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First attempt of barrier functions for Caputo's fractional-order nonlinear dynamical systems

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Dear editor,

In the past two decades, scholars have begun to use system dynamical models and state motions to study the safety of a system to realize the analysis, diagnosis, and control of system safety. This safety is called state safety, which is different but closely related to state stability. State safety means a system $\dot{x} = f(x)$ with a set $\mathcal{C}$ called safe. If for any initial $x(t_0) \in \mathcal{C}$, the solution $x(t)$ will stay in the set $\mathcal{C}$ all time, indicating that state safety focuses on both the results and the entire process. Under this definition, several barrier functions (BFs) are proposed and have become the hot ones for studying state safety in robotics and automation control, e.g., [1–5], which belong to the integer-order system.

However, for fractional-order systems, the research field of their safety analysis and control is still an "uncultivated land". In the past decades, fractional-order differential equations have been used to describe systems more accurately than integer differential ones [6]. Therefore, it is essential to extend and migrate the state safety theories to fractional-order systems due to their different laws of integration and differentiation from integer-order systems. Thus, this study is a preliminary attempt at some safety analysis and control problems through BFs for such systems. Our main contributions in this study are presented as follows. The less-zero BF [1] and exponential-alpha BF [2] are extended to the fractional-order field. Under Caputo's description of Caputo's fractional derivative, we proposed the above two Caputo's BFs. Then, we used these to present analysis and diagnosis theorems of state safety for fractional-order nonlinear dynamic systems, which guarantees that all states will keep in an available state set.

Model and methodology. This study mainly presents an extension of safety theories through BFs [1, 2] for Caputo's fractional-order nonlinear dynamical systems with the order α satisfying $\alpha \in (0, 1)$. Thus, a Caputo's fractional-order system (CFOS) with Definition 2 [7] can be expressed as

$$ {}^{C}D^{\alpha}x(t) = f(x(t)) \tag{1} $$

with $\alpha \in (0, 1)$. For the CFOS (1), there is an available state set $\mathcal{C}$, where every state indicates that the system is in a safe operating state, defined by

$$ \mathcal{C} = \{x \in \mathbb{R}^n : h(x) \geqslant 0\}, \tag{2} $$
$$ \partial\mathcal{C} = \{x \in \mathbb{R}^n : h(x) = 0\}, \tag{3} $$
$$ \mathrm{Int}(\mathcal{C}) = \{x \in \mathbb{R}^n : h(x) > 0\} \tag{4} $$

for a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$. If any state is outside of set $\mathcal{C}$, it satisfies $\forall x \in \mathbb{R}^n \setminus \mathcal{C}$, $h(x) < 0$. The definition of set $\mathcal{C}$ mainly consisting of (2)–(4) are followed by [3, 4].

Definition 1. Given a CFOS (1) with an available set $\mathcal{C}$ defined by (2)–(4), satisfying $x(t_0) \in \mathcal{C}$, if the set $\mathcal{C}$ is forward invariant, the CFOS (1) is safe.

Next, we will propose two Caputo's BFs. One is Caputo's less-zero BF via Theorem 1, and the other is Caputo's exponential-beta BF via Theorem 2.

Theorem 1. For a CFOS (1) with a set $\mathcal{C}$ defined by (2)–(4) for some continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, $\forall x(t_0) \in \mathrm{Int}(\mathcal{C})$, if there exists a Caputo's less-zero BF $B : \mathcal{C} \to \mathbb{R}$ satisfying

$$ B(x) = -\beta(h(x)), \tag{5} $$
$$ {}^{C}D^{\alpha}B(x(t)) \leqslant 0, \tag{6} $$

where β is a locally Lipschitz class $\mathcal{K}$ function, the set $\mathcal{C}$ is forward invariant, and the CFOS (1) can be said to be safe.

Proof. Using fractional-order integral for (6), we have $I^{\alpha}({}^{C}D^{\alpha}B) \leqslant I^{\alpha}0$. According to Lemma 3 [7] and Theorem 2.4 [8], we obtain $B(x(t)) - B(x(t_0)) \leqslant 0 \Rightarrow B(x(t)) \leqslant$

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$B(x(t_0)) \Rightarrow h(x(t)) \geqslant h(x(t_0)) > 0$. It means for any $x(t_0) \in \mathrm{Int}(\mathcal{C})$, the set $\mathcal{C}$ is forward invariant. Therefore, the CFOS (1) is safe.

Theorem 2. For a CFOS (1) with a set $\mathcal{C}$ defined by (2)–(4) for some continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, $\forall x(t_0) \in \mathrm{Int}(\mathcal{C})$, if there exists a Caputo's exponential-beta BF $B : \mathcal{C} \to \mathbb{R}$ satisfying

$$B(x) = -\gamma(h(x)), \tag{7}$$

$$^{C}D^{\alpha}B(x(t)) \leqslant \lambda\,\mathrm{sgn}(B)|B(x(t))|^{\beta}, \tag{8}$$

where γ is a locally Lipschitz class $\mathcal{K}$ function, $\lambda \in \mathbb{R}$ and $\forall c > 0$, $\beta \in (0, c)$ with c a finite large real number. The set $\mathcal{C}$ is forward invariant and the CFOS (1) can be said to be safe.

Proof. The proof can be divided into two parts as $\beta \in (0, 1]$, and $\beta > 1$.

(I) Assume $\beta \in (0, 1]$. Let $W(t) = B(x(t))$. By Lemma 3 [7], Theorems 2.2 and 2.4 [8], we have $W(t) \leqslant W(t_0) + \frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\mathrm{sgn}(W)|W(\tau)|^{\beta}(t-\tau)^{\alpha-1}\mathrm{d}\tau \leqslant \mathrm{sgn}(W(t_0))|W(t_0)|^{\beta} + \frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\mathrm{sgn}(W)|W(\tau)|^{\beta}(t-\tau)^{\alpha-1}\mathrm{d}\tau \leqslant \mathrm{sgn}(W(t_0)) \exp[\frac{\lambda}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}] \leqslant 0$. Thus, the set $\mathcal{C}$ is forward invariant.

(II) Assume $\beta > 1$. (a) $\lambda \geqslant 0$. By Lemma 3 [7], and Theorem 2.4 [8], we have $B(x(t)) \leqslant B(x(t_0)) + \frac{\lambda}{\Gamma(\alpha)}\int_{t_0}^{t}\mathrm{sgn}(B) \cdot|B(xt\tau))|^{\beta}(t-\tau)^{\alpha-1}\mathrm{d}\tau \leqslant B(x(t_0))$. Thus, the set $\mathcal{C}$ is forward invariant. (b) $\lambda < 0$. Let $\lambda\,\mathrm{sgn}(B)|B(x(t))|^{\beta} = \phi(-B)$ with ϕ a class $\mathcal{K}$ function. Then, we obtain $^{C}D^{\alpha}B \leqslant \phi(-B)$. Let $-B = y$; then $-{}^{C}D^{\alpha}y \leqslant \phi(y) \Rightarrow {}^{C}D^{\alpha}y \geqslant -\phi(y)$. According to the inequalities (27) and (35) in the proof of Theorem 3.1 [8] and Lemma 1.1, we have $y \geqslant \sigma(y_0, t-t_0)$, with σ a class $\mathcal{KL}$ function. As $y = -B$ and $B = -\gamma(h)$, we have $h(x) \geqslant \gamma^{-1}(\sigma(-B(x(t_0)), t-t_0))$, for all $t \in I(x(t_0))$, where γ^{-1} is the inverse of γ and a class $\mathcal{K}$ function. Since $x(t_0) \in \mathcal{C}$, then $B(x(t_0)) < 0$, and $h(x(t)) \geqslant 0$, for all $t \in I(x(t_0))$. Thus, the set $\mathcal{C}$ is forward invariant.

In summary, the CFOS (1) is safe with $\lambda \in \mathbb{R}$ and $\forall c > 0$, $\beta \in (0, c)$ with c a finite large real number for $\forall x(t_0) \in \mathrm{Int}(\mathcal{C})$.

Remark 1. Here, the definition of forward invariant can be seen in [3] and the definitions of class $\mathcal{K}$ and $\mathcal{KL}$ functions can be seen in [9].

Example 1. Consider a CFOS as

$$\begin{aligned} {}^{C}D^{\alpha}x_1 &= \frac{x_1 \sin x_2}{2} + x_1 + u_1, \\ {}^{C}D^{\alpha}x_2 &= \frac{x_2 \sin x_1}{2} + x_2 + u_2, \end{aligned} \tag{9}$$

with $\alpha \in (0, 1)$, $x = (x_1, x_2)^{\mathrm{T}}$ and $u = (u_1, u_2)^{\mathrm{T}}$. Set $\mathcal{C} = \{x \in \mathbb{R}^2 : r^2 - \|x\|^2 \geqslant 0\}$ with $h(x) = r^2 - \|x\|^2$. Then, we choose $\alpha = 0.5$ to conduct the simulations with the common parameter $r = 10$ and four initial states $(8, 5)^{\mathrm{T}}, (9, -3)^{\mathrm{T}}, (-6, -7)^{\mathrm{T}}, (-4, 8)^{\mathrm{T}}$. Using Theorem 2, we can design a controller as $u = \arg\min u^{\mathrm{T}}u$, such that $2x^{\mathrm{T}}u \leqslant -\mathrm{sgn}(x^{\mathrm{T}}x - r^2)|x^{\mathrm{T}}x - r^2|^{\frac{1}{2}} - 2x^{\mathrm{T}}[\frac{x_1 \sin x_2}{2} - x_1, \frac{x_2 \sin x_1}{2} - x_2]^{\mathrm{T}}$. Here, set $B(x) = -h(x)$ and $\beta = \frac{1}{2}$. The solver of u needs quadratic programs, using "quadprog" function in MATLAB. The details of the results can be shown in Figure 1. Obviously, the safety controller works well.

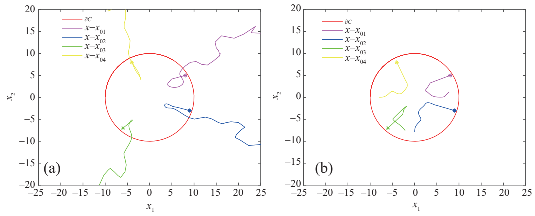

Figure 1 (Color online) Relationship between x and set $\mathcal{C}$ with $\alpha = 0.5$ for CFOS, where "*" is the initial state. (a) Without control; (b) using the controller based on Caputo's exponential-beta BF.

Conclusion. This study presented extended two Caputo's BFs and proposed two kinds of safety criteria for CFOS. Additionally, we incompletely verified the validity of the theory using a simulation example.

Moreover, using Theorem 2, we obtained that when $\beta = 0$ and $\lambda < 0$, its proof has some doubt because the Caputo's derivative is a nonlocal operator. It implies that fractional-order systems are quite different from integer-order systems, such that the safety control theory applicable to integer-order systems is not necessarily applicable to fractional-order systems. Thus, we need to have more effort for other extensions of safety control theory in fractional-order systems, which we have to find another chance to supplement.

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