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Consensus of hybrid linear multi-agent systems with periodic jumps

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Appendix A Notations

$\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{n\times n}$ denotes the set of all $n \times n$ matrices with real entries. $\mathbb{C}$ denotes the set of all complex numbers, $\mathbb{C}^n$ denotes complex vector space of complex n-vectors, and $\mathbb{C}^{n\times n}$ denotes the set of all $n \times n$ matrices with complex entries. I_n denotes the $n \times n$ identity matrix, and $\mathbf{1}_n$ denotes the n-dimensional column vector with every element being 1. $\mathbb{Z}^+$ denotes the set of all positive integers. Let $\mathbb{N} = \{0, \mathbb{Z}^+\}$. Define $\mathbb{C}_g = \{s \in \mathbb{C} : |s| < 1\}$. $\mathrm{Re}(\lambda)$ denotes the real part of the complex number λ.

Appendix B Literature review

Over the past decades, multi-agent systems have attracted extensive attention in the control communities due to their wide range of applications, such as smart power grids [1], multiple spacecraft systems [2], wireless sensors [29], etc. In these practical applications, the main purpose is to design distributed control law to achieve some common tasks for all agents, but in which each agent can only make use of the information about itself and its neighboring agents [5, 24, 27] without knowing the full information of the whole group. This control scheme is called the cooperative/distributed control. Consensus problem, which aims to drive the states of all agents to a common trajectory asymptotically, is one of the most fundamental cooperative control problems, including many practical cooperative control problems such as formation, swarm, and clustering control [4, 14, 21].

So far, most results on consensus problem mainly have focused on either continuous-time or discrete-time multi-agent systems individually. The Riccati-type equation/inequality has been shown to be one of the useful tools for the consensus algorithm design for general linear systems. Specifically, for continuous-time multi-agent systems, the Riccati equation was first employed in [23] to provide an effective analysis for the state feedback design. Later, the static output feedback control was proposed in [13] and the observer based output feedback control was presented in [26] both for directed communication graph also using Riccati equations. In these results, the stabilizability and detectability conditions have been shown sufficient for ensuring the solvability of corresponding Riccati equation/inequality and obtaining the efficient control gain. Similarly, for discrete-time multi-agent systems, a necessary and sufficient condition was initially given by using the discrete-time H_∞ type Riccati inequality design method to achieve consensus problem for a class of unstable linear multi-agent systems with single inputs [25]. This method was then extended to systems with multi inputs in case of an input matrix with full column rank [7]. Based on a modified algebraic Riccati equation, reference [11] gives a general distributed state feedback control law, which is easier to be implemented. Notice that, for discrete-time multi-agent systems, even though the stabilizability and detectability conditions are sufficient for the solvability of corresponding Riccati equation/inequality, one still needs additional eigenvalue conditions of the plant so as to obtain the control gains [7, 11, 25].

In modeling real-world phenomena, it is more usual to consider a type of system that exhibits characteristics of both continuous-time (flow) dynamics and discrete-time (jump) dynamics, called the hybrid system [6]. The hybrid system can not only describe various mechanical systems, such as bouncing balls, power control with a thyristor, RC circuit, etc, but also embrace a number of control areas: sampled-data control systems, reset linear control systems, and impulsive systems, etc, hence has been widely studied recently [16, 17, 19, 22]. In recent years, many efforts have been devoted into a class of hybrid linear systems with periodic flows and jumps [16, 22]. The structural properties of this class of systems are much more general than either purely continuous-time or discrete-time systems in the sense that both stabilizability and detectability conditions have to be defined in the hybrid time domain, while neither its flow dynamics nor jump dynamics need to be stabilizable and detectable, giving rise to the so-called hybrid control scheme.

To the authors' knowledge, there is no result on the research of this class of hybrid multi-agent systems, where the influence of communication graphs has to be taken into account. Comparing with the previous studies on continuous-time or discrete-time multi-agent systems, the major difficulty lies in that neither the flow dynamics nor jump dynamics are stabilizable or detectable, which prevents the use of purely distributed continuous or discrete controllers in previous references [7, 11, 13, 23, 25, 26].

Appendix C Motivating examples

Four motivating examples are collected to further illustrate the practical motivations. Examples 1 and 2 are mechanical models that have to be depicted in the hybrid system due to their practical meaning, while Examples 3 and 4 are control designs that can be converted into the hybrid scheme with some transformations.

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• **Example 1:** This example is about multiple spinning and bouncing disks moving on a horizontal plane between parallel walls [17, 18]. Each disk can be modeled in the hybrid scheme as follows, where the subscript i denoting the i-th agent is omitted for simplicity:

$$\dot{\tau} = 1, \quad \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ 0 \end{bmatrix} u_F$$

whether $(\tau, x) \in [0, \tau_d] \times \mathbb{R}^4$ and

$$\tau^+ = 0, \ x^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\omega^{-1} & 0 & -\omega^{-1} r \\ 0 & 0 & 1 & 0 \\ 0 & -r^{-1}(1-\omega^{-1}) & 0 & \omega^{-1} \end{bmatrix} x$$

whether $(\tau, x) \in \{\tau_d\} \times \mathbb{R}^4$, with system output

$$y_F(t,k) = 0, \ y_J(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t_k, k-1)$$

where $\tau(0,0) = 0$, and $\omega = \frac{r^2 M}{\mathcal{I}}$ with r, M, $\mathcal{I}$ denoting radius, total mass, inertia of each disk, respectively. Appendix K has given a detailed design process for the consensus of four bouncing disks moving on a horizontal plane between parallel walls.

• **Example 2:** This example is about multiple RC circuits, where the switches are closed periodically at integer multiples of τ_d. Each circuit can be defined by $x = \mathrm{col}(x_1, x_2, x_3)$ and $u_F = \mathrm{col}(u_1, u_2)$, where the subscript i denoting the i-th agent is omitted for simplicity, and modeled with flow dynamics,

$$\dot{\tau} = 1, \ \dot{x} = \begin{bmatrix} -\frac{1}{R_1 C_1} & 0 & 0 \\ 0 & -\frac{1}{R_2 C_2} & 0 \\ 0 & 0 & -\frac{1}{R_2 C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1 C_1} & 0 \\ 0 & \frac{1}{R_2 C_2} \\ 0 & \frac{1}{R_3 C_3} \end{bmatrix} u_F$$

whether $(\tau, x) \in [0, \tau_d] \times \mathbb{R}^4$ and the jump dynamics

$$\tau^+ = 0, \ x^+ = \frac{1}{C_1 + C_2 + C_3} \begin{bmatrix} C_1 & C_2 & C_3 \\ C_1 & C_2 & C_3 \\ C_1 & C_2 & C_3 \end{bmatrix} x$$

whether $(\tau, x) \in \{\tau_d\} \times \mathbb{R}^4$, where for $k = 1, 2, 3$, x_k, C_k, and R_k denote the voltage across the k-th capacitor, k-th capacitance and k-th resistance, respectively. $u_s \in \mathbb{R}$ with $s = 1, 2$, denotes the (controlled) tension of the s-th voltage generator. For more details, please refer to [3].

• **Example 3:** This example is to relate the sampled-data control for the linear system

$$\dot{x} = Ax + Bu_F$$

where $x \in \mathbb{R}^n$ is the state, $u_F \in \mathbb{R}^m$ is the input, A and B are the constant matrices with compatible dimensions. The system, with a periodical sampled-data control, leads to a hybrid system as follows: with the flow dynamics $\dot{\tau} = 1, \ \dot{x} = Ax + Bu_F$ whether $(\tau, x) \in [0, \tau_d] \times \mathbb{R}^n$, and the jump dynamics $\tau^+ = 0, \ x^+ = x$ whether $(\tau, x) \in \{\tau_d\} \times \mathbb{R}^n$, where τ_d denotes sampling period. Corollary 1 in Appendix J gives two sample-data controllers for this special case.

• **Example 4:** This example is to relate the periodic linear impulsive system given by

$$\begin{aligned} \dot{x}(t) &= Ax(t), \ t \neq k\tau_d, \ k \in \{1, 2, \ldots\} \\ x(t) &= u_J(t^-), \ t = k\tau_d, \ k \in \{1, 2, \ldots\} \end{aligned}$$

where $x \in \mathbb{R}^n$ is the state, $u_J \in \mathbb{R}^m$ is the input, and $u(t^-) = \lim_{\tau \to t, \tau < t} u(\tau)$. Such impulsive system can be modeled in the format (1) as follows: with the flow dynamics $\dot{\tau} = 1, \ \dot{x} = Ax$ whether $(\tau, x) \in [0, \tau_d] \times \mathbb{R}^n$, and the jump dynamics $\tau^+ = 0, \ x^+ = u_J$ whether $(\tau, x) \in \{\tau_d\} \times \mathbb{R}^n$.

Appendix D Summary of main contributions

The distinguishing features for studying the hybrid linear multi-agent systems with periodic jumps are of the following two aspects.

On one hand, this class of hybrid systems can ensure us to study the cooperative control of many interesting mechanical systems, such as multiple spinning and bouncing disks [17] and multiple RC circuits [3]. All of these systems can neither be modeled by the continuous-time models nor the discrete-time models, but have to be described in the hybrid sense, containing both continuous-time (flow) dynamics and discrete-time (jump) dynamics, particularly, neither of which are necessary to be stabilizable and detectable. This situation makes the previous consensus studies based only on either continuous-time model or discrete-time model no longer work on them. Thus, our study provides a systematic method for handling these practical models.

On the other hand, this class of hybrid systems can also embrace a number of control areas [6], such as sampled-data control systems, periodic impulsive systems, and so forth. Thus, our study gives a unified synthesis process for handling these cooperative control problems.

Technically, we establish the consensus protocol in the hybrid sense containing both flow and jump dynamics. The novelties are of the following two aspects.

Firstly, both hybrid distributed state feedback and output feedback control laws are developed so as to deal with the hybrid structure of the plant, which is stabilizable and detectable in the sense of hybrid dynamics, while neither its flow dynamics nor jump dynamics need to be. In particular, the hybrid model prevents us to adopt the traditional Luenberger observer when considering the output feedback control. Accordingly, a novel hybrid distributed observer combining both continuous output and discrete output is developed without requiring either of them to be detectable.

Secondly, novel feedback and observer gain assignment algorithms that involve the modified H_∞ type Riccati inequality are proposed under stabilizability and detectability conditions in the hybrid time domain, respectively. The feedback gain design and the observer gain design are both highly relative to a so-called monodromy discrete-time system. However, the input matrix (resp., the output matrix) of this monodromy discrete-time system is just of a very low column rank (resp., row rank), hence preventing the direct use of the existing discrete-time Riccati inequality technique. In order to handle this difficulty, we first find their full rank parts by using elementary transformation, then reset the gain matrices of flow dynamics by utilizing properties of controllable and observable subspaces.

Appendix E Communication graph

Given the hybrid linear multi-agent system (1), the information exchange among the agents is described by a directed communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, N\}$ represents the N agents and $\mathcal{E} = \{(i,j) : i,j \in \mathcal{V}, i \neq j\} \subseteq \mathcal{V} \times \mathcal{V}$ denotes edge set. The edge $(j,i) \in \mathcal{E}$ if and only if the control inputs u_{Fi} and u_{Ji} can make use of the information of the j-th agent for feedback. A matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is said to be a weighted adjacency matrix of the graph $\mathcal{G}$ if $a_{ii} = 0$ and $a_{ij} > 0 \Leftrightarrow (j,i) \in \mathcal{E}$. As in [12], we define the normalized weighted adjacency matrix $\Omega = [\omega_{ij}] \in \mathbb{R}^{N\times N}$, in which for $i,j = 1,\dots,N$, $\omega_{ij} = 1/(1+\sum_{j=1}^{N} a_{ij})$ if $i = j$; and $\omega_{ij} = a_{ij}/(1+\sum_{j=1}^{N} a_{ij})$ otherwise.

Let $\mathcal{L}_N = I_N - \Omega$. It has been shown in [20, Corollary 3.5] that all the eigenvalues of Ω lie inside or on the unit circle, and zero is a simple eigenvalue of $\mathcal{L}_N$ if and only if Assumption 2 is satisfied. Thus under Assumption 2, there exists a vector $r \in \mathbb{R}^n$ satisfying $r^\mathsf{T}\mathcal{L}_N = 0$ and $r^\mathsf{T}\mathbf{1}_N = 1$. Moreover, One can choose $U_1 \in \mathbb{C}^{N\times(N-1)}$, $U_2 \in \mathbb{C}^{(N-1)\times N}$, $U \in \mathbb{C}^{N\times N}$, and an upper triangular $\Delta \in \mathbb{C}^{(N-1)\times(N-1)}$ such that

$$U = \begin{bmatrix} \mathbf{1}_N & U_1 \end{bmatrix},\ U^{-1} = \begin{bmatrix} r^\mathsf{T} \\ U_2 \end{bmatrix},\ J_L = U^{-1}\mathcal{L}_N U = \begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix} \tag{E1}$$

where the diagonal entries of Δ are the nonzero eigenvalues of $\mathcal{L}_N$, denoting by $\lambda_2,\dots,\lambda_N$.

Appendix F PBH tests for the stabilizability and detectability of hybrid system (1)

The hybrid linear multi-agent system (1) contains both the continuous-time dynamics (1a) with the flow output (1c) and the discrete-time dynamics (1b) with the jump output (1d). Recall from [16, Theorems 4 & 7], the PBH tests for the stabilizability and detectability of each subsystem of (1) are given by

$$\operatorname{rank}\left(\begin{bmatrix} Ee^{A\tau_d} - sI & F & R_{A,B} \end{bmatrix}\right) = n,\ \forall\ s \in \Lambda(Ee^{A\tau_d}),\ s \notin \mathbb{C}_g \tag{F1}$$

and

$$\operatorname{rank}\left(\begin{bmatrix} (Ee^{A\tau_d})^\mathsf{T} - sI & (C_J e^{A\tau_d})^\mathsf{T} & O_{A,C_F}^\mathsf{T} \end{bmatrix}^\mathsf{T}\right) = n,\ \forall\ s \in \Lambda(Ee^{A\tau_d}),\ s \notin \mathbb{C}_g \tag{F2}$$

respectively, where $R_{A,B} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$, $O_{A,C_F} = \begin{bmatrix} C_F^\mathsf{T} & (C_F A)^\mathsf{T} & \cdots & (C_F A^{n-1})^\mathsf{T} \end{bmatrix}^\mathsf{T}$. One can conclude from (F1) and (F2) that, the stabilizability and detectability of each subsystem of (1) cover but cannot imply the stabilizability and detectability of either the continuous-time dynamics (1a) with the output (1c) or the discrete-time dynamics (1b) with the output (1d). Our simulation example in Appendix K does give such a situation. As a result, using only the flow input u_{Fi} or the jump input u_{Ji} cannot achieve the consensus, making the previous designs for purely continuous-time or discrete-time multi-agent systems are inadmissible, say [7, 11, 13, 23, 25, 26].

Appendix G Some technical lemmas

This appendix collects some technical lemmas that will be frequently used in this paper.

Lemma 1. Given any $\mathbf{A} \in \mathbb{R}^{n_1\times n_1}$, $\mathbf{B} \in \mathbb{R}^{n_1\times n_2}$ with $n_2 > n_1$ and $\mathbf{C} \in \mathbb{R}^{q_1\times n_1}$ with $q_1 > n_1$,

1) there exists a nonsingular matrix $V_c \in \mathbb{R}^{n_2\times n_2}$ such that $\mathbf{B}V_c = [\mathbf{B}_v\ 0]$, where $\mathbf{B}_v$ has full column rank. Moreover, $(\mathbf{A}, \mathbf{B}_v)$ is stabilizable if and only if $(\mathbf{A}, \mathbf{B})$ is stabilizable;

2) there exists a nonsingular matrix $V_o \in \mathbb{R}^{q_1\times q_1}$ such that $\mathbf{C}^\mathsf{T} V_o^\mathsf{T} = [\mathbf{C}_v^\mathsf{T}\ 0]$, where $\mathbf{C}_v$ has full row rank. Moreover, $(\mathbf{C}_v, \mathbf{A})$ is detectable if and only if $(\mathbf{C}, \mathbf{A})$ is detectable.

Lemma 1 is a direct result of the elementary transformation as well as PBH test for the discrete-time system [8, Theorem 14.2].

Lemma 2 ([7])**.** Assume that $(\mathbf{A}, \mathbf{B})$ is stabilizable and $\mathbf{B}$ has full column rank. Consider the modified H_∞ type Riccati inequality

$$\mathbf{A}^\mathsf{T}\mathbf{P}\mathbf{A} - \mathbf{P} - (1-\delta^2)\mathbf{A}^\mathsf{T}\mathbf{P}\mathbf{B}(\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{B})^{-1}\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{A} < 0. \tag{G1}$$

Let $\delta_c = \sup_{\delta>0}\{\delta | \exists \mathbf{P}$ s.t. the inequality (G1) holds$\}$. Given a complex number λ. If there exists $\alpha \in \mathbb{R}$ such that $|1-\alpha\lambda| < \delta_c$, then, there exists a positive definite matrix $\mathbf{P}$ solving (G1) with $\delta = |1-\alpha\lambda|$. Moreover, $\mathbf{K} = -\alpha(\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{B})^{-1}\mathbf{B}\mathbf{P}\mathbf{A}$ is such that $\mathbf{A} + \lambda\mathbf{B}\mathbf{K}$ is Schur stable.

The following lemma follows from duality and Lemma 2.

Lemma 3. Assume that $(\mathbf{C}, \mathbf{A})$ is detectable and $\mathbf{C}$ has full row rank. Consider the modified H_∞ type Riccati inequality

$$\mathbf{A}\mathbf{Q}\mathbf{A}^\mathsf{T} - \mathbf{Q} - (1-\delta^2)\mathbf{A}\mathbf{Q}\mathbf{C}^\mathsf{T}(\mathbf{C}\mathbf{Q}\mathbf{C}^\mathsf{T})^{-1}\mathbf{C}\mathbf{Q}\mathbf{A}^\mathsf{T} < 0. \tag{G2}$$

Let $\delta_o = \sup_{\delta>0}\{\delta | \exists \mathbf{Q}$ s.t. the inequality (G2) holds$\}$. Given a complex number λ, if there exists $\alpha \in \mathbb{R}$ such that $|1-\alpha\lambda| < \delta_o$, then, there exists a positive definite matrix $\mathbf{Q}$ solving (G2) with $\delta = |1-\alpha\lambda|$. Moreover, $\mathbf{L} = -\alpha\mathbf{A}\mathbf{Q}\mathbf{C}^\mathsf{T}(\mathbf{C}\mathbf{Q}\mathbf{C}^\mathsf{T})^{-1}$ is such that $\mathbf{A}+\lambda\mathbf{L}\mathbf{C}$ is Schur stable.

Remark 1. It has been shown in [7, Lemma 6] that the lower bound of δ_c (and correspondingly δ_o) always exists, and is relative to the unstable eigenvalues of $\mathbf{A}$. In particular, if $\mathbf{A}$ contains no unstable eigenvalues, then δ_c and δ_o can always be chosen as one.

Appendix H Gain assignment algorithms based on monodromy discrete-time system

For the hybrid linear multi-agent system (1), we define the following so-called monodromy discrete-time system [16]

$$\begin{aligned} z_i(k+1) &= \bar{A}z_i(k) + \bar{B}v_i(k) \\ w_i(k) &= \bar{C}z_i(k),\ i = 1,\dots,N \end{aligned} \tag{H1}$$

where $z_i \in \mathbb{R}^n$, $v_i \in \mathbb{R}^{nm_1+m_2}$, and $w_i \in \mathbb{R}^{nq_1+q_2}$. Here $\bar{A} = Ee^{A\tau_d} \in \mathbb{R}^{n\times n}$, $\bar{B} = \begin{bmatrix} F & R_{A,B} \end{bmatrix} \in \mathbb{R}^{n\times(nm_1+m_2)}$, $\bar{C} = \begin{bmatrix} (C_J e^{A\tau_d})^\mathsf{T} & O_{A,C_F}^\mathsf{T} \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{(nq_1+q_2)\times n}$, and $Ee^{A\tau_d}$ is the so-called monodromy matrix. Recall from [16, Proposition 7], each subsystem of the monodromy discrete-time system (H1) is stabilizable (resp., detectable) if and only if each subsystem of the hybrid linear multi-agent system (1) is stabilizable (resp., detectable). Moreover, let $\tilde{A} = e^{A\tau_d}E \in \mathbb{R}^{n\times n}$ and $\tilde{B} = \begin{bmatrix} e^{A\tau_d}F & R_{A,B} \end{bmatrix} \in \mathbb{R}^{n\times(nm_1+m_2)}$. From PBH test (F2) and the property $\mathrm{Im}(e^{A\tau_d}R_{A,B}) = \mathrm{Im}(R_{A,B})$, it can be verified that $(\tilde{A}, \tilde{B})$ is stabilizable if and only if $(\bar{A}, \bar{B})$ is stabilizable [16]. Such results can be summarized as the following lemma.

Lemma 4. The following properties are satisfied.

1) $(\tilde{A}, \tilde{B})$ is stabilizable if and only if each subsystem of the hybrid linear multi-agent system (1) is stabilizable.

2) $(\bar{C}, \bar{A})$ is detectable if and only if each subsystem of the hybrid linear multi-agent system (1) is detectable.

With the aid of Lemmas 1 and 4, the feedback gain and the observer gain can be assigned by Algorithms H1 and H2, respectively.

Algorithm H1 The design for matrices $\bar{K}_F$ and K_J

Require: Assumptions 1 and 2 are satisfied. Denote $\lambda_2, \dots, \lambda_N$ as nonzero eigenvalues of $\mathcal{L}_N$.
Require: Given $\delta_c \in (0, 1]$, there exists $\alpha_c \in \mathbb{R}$ such that

$$|1-\alpha_c\lambda_i| < \delta_c. \tag{H2}$$

1: Let the matrix $V_c \in \mathbb{R}^{(nm_1+m_2)\times(nm_1+m_2)}$ such that $\tilde{B}V_c = [\tilde{B}_v, 0]$, where $\tilde{B}_v$ has full column rank;
2: Denote $\delta(\alpha_c) \triangleq \max_{i=2,\dots,N} |1-\alpha_c\lambda_i|$. Find a positive definite matrix P satisfying

$$\tilde{A}^\mathsf{T}P\tilde{A} - P - (1-\delta(\alpha_c)^2)\tilde{A}^\mathsf{T}P\tilde{B}_v(\tilde{B}_v^\mathsf{T}P\tilde{B}_v)^{-1}\tilde{B}_v^\mathsf{T}P\tilde{A} < 0; \tag{H3}$$

3: Define $[K_J^\mathsf{T}, K_F^\mathsf{T}]^\mathsf{T} = [K_v^\mathsf{T}, 0]^\mathsf{T}V_c^\mathsf{T}$, where $K_v = -\alpha_c(\tilde{B}_v^\mathsf{T}P\tilde{B}_v)^{-1}\tilde{B}_v^\mathsf{T}P\tilde{A}$;
4: Obtain $\bar{K}_F$ by solving $R_{A,B}K_F = G(\tau_d)\bar{K}_F$ with $G(\tau_d) = \int_0^{\tau_d} e^{A(\tau_d-\tau)}BB^\mathsf{T}e^{A^\mathsf{T}(\tau_d-\tau)}\mathrm{d}\tau$.

Algorithm H2 The design for matrices $\bar{L}_F$ and L_J

Require: Assumptions 1 and 2 are satisfied. Denote $\lambda_2, \dots, \lambda_N$ as nonzero eigenvalues of $\mathcal{L}_N$.
Require: Given $\delta_o \in (0, 1]$, there exists $\alpha_o \in \mathbb{R}$ such that

$$|1-\alpha_o\lambda_i| < \delta_o. \tag{H4}$$

1: Let the matrix $V_o \in \mathbb{R}^{(nq_1+q_2)\times(nq_1+q_2)}$ such that $\bar{C}^\mathsf{T}V_o^\mathsf{T} = [\bar{C}_v^\mathsf{T}\ 0]$, where $\bar{C}_v$ has full row rank;
2: Denote $\delta(\alpha_o) \triangleq \max_{i=2,\dots,N} |1-\alpha_o\lambda_i|$. Find a positive definite matrix Q satisfying

$$\bar{A}Q\bar{A}^\mathsf{T} - Q - (1-\delta(\alpha_o)^2)\bar{A}Q\bar{C}_v^\mathsf{T}(\bar{C}_vQ\bar{C}_v^\mathsf{T})^{-1}\bar{C}_vQ\bar{A}^\mathsf{T} < 0; \tag{H5}$$

3: Define $[L_J, L_F] = [L_v, 0]V_o$, where $L_v = -\alpha_o\bar{A}Q\bar{C}_v^\mathsf{T}(\bar{C}_vQ\bar{C}_v^\mathsf{T})^{-1}$;
4: Obtain $\bar{L}_F$ by solving $L_FO_{A,C_F} = \bar{L}_FW(\tau_d)$ with $W(\tau_d) = \int_0^{\tau_d} e^{A^\mathsf{T}(\tau_d-t)}C_F^\mathsf{T}C_Fe^{A(\tau_d-t)}\mathrm{d}\tau$.

Remark 2. It is interesting to see that the common hybrid time domain $\mathcal{T}$ has been assumed for each subsystem of (1), that is to say, all the subsystems have to jump simultaneously. Such a condition is not restrictive but necessary for the hybrid linear multi-agent system with state-driven jumps along periodic (nonzero) trajectories. Notice that, at the steady-state, the states of all subsystems must be synchronized when the consensus is achieved, and hence the jump must happen simultaneously, see a similar discussion in [3].

Remark 3. Notice that condition (H2) has to be imposed in Algorithm H1 according to the existence of the jump dynamics in (1). Such a condition is mild and consistent with that of the system matrix in the discrete-time consensus algorithm, see [7, 25]. Furthermore, the existence of V_c is guaranteed by Lemma 1, while the existence of P is guaranteed by Lemmas 2 and 4. Moreover, $\bar{K}_F$ can be obtained by the fact that $\mathrm{Im}(R_{A,B}) = \mathrm{Im}(G(\tau_d))$, see [8, Theorem 11.5].

Remark 4. The purpose of condition (H4) in Algorithm H2 is the same as that of condition (H2) in Algorithm H1. Similarly with Algorithm H1, the existence of V_o is also guaranteed by Lemma 1, the existence of Q is guaranteed by Lemmas 3 and 4, and $\bar{L}_F$ can be obtained by the fact that $\mathrm{Im}(O_{A,C_F}^\mathsf{T}) = \mathrm{Im}(W(\tau_d))$, see [8, Theorem 15.1].

Remark 5. Lemma 4 has provided some equivalent properties between the hybrid linear multi-agent system (1) and its monodromy discrete-time system. These properties inspire us to adopt the modified H_∞ type Riccati inequality based design proposed in recent references, say [7], for the gain matrices. However, as can be seen, the input matrix $\tilde{B}$ has a huge number of columns, so it is of very low column rank. Similarly, the output matrix $\bar{C}$ has a huge number of rows, so it is of very low row rank. Both facts prevent the direct use of Lemma 2 or 3 (i.e., the modified H_∞ type Riccati inequality (G1) or (G2)), where it is required that the input matrix is of full column rank or the output matrix is of full row rank. So we have to introduce Lemma 1 in order to find the full column rank part of the input matrix (i.e., $\tilde{B}_v$) and the full row rank part of the output matrix (i.e., $\bar{C}_v$).

Remark 6. The design process of the distributed dynamic state feedback control law (2) is inspired by the hybrid stabilization design [16]. Indeed, besides the jump input u_{Ji}, the flow input u_{Fi} has to be added to reflect the influence of flow dynamics, leading to a dynamic state feedback control with a hybrid internal dynamics ξ_i and a redesign of feedback gain $\bar{K}_F$. Moreover, a novel hybrid distributed observer combining both continuous output and discrete output has been developed in the design process of the distributed dynamic output feedback control law (3).

Appendix I Proof of Theorem 1

Appendix I.1 Control via the distributed dynamic state feedback

The proof requires the following result, which depicts the relation between the states of agents and their consensus center under the distributed dynamic state feedback control law (2). Let $x=[x_1^\mathsf{T},\dots,x_N^\mathsf{T}]^\mathsf{T}$ and $\xi=[\xi_1^\mathsf{T},\dots,\xi_N^\mathsf{T}]^\mathsf{T}$. We define the consensus center as: for any $(t,k)\in\mathcal{T}$,

$$x_c(t,k)=\left(r^\mathsf{T}\otimes e^{A(t-t_k)}\right)x(t_k,k),\ x_c(t_k,k)=Ex_c(t_k,k-1). \tag{I1}$$

Lemma 5. Under Assumption 2, the closed-loop system composed of (1) and (2) has the property that $x_c(t_k,k)=(r^\mathsf{T}\otimes I_n)x(t_k,k)$.

Proof. Under the dynamic state feedback control law (2), the closed-loop system can be written as

$$\dot{\tau}=1,\ \dot{x}=(I_N\otimes A)x+(I_N\otimes BB^\mathsf{T})\xi,\ \dot{\xi}=-(I_N\otimes A^\mathsf{T})\xi \tag{I2a}$$

whether $(\tau,x,\xi)\in[0,\tau_d]\times\mathbb{R}^{Nn}\times\mathbb{R}^{Nn}$, and

$$\tau^+=0,\ x^+=(I_N\otimes E+\mathcal{L}_N\otimes FK_J)x,\ \xi^+=(\mathcal{L}_N\otimes e^{A^\mathsf{T}\tau_d}\bar{K}_F)x \tag{I2b}$$

whether $(\tau,x,\xi)\in\{\tau_d\}\times\mathbb{R}^{Nn}\times\mathbb{R}^{Nn}$. From (I2a), for any $(t,k)\in\mathcal{T}$, $\xi(t,k-1)=\left(I_N\otimes e^{-A^\mathsf{T}(t-t_{k-1})}\right)\xi(t_{k-1},k-1)$, thus, we have

$$\begin{aligned}x(t_k,k-1)&=\left(I_N\otimes e^{A\tau_d}\right)x(t_{k-1},k-1)+\int_{t_{k-1}}^{t_k}\left(I_N\otimes e^{A(t_k-\tau)}BB^\mathsf{T}\right)\xi(\tau,k-1)\mathrm{d}\tau\\&=\left(I_N\otimes e^{A\tau_d}\right)x(t_{k-1},k-1)+\int_{t_{k-1}}^{t_k}\left(I_N\otimes e^{A(t_k-\tau)}BB^\mathsf{T}e^{-A^\mathsf{T}(\tau-t_{k-1})}\right)\mathrm{d}\tau\cdot\xi(t_{k-1},k-1).\end{aligned}$$

Then, according to (I2b), we have $(r^\mathsf{T}\otimes I_n)\xi(t_k,k)=0$, for any $k\in\mathbb{N}$. This together with $r^\mathsf{T}\mathcal{L}_N=0$ gives

$$\begin{aligned}&(r^\mathsf{T}\otimes I_n)x(t_k,k)=(r^\mathsf{T}\otimes E)x(t_k,k-1)\\&=E\left(r^\mathsf{T}\otimes e^{A\tau_d}\right)x(t_{k-1},k-1)+\int_{t_{k-1}}^{t_k}\left(r^\mathsf{T}\otimes Ee^{A(t_k-\tau)}BB^\mathsf{T}e^{-A^\mathsf{T}(\tau-t_{k-1})}\right)\mathrm{d}\tau\cdot\xi(t_{k-1},k-1)\\&=E\left(r^\mathsf{T}\otimes e^{A\tau_d}\right)x(t_{k-1},k-1)+\int_{t_{k-1}}^{t_k}Ee^{A(t_k-\tau)}BB^\mathsf{T}e^{-A^\mathsf{T}(\tau-t_{k-1})}\mathrm{d}\tau\cdot\left(r^\mathsf{T}\otimes I_n\right)\xi(t_{k-1},k-1)\\&=E\left(r^\mathsf{T}\otimes e^{A\tau_d}\right)x(t_{k-1},k-1).\end{aligned}$$

It follows from (I1) that

$$x_c(t_k,k)=Ex_c(t_k,k-1)=E(r^\mathsf{T}\otimes e^{A\tau_d})x(t_{k-1},k-1)=(r^\mathsf{T}\otimes I_n)x(t_k,k).$$

This completes the proof. □

Let us show that Problem 1 is solved by the distributed dynamic state feedback control law (2).

Proof. Define the consensus error as $\bar{x}_i=x_i-x_c$. According to (1), (2), and (I1),

$$\dot{\bar{x}}_i=\dot{x}_i-\dot{x}_c=Ax_i+BB^\mathsf{T}\xi_i-Ax_c=A\bar{x}_i+BB^\mathsf{T}\xi_i$$

whether $(\tau,x_i,x_c,\xi_i)\in[0,\tau_d]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$, and

$$\bar{x}_i^+=x_i^+-x_c^+=Ex_i+FK_J\sum_{j=1}^N\omega_{ij}(x_i-x_j)-Ex_c=E\bar{x}_i+FK_J\sum_{j=1}^N\omega_{ij}(\bar{x}_i-\bar{x}_j)$$

whether $(\tau,x_i,x_c,\xi_i)\in\{\tau_d\}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$. Letting $\bar{x}=[\bar{x}_1^\mathsf{T},\dots,\bar{x}_N^\mathsf{T}]^\mathsf{T}$, and $\chi=[\bar{x}^\mathsf{T},\xi^\mathsf{T}]^\mathsf{T}$ yields that

$$\dot{\tau}=1,\ \dot{\chi}=A_\chi\chi \tag{I3a}$$

whether $(\tau, \chi) \in [0, \tau_d] \times \mathbb{R}^{2Nn}$, and

$$\tau^+ = 0,\ \chi^+ = E_\chi \chi \tag{I3b}$$

whether $(\tau, \chi) \in \{\tau_d\} \times \mathbb{R}^{2Nn}$, where the matrices A_χ and E_χ are of the following form

$$A_\chi = \begin{bmatrix} I_N \otimes A & I_N \otimes BB^\mathsf{T} \\ 0 & -(I_N \otimes A^\mathsf{T}) \end{bmatrix},\quad E_\chi = \begin{bmatrix} I_N \otimes E + \mathcal{L}_N \otimes FK_J & 0 \\ \mathcal{L}_N \otimes e^{A^\mathsf{T}\tau_d} \bar{K}_F & 0 \end{bmatrix}. \tag{I4}$$

Thus, for all $k \in \mathbb{N}$, $\chi(t_{k+1}) = E_\chi e^{A_\chi \tau_d} \chi(t_k)$, where $t_0 = 0$ and $\chi(t_k) \triangleq \chi(t_k, k)$ for simplicity. Let $\bar{\chi}(t_k) = e^{A_\chi \tau_d}\chi(t_k)$. Then $\bar{\chi}(t_{k+1}) = e^{A_\chi \tau_d} E_\chi \bar{\chi}(t_k)$. Noticing that $e^{A_\chi \tau_d} = \begin{bmatrix} I_N \otimes e^{A\tau_d} & I_N \otimes G(\tau_d) e^{-A^\mathsf{T}\tau_d} \\ 0 & I_N \otimes e^{-A^\mathsf{T}\tau_d} \end{bmatrix}$ and $R_{A,B}K_F = G(\tau_d)\bar{K}_F$, one has

$$\begin{aligned} e^{A_\chi \tau_d} E_\chi &= \begin{bmatrix} I_N \otimes e^{A\tau_d} E + \mathcal{L}_N \otimes e^{A\tau_d} FK_J + \mathcal{L}_N \otimes G(\tau_d)\bar{K}_F & 0 \\ \mathcal{L}_N \otimes \bar{K}_F & 0 \end{bmatrix} \\ &= \begin{bmatrix} I_N \otimes e^{A\tau_d} E + \mathcal{L}_N \otimes e^{A\tau_d} FK_J + \mathcal{L}_N \otimes R_{A,B}K_F & 0 \\ \mathcal{L}_N \otimes \bar{K}_F & 0 \end{bmatrix} = \begin{bmatrix} I_N \otimes \tilde{A} + \mathcal{L}_N \otimes \tilde{B}K & 0 \\ \mathcal{L}_N \otimes \bar{K}_F & 0 \end{bmatrix} \end{aligned}$$

where $K = [K_J^\mathsf{T}, K_F^\mathsf{T}]^\mathsf{T}$. Let U, J_L, and Δ be given in (E1), and

$$\tilde{\chi}(t_k) = \begin{bmatrix} U^{-1} \otimes I_n & 0 \\ 0 & U^{-1} \otimes I_n \end{bmatrix} \bar{\chi}(t_k).$$

Then it holds that

$$\tilde{\chi}(t_{k+1}) = \begin{bmatrix} I_N \otimes \tilde{A} + J_L \otimes \tilde{B}K & 0 \\ J_L \otimes \bar{K}_F & 0 \end{bmatrix} \tilde{\chi}(t_k). \tag{I5}$$

Decompose $\tilde{\chi}(t_k)$ as $\tilde{\chi}(t_k) \triangleq [\tilde{\chi}_1(t_k)^\mathsf{T}, \tilde{\chi}_2(t_k)^\mathsf{T}, \tilde{\chi}_3(t_k)^\mathsf{T}, \tilde{\chi}_4(t_k)^\mathsf{T}]^\mathsf{T}$, where $\tilde{\chi}_1(t_k), \tilde{\chi}_3(t_k) \in \mathbb{R}^n$, $\tilde{\chi}_2(t_k), \tilde{\chi}_4(t_k) \in \mathbb{C}^{(N-1)n}$. From (I5), one has $\tilde{\chi}_3(t_k) \equiv 0$. From (E1) and Lemma 5,

$$\begin{aligned} \tilde{\chi}_1(t_k) &= \begin{bmatrix} r^\mathsf{T} \otimes I_n & 0 \end{bmatrix} \bar{\chi}(t_k) \\ &= \begin{bmatrix} r^\mathsf{T} \otimes I_n & 0 \end{bmatrix} \begin{bmatrix} I_N \otimes e^{A\tau_d} & I_N \otimes G(\tau_d) e^{-A^\mathsf{T}\tau_d} \\ 0 & I_N \otimes e^{-A^\mathsf{T}\tau_d} \end{bmatrix} \chi(t_k) \\ &= \left(r^\mathsf{T} \otimes e^{A\tau_d}\right) \bar{x}(t_k) + \left(r^\mathsf{T} \otimes G(\tau_d) e^{-A^\mathsf{T}\tau_d}\right) \xi(t_k) \\ &= \left(r^\mathsf{T} \otimes e^{A\tau_d}\right) (x(t_k) - \mathbf{1}_N \otimes x_c(t_k)) + G(\tau_d) e^{-A^\mathsf{T}\tau_d} \left(r^\mathsf{T} \otimes I_n\right) \xi(t_k) \\ &= \left(r^\mathsf{T} \otimes e^{A\tau_d}\right) x(t_k) - e^{A\tau_d} x_c(t_k) \\ &= \left(r^\mathsf{T} \otimes e^{A\tau_d}\right) x(t_k) - e^{A\tau_d}(r^\mathsf{T} \otimes I_n) x(t_k) = 0. \end{aligned}$$

Again from (E1) and (I5),

$$\begin{bmatrix} \tilde{\chi}_2(t_{k+1}) \\ \tilde{\chi}_4(t_{k+1}) \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes \tilde{A} + \Delta \otimes \tilde{B}K & 0 \\ \Delta \otimes \bar{K}_F & 0 \end{bmatrix} \begin{bmatrix} \tilde{\chi}_2(t_k) \\ \tilde{\chi}_4(t_k) \end{bmatrix}. \tag{I6}$$

Notice that $I_{N-1} \otimes \tilde{A} + \Delta \otimes \tilde{B}K$ is a block upper triangular matrix whose diagonal matrices are

$$\tilde{A} + \lambda_i \tilde{B}K = \tilde{A} + \lambda_i \tilde{B} V_c \begin{bmatrix} K_v \\ 0 \end{bmatrix} = \tilde{A} + \lambda_i \begin{bmatrix} \tilde{B}_v & 0 \end{bmatrix} \begin{bmatrix} K_v \\ 0 \end{bmatrix} = \tilde{A} + \lambda_i \tilde{B}_v K_v, i = 2, \dots, N.$$

From Lemma 2, by taking $K_v = -\alpha_c (\tilde{B}_v^\mathsf{T} P \tilde{B}_v)^{-1} \tilde{B}_v^\mathsf{T} P \tilde{A}$, $\tilde{A} + \lambda_i \tilde{B}_v K_v$ and hence $\tilde{A} + \lambda_i \tilde{B}K$ is Schur stable for all $i \in \{2, \dots, N\}$. Thus, system (I6) is asymptotically stable. Then we have $\lim_{k\to\infty} ||\tilde{\chi}(t_k)|| = 0$ and hence $\lim_{k\to\infty} ||\chi(t_k)|| = 0$.

Consequently, from (I3a), $\lim_{t+k\to\infty} ||\chi(t,k)|| \leqslant e^{||A_\chi||\tau_d} \lim_{k\to\infty} ||\chi(t_k)|| = 0$, which, in turn, implies that $\lim_{t+k\to\infty} \bar{x}(t,k) = 0$, i.e., $\lim_{t+k\to\infty}(x_i(t,k) - x_j(t,k)) = 0$ for any $i, j = 1, \dots, N$. This completes the proof. □

Appendix I.2 Control via the distributed dynamic output feedback

The proof requires the following result, which depicts the relation between the states of observer and their consensus center under the distributed dynamic output feedback control law (3). Let $\hat{x} = [\hat{x}_1^\mathsf{T}, \dots, \hat{x}_N^\mathsf{T}]^\mathsf{T}$ and $\zeta = [\zeta_1^\mathsf{T}, \dots, \zeta_N^\mathsf{T}]^\mathsf{T}$. Besides the consensus error defined in (I1), we also define the observer center as: for any $(t,k) \in \mathcal{T}$,

$$\hat{x}_c(t,k) = \left(r^\mathsf{T} \otimes e^{A(t-t_k)}\right) \hat{x}(t_k, k),\ \hat{x}_c(t_k, k) = E\hat{x}_c(t_k, k-1). \tag{I7}$$

Lemma 6. Under Assumption 2, the closed-loop system composed of (1) and (3) has the properties that $x_c(t_k,k) = (r^{\mathsf{T}} \otimes I_n)x(t_k,k)$ and $\hat{x}_c(t_k,k) = (r^{\mathsf{T}} \otimes I_n)\hat{x}(t_k,k)$.

Proof. Under the dynamic output feedback control law (3), the closed-loop system can be written as

$$\dot{\tau} = 1,\ \dot{x} = (I_N \otimes A)x + (I_N \otimes BB^{\mathsf{T}})\xi,\ \dot{\xi} = -(I_N \otimes A^{\mathsf{T}})\xi$$
$$\dot{\hat{x}} = (I_N \otimes A)\hat{x} + (I_N \otimes BB^{\mathsf{T}})\xi,\ \dot{\zeta} = \left(\mathcal{L}_N \otimes C_F^{\mathsf{T}} C_F\right)(\hat{x} - x) - \left(I_N \otimes A^{\mathsf{T}}\right)\zeta \tag{I8a}$$

whether $(\tau, x, \hat{x}, \xi, \zeta) \in [0, \tau_d] \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn}$, and

$$\tau^+ = 0,\ x^+ = (I_N \otimes E)x + (\mathcal{L}_N \otimes FK_J)\hat{x},\ \xi^+ = \left(\mathcal{L}_N \otimes e^{A^{\mathsf{T}}\tau_d}\bar{K}_F\right)\hat{x}$$
$$\hat{x}^+ = (I_N \otimes E + \mathcal{L}_N \otimes FK_J)\hat{x} + (\mathcal{L}_N \otimes L_J C_J)(\hat{x} - x) + (I_N \otimes L_F e^{A^{\mathsf{T}}\tau_d})\zeta,\ \zeta^+ = 0 \tag{I8b}$$

whether $(\tau, x, \hat{x}, \xi, \zeta) \in \{\tau_d\} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn}$. Since the flow dynamics of x and ξ are the same as those in (I2a), together with (I8b), we directly have, for any $(t,k) \in \mathcal{T}$, $(r^{\mathsf{T}} \otimes I_n)\xi(t_k,k) = 0$ and $(r^{\mathsf{T}} \otimes I_n)x(t_k,k) = (r^{\mathsf{T}} \otimes E)x(t_k,k-1)$. Therefore, by Lemma 5, $x_c(t_k,k) = (r^{\mathsf{T}} \otimes I_n)x(t_k,k)$. In addition, from (I8a), for any $(t,k) \in \mathcal{T}$,

$$\begin{aligned}\hat{x}(t_k,k-1) &= \left(I_N \otimes e^{A\tau_d}\right)\hat{x}(t_{k-1},k-1) + \int_{t_{k-1}}^{t_k} \left(I_N \otimes e^{A(t_k-\tau)}BB^{\mathsf{T}}\right)\xi(\tau,k-1)\mathrm{d}\tau \\ &= \left(I_N \otimes e^{A\tau_d}\right)\hat{x}(t_{k-1},k-1) + \int_{t_{k-1}}^{t_k} \left(I_N \otimes e^{A(t_k-\tau)}BB^{\mathsf{T}}e^{-A^{\mathsf{T}}(\tau-t_{k-1})}\right)\mathrm{d}\tau \cdot \xi(t_{k-1},k-1).\end{aligned}$$

Since $r^{\mathsf{T}}\mathcal{L}_N = 0$ and $\zeta(t_k,k) = 0$ for any $k \in \mathbb{N}$, we further obtain

$$\begin{aligned}(r^{\mathsf{T}} \otimes I_n)\zeta(t_k,k-1) &= \left(r^{\mathsf{T}} \otimes e^{-A^{\mathsf{T}}\tau_d}\right)\zeta(t_{k-1},k-1) + \int_{t_{k-1}}^{t_k} \left(r^{\mathsf{T}}\mathcal{L}_N \otimes e^{-A^{\mathsf{T}}(t_k-\tau)}C_F^{\mathsf{T}}C_F\right)(\hat{x}(\tau,k-1) - x(\tau,k-1))\mathrm{d}\tau \\ &= \left(r^{\mathsf{T}} \otimes e^{-A^{\mathsf{T}}\tau_d}\right)\zeta(t_{k-1},k-1) = 0.\end{aligned}$$

thus, we have

$$\begin{aligned}&(r^{\mathsf{T}} \otimes I_n)\hat{x}(t_k,k) = (r^{\mathsf{T}} \otimes E)\hat{x}(t_k,k-1) + (r^{\mathsf{T}} \otimes L_F e^{A^{\mathsf{T}}\tau_d})\zeta(t_k,k-1) \\ &= \left(r^{\mathsf{T}} \otimes Ee^{A\tau_d}\right)\hat{x}(t_{k-1},k-1) + \int_{t_{k-1}}^{t_k} Ee^{A(t_k-\tau)}BB^{\mathsf{T}}e^{-A^{\mathsf{T}}(\tau-t_{k-1})}\mathrm{d}\tau \cdot \left(r^{\mathsf{T}} \otimes I_n\right)\xi(t_{k-1},k-1) + L_F e^{A^{\mathsf{T}}\tau_d}(r^{\mathsf{T}} \otimes I_n)\zeta(t_k,k-1) \\ &= \left(r^{\mathsf{T}} \otimes Ee^{A\tau_d}\right)\hat{x}(t_{k-1},k-1).\end{aligned}$$

It follows from (I8b) that

$$\hat{x}_c(t_k,k) = E\hat{x}_c(t_k,k-1) = E(r^{\mathsf{T}} \otimes e^{A\tau_d})\hat{x}(t_{k-1},k-1) = (r^{\mathsf{T}} \otimes I_n)\hat{x}(t_k,k).$$

This completes the proof. □

We now show that Problem 1 is solved by the dynamic output feedback control law (3).

Proof. Define the consensus error as $\bar{x}_i = x_i - x_c$ and the estimation error as $e_i(t) = w_i(t) - \bar{x}_i(t)$, where $w_i(t) = \hat{x}_i(t) - \hat{x}_c(t)$. According to (1), (I1), (3), and (I7)

$$\dot{\bar{x}}_i = A\bar{x}_i + BB^{\mathsf{T}}\xi_i,\ \dot{e}_i = Ae_i$$

whether $(\tau, x_i, \hat{x}_i, x_c, \hat{x}_c, \xi_i, \zeta_i) \in [0, \tau_d] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, and

$$\bar{x}_i^+ = E\bar{x}_i + FK_J \sum_{j=1}^N \omega_{ij}(\bar{x}_j - \bar{x}_i) + FK_J \sum_{j=1}^N \omega_{ij}(e_j - e_i),\ e_i^+ = Ee_i + L_J C_J \sum_{j=1}^N \omega_{ij}(e_j - e_i) + \bar{L}_F e^{A^{\mathsf{T}}\tau_d}\zeta_i$$

whether $(\tau, x_i, \hat{x}_i, x_c, \hat{x}_c, \xi_i, \zeta_i) \in \{\tau_d\} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. Letting $e = [e_1^{\mathsf{T}}, \ldots, e_N^{\mathsf{T}}]^{\mathsf{T}}$, $\chi = [\bar{x}^{\mathsf{T}}, \xi^{\mathsf{T}}]^{\mathsf{T}}$, and $\eta = [e^{\mathsf{T}}, \zeta^{\mathsf{T}}]^{\mathsf{T}}$ yields that

$$\dot{\tau} = 1,\ \dot{\chi} = A_\chi \chi,\ \dot{\eta} = A_\eta \eta \tag{I9a}$$

whether $(\tau, \chi, \eta) \in [0, \tau_d] \times \mathbb{R}^{2Nn} \times \mathbb{R}^{2Nn}$, and

$$\tau^+ = 0,\ \chi^+ = E_\chi \chi + F_\chi \eta,\ \eta^+ = E_\eta \eta \tag{I9b}$$

whether $(\tau, \chi, \eta) \in \{\tau_d\} \times \mathbb{R}^{2Nn} \times \mathbb{R}^{2Nn}$, where the matrices A_χ and E_χ are given by (I4), and the matrices A_η, F_χ, and E_η are of the following form

$$A_\eta = \begin{bmatrix} I_N \otimes A & 0 \\ \mathcal{L}_N \otimes C_F^{\mathsf{T}}C_F & -(I_N \otimes A^{\mathsf{T}}) \end{bmatrix},\ F_\chi = \begin{bmatrix} \mathcal{L}_N \otimes FK_J & 0 \\ \mathcal{L}_N \otimes e^{A^{\mathsf{T}}\tau_d}\bar{K}_F & 0 \end{bmatrix},\ E_\eta = \begin{bmatrix} I_N \otimes E + \mathcal{L}_N \otimes L_J C_J & I_N \otimes \bar{L}_F e^{A^{\mathsf{T}}\tau_d} \\ 0 & 0 \end{bmatrix}.$$

Thus for all $k \in \mathbb{N}$,

$$\begin{bmatrix} \chi(t_{k+1}) \\ \eta(t_{k+1}) \end{bmatrix} = \begin{bmatrix} E_\chi e^{A_\chi \tau_d} & F_\chi e^{A_\eta \tau_d} \\ 0 & E_\eta e^{A_\eta \tau_d} \end{bmatrix} \begin{bmatrix} \chi(t_k) \\ \eta(t_k) \end{bmatrix}$$

where $t_0 = 0$, $\chi(t_k) \triangleq \chi(t_k, k)$, and $\eta(t_k) \triangleq \eta(t_k, k)$ for simplicity. Let $\bar{\chi}(t_k) = e^{A_\chi \tau_d} \chi(t_k)$. Then

$$\begin{bmatrix} \bar{\chi}(t_{k+1}) \\ \eta(t_{k+1}) \end{bmatrix} = \begin{bmatrix} e^{A_\chi \tau_d} E_\chi & e^{A_\chi \tau_d} F_\chi e^{A_\eta \tau_d} \\ 0 & E_\eta e^{A_\eta \tau_d} \end{bmatrix} \begin{bmatrix} \bar{\chi}(t_k) \\ \eta(t_k) \end{bmatrix}.$$

Noticing that $e^{A_\eta \tau_d} = \begin{bmatrix} I_N \otimes e^{A\tau_d} & 0 \\ \mathcal{L}_N \otimes e^{-A^\mathsf{T}\tau_d} W(\tau_d) & I_N \otimes e^{-A^\mathsf{T}\tau_d} \end{bmatrix}$ and $L_F O_{A,C_F} = \bar{L}_F W(\tau_d)$, one has

$$\begin{aligned} E_\eta e^{A_\eta \tau_d} &= \begin{bmatrix} I_N \otimes E e^{A\tau_d} + \mathcal{L}_N \otimes L_J C_J e^{A\tau_d} + \mathcal{L}_N \otimes \bar{L}_F W(\tau_d) & I_N \otimes \bar{L}_F \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I_N \otimes E e^{A\tau_d} + \mathcal{L}_N \otimes L_J C_J e^{A\tau_d} + \mathcal{L}_N \otimes L_F O_{A,C_F} & I_N \otimes \bar{L}_F \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I_N \otimes \bar{A} + \mathcal{L}_N \otimes L\bar{C} & I_N \otimes \bar{L}_F \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where $L = [L_J, L_F]$. Let U, J_L, and Δ be given in (E1), and

$$\tilde{\chi}(t_k) = \begin{bmatrix} U^{-1} \otimes I_n & 0 \\ 0 & U^{-1} \otimes I_n \end{bmatrix} \bar{\chi}(t_k), \ \tilde{\eta}(t_k) = \begin{bmatrix} U^{-1} \otimes I_n & 0 \\ 0 & U^{-1} \otimes I_n \end{bmatrix} \eta(t_k).$$

Then it holds that

$$\begin{aligned} \tilde{\chi}(t_{k+1}) &= \begin{bmatrix} I_N \otimes \tilde{A} + J_L \otimes \tilde{B}K & 0 \\ J_L \otimes \bar{K}_F & 0 \end{bmatrix} \tilde{\chi}(t_k) + \begin{bmatrix} J_L \otimes e^{A\tau_d} F K_J e^{A\tau_d} + J_L \otimes G(\tau_d)\bar{K}_F e^{A\tau_d} & 0 \\ J_L \otimes \bar{K}_F e^{A\tau_d} & 0 \end{bmatrix} \tilde{\eta}(t_k) \\ \tilde{\eta}(t_{k+1}) &= \begin{bmatrix} I_N \otimes \bar{A} + J_L \otimes L\bar{C} & I_N \otimes \bar{L}_F \\ 0 & 0 \end{bmatrix} \tilde{\eta}(t_k). \end{aligned} \tag{I10}$$

Decompose $\tilde{\chi}(t_k)$ as $\tilde{\chi}(t_k) \triangleq [\tilde{\chi}_1(t_k)^\mathsf{T}, \tilde{\chi}_2(t_k)^\mathsf{T}, \tilde{\chi}_3(t_k)^\mathsf{T}, \tilde{\chi}_4(t_k)^\mathsf{T}]^\mathsf{T}$, where $\tilde{\chi}_1(t_k), \tilde{\chi}_3(t_k) \in \mathbb{R}^n$, $\tilde{\chi}_2(t_k), \tilde{\chi}_4(t_k) \in \mathbb{C}^{(N-1)n}$, and $\tilde{\eta}(t_k)$ as $\tilde{\eta}(t_k) \triangleq [\tilde{\eta}_1(t_k)^\mathsf{T}, \tilde{\eta}_2(t_k)^\mathsf{T}, \tilde{\eta}_3(t_k)^\mathsf{T}, \tilde{\eta}_4(t_k)^\mathsf{T}]^\mathsf{T}$, where $\tilde{\eta}_1(t_k), \tilde{\eta}_3(t_k) \in \mathbb{R}^n$, $\tilde{\eta}_2(t_k), \tilde{\eta}_4(t_k) \in \mathbb{C}^{(N-1)n}$. From (I10), one has $\tilde{\chi}_3(t_k) \equiv 0$, $\tilde{\eta}_3(t_k) \equiv 0$, and $\tilde{\eta}_4(t_k) \equiv 0$. From (E1) and Lemma 6, we have $\tilde{\chi}_1(t_k) = 0$, and

$$\begin{aligned} \tilde{\eta}_1(t_k) &= \begin{bmatrix} r^\mathsf{T} \otimes I_n & 0 \end{bmatrix} \eta(t_k) = (r^\mathsf{T} \otimes I_n) e(t_k) = (r^\mathsf{T} \otimes I_n)\left(\hat{x}(t_k) - \mathbf{1}_N \otimes \hat{x}_c(t_k) - x(t_k) + \mathbf{1}_N \otimes x_c(t_k)\right) \\ &= \left(r^\mathsf{T} \otimes I_n\right) \hat{x}(t_k) - \hat{x}_c(t_k) - \left(r^\mathsf{T} \otimes I_n\right) x(t_k) + x_c(t_k) = 0. \end{aligned}$$

Again from (E1) and (I10),

$$\begin{bmatrix} \tilde{\chi}_2(t_{k+1}) \\ \tilde{\chi}_4(t_{k+1}) \\ \tilde{\eta}_2(t_{k+1}) \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes \tilde{A} + \Delta \otimes \tilde{B}K & 0 & \Delta \otimes \left(e^{A\tau_d} F K_J + G(\tau_d)\bar{K}_F\right) e^{A\tau_d} \\ \Delta \otimes \bar{K}_F & 0 & \Delta \otimes \bar{K}_F e^{A\tau_d} \\ 0 & 0 & I_{N-1} \otimes \bar{A} + \Delta \otimes L\bar{C} \end{bmatrix} \begin{bmatrix} \tilde{\chi}_2(t_k) \\ \tilde{\chi}_4(t_k) \\ \tilde{\eta}_2(t_k) \end{bmatrix}. \tag{I11}$$

By taking $K_v = -\alpha_c (\tilde{B}_v^\mathsf{T} P \tilde{B}_v)^{-1} \tilde{B}_v^\mathsf{T} P \tilde{A}$, Theorem 1 has shown that $\tilde{A} + \lambda_i \tilde{B} K$ is Schur stable for all $i \in \{2, \ldots, N\}$. Notice that $I_{N-1} \otimes \bar{A} + \Delta \otimes L\bar{C}$ is a block upper matrix whose diagonal matrices are

$$\bar{A} + \lambda_i L\bar{C} = \bar{A} + \lambda_i \begin{bmatrix} L_v & 0 \end{bmatrix} V_o \bar{C} = \bar{A} + \lambda_i \begin{bmatrix} L_v & 0 \end{bmatrix} \begin{bmatrix} \bar{C}_v \\ 0 \end{bmatrix} = \bar{A} + \lambda_i L_v \bar{C}_v, \ i = 2, \ldots, N.$$

From Lemma 3, by taking $L_v = -\alpha_o \bar{A} Q \bar{C}_v^\mathsf{T} (\bar{C}_v Q \bar{C}_v^\mathsf{T})^{-1}$, $\bar{A} + \lambda_i L_v \bar{C}_v$ and hence $\bar{A} + \lambda_i L\bar{C}$ is Schur stable for all $i \in \{2, \ldots, N\}$. Thus, system (I11) is asymptotically stable. Then we have $\lim_{k\to\infty} ||\tilde{\chi}(t_k)|| = 0$, $\lim_{k\to\infty} ||\tilde{\eta}(t_k)|| = 0$, and hence $\lim_{k\to\infty} ||\chi(t_k)|| = 0$, $\lim_{k\to\infty} ||\eta(t_k)|| = 0$.

Consequently, from (I9a),

$$\lim_{t+k\to\infty} ||\chi(t,k)|| \leqslant e^{||A_\chi||\tau_d} \lim_{k\to\infty} ||\chi(t_k)|| = 0 \text{ and } \lim_{t+k\to\infty} ||\eta(t,k)|| \leqslant e^{||A_\eta||\tau_d} \lim_{k\to\infty} ||\eta(t_k)|| = 0$$

which, in turn, imply that $\lim_{t+k\to\infty} \bar{x}(t,k) = 0$ and $\lim_{t+k\to\infty} e(t,k) = 0$, i.e.,

$$\lim_{t+k\to\infty} (x_i(t,k) - x_j(t,k)) = 0 \text{ and } \lim_{t+k\to\infty} (\hat{x}_i(t,k) - x_i(t,k)) = 0$$

for any $i, j = 1, \ldots, N$. This completes the proof. □

Remark 7. Our consensus algorithm can be applied to any hybrid multi-agent system of the form (1) with arbitrary dimension and number of agents. Indeed, since our controllers do not depend on the system dimension n and the numbers of agents N, the increasing of system scale does not affect the algorithm complexity, but only results in a large amount of calculation.

Remark 8. Our controllers are able to tolerate some plant uncertainties. Indeed, controller gain matrices $\bar{K}_F$ and K_J as well as observer gain matrices $\bar{L}_F$ and L_J are chosen such that the monodromy matrices of closed-loop systems (I6) and (I11) are Schur stable. In this sense, for any uncertainties perturbed in some open neighborhood of origin, both monodromy matrices can be still Schur stable with the same gain matrices.

Appendix J Two corollaries

Two special cases of Theorem 1 are considered in this appendix:

• Case 1: $E = I_n$, $F = 0$, $C_F = 0$, (A, B) is stabilizable, and (C_J, A) is detectable. For this case, system (1) reduces to the continuous-time linear multi-agent system

$$\dot{x}_i = Ax_i + Bu_{Fi}, \quad i = 1, \ldots, N \tag{J1}$$

with periodic sampled output $y_i(k\tau_d) = C_J x_i(k\tau_d)$.

• Case 2: $A = 0$, $B = 0$, $C_F = 0$, (E, F) is stabilizable, and (C_J, E) is detectable. For this case, system (1) reduces to the discrete-time linear multi-agent system

$$x_i^+ = Ex_i + Fu_{Ji}, \quad i = 1, \ldots, N \tag{J2}$$

with discrete-time output $y_i(k) = C_J x_i(k)$, where we omit τ_d for simplicity.

We have the following lemma.

Lemma 7. The PBH tests (F1) and (F2) are both satisfied provided that either Case 1 or Case 2 holds.

Proof. Firstly, we prove that if Case 1 holds, then the PBH tests (F1) and (F2) are both satisfied. By doing so, we should prove that for any $\tau_d > 0$, all eigenvectors of $e^{A\tau_d}$ corresponding to eigenvalues with magnitudes larger or equal to 1 are the same as those of A corresponding to eigenvalues with positive or zero real parts. Let $x \neq 0$ be an eigenvector of $e^{A\tau_d}$ associated with the eigenvalue λ with magnitude larger or equal to 1, i.e., $e^{A\tau_d}x = \lambda x$, $\lambda \notin \mathbb{C}_g$. From [9, Corollary 6.2.11], we have $\ln e^{A\tau_d} = \tau_d A$. Then $(\ln e^{A\tau_d})x = (\ln \lambda)x$, and therefore $Ax = \frac{\ln \lambda}{\tau_d}x$. Since $\lambda \notin \mathbb{C}_g$ and $\tau_d > 0$, we conclude that $\mathrm{Re}(\frac{\ln \lambda}{\tau_d}) \geqslant 0$. This implies that x is also an eigenvector of A corresponding to an eigenvalue with positive or zero real part. Through a similar discussion, we can also conclude that all eigenvectors of $e^{A^\mathsf{T}\tau_d}$ corresponding to eigenvalues with magnitudes larger or equal to 1 are those of A^T corresponding to eigenvalues with positive or zero real parts.

Notice that for case 1, the PBH test (F1) reduces to

$$\mathrm{rank}\left(\begin{bmatrix} e^{A\tau_d} - sI & 0 & R_{A,B} \end{bmatrix}\right) = n, \quad \forall\ s \in \Lambda(e^{A\tau_d}),\ s \notin \mathbb{C}_g. \tag{J3}$$

To prove by contradiction, assume that (J3) does not hold. From eigenvector test for stabilizability of discrete-time linear system [8, Theorem 14.1], there exists an eigenvalue λ with $\lambda \notin \mathbb{C}_g$ such that $e^{A^\mathsf{T}\tau_d}x = \lambda x$ with $x \neq 0$, for which $R_{A,B}^\mathsf{T} x = 0$. This means that $\begin{bmatrix} x^\mathsf{T}B & x^\mathsf{T}AB & \cdots & x^\mathsf{T}A^{n-1}B \end{bmatrix}^\mathsf{T} = 0$. Thus, we have $B^\mathsf{T}x = 0$. Notice that x is also an eigenvector of A^T corresponding to an eigenvalue with positive or zero real part, which contradicts the stabilizability of (A, B).

Similarly, for case 1, the PBH test (F2) reduces to

$$\mathrm{rank}\left(\begin{bmatrix} (e^{A\tau_d})^\mathsf{T} - sI & (C_J e^{A\tau_d})^\mathsf{T} & 0 \end{bmatrix}^\mathsf{T}\right) = n, \quad \forall\ s \in \Lambda(e^{A\tau_d}),\ s \notin \mathbb{C}_g. \tag{J4}$$

To prove by contradiction, assume that (J4) does not hold. From eigenvector test for detectability of discrete-time linear system [8, Theorem 16.4], there exists an eigenvalue λ such that $e^{A\tau_d}x = \lambda x$ with $x \neq 0$ and $\lambda \notin \mathbb{C}_g$, for which $C_J e^{A\tau_d}x = 0$. Then $0 = C_J e^{A\tau_d}x = \lambda C_J x$. Since $\lambda \neq 0$, we further obtain $C_J x = 0$. Notice that x is also an eigenvector of A corresponding to an eigenvalue with positive or zero real part, which contradicts the detectability of (A, C_J).

Next, we prove that if Case 2 holds, then the PBH tests (F1) and (F2) are both satisfied. In this case, since (E, F) is stabilizable and (C_J, E) is detectable, we have

$$\mathrm{rank}\left(\begin{bmatrix} E - sI & F \end{bmatrix}\right) = n, \quad \forall\ s \in \Lambda(E),\ s \notin \mathbb{C}_g, \text{ and } \mathrm{rank}\left(\begin{bmatrix} E - sI \\ C_J \end{bmatrix}\right) = n, \quad \forall\ s \in \Lambda(E),\ s \notin \mathbb{C}_g.$$

These directly give that

$$\mathrm{rank}\left(\begin{bmatrix} E - sI & F & 0 \end{bmatrix}\right) = n, \quad \forall\ s \in \Lambda(E),\ s \notin \mathbb{C}_g, \text{ and } \mathrm{rank}\left(\begin{bmatrix} E - sI \\ C_J \\ 0 \end{bmatrix}\right) = n, \quad \forall\ s \in \Lambda(E),\ s \notin \mathbb{C}_g$$

which are exactly the PBH tests (F1) and (F2) with $A = 0$, $B = 0$, and $C_F = 0$, respectively. □

With the aid of Lemma 7, it is possible to apply Theorem 1 to both cases, respectively, giving rise to the following corollaries.

Corollary 1. Assume that Case 1 holds. Let τ_d be chosen such that inequalities (H2) and (H4) hold. Then under Assumption 2, the consensus for continuous-time system (J1) can be achieved by either the distributed sampled state feedback controller

$$u_{Fi}(t) = B^\mathsf{T} e^{-A^\mathsf{T}(t-k\tau_d)} e^{A^\mathsf{T}\tau_d} \bar{K}_F \sum_{j=1}^{N} \omega_{ij}(x_i(k\tau_d) - x_j(k\tau_d)),\ t \in [k\tau_d, (k+1)\tau_d)$$

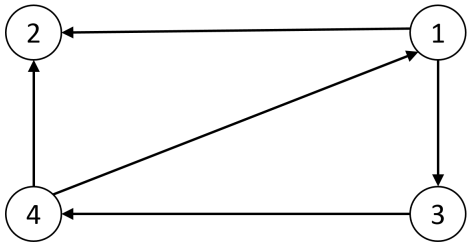

Figure J1 Network graph $\mathcal{G}$

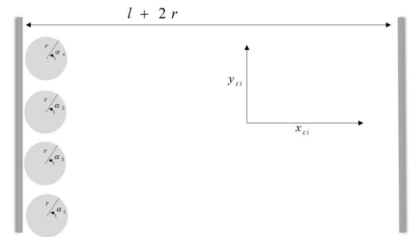

Figure J2 Four rotating disks bouncing between two walls

or the distributed sampled output feedback controller

$$u_{Fi}(t) = B^{\mathsf{T}} e^{-A^{\mathsf{T}}(t-k\tau_d)} e^{A^{\mathsf{T}}\tau_d} \bar{K}_F \sum_{j=1}^{N} \omega_{ij}(\hat{x}_i(k\tau_d^-) - \hat{x}_j(k\tau_d^-)),\ t \in [k\tau_d, (k+1)\tau_d) \tag{J5a}$$

$$\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu_{Fi}(t),\ t \in [k\tau_d, (k+1)\tau_d) \tag{J5b}$$

$$\hat{x}_i(k\tau_d) = \hat{x}_i(k\tau_d^-) + L_J \sum_{j=1}^{N} \omega_{ij}(C_J\hat{x}_i(k\tau_d^-) - y_{Ji}(k\tau_d) - C_J\hat{x}_j(k\tau_d^-) + y_{Jj}(k\tau_d)) \tag{J5c}$$

where $\hat{x}_i(k\tau_d^-) = \lim_{t\to k\tau_d, t<k\tau_d} \hat{x}_i(t)$, $i = 1, \ldots, N$, and $\bar{K}_F$, L_J are determined by Algorithms H1 and H2, respcetively.

Corollary 2. Assume that Case 2 holds. Then under Assumption 2, the consensus for discrete-time system (J2) can be achieved by either the distributed discrete-time state feedback controller

$$u_{Ji}(k) = K_J \sum_{j=1}^{N} \omega_{ij}(x_i(k) - x_j(k)) \tag{J6}$$

or the distributed discrete-time output feedback controller

$$u_{Ji}(k) = K_J \sum_{j=1}^{N} \omega_{ij}(\hat{x}_i(k) - \hat{x}_j(k)) \tag{J7a}$$

$$\hat{x}_i(k+1) = E\hat{x}_i(k) + Fu_{Ji}(k) + L_J \sum_{j=1}^{N} \omega_{ij}(C_J\hat{x}_i(k) - C_J\hat{x}_j(k)) - L_J \sum_{j=1}^{N} \omega_{ij}(y_{Ji}(k) - y_{Jj}(k)). \tag{J7b}$$

Remark 9. Corollary 1 provides the solvability of the sampling consensus problem (e.g., [28]) of system (J1). Moreover, it is interesting to see that the observer composed of (J5b) and (J5c) here is a continuous-discrete time observer [15].

Remark 10. Corollary 2 provides the solvability of the consensus problem of system (J2). In particular, the distributed state feedback controller (J6) is consistent with that in [7], while the distributed output feedback controller (J7) has removed some extra conditions in the previous references, such as the marginal stability of system matrix [10].

Remark 11. The jumping period τ_d may affect the consensusability from the following two aspects. On one hand, it has to guarantee the PBH tests (F1) and (F2) hold so that the hybrid linear system (1) is stabilizable and detectable in the hybrid sense. As mentioned in Lemma 7, for the special case that either its continuous-time linear dynamics or discrete-time linear dynamics are stabilizable and detectable, PBH tests (F1) and (F2) hold automatically, and hence τ_d is allowed to be any arbitrary positive for this matter. On the other hand, it should make sense of the selection of δ_c (resp., δ_o), which is, in turn, to guarantee the existence of a suitable α_c (resp., α_o) satisfying inequality (H2) (resp., (H4)), where α_c (resp., α_o) is a key part of the controller gain (resp., observer gain). As can be seen in Remark 1 of Appendix G, for the special case that the matrix A contains no unstable eigenvalues, δ_c (resp., δ_o) can always be chosen as one, and hence α_c (resp., α_o) exists for all $\tau_d > 0$. It is worth mentioning that, for the mechanical models that have to be described by the hybrid system, the jumping period τ_d is undesirable but has to satisfy these two conditions, while for some control areas like the sampling feedback control, the jumping period τ_d can be selected appropriately so as to guarantee these two conditions hold automatically, see Corollary 1.

Appendix K Simulation results

Consider a group of disks moving on a horizontal plane between parallel walls, orthogonal to the plane of motion and infinitely massive, see Figure J2 for an example with four disks. Let (x_{ci}, y_{ci}) be the coordinate of the center mass of i-th disk, and α_i be angular position of i-th disk. By assuming that $|\dot{x}_{ci}(t)| = |\dot{x}_{ci}(0)| = v$ is constant for $i = 1, \ldots, N$ and $x_{ci}(0) = x_{cj}(0) = 0$ for $i, j = 1, \ldots, N$, as discussed in [17, 18], the state-space equation of this mechanical system with state $x_i = [y_{ci}, \dot{y}_{ci}, \alpha_i, \dot{\alpha}_i]^{\mathsf{T}} \in \mathbb{R}^4$ and input $u_{Fi} \in \mathbb{R}$, is described by

$$\dot{\tau} = 1,\ \dot{x}_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ 0 \end{bmatrix} u_{Fi},\ i = 1, 2, 3, 4 \tag{K1a}$$

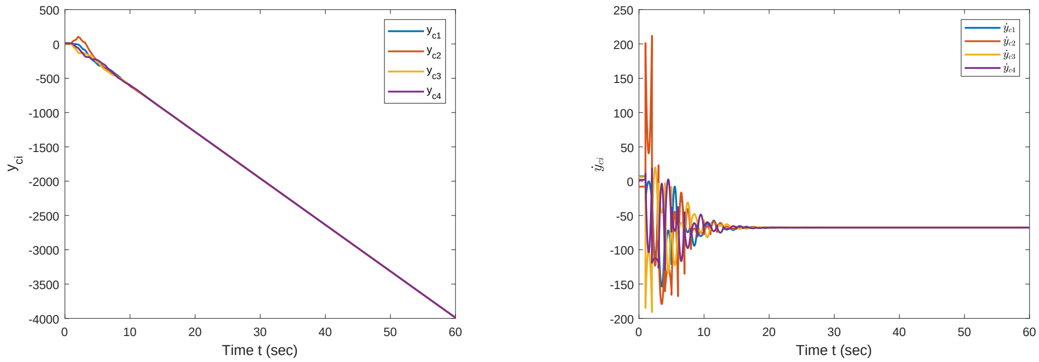

Figure J3 Trajectories of y_{ci} and $\dot{y}_{ci}$ for control law (2)

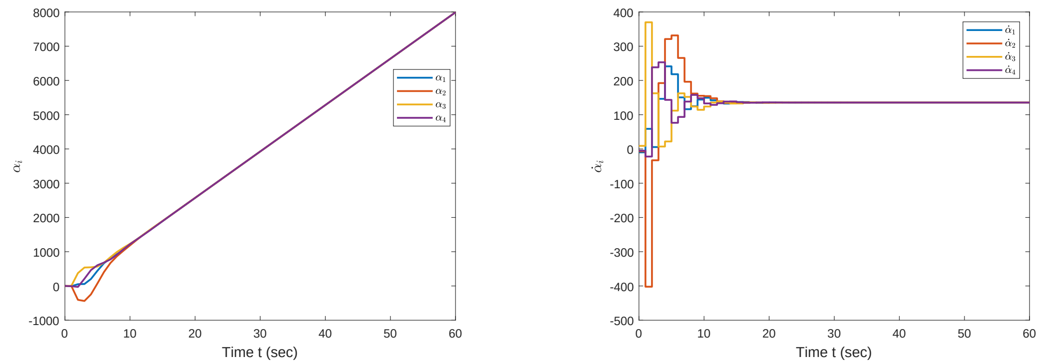

Figure J4 Trajectories of α_i and $\dot{\alpha}_i$ for control law (2)

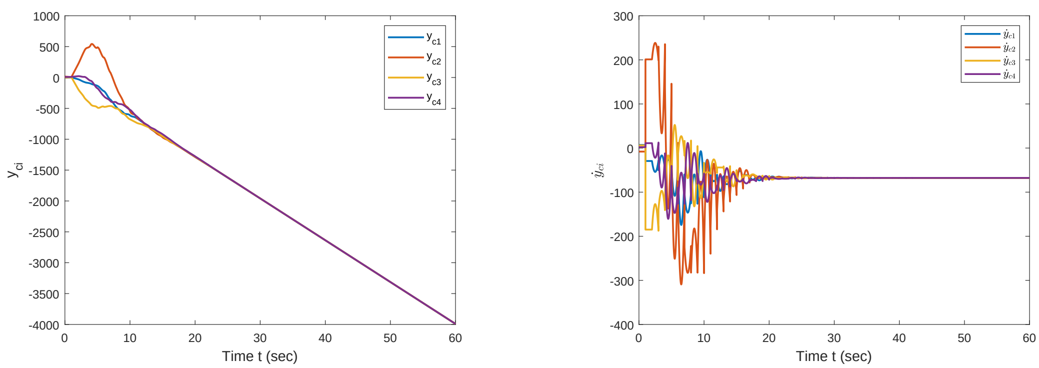

Figure J5 Trajectories of y_{ci} and $\dot{y}_{ci}$ for control law (3)

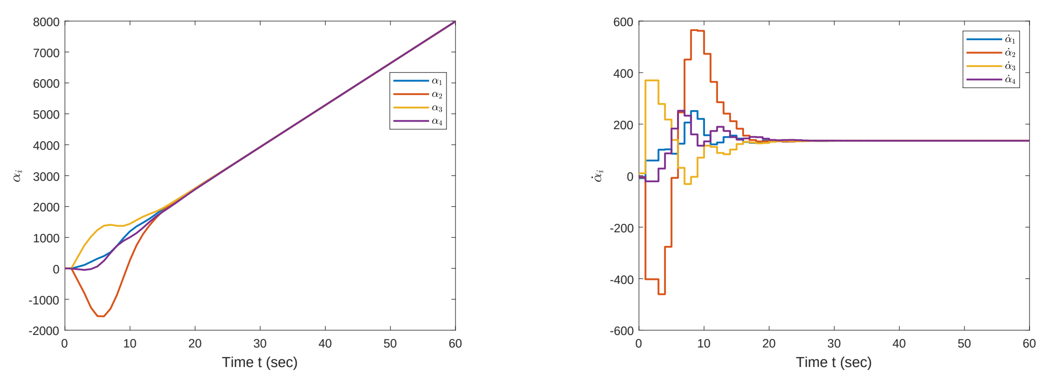

Figure J6 Trajectories of α_i and $\dot{\alpha}_i$ for control law (3) whether $(\tau, x_i) \in [0, \tau_d] \times \mathbb{R}^4$ and

$$\tau^+ = 0,\ x_i^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\omega^{-1} & 0 & -\omega^{-1}r \\ 0 & 0 & 1 & 0 \\ 0 & -r^{-1}(1-\omega^{-1}) & 0 & \omega^{-1} \end{bmatrix} x_i,\ i = 1,2,3,4 \tag{K1b}$$

whether $(\tau, x_i) \in \{\tau_d\} \times \mathbb{R}^4$, where $\tau(0,0) = 0$, $\tau_d = \frac{l}{v}$ and $\omega = \frac{r^2 M}{\mathcal{I}}$ with r, M, $\mathcal{I}$ denoting radius, total mass, inertia of each disk, respectively. Here, the distance between two walls is denoted by $l + 2r$ as shown in Figure J2. We assume that only the pre-impact verticals $y_{ci}(t_k, k-1)$ and $\alpha_i(t_k, k-1)$ are measurable, that is, $C_F = 0$ and $C_J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. It can be easily checked that neither (A, B) nor (E, F) is stablizable and (C_F, A) is not detectable, while each subsystem of the hybrid linear multi-agent system (K1) is stablizable and detectable according to PBH tests (F1) and (F2).

In our simulation, we assume that the parameters in (K1) are given as $M = 0.22kg$, $r = 0.5m$, $\mathcal{I} = 1$, $l = 1m$, and $v = 1m/s$. The network digraph $\mathcal{G}$ is described in Figure J1, where Assumption 2 is satisfied. Moreover, with some calculation, we get $\mathcal{L}_N = [1/2, 0, -1/2, 0;\ -1/3, 2/3, 0, -1/3;\ -1/2, 0, 1/2, 0;\ 0, 0, -1/2, 1/2]$ with eigenvalues $\{0, 2/3, 1/2, 1\}$. According to Algorithms H1 - H2, for given $\delta_c = \delta_o = 1$, we can take $\alpha_c = \alpha_o = 1$ so that conditions (H2) and (H4) hold with $\delta(\alpha_c) = \delta(\alpha_o) = 1/3$. From Algorithm H1,

$$P = \begin{bmatrix} 0.0180 & -0.2474 & 0 & -0.1309 \\ -0.2474 & 210.5812 & 1.9311 & 111.4133 \\ 0 & 1.9311 & 0.0412 & 1.0218 \\ -0.1309 & 111.4133 & 1.0218 & 58.9588 \end{bmatrix},\ \text{then } \bar{K}_F = \begin{bmatrix} -0.5808 & 7.8123 & -0.0717 & 4.1335 \\ 0.2904 & -3.9700 & 0.0354 & -2.1005 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for flow dynamics and $K_J = 0$ for the jump dynamics. The simulation results based on the distributed dynamic state feedback control law (2) are shown in Figures J3 - J4. Similarly, from Algorithm H2,

$$Q = \begin{bmatrix} 15.0110 & 5.7106 & -30.2481 & -11.4212 \\ 5.7106 & 7.3201 & -11.7688 & -14.6201 \\ -30.2481 & -11.7688 & 61.0784 & 23.5376 \\ -11.4212 & -14.6201 & 23.5376 & 29.2502 \end{bmatrix},\ \text{then } L_J = \begin{bmatrix} -1.0000 & 0 \\ 6.3603 & 3.3431 \\ 0 & -1.0000 \\ -12.7206 & -6.6861 \end{bmatrix}$$

for jump dynamics and $\bar{L}_F = 0$ for flow dynamics. The simulation results based on the distributed dynamic output feedback control law (3) are depicted in Figure J5 - J6. It is observed that satisfactory consensus has been achieved.

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