SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

July 2023, Vol. 66 172205:1–172205:14
https://doi.org/10.1007/s11432-022-3607-6

Results on the realization of Boolean control networks by the vertex partition method

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Received 29 May 2022/Revised 16 August 2022/Accepted 15 September 2022/Published online 21 June 2023

Abstract Realization problems, including the minimum realization of Boolean control networks (BCNs), are investigated by the vertex partition method. A general definition of the realization for BCNs that does not involve any subspaces is proposed. For the new and existing realization definitions, by a common concolorous perfect vertex partition (CCP-VP) of vertex-colored state transition graphs, an equivalence relation and a quotient mapping are defined, which induce a quotient system, i.e., the realization of original BCNs. In addition, if the CCP-VP is an equal partition, a realization induced by a $\mathcal{Y}$-friendly controlled invariant regular subspace is constructed. Finally, an algorithm is designed to construct the minimum realization of BCNs for different realization definitions, and three examples are given to illustrate the obtained results.

Keywords Boolean control networks, realization, semi-tensor product of matrices, vertex partition, state transition graph

Citation Li Y F, Zhu J D, Liu X W. Results on the realization of Boolean control networks by the vertex partition method. Sci China Inf Sci, 2023, 66(7): 172205, https://doi.org/10.1007/s11432-022-3607-6

1 Introduction

Genetic regulatory networks are essential networks in systems biology. Various models have been proposed to model intricate genetic regulatory networks, such as ordinary and partial differential equations, chemical master equations, Bayesian networks, and Boolean networks (BNs) (see [1–3] and the references therein). Among them, BNs have been proven to be a powerful tool in modeling, analyzing, and simulating genetic regulatory networks [4–6]. BNs were first introduced by Kauffman in 1969 [1], wherein each gene is characterized by a Boolean variable (active/inactive:1/0), and the evolution of each node is determined by logical functions composed of logical operators. Genetic regulatory networks are susceptible to the external input, and to model the external input and its influence on the system's output, BNs have been extended to a Boolean control network (BCN) [7]. Cheng et al. [8] established the algebraic representation of BNs/BCNs based on the semi-tensor product (STP) of matrices, which makes it relatively easy to formulate and solve classical control-theoretic problems for BNs/BCNs, and thereby many fundamental problems in the control theory of BNs and BCNs have been investigated, such as controllability and observability [9–16], optimal control [17, 18], stability and stabilization [19–23], system decomposition [24,25], decoupling [26–29], output regulation [30], and synchronization [31,32]. Moreover, as BNs from genetic regulatory networks are usually of large scale, some approaches have recently been proposed to study large-scale BNs [14, 33–39].

Realization problems, including the minimum realization problem, are among the fundamental issues of the traditional control system theory [40]. For BCNs, the pioneering work on this issue has been performed in [24] in which the idea for solving the realization problem is to construct a coordinate transformation to decompose a BCN into a Kalman decomposition form, and subsequently, the minimum realization is defined with a value assumption. However, in [24], the Kalman decomposition form of BCNs is obtained with the regularity assumption of the largest unobservable subspace and the largest

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uncontrollable subspace. Thus, there are two additional restrictive conditions, i.e., a value condition and a regularity condition. In [41], the realization of BCNs is defined by resorting to controlled invariant subspaces without the regularity assumption, and an algorithm is provided to construct the minimum realization. Compared with [24], although the regularity assumption is removed, the realization is still induced by a subspace of the original BCN; i.e., the realization is a sub-BCN of the original BCN. An interesting question is whether the realization must be a sub-BCN of the original BCN and, if not, how to propose a more general realization definition for BCNs.

Moreover, for BCNs, the state transition graph (STG) constructed by a structure matrix is an important tool. Compared with the algebraic view, many control-theoretic problems of BCNs can be characterized intuitively, and their essences can be clearly revealed from the perspective of the STG. Many excellent results have been obtained from the perspective of the STG, such as [16, 26, 29, 42–44]. However, for the realization problem of BCNs, the STG has not been taken into account.

Motivated by the above discussions, three problems need to be further discussed: (1) how to construct the minimum realization of BCNs from the perspective of the STG; (2) how to propose a more general realization definition in which the realization is not necessarily a sub-BCN of the original BCN; and (3) how to construct the minimum realization defined by the general definition. Thus, we reconsider the realization problem of BCNs from the perspective of the STG. Three kinds of realization problems for BCNs, i.e., defined by subspaces with the regularity assumption, defined by subspaces without the regularity assumption, and defined without involving subspaces, are studied by the vertex partition method. The main contributions of this study are as follows.

(1) A general definition of the realization that does not involve subspaces is proposed. For this definition, the minimum realization problem is solved by the vertex partition method by constructing quotient mappings. By the vertex partition method, the two restrictions in [24] are removed.

(2) A new method is provided for constructing $\mathcal{Y}$-friendly controlled invariant regular subspaces of BCNs. Furthermore, by the constructed $\mathcal{Y}$-friendly controlled invariant regular subspace, the minimum realization with the regularity assumption is induced.

(3) An algorithm is designed to solve three kinds of realization problems for BCNs. As the three realization problems are solved in a unified framework, the relationship among them is clearly revealed. Moreover, the method used in constructing the realization of BCNs may provide a way to study large-scale BCNs.

The rest of this paper is organized as follows. In Section 2, some necessary preparations, definitions of the realization, and the motivation of this paper are given. In Section 3, three types of realization problems are studied, and a comparison with the existing results is given. In Section 4, examples are presented, showing the obtained theoretical results. In Section 5, a brief conclusion is provided.

2 Preliminaries and problem statement

2.1 Preliminaries

In this subsection, some necessary preparations are presented, including the algebraic expression of BCNs, the vertex-colored STG of BCNs, and some definitions of vertex partitions. Throughout the whole paper, we use the following notations.

- $|S|$: cardinal number of the set S;
- $\mathbb{Z}$: set of integers;
- $[a, b] := \{a, a+1, \dots, b\}$, where a and b are two integers;
- $\mathcal{R}_{m\times n}$: the set of real matrices of $m \times n$;
- $\mathrm{Col}_i(A)$: the ith column of the matrix A; $\mathrm{Col}(A)$ denotes the column set of A;
- A^{T}: transposition of the matrix A;
- $\mathbf{1}_k$: k-dimensional column vector whose every element is 1;
- δ_k^i: the ith column of the identity matrix I_k, where I_k is the $k \times k$ identity matrix;
- $\Delta_k := \{ \delta_k^i : i = 1, 2, \dots, k \}$, especially let Δ denote $\Delta_2 = \{ \delta_2^1, \delta_2^2 \}$;
- $\mathcal{B} := \{ 0, 1 \}$; $\mathcal{B}^n$: n-dimensional Boolean vector set;
- $\mathcal{L}_{m\times n} := \{ L \in \mathcal{R}_{m\times n} \,|\, \mathrm{Col}(L) \subset \Delta_m \}$. $L \in \mathcal{L}_{m\times n}$ is called a logical matrix;
- $\delta_m[i_1\ i_2\ \cdots\ i_r] := [\delta_m^{i_1}\ \delta_m^{i_2}\ \cdots\ \delta_m^{i_r}]$;
- $\wedge$, $\vee$, $\neg$ denote the logical operator conjunction, disjunction, and negation, respectively.

Definition 1 ([8])**.** Set $A \in \mathcal{R}_{m\times n}$, $B \in \mathcal{R}_{p\times q}$. Let $\alpha = \mathrm{lcm}(n,p)$ be the least common multiple of n and p. Then, the STP of A and B is defined as

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}),$$

where $\otimes$ is the Kronecker product.

If $n = p$, the STP is reduced to the traditional matrix product. Throughout this paper, for simplicity of presentation, let AB denote $A \ltimes B$. Because $\mathbf{1}_m^{\mathrm{T}} x = 1$ and $\mathbf{1}_n^{\mathrm{T}} y = 1$, for any $x \in \Delta_m$ and $y \in \Delta_n$, we have

$$x = (I_m x)\otimes(\mathbf{1}_n^{\mathrm{T}} y) = (I_m \otimes \mathbf{1}_n^{\mathrm{T}})(x \otimes y) = (I_m \otimes \mathbf{1}_n^{\mathrm{T}})xy, \tag{1}$$

$$y = (\mathbf{1}_m^{\mathrm{T}} x)\otimes(I_n y) = (\mathbf{1}_m^{\mathrm{T}} \otimes I_n)(x \otimes y) = (\mathbf{1}_m^{\mathrm{T}} \otimes I_n)xy. \tag{2}$$

We identify Boolean variables $X \in \mathcal{B}$ with logical vectors $x \in \Delta_2$ as $X = 1 \sim x = \delta_2^1$ and $X = 0 \sim x = \delta_2^2$. Let x_i be the vector form of the logical variable X_i, i.e., $X_i \sim x_i$, $i = 1, 2, \ldots, n$. Then, there is a one-to-one correspondence between $X = (X_1, X_2, \ldots, X_n)^{\mathrm{T}} \in \mathcal{B}^n$ and $x = \ltimes_{i=1}^n x_i \in \Delta_{2^n}$.

A BCN can be described as

$$\begin{aligned} &X(t+1) = f(X(t), U(t)),\\ &Y(t) = h(X(t)), \end{aligned} \tag{3}$$

where $X = [X_1, X_2, \ldots, X_n]^{\mathrm{T}} \in \mathcal{B}^n$, $U = [U_1, U_2, \ldots, U_m]^{\mathrm{T}} \in \mathcal{B}^m$, $Y = [Y_1, Y_2, \ldots, Y_p]^{\mathrm{T}} \in \mathcal{B}^p$ are the state vector, input vector, and output vector, respectively, $f = [f_1, f_2, \ldots, f_n]^{\mathrm{T}} : \mathcal{B}^{n+m} \to \mathcal{B}^n$ and $h = [h_1, h_2, \ldots, h_p]^{\mathrm{T}} : \mathcal{B}^n \to \mathcal{B}^p$ are the system mapping and output mapping, respectively. The BCN (3) can be converted into the following algebraic form:

$$\begin{aligned} &x(t+1) = Lu(t)x(t),\\ &y(t) = Hx(t), \end{aligned} \tag{4}$$

where $L \in \mathcal{L}_{2^n \times 2^{n+m}}$, $H \in \mathcal{L}_{2^p \times 2^n}$, $x = \ltimes_{i=1}^n x_i$, $u = \ltimes_{i=1}^m u_i$, $y = \ltimes_{i=1}^p y_i$, $X_i \sim x_i$, $U_i \sim u_i$, and $Y_i \sim y_i$. The conversion process between the logical form (3) and algebraic form (4) can be found in [8].

Consider a digraph $\mathcal{G} = (V, \mathcal{E})$, where V and $\mathcal{E}$ are the vertex set and directed edge set, respectively. A nonnegative matrix $A = (a_{ij})$ is called the adjacency matrix. Each $(i,j) \in \mathcal{E}$ denotes the directed edge from i to j, and $(i,j) \in \mathcal{E}$ if and only if $a_{ji} > 0$, where j is called an out-neighbor of i. Let $\mathcal{N}(\cdot)$ denote the out-neighborhood of a set or a vertex. Let P_l, $l = 1, 2, \ldots, r$ be some subsets of V. $\{P_l\}_{l=1}^r$ is called a vertex partition of the digraph $\mathcal{G}$ if $\cup_{l=1}^r P_l = V$ and $P_i \cap P_j = \emptyset$ for any $i \neq j$. A vertex partition $\{P_l\}_{l=1}^r$ is called a perfect vertex partition (P-VP) if for every $l \in [1, r]$, there exists α_l, such that $\mathcal{N}(P_l) \subset P_{\alpha_l}$. A vertex partition $\{P_l\}_{l=1}^r$ is called an equal vertex partition (E-VP) if $|P_l| = |V|/r$ for every $l \in [1, r]$. A vertex-colored digraph is obtained by assigning a color to every vertex of a digraph. A vertex partition $\{P_l\}_{l=1}^r$ of a vertex-colored digraph is concolorous if for each P_l, $l \in [1, r]$, all the vertices in P_l have the same color. A perfect E-VP is denoted by PE-VP. A concolorous P-VP is denoted by CP-VP whereas a concolorous PE-VP is denoted by CPE-VP.

Definition 2 ([45])**.** Let $\mathcal{P}$ and $\mathcal{S}$ be two partitions of a set V. Assume that for every $P \in \mathcal{P}$, there exists an $S \in \mathcal{S}$, such that $P \subset S$. Then, the partition $\mathcal{P}$ is a finer partition than $\mathcal{S}$, and the partition $\mathcal{S}$ is a coarser partition than $\mathcal{P}$, denoted by $\mathcal{P} \sqsubset \mathcal{S}$. A partition $\mathcal{S}$ is called the coarsest partition if there is no partition coarser than it.

Consider BCN (4). Let

$$L = [L_1\ L_2 \cdots L_{2^m}], \quad B = (b_{ij}) = \sum_{i=1}^{2^m} L_i, \tag{5}$$

where each $L_i \in \mathcal{L}_{2^n \times 2^n}$. Then, B is a nonnegative square matrix, which can be a weighted adjacency matrix of a digraph $\mathcal{G}$. From (5), $b_{ij} > 0$ if and only if there exists $k \in [1,\ 2^m]$, such that $(L_k)_{ij} = 1$, i.e., $\delta_{2^n}^i = L\delta_{2^m}^k \delta_{2^n}^j$, which implies that a control $u = \delta_{2^m}^k$ exists, such that the state $\delta_{2^n}^j$ is transferred to $\delta_{2^n}^i$ by the state equation of (4). Thus, we call digraph $\mathcal{G}$ the STG associated with B. Similarly, one can construct the STG associated with L_i and define it as $\mathcal{G}_i$, $i \in [1, 2^m]$. We use the output function of the BCN (4) to assign a color for every vertex of the STG $\mathcal{G}$ in such a way that $\delta_{2^n}^i$ and $\delta_{2^n}^j$ are of the same color if and only if $H\delta_{2^n}^i = H\delta_{2^n}^j$. Then, the constructed STG $\mathcal{G}$ is a vertex-colored STG

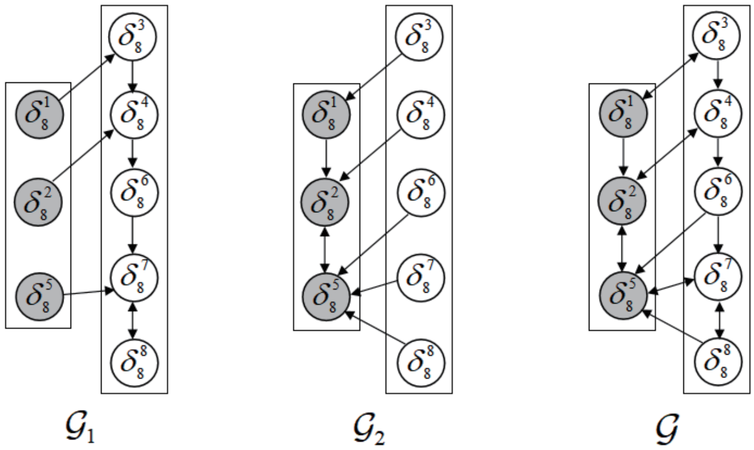

Figure 1 Vertex-colored STG $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}$.

associated with H and B. Similarly, using H and L_i, one can construct a vertex-colored STG denoted by $\mathcal{G}_i$, $i = 1, 2, \ldots, 2^m$ and clearly $\mathcal{G} = \cup_{i=1}^{2^m} \mathcal{G}_i$.

Here, we present a simple example to illustrate the above concepts.

Consider a BCN with an algebraic form as (4), where $H = \delta_2[1\ 1\ 2\ 2\ 1\ 2\ 2\ 2]$, $L = [L_1\ L_2] = \delta_{2^3}[3\ 4\ 4\ 6\ 7\ 7\ 8\ 7\ 2\ 5\ 1\ 2\ 2\ 5\ 5\ 5]$, $x \in \Delta_8$, $u \in \Delta$, and $y \in \Delta$. Let $y = \delta_2^1, \delta_2^2$, represented in gray and white. Then, based on L_1, L_2, L, and H, the vertex-colored STG $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}$ can be obtained, as shown in Figure 1. From Figure 1, considering $\mathcal{G}_1$ and partition $\mathcal{S} = \{S_1, S_2\}$, with $S_1 = \{1, 2, 5\}$, $S_2 = \{3, 4, 6, 7, 8\}$, we have $\mathcal{N}(S_1) \subset S_2$, $\mathcal{N}(S_2) \subset S_2$, and for each S_i, $i = 1, 2$, the vertices in S_i have the same color. Thus, $\mathcal{S} = \{S_1, S_2\}$ is a CP-VP for $\mathcal{G}_1$. Similarly, we have $\mathcal{S} = \{S_1, S_2\}$, which is also a CP-VP for $\mathcal{G}_2$. Thus, $\mathcal{S} = \{S_1, S_2\}$ is a common CP-VP for G_1 and G_2.

2.2 Problem statement

Consider the logical mapping $g : \mathcal{B}^n \to \mathcal{B}^n$ defined by

$$g : (X_1, X_2, \ldots, X_n) \mapsto (Z_1, Z_2, \ldots, Z_n). \tag{6}$$

If g is a bijection, it is called a logical coordinate transformation [46]. Let $z = Tx$ be the algebraic form of the logical coordinate transformation (6), where $T \in \mathcal{L}_{2^n \times 2^n}$ is the structure matrix of g, $x = \ltimes_{i=1}^n x_i$, $z = \ltimes_{i=1}^n z_i$, $X_i \sim x_i$, $Z_i \sim z_i$. Then, g is a logical coordinate transformation if and only if T is a nonsingular logical matrix, i.e., a permutation matrix [46].

Definition 3 ([46])**.** Consider BCN (4). The state space $\mathcal{X}$ is defined as the set of all logical functions of $\{X_1, X_2, \ldots, X_n\}$, denoted by $\mathcal{F}_l\{X_1, X_2, \ldots, X_n\}$. Let $Z_1, Z_2, \ldots, Z_n \in \mathcal{X}$. The subspace generated by $\{Z_1, Z_2, \ldots, Z_n\}$ is defined as the set of logical functions of $\{Z_1, Z_2, \ldots, Z_n\}$, denoted by $\mathcal{Z} = \mathcal{F}_l\{Z_1, Z_2, \ldots, Z_n\}$. A subspace $\mathcal{Z} = \mathcal{F}_l\{Z_1, Z_2, \ldots, Z_k\} \subset \mathcal{X}$ is called a regular subspace of the dimension k, if there are $Z_{k+1}, z_{k+2}, \ldots, Z_n \in \mathcal{X}$, such that $Z = (Z_1, Z_2, \ldots, Z_n)^{\mathrm{T}}$ is a logical coordinate transformation.

Definition 4 ([41])**.** Consider BCN (4). If there exists a subspace $\mathcal{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \ldots, Z_s\}$, such that

$$\begin{cases} z^1(t+1) = G_1 u(t) z^1(t), \\ y(t) = K z^1(t), \end{cases} \tag{7}$$

then Eq. (7) is a realization of (4), where $Z_i \sim z_i$, $i \in [1, s]$.

In [41], the subspace $\mathcal{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \ldots, Z_s\}$ is a controlled invariant subspace containing $\mathcal{Y}$. Here we call it a $\mathcal{Y}$-friendly controlled invariant subspace, and we call (7) a realization induced by a $\mathcal{Y}$-friendly controlled invariant subspace. Eq. (7) is called the minimum realization if $\mathcal{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \ldots, Z_s\}$ is the smallest $\mathcal{Y}$-friendly controlled invariant subspace. Essentially, $\mathcal{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \ldots, Z_s\}$ in Definition 4 is not required to be a regular subspace. In the following, we present a definition called a $\mathcal{Y}$-friendly controlled invariant regular subspace.

Definition 5. Consider BCN (4). Assume that $Z^1 = \mathcal{F}_l(Z_1, Z_2, \ldots, Z_s)$ is a regular subspace and $Z = (Z_1, Z_2, \ldots, Z_n)$ is a new coordinate frame. Z^1 is called a $\mathcal{Y}$-friendly controlled invariant regular subspace, if under Z, Eq. (4) can be expressed as

$$\begin{cases} z^1(t+1) = G_1 u(t) z^1(t), \\ z^2(t+1) = G_2 u(t) z(t), \\ y(t) = K z^1(t), \end{cases} \tag{8}$$

where $K \in \mathcal{L}_{2^p \times 2^s}$, $G_1 \in \mathcal{L}_{2^s \times 2^{s+m}}$, $G_2 \in \mathcal{L}_{2^{n-s} \times 2^{n+m}}$, $z = z^1 z^2$, $z^1 = \ltimes_{i=1}^s z_i$, $Z_i \sim z_i$, $i \in [1, n]$.

According to Definition 4,

$$\begin{cases} z^1(t+1) = G_1 u(t) z^1(t), \\ y(t) = K z^1(t) \end{cases} \tag{9}$$

in (8) is a realization of the BCN (4), which is a special case of (7). Eq. (9) is called the minimum realization if $\mathcal{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \ldots, Z_s\}$ is the smallest $\mathcal{Y}$-friendly controlled invariant regular subspace. Similarly, Eq. (9) is a realization induced by the $\mathcal{Y}$-friendly controlled invariant regular subspace. Based on the above discussion, the realizations (8) and (7) are the sub-BCNs of the original BCN. An interesting question is whether the realization must be a BCN. Actually, the answer is negative. In the following, we propose a general definition in which the realization is extended to a general logical system.

Definition 6. A logical control system Σ :

$$\begin{cases} \tilde{x}(t+1) = \tilde{L} u(t) \tilde{x}(t), \\ \tilde{y}(t) = \tilde{H} \tilde{x}(t) \end{cases}$$

is a realization of the BCN (4) if, for any initial state x_0 of BCN (4), there is an initial state $\tilde{x}_0$ of Σ, such that the outputs $\{y(t)\}$ and $\{\tilde{y}(t)\}$ satisfy

$$\tilde{y}(t) = \varphi(y(t)), \quad t = 0, 1, 2, \ldots$$

for any input sequence $u(t)$, $t = 0, 1, 2, \ldots$, where φ is a one-to-one correspondence from $\mathrm{Col}(H)$ to $\mathrm{Col}(\tilde{H})$. It is called the minimum realization if the dimension of the state vector of Σ is the smallest.

The main problem considered in this paper is how to construct the minimum realization for the above three realization definitions, i.e., the realization induced by the $\mathcal{Y}$-friendly controlled invariant regular subspace, the realization induced by the $\mathcal{Y}$-friendly controlled invariant subspace, and Definition 6.

3 Main results

In this section, the realization problem, including the minimum realization of BCNs, is investigated using the vertex partition method. First, we study how to construct the realization of BCNs for Definition 6.

Consider the BCN (4). Suppose that $\mathcal{S} = \{S_l\}_{l=1}^r$ is a common CP-VP of the vertex-colored STG $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{2^m}$. Using the vertex partition $\mathcal{S}$, we define an equivalence relationship $\sim$ on Δ_{2^n} as follows: $\delta_{2^n}^i$ and $\delta_{2^n}^j$ are equivalent, i.e., $\delta_{2^n}^i \sim \delta_{2^n}^j$, if and only if $\delta_{2^n}^i$ and $\delta_{2^n}^j$ belong to the same S_l, $l = 1, 2, \ldots, r$. The equivalence class of $\delta_{2^n}^i$ denoted by $\tilde{\delta}_{2^n}^i$, is defined as

$$\tilde{\delta}_{2^n}^i := \{\delta_{2^n}^j | \delta_{2^n}^j \sim \delta_{2^n}^i\}.$$

Evidently, for any $\delta_{2^n}^i \in S_l$, $\tilde{\delta}_{2^n}^i = S_l$. Thus, $\Delta_{2^n}/\sim = \mathcal{S} = \{S_l\}_{l=1}^r$ and $|\Delta_{2^n}/\sim| = r$. Because $|\Delta_{2^n}/\sim| = |\Delta_r|$, a one-to-one correspondence $\phi : \Delta_{2^n}/\sim \mapsto \Delta_r$ can be defined as

$$\phi(S_l) = \delta_r^l, \ l = 1, 2, \ldots, r.$$

With a mild abuse of notation, we still use the symbol $\sim$ to denote the mapping from Δ_{2^n} to $\Delta_{2^n}/\sim$ induced by the equivalence relationship $\sim$, which is defined as follows: for any $\delta_{2^n}^i \in S_l$, $\sim(\delta_{2^n}^i) = S_l$. Then, we call the composite mapping $\phi \circ \sim : \Delta_{2^n} \mapsto \Delta_r$ the quotient mapping and define

$$z = \phi \circ \sim (x), \ \forall x \in \Delta_{2^n}. \tag{10}$$

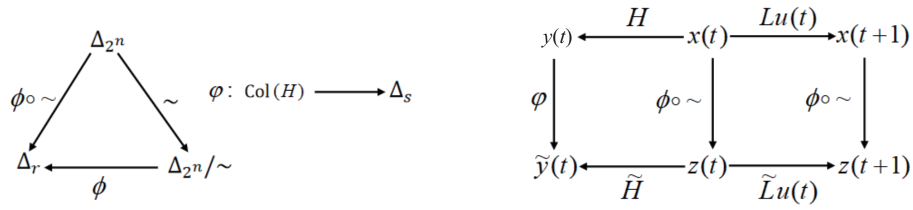

Figure 2 The mappings φ, $\phi\circ\sim$ and the relationship between the systems (13) and (4).

We set $\mathrm{Col}(H) = \{\delta_{2^p}^{c_1}, \delta_{2^p}^{c_2}, \dots, \delta_{2^p}^{c_s}\}$. As $|\mathrm{Col}(H)| = s$, we can reduce the order of column of H from 2^p to s by defining a one-to-one correspondence φ from $\mathrm{Col}(H)$ to Δ_s as

$$\varphi(\delta_{2^p}^{c_j}) = \delta_s^j, \ j = 1, 2, \dots, s. \tag{11}$$

Let

$$\tilde{y} = \varphi(y), \ \forall y \in \mathrm{Col}(H). \tag{12}$$

Because there is a one-to-one correspondence between y and $\tilde{y}$, the outputs $y = \delta_{2^n}^{c_j}$ and $\tilde{y} = \delta_s^j$ are seen as the same in the following.

Suppose that $\mathcal{S} = \{S_l\}_{l=1}^r$ is a common CP-VP of the vertex-colored STG $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_{2^m}$ and $\mathrm{Col}(H) = \{\delta_{2^p}^{c_1}, \delta_{2^p}^{c_2}, \dots, \delta_{2^p}^{c_s}\}$. By resorting to the defined mappings φ and $\phi\circ\sim$, the quotient logical system can be constructed as

$$\begin{cases} z(t+1) = \tilde{L}u(t)z(t), \\ \tilde{y}(t) = \tilde{H}z(t), \end{cases} \tag{13}$$

where $z \in \Delta_r, u \in \Delta_{2^m}, \tilde{y} \in \Delta_s, \tilde{L} \in \mathcal{L}_{r\times r2^m}$, and $\tilde{H} \in \mathcal{L}_{s\times r}$. The defined mappings and the relationship between the quotient logical system and the original system are shown in Figure 2.

In the following, by resorting to L and H of the BCN (4) and defined mappings, we show how to compute matrices $\tilde{L}$ and $\tilde{H}$ of the quotient logical system (13).

For any $l \in [1, r]$ and any $x(t) = \delta_{2^n}^i \in S_l$,

$$z(t) = \phi\circ\sim(x(t)) = \phi\circ\sim(\delta_{2^n}^i) = \delta_r^l. \tag{14}$$

For any given $u(t) = \delta_{2^m}^j$, $j = 1, 2, \dots, 2^m$, let

$$x(t+1) = Lu(t)x(t) = L\delta_{2^m}^j\delta_{2^n}^i := \delta_{2^n}^{\bar{i}}. \tag{15}$$

Because $\mathcal{S} = \{S_l\}_{l=1}^r$ is perfect for each $\mathcal{G}_j$, $j = 1, 2, \dots 2^m$, there exists an α_l^j, such that $\delta_{2^n}^{\bar{i}} \in S_{\alpha_l^j}$. Thus,

$$z(t+1) = \phi\circ\sim(x(t+1)) = \phi\circ\sim(\delta_{2^n}^{\bar{i}}) = \delta_r^{\alpha_l^j}. \tag{16}$$

Combining the formulas (14)–(16), we have

$$\delta_r^{\alpha_l^j} = \tilde{L}\delta_{2^m}^j\delta_r^l. \tag{17}$$

Let $\tilde{L} = [\tilde{L}_1 \ \tilde{L}_2 \ \cdots \ \tilde{L}_{2^m}]$. By (17), $\mathrm{Col}_l(\tilde{L}_j) = \delta_r^{\alpha_l^j}$. Due to the arbitrariness of l and j, $\tilde{L}$ is obtained.

For any l and any $x(t) = \delta_{2^n}^i \in S_l$, let

$$y(t) = Hx(t) = H\delta_{2^n}^i := \delta_{2^p}^{c_{j_l}}.$$

Then,

$$\tilde{y}(t) = \varphi(y(t)) = \varphi(\delta_{2^p}^{c_{j_l}}) = \delta_s^{j_l}. \tag{18}$$

By (14) and (18), given that $\mathcal{S} = \{S_l\}_{l=1}^r$ is concolorous,

$$\tilde{y}(t) = \tilde{H}z(t) = \tilde{H}\delta_r^l = \delta_s^{j_l}. \tag{19}$$

Thus $\mathrm{Col}_l(\tilde{H}) = \delta_s^{j_l}$. Due to the arbitrariness of l, $\tilde{H}$ is derived.

Theorem 1. Consider the BCN (4). Suppose that $\mathcal{S} = \{S_l\}_{l=1}^r$ is a common CP-VP of the vertex-colored STG $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{2^m}$ and $\mathrm{Col}(H) = \{\delta_{2^p}^{c_1}, \delta_{2^p}^{c_2}, \ldots, \delta_{2^p}^{c_s}\}$. Then, the quotient logical system (13) is a realization of the BCN (4) defined by Definition 6.

Proof. For any $l_0 \in [1, r]$, any $x(0) = x_0 \in S_{l_0}$, any input sequence

$$[u(0), u(1), \ldots, u(k-1)] = [\delta_{2^m}^{j_0}, \delta_{2^m}^{j_1} \ldots, \delta_{2^m}^{j_{k-1}}],\ k > 0,$$

where $j_0, j_1, \ldots, j_{k-1} \in [1, 2^m]$, as $\mathcal{S} = \{S_l\}_{l=1}^r$ is perfect for each $\mathcal{G}_j$, $j \in [1, 2^m]$, without loss of generality, we set $x(t) \in S_{l_t}$, $t = 0, 1, \ldots, k$. Given that $\mathcal{S} = \{S_l\}_{l=1}^r$ is concolorous, i.e., the output of each point is only dependent on S_l to which the point belongs, the output sequence corresponding to the initial x_0 and the above input sequence can be set as

$$[y(0), y(1), \ldots, y(k)] = [\delta_{2^p}^{c_{j_{l_0}}}, \delta_{2^p}^{c_{j_{l_1}}}, \ldots, \delta_{2^p}^{c_{j_{l_k}}}],$$

where $j_{l_0}, j_{l_1}, \ldots, j_{l_k} \in [1, s]$.

Consider the system (13). Let $z(0) = z_0 = \phi \circ \sim (x_0)$. Given that $x(t) \in S_{l_t}$, $t = 0, 1, \ldots, k$, under the input sequence $[u(0), u(1), \ldots, u(k-1)] = [\delta_{2^m}^{j_0}, \delta_{2^m}^{j_1} \ldots, \delta_{2^m}^{j_{k-1}}]$, $k > 0$, the corresponding $z(t) = \phi \circ \sim (x(t)) = \delta_r^{l_t}$. Then,

$$\tilde{y}(t) = \tilde{H} z(t) = \tilde{H} \delta_r^{l_t} = \delta_s^{j_{l_t}}, \quad t = 0, 1, \ldots, k.$$

Thus, the output sequence corresponding to the initial z_0 under the input sequence $[u(0), u(1), \ldots, u(k-1)] = [\delta_{2^m}^{j_0}, \delta_{2^m}^{j_1}, \ldots, \delta_{2^m}^{j_{k-1}}]$, $k > 0$ is

$$[\tilde{y}(0), \tilde{y}(1), \ldots, \tilde{y}(k)] = [\delta_s^{j_{l_0}}, \delta_s^{j_{l_1}}, \ldots, \delta_s^{j_{l_k}}].$$

Note that the outputs $y = \delta_{2^n}^{c_{l_j}}$ and $\tilde{y} = \varphi(y) = \delta_s^{l_j}$ can be seen as the same. Thus, the system (13) is a realization of the BCN (4).

Corollary 1. Consider the BCN (4). Suppose that $\mathcal{S} = \{S_l\}_{l=1}^r$ is the coarsest common CP-VP of the vertex-colored STG $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{2^m}$. Then, the quotient logical system (13) constructed by the quotient mapping $\phi \circ \sim$ and the mapping φ is the minimum realization of the BCN (4) defined by Definition 6.

Corollary 2. Suppose that $\mathcal{S} = \{S_l\}_{l=1}^r$ is a common CP-VP of the vertex-colored STG $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{2^m}$ with $r = 2^a$ and $\mathrm{Col}(H) = \{\delta_{2^p}^{c_1}, \delta_{2^p}^{c_2}, \ldots, \delta_{2^p}^{c_s}\}$ with $s = 2^b$, where $a, b \in \mathbb{Z}$, then the realization form as (7) in Definition 4 can be constructed through the same construction process as (13).

In the following, we provide another way to construct the realization of the BCN (4) induced by the $\mathcal{Y}$-friendly controlled invariant regular subspace.

Theorem 2. Consider the BCN (4). Suppose that $\mathcal{S} = \{S_l\}_{l=1}^{2^s}$ is a common CPE-VP of the vertex-colored STG $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{2^m}$ with $|S_l| = 2^{n-s}$. Then, there is a $\mathcal{Y}$-friendly controlled invariant regular subspace of the dimension s.

Proof. Consider $\mathcal{S} = \{S_l\}_{l=1}^{2^s}$ with $|S_l| = 2^{n-s}$. Construct a matrix M as follows: for each $l \in [1, 2^s]$, any $\delta_{2^n}^i \in S_l$, let

$$M \delta_{2^n}^i = \delta_{2^s}^l. \tag{20}$$

Because $\mathcal{S}$ is an E-VP, every row of M has exactly 2^{n-s} ones. Thus, one can obtain M by rearranging the columns of $I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}}$. In other words, a permutation matrix T exists, such that

$$M = (I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}}) T. \tag{21}$$

Let the logical coordinate transformation be $z = z^1 z^2 = Tx$ with $z^1 = \ltimes z_{i=1}^s \in \Delta_{2^s}$, where $Z_i \sim z_i$, $i \in [1, n]$. In the rest of the proof, we derive the form of (8).

Given that $\mathcal{S} = \{S_l\}_{l=1}^{2^s}$ is concolorous, for any $l \in [1, 2^s]$, there exists β_l, such that

$$H \delta_{2^n}^a = \delta_{2^p}^{\beta_l}, \quad \forall \delta_{2^n}^a \in S_l. \tag{22}$$

Construct a logical matrix K as follows:

$$\mathrm{Col}_l(K) = \delta_{2^p}^{\beta_l}, \quad \forall l \in [1, 2^s]. \tag{23}$$

From (20), (22), and (23), for any $\delta_{2^n}^a \in S_l$,

$$KM\delta_{2^n}^a = K\delta_{2^s}^l = \delta_{2^p}^{\beta_l} = H\delta_{2^n}^a,$$

which implies that

$$KM = H. \tag{24}$$

Therefore,

$$y = Hx = KMx = Kz^1. \tag{25}$$

Considering that for each $\mathcal{G}_i$, $i = 1, 2, \ldots, 2^m$, $\mathcal{S}$ is perfect, for each $l \in [1, 2^s]$, there exists α_l^i, such that $\mathcal{N}_i(S_l) \subset S_{\alpha_l^i}$, where $\mathcal{N}_i(\cdot)$ denotes the neighborhood of a point or a set for the digraph $\mathcal{G}_i$. Construct the logical matrix $G_1 = [G_{11}\ G_{12}\ \cdots\ G_{12^m}]$ as follows:

$$\mathrm{Col}_l(G_{1i}) = \delta_{2^s}^{\alpha_l^i}, \quad \forall l \in [1, 2^s], \tag{26}$$

where $G_{1i} \in \mathcal{L}_{2^s \times 2^s}$. For any $\delta_{2^n}^a \in S_l$ and the digraph $\mathcal{G}_i$, let the out-neighbor of $\delta_{2^n}^a$ be $\delta_{2^n}^{b_i}$, i.e., $\delta_{2^n}^{b_i} = L_i\delta_{2^n}^a$. Then, $\delta_{2^n}^{b_i} \in S_{\alpha_l^i}$. Thus, for any $\delta_{2^n}^a \in S_l$, $l \in [1, 2^s]$,

$$ML_i\delta_{2^n}^a = M\delta_{2^n}^{b_i} \xlongequal{\text{by (20)}} \delta_{2^s}^{\alpha_l^i} \xlongequal{\text{by (26)}} G_{1i}\delta_{2^s}^l \xlongequal{\text{by (20)}} G_{1i}M\delta_{2^n}^a, \tag{27}$$

which implies that

$$ML_i = G_{1i}M. \tag{28}$$

Substituting (21) into (28) yields

$$(I_{2^s} \otimes 1_{2^{n-s}}^{\mathrm{T}})TL_iT^{\mathrm{T}} = G_{1i}(I_{2^s} \otimes 1_{2^{n-s}}^{\mathrm{T}}).$$

Therefore,

$$\begin{aligned} z^1(t+1) &= (I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})z^1(t+1)z^2(t+1) \\ &= (I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})Tx(t+1) \\ &= (I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})TL_ix(t) \\ &= (I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})TL_iT^{\mathrm{T}}z(t) \\ &= G_{1i}(I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})z^1(t)z^2(t) \\ &= G_{1i}z^1(t). \end{aligned}$$

Thus, we have

$$z^1(t+1) = G_1u(t)z^1(t). \tag{29}$$

Considering (25) and (29), the form (8) is derived. Thus, $\mathscr{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \ldots, Z_s\}$ is a $\mathcal{Y}$-friendly controlled invariant regular subspace of the dimension s.

Remark 1. The system consisting of (25) and (29) is just the realization induced by the $\mathcal{Y}$-friendly controlled invariant regular subspace $\mathscr{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \ldots, Z_s\}$. If $\mathcal{S} = \{S_l\}_{l=1}^{2^s}$ is the coarsest common CPE-VP of the vertex-colored STG $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{2^m}$, then the system consisting of (25) and (29) is the minimum realization of the BCN (4).

According to the proof of Theorem 2, we provide an algorithm (see Algorithm 1) to construct the minimum realization of the BCN (4), where the matrix M is constructed as follows:

$$\forall \delta_{2^n}^i \in S_l,\ \mathrm{Col}_i(M) = \delta_{2^s}^l,\ l = 1, 2, \ldots, 2^s. \tag{30}$$

Consider the BCN (4). To construct the minimum realization (defined by Definition 6) of a BCN, we first need to find the coarsest common CP-VP of the vertex-colored STG $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{2^m}$ and then construct the quotient logical system (13) by the quotient mapping $\phi\circ \sim$ and the mapping φ. In the following, for a BCN, we introduce an algorithm to compute the coarsest common CP-VP of the vertex-colored STG $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{2^m}$.

Define a sequence of the sets of $2^p \times 2^n$ logical matrices as follows:

$$\Gamma^0 = \{H\}, \quad \Gamma^\mu = \{HL_{i_\mu} \cdots L_{i_1} \mid i_\mu, \ldots, i_1 \in [1, 2^m]\}, \tag{31}$$

Algorithm 1 Construction of the minimum realization induced by the $\mathcal{Y}$-friendly controlled invariant regular subspace

Step 1. Consider the BCN (4). Compute L and H by the STP and suppose that $\mathcal{S} = \{S_l\}_{l=1}^{2^s}$ is the coarsest common CPE-VP of $\mathcal{G}_1, \ldots, \mathcal{G}_{2^m}$.
Step 2. Construct M according to (30) to compute K.
Step 3. Compute K from (24) using M.
Step 4. Let $L = [L_1\ L_2\ \cdots\ L_{2^m}]$. Compute G_i from (28). Then, G_1 can be obtained.
Step 5. By K and G_1, the realization system consisting of (25) and (29) is obtained.

where $\mu = 1, 2, \ldots$. For notational ease, let Γ^μ denote the matrix consisting of its elements and arrange them in a column. Let $\mathcal{O}^0 = \Gamma^0$ and

$$\mathcal{O}^{s+1} = \begin{bmatrix} \mathcal{O}^s \\ \Gamma^{s+1} \end{bmatrix} \tag{32}$$

for every $s = 0, 1, 2, \ldots$ and construct a partition $\mathcal{P}^\tau = \{P_l^\tau\}_{l=1}^{s_\tau}$ of Δ_{2^n} according to

$$a, b \in P_l^\tau \Leftrightarrow \mathrm{Col}_a(\mathcal{O}^\tau) = \mathrm{Col}_b(\mathcal{O}^\tau) \tag{33}$$

for any $\tau = 0, 1, 2, \ldots$.

Lemma 1 ([47]). The partition sequence $\mathcal{P}^\tau$, $\tau = 0, 1, 2, \ldots$, satisfies

(I) $\mathcal{P}^{\tau+1} \sqsubset \mathcal{P}^\tau$;

(II) $\mathcal{P}^{\tau+1} = \mathcal{P}^\tau \ \Rightarrow\ \mathcal{P}^{\tau+2} = \mathcal{P}^{\tau+1}$;

(III) there exists $\tau^* \leqslant 2^n - 2$, such that $\mathcal{P}^\tau = \mathcal{P}^{\tau^*}$ for all $\tau > \tau^*$ and $\mathcal{P}^{\tau^*}$ is the coarsest common CP-VP of $\mathcal{G}_1, \ldots, \mathcal{G}_{2^m}$.

Based on Lemma 1, an algorithm is provided to search for the coarsest common CP-VP $\mathcal{P}^{\tau^*}$ of $\mathcal{G}_1$, $\ldots$, $\mathcal{G}_{2^m}$ in [47]. In the following, we present an algorithm to construct the minimum realization of the BCN (4) for different realization definitions (see Algorithm 2).

Algorithm 2 Construction of the minimum realization of BCNs

Step 1. Consider the BCN (4). Compute L and H by the STP.
Step 2. Compute the coarsest common CP-VP of $\mathcal{G}_1, \ldots, \mathcal{G}_{2^m}$ based on L and H, denoted by $\mathcal{S} = \{S_l\}_{l=1}^r$.
Step 3.
- If $\mathcal{S} = \{S_l\}_{l=1}^r$ is an E-VP, then the minimum realization induced by the $\mathcal{Y}$-friendly controlled invariant regular subspace can be constructed according to Algorithm 1.
- If $\mathcal{S} = \{S_l\}_{l=1}^r$ satisfies $r = 2^a$, and $\mathrm{Col}(H) = \{\delta_{2^p}^{c_1}, \delta_{2^p}^{c_2}, \ldots, \delta_{2^p}^{c_s}\}$ with $s = 2^b$, where $a, b \in \mathcal{Z}$, then the realization as form (7) in Definition 4 can be constructed. The construction process is the same as that of constructing (13).
- Otherwise, we construct the quotient mapping $\phi \circ \sim$ and mapping φ by $\mathcal{S}$ and subsequently construct the quotient logical system (13) by $\phi \circ \sim$ and φ. The quotient logical system (13) is the minimum realization of the BCN (4) defined by Definition 6.

Remark 2. The first step of Algorithms 1 and 2 is to compute matrices L and H in (4). Considering the sizes of L and H, Algorithms 1 and 2 have at least exponential time complexity.

Comparison. In [24], the minimum realization problem of BCNs is investigated by Kalman decomposition. Specifically, suppose that the largest unobservable subspace and largest uncontrollable subspace of the BCN (4) are regular and its Kalman decomposition is, under a suitable coordinate transformation $z = Tx$,

$$\begin{cases} z^1(t+1) = f_1(z^1(t), z^3(t), u(t)), \\ z^2(t+1) = f_2(z^1(t), z^2(t), z^3(t), z^4(t), u(t)), \\ z^3(t+1) = f_1(z^3(t)), \\ z^4(t+1) = f_2(z^3(t), z^4(t)), \\ y(t+1) = \tilde{h}(z^1(t), z^3(t)), \end{cases} \tag{34}$$

where $z = z^1 z^2 z^3 z^4$. Then, the minimum realization system is defined as

$$\begin{cases} z^1(t+1) = f_1(z^1(t), z^3(t), u(t)), \\ y(t+1) = \tilde{h}(z^1(t), z^3(t)), \end{cases} \tag{35}$$

with a fixed value $z^3 = z_0^3$. Compared with [24], in this study, based on the vertex partition method, the minimum realization is constructed, and the regularity condition of the largest unobservable subspace and largest uncontrollable subspace and the value condition are removed. Compared with the realization definition in [41], we propose a new realization definition, which is a generalization of the definition in [41].

Remark 3. For a large-scale BCN, an essential issue is to reduce computational complexity. In this study, by constructing quotient mappings, the quotient system, i.e., the minimum realization of the original BCN, is derived, which might be a much smaller sub-BCN or a much smaller logical dynamic system. For certain properties, the realization system and the original system may have similar behaviors. Thus, this study may provide a way to study large-scale BCNs.

4 Examples

In this section, three examples are presented to validate the obtained theoretical results. The first example is given to illustrate the minimum realization induced by a regular subspace of the original BCN. The second example is taken to show the minimum realization induced by a subspace that does not have to be a regular subspace of the original BCN. The last example is provided to show that the minimum realization does not have to be a BCN, and a BCN can be realized under Definition 6 proposed in this paper, but it cannot be realized under the definitions in [24, 41].

Example 1 ([48])**.** Consider a simplified cell apoptosis Boolean model expressed as follows:

$$\begin{aligned} X_1(t+1) &= \neg X_2(t) \wedge U(t), \\ X_2(t+1) &= \neg X_1(t) \wedge X_3(t), \\ X_3(t+1) &= X_2(t) \vee U(t), \end{aligned} \tag{36}$$

where X_1, X_2, X_3, and $U \in \mathcal{B}$ represent the inhibitor of apoptosis proteins, active caspase 3, active caspase 8, and tumor necrosis factor, respectively. The output equation is given by

$$Y(t) = (X_1(t) \wedge X_2(t)) \vee [\neg X_1(t) \wedge (X_2(t) \bar{\vee} X_3(t))], \tag{37}$$

where $Y \in \mathcal{B}$. Based on the STP, the BCN (36) and (37) can be converted into the algebraic form (4) with $x \in \Delta_8$, $u \in \Delta$, $y \in \Delta$, $L = \delta_8[7\ 7\ 3\ 3\ 5\ 7\ 1\ 3\ 7\ 7\ 8\ 8\ 5\ 7\ 6\ 8]$, $H = \delta_2[1\ 1\ 2\ 2\ 2\ 1\ 1\ 2]$. $L = [L_1, L_2]$ and $y = \delta_2^1$ (δ_2^2) are represented in gray (white). Then, the vertex-colored STG $\mathcal{G}_1$ and $\mathcal{G}_2$ are constructed by L_1, L_2, and H, as shown in Figure 3. With a straightforward computation, we have

$$\mathcal{O}^0 = \Gamma^0 = \begin{bmatrix} \delta_2[1\ 1\ 2\ 2\ 2\ 1\ 1\ 2] \end{bmatrix}.$$

Then, by (33), $\mathcal{P}^0 = \{\{1, 2, 6, 7\}, \{3, 4, 5, 8\}\}$. Let $\Gamma^1 = \left[\begin{smallmatrix} HL_1 \\ HL_2 \end{smallmatrix}\right]$ and construct

$$\mathcal{O}^1 = \begin{bmatrix} \mathcal{O}^0 \\ \Gamma^1 \end{bmatrix} = \begin{bmatrix} \delta_2[1\ 1\ 2\ 2\ 2\ 1\ 1\ 2] \\ \delta_2[1\ 1\ 2\ 2\ 2\ 1\ 1\ 2] \\ \delta_2[1\ 1\ 2\ 2\ 2\ 1\ 1\ 2] \end{bmatrix}.$$

Then,

$$\mathcal{P}^1 = \{\{1, 2, 6, 7\}, \{3, 4, 5, 8\}\}.$$

Given that $\mathcal{P}^0 = \mathcal{P}^1$, from Lemma 1, $\mathcal{P}^1$ is the coarsest common CP-VP of $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $P_1 = \{3, 4, 5, 8\}$, $P_2 = \{1, 2, 6, 7\}$. As $\mathcal{P}^1$ is an E-VP, $|\mathcal{P}^1| = 2$. According to (30), M is constructed as

$$M = \delta_2[2\ 2\ 1\ 1\ 1\ 2\ 2\ 1]. \tag{38}$$

Omitting the computation process, by (24) and (28), one can obtain

$$K = \delta_2[2\ 1],\ G_1 = \delta_2[1\ 2\ 1\ 2].$$

Then,

$$\begin{cases} z^1(t+1) = \delta_2[1\ 2\ 1\ 2]u(t)z^{[1]}(t), \\ y(t+1) = \delta_2[2\ 1]z^{[1]}(t) \end{cases} \tag{39}$$

is the minimum realization of the original BCN.

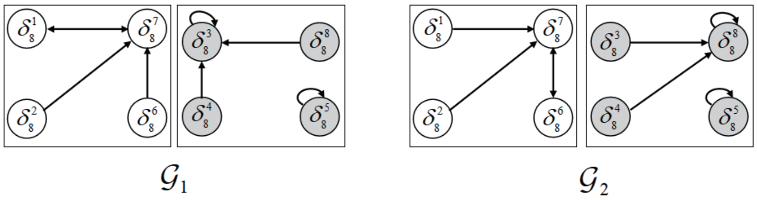

Figure 3 Vertex-colored STGs $\mathcal{G}_1$ and $\mathcal{G}_2$ of the system (36) with (37).

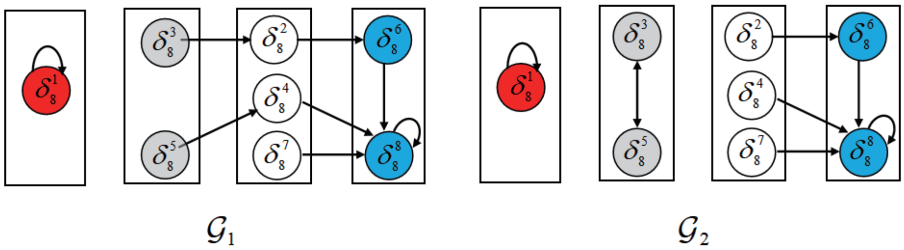

Figure 4 (Color online) Vertex-colored STGs $\mathcal{G}_1$ and $\mathcal{G}_2$ of the system (40).

Example 2. Consider a BCN with 1 input, 2 outputs, and 3 state variables with the algebraic form:

$$\begin{aligned} x(t+1) &= Lu(t)x(t), \\ y(t) &= Hx(t), \end{aligned} \tag{40}$$

where $L = \delta_8[1\ 6\ 2\ 8\ 4\ 8\ 8\ 8\ 1\ 6\ 5\ 8\ 3\ 8\ 8\ 8]$, $H = \delta_4[1\ 3\ 2\ 3\ 2\ 4\ 3\ 4]$, $x \in \Delta_8$, $u \in \Delta$, and $y \in \Delta_4$.

Let $L = [L_1, L_2]$ and $y = \delta_4^1, \delta_4^2, \delta_4^3, \delta_4^4$, represented in red, gray, white, and blue, respectively. Then, the vertex-colored STG $\mathcal{G}_1$ and $\mathcal{G}_2$ are constructed by L_1, L_2, and H, as shown in Figure 4. Omitting the same processes as Example 1, one can get the coarsest common CP-VP of $\mathcal{G}_1$ and $\mathcal{G}_2$ $\mathcal{S} = \{S_l\}_{l=1}^4$ with $S_1 = \{1\}, S_2 = \{3,5\}, S_3 = \{2,4,7\}, S_4 = \{6,8\}$. Then,

$$\Delta_8/\sim = \mathcal{S} = \{S_l\}_{l=1}^4 := \{\tilde{\delta}_8^1, \tilde{\delta}_8^3, \tilde{\delta}_8^2, \tilde{\delta}_8^6\} \tag{41}$$

and $|\Delta_8/\sim| = 4$. Define $\phi : \Delta_8/\sim \mapsto \Delta_4$ as $\phi(S_l) = \delta_4^l, l = 1,2,3,4$. For any $\delta_8^i \in S_l$, let $\sim(\delta_8^i) = S_l$. Then, the quotient mapping is $\phi\circ\sim: \Delta_8 \mapsto \Delta_4$.

Given that $\mathrm{Col}(H) = \Delta_4$, then $|\mathrm{Col}(H)| = 4$. Thus, $\varphi : y \in \Delta_4 \mapsto \tilde{y} \in \Delta_4$ can be defined as

$$\varphi(\delta_4^i) = \delta_4^i,\ i = 1,2,3,4.$$

Let $z = \phi\circ\sim(x)$ and $\tilde{y} = \varphi(y)$. Then, the quotient system with the algebraic form is

$$\begin{aligned} z(t+1) &= \tilde{L}u(t)z(t), \\ \tilde{y}(t) &= \tilde{H}z(t), \end{aligned} \tag{42}$$

where $\tilde{L} \in \mathcal{L}_{4\times8}$ and $\tilde{H} \in \mathcal{L}_{4\times4}$. In the following, we compute $\tilde{L}$ and $\tilde{H}$.

Let $\tilde{L} = [\tilde{L}_1, \tilde{L}_2]$. Take $x(t) = \delta_8^5$ as an example. According to L in (40),

$$x(t+1) = \begin{cases} \delta_8^4, & u(t) = \delta_2^1, \\ \delta_8^3, & u(t) = \delta_2^2. \end{cases} \tag{43}$$

Given

$$\phi\circ\sim(\delta_8^5) = \delta_4^2,\quad \phi\circ\sim(\delta_8^4) = \delta_4^3,\quad \phi\circ\sim(\delta_8^3) = \delta_4^2,$$

then $z(t) = \delta_4^2$, and

$$z(t+1) = \begin{cases} \delta_4^3, & u(t) = \delta_2^1, \\ \delta_4^2, & u(t) = \delta_2^2, \end{cases} \tag{44}$$

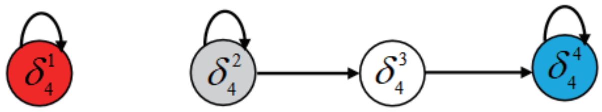

Figure 5 (Color online) Vertex-colored STG of the quotient system (42).

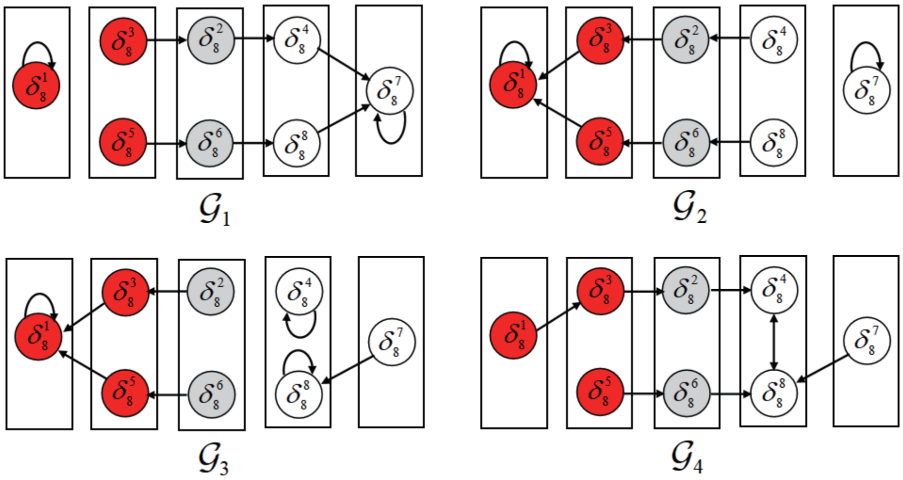

Figure 6 (Color online) Vertex-colored STG $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$.

corresponding to $x(t)$ and $x(t+1)$ by $\phi\circ\sim$. Thus, we have $\mathrm{Col}_2(\tilde{L}_1)=\delta_4^3$, $\mathrm{Col}_2(\tilde{L}_2)=\delta_4^2$. Repeating this process for $x(t)=\delta_8^i$, $i\in[1,8]$, we have $\tilde{L}=\delta_4[1\ 3\ 4\ 4\ 1\ 2\ 4\ 4]$. Consider $x(t)=\delta_8^5$ and $z(t)=\phi\circ\sim(x(t))=\delta_4^2$. Given that

$$y(t)=Hx(t)=H\delta_8^5=\delta_4^2,$$

then

$$\tilde{y}(t)=\varphi(y(t))=\varphi(\delta_4^2)=\delta_4^2.$$

From $\tilde{y}(t)=\tilde{H}z(t)$, it follows that $\mathrm{Col}_2(\tilde{H})=\delta_4^2$. By repeating this process for $x(t)=\delta_8^i$, $i\in[1,8]$, we obtain $\tilde{H}=\delta_4[1\ 2\ 3\ 4]$. The quotient system (42) is the minimum realization of the BCN (40). The vertex-colored STG of the quotient system is shown in Figure 5.

Example 3. Consider a BCN with 2 inputs, 2 outputs, and 3 state variables with an algebraic form:

$$\begin{aligned} x(t+1)&=Lu(t)x(t),\\ y(t)&=Hx(t), \end{aligned} \tag{45}$$

where

$$\begin{aligned} L&=\delta_{2^3}[1\ 4\ 2\ 7\ 6\ 8\ 7\ 7\ 1\ 3\ 1\ 2\ 1\ 5\ 7\ 6\ 1\ 3\ 1\ 4\ 1\ 5\ 8\ 8\ 3\ 4\ 2\ 8\ 6\ 8\ 8\ 4],\\ H&=\delta_4[1\ 2\ 1\ 3\ 1\ 2\ 3\ 3], \end{aligned}$$

$x\in\Delta_8$, $u\in\Delta_4$, and $y\in\Delta_4$. Let $L=[L_1,L_2,L_3,L_4]$ and $y=\delta_4^1,\delta_4^2,\delta_4^3$, represented in red, gray, and white, respectively. Then, the vertex-colored STG $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$ are constructed as shown in Figure 6. Omitting the same processes as Example 1, we obtain the coarsest common CP-VP of $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$ $\mathcal{S}=\{S_l\}_{l=1}^5$ with $S_1=\{1\},S_2=\{3,5\},S_3=\{2,6\},S_4=\{4,8\},S_5=\{7\}$. Then,

$$\Delta_8/\sim=\mathcal{S}=\{S_l\}_{l=1}^5:=\{\tilde{\delta}_8^1,\tilde{\delta}_8^3,\tilde{\delta}_8^2,\tilde{\delta}_8^4,\tilde{\delta}_8^7\} \tag{46}$$

and $|\Delta_8/\sim|=5$. Define $\phi:\Delta_8/\sim\mapsto\Delta_5$ as $\phi(S_l)=\delta_5^l, l=1,2,3,4,5$. For any $\delta_8^i\in S_l$, let $\sim(\delta_8^i)=S_l$. Then, the quotient mapping is $\phi\circ\sim:\Delta_8/\sim\mapsto\Delta_5$. Given that $\mathrm{Col}(H)=\{\delta_4^1,\delta_4^2,\delta_4^3\}$, then $|\mathrm{Col}(H)|=3$. Thus, $\varphi:y\in\Delta_4\mapsto\tilde{y}\in\Delta_3$ can be defined as $\varphi(\delta_4^i)=\delta_3^i$, $i=1,2,3$. Let $z=\phi\circ\sim(x)$ and $\tilde{y}=\varphi(y)$. Then, the quotient system with the algebraic form is

$$\begin{aligned} z(t+1)&=\tilde{L}u(t)z(t),\\ y(t)&=\tilde{H}z(t), \end{aligned} \tag{47}$$

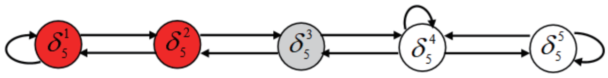

Figure 7 (Color online) Vertex-colored STG of the quotient system (47).

where $\tilde{L} \in \mathcal{L}_{5\times 10}$ and $\tilde{H} \in \mathcal{L}_{3\times 5}$. Omitting the same processes as Example 2, one can obtain $\tilde{L}$ and $\tilde{H}$ as

$$\tilde{L} = \delta_5[1\ 3\ 4\ 5\ 5\ 1\ 1\ 2\ 3\ 5\ 1\ 1\ 2\ 4\ 4\ 2\ 3\ 4\ 4\ 4],$$
$$\tilde{H} = \delta_3[1\ 1\ 2\ 3\ 3].$$

The quotient system (47) is the minimum realization of the BCN (45). The vertex-colored STG of the quotient system is shown in Figure 7.

Thus, under Definition 6, this example is realizable. However, because in this example, $|\Delta_8/\sim| = 5$ and $|\mathrm{Col}(H)| = 3$, according to Corollary 2, this example cannot be realized under the definition in [41]. As the definition in [41] is more general than that in [24], this example cannot be realized under the definition in [24].

5 Conclusion

Three types of realization problems, including the minimum realization of BCNs, are investigated by the vertex partition method from the perspective of STG. We propose a new definition of the realization for BCNs. Compared with the existing definitions, the new definition extends the realization system to a general logical dynamic system that does not have to be a BCN. For the general definition, by constructing a quotient mapping, the realization of BCNs is constructed. For the existing realization definitions, the realizations of a BCN have also been constructed by the vertex partition method, and two additional assumptions in [24] have been removed. As the realization of a BCN may be a much smaller logical dynamic system compared with the original BCN, especially a much smaller sub-BCN, the method used in this paper may be applied to study large-scale BCNs, which is one of our future work.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11991020, 11991024, 61673012, 11971240), Research Project of Chongqing National Center for Applied Mathematics (Grant No. ncamc2022-msxm05), and Chongqing Normal University Foundation (Grant No. 21XLB045).

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