ on the input state ρ_{in} , and the unknown parameter φ of the black box must be estimated by measuring $\rho_{\text{out}} = U_\varphi \rho_{\text{in}} U_\varphi^\dagger := \rho_\varphi$. Assuming that the Hamiltonian H is degenerate and already known, the quantum parameter estimation task is to estimate the unknown parameter φ from the output state ρ_{out} .

Remark 3. It is important to note that witnesses for standard coherence may be useless in estimating the unknown parameter when the Hamiltonian H is degenerate, even though the input state has standard coherence. Consider the following special case: we use an input state that is block-incoherent, but its density matrix contains off-diagonal nonzero elements in some degenerate subspaces under the chosen eigenvectors of H as the reference basis, that is, this input state contains standard coherence but no block coherence. The expected values of standard coherence witnesses have no information regarding the parameter, and they cannot be estimated by measuring the witnesses of standard coherence, even though the input state contains standard coherence.

5.2 Quantum Fisher information of block incoherent states

The quantum Fisher information F_q can be obtained for an arbitrary output state ρ_φ according to [29–32]

$$\begin{aligned} F_q &= \text{Tr}[\rho_\varphi L_\varphi^2] \\ &= \sum_{m,n} 4c_m \left(\frac{c_n - c_m}{c_n + c_m} \right)^2 |\langle m|H|n\rangle|^2, \end{aligned} \quad (31)$$

where H is the corresponding Hermitian Hamiltonian, and the output state $\rho_\varphi = U_\varphi \rho_{\text{in}} U_\varphi^\dagger = U_\varphi (\sum_n c_n |n\rangle \langle n|) U_\varphi^\dagger$, $\rho_{\text{in}} = \sum_n c_n |n\rangle \langle n|$. c_n and $|n\rangle$ are the eigenvalues and eigenvectors of ρ_{in} , respectively. The symmetric logarithmic derivative operator is $L_\varphi = U_\varphi (-2i \sum_{m,n} \frac{\langle m|[H, \rho_{\text{in}}]|n\rangle}{c_n + c_m} |m\rangle \langle n|) U_\varphi^\dagger$ [29–32]. Furthermore, we discuss quantum Fisher information with a block-incoherent state.

Proposition 2. Consider a degenerate Hamiltonian H in (27) and the quantum parameter estimation task in Figure 1; in this case, one can choose the degenerate subspaces of H as the reference subspaces. If the input state ρ_{in} is a block-incoherent state, the quantum Fisher information for the output state is $F_q = 0$.

Proof. Consider an arbitrary block-incoherent state as the input state ρ_{in} ,

$$\begin{aligned} \rho_{\text{in}} &= \sum_i \sum_{g,g'} \rho_{(i,g),(i,g')} |i, g\rangle \langle i, g'| \\ &= \sum_i \sum_{\tilde{g}} A_{\tilde{g}}^{(i)} |i, \tilde{g}\rangle \langle i, \tilde{g}|, \end{aligned} \quad (32)$$

where the second equation holds, because we diagonalize ρ_{in} in each subspace. Thus, the eigenvalues of ρ_{in} are $\{A_{\tilde{g}}^{(i)}\}$ with the corresponding eigenvectors $\{|i, \tilde{g}\rangle\}$. The output state ρ_φ can be expressed as

$$\begin{aligned} \rho_\varphi &= U_\varphi \rho_{\text{in}} U_\varphi^\dagger \\ &= U_\varphi \sum_i \sum_{\tilde{g}} A_{\tilde{g}}^{(i)} |i, \tilde{g}\rangle \langle i, \tilde{g}| U_\varphi^\dagger, \end{aligned} \quad (33)$$

where $U_\varphi = e^{-iH\varphi}$ denotes a unitary operator. Furthermore, the symmetric logarithmic derivative operator L_φ can be expressed as [29–32]

$$L_\varphi = U_\varphi \left(-2i \sum_{i,j} \sum_{\tilde{g}, \tilde{h}} \frac{\langle i, \tilde{g}|[H, \rho_{\text{in}}]|j, \tilde{h}\rangle}{A_{\tilde{g}}^{(i)} + A_{\tilde{h}}^{(j)}} |i, \tilde{g}\rangle \langle j, \tilde{h}| \right) U_\varphi^\dagger$$

$$= U_\varphi \left(-2i \sum_{i,j} \sum_{\tilde{g},\tilde{h}} \frac{A_h^{(j)} - A_g^{(i)}}{A_g^{(i)} + A_h^{(j)}} \langle i, \tilde{g} | H | j, \tilde{h} \rangle | i, \tilde{g} \rangle \langle j, \tilde{h} | \right) U_\varphi^\dagger.$$

Finally, we obtain the quantum Fisher information F_q as

$$\begin{aligned} F_q &= \text{Tr}(\rho_\varphi L_\varphi^2) \\ &= 4 \sum_{i,j} \sum_{\tilde{g},\tilde{h}} A_{\tilde{g}}^{(i)} |\langle i, \tilde{g} | H | j, \tilde{h} \rangle|^2 \left(\frac{A_h^{(j)} - A_g^{(i)}}{A_g^{(i)} + A_h^{(j)}} \right)^2 \\ &= 4 \sum_{\substack{i,j \\ i \neq j}} \sum_{\tilde{g},\tilde{h}} A_{\tilde{g}}^{(i)} |\langle i, \tilde{g} | H | j, \tilde{h} \rangle|^2 \left(\frac{A_h^{(j)} - A_g^{(i)}}{A_g^{(i)} + A_h^{(j)}} \right)^2 \\ &\quad + 4 \sum_i \sum_{\tilde{g},\tilde{h}} A_{\tilde{g}}^{(i)} |\langle i, \tilde{g} | H | i, \tilde{h} \rangle|^2 \left(\frac{A_h^{(i)} - A_g^{(i)}}{A_g^{(i)} + A_h^{(i)}} \right)^2 \\ &= 0, \end{aligned} \tag{34}$$

where we used $\langle i, \tilde{g} | H | j, \tilde{h} \rangle = 0$ with $i \neq j$, and $\langle i, \tilde{g} | H | i, \tilde{h} \rangle = E_i \delta_{\tilde{g}\tilde{h}}$.

Remark 4. When the input state is a block-incoherent state, the quantum Fisher information of the output state is equal to zero, which means that the parameter cannot be estimated from the output state; that is, the output state contains no information about the parameter. This is in agreement with the results of Proposition 1.

6 Discussion and conclusion

Besides the Fisher information estimation mentioned in the above example, when performing quantum coherence operations on any degenerate quantum system, we must also consider the measure of coherence. In the practical systems, degeneracy is a common situation and has been widely reported in the microscopic field, e.g., orbital degenerate states and spin states in λ -type and chainlike structured atomic systems [33], and near-degenerate states in few-electron ion system [34]. Moreover, degeneracy exists in macroscopic systems, e.g., macroscopically degenerate states in intrinsic quasi-crystals [35], and allegedly appears during the conversion or transition of black holes [36]. In addition, degeneracy plays an important role in both quantum computing and quantum network construction [37, 38]. It is often used as a protected qubit in a fault-tolerant quantum operation [37]. Moreover, it can also be used as a control device to control the coherence between quantum systems [39]. The study of degeneracy and near-degeneracy also plays an important role in the discussion of topological structures and phase transitions [40–42] and involves determining how to correctly measure the coherence of degenerate quantum systems. Therefore, our coherence detection scheme has potential value in the measurement and application of degenerate states.

In conclusion, we have discussed witnesses for block and POVM-based coherence. The necessary and sufficient conditions for arbitrary block and POVM-based coherence witnesses have been obtained. Furthermore, we have shown that block-coherent states can be used in a quantum-parameter estimation task with a degenerate Hamiltonian if the input state is block coherent. The quantum Fisher information of block-incoherent states is equal to zero, in agreement with the results from the block coherence witness.

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