SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

July 2023, Vol. 66 172201:1–172201:15
https://doi.org/10.1007/s11432-022-3648-8

Event-triggered tracking control for a class of nonholonomic systems in chained form

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Received 15 July 2022/Revised 21 September 2022/Accepted 23 October 2022/Published online 10 May 2023

Abstract In this study, the event-triggered asymptotic tracking control problem is considered for a class of nonholonomic systems in chained form for the time-varying reference input. First, to eliminate the ripple phenomenon caused by the imprecise compensation of the time-varying reference input, a novel time-varying event-triggered piecewise continuous control law and a triggering mechanism with a time-varying triggering function are developed. Second, an explicit integral input-to-state stable Lyapunov function is constructed for the time-varying closed-loop system regarding the sampling error as the external input. The origin of the closed-loop system is shown to be uniformly globally asymptotically stable for any global exponential decaying threshold signals, which in turn rules out the Zeno behavior. Moreover, infinitely fast sampling can be avoided by appropriately tuning the exponential convergence rate of the threshold signal. A numerical simulation example is provided to illustrate the proposed control approach.

Keywords event-triggered, nonholonomic systems, strict Lyapunov function, tracking, integral input-to-state stable

Citation Xu L, Su Y F, Cai H. Event-triggered tracking control for a class of nonholonomic systems in chained form. Sci China Inf Sci, 2023, 66(7): 172201, https://doi.org/10.1007/s11432-022-3648-8

1 Introduction

Nonholonomic systems, such as mobile robots [1], snake-like robots [2], and underactuated surface vessels [3], are common and important in practice and have attracted extensive attention from academia and industry [4, 5]. Brockett [6] showed that because of the nonintegrable motion constraints [7], nonholonomic systems cannot be stabilized by any continuously differentiable time-invariant control law. Instead, various discontinuous and/or time-varying control laws have been developed. The stabilization of nonholonomic systems has been well studied in the literature [8–12].

The tracking problem for nonholonomic systems was then further investigated [13–20]. The early work [13] solved the local tracking problem of a mobile robot based on a linearized error system. Later, local and global tracking problems were considered in [14, 15], where Ref. [14] proposed a time-varying state feedback control law using the backstepping technique for a dynamic mobile robot and Ref. [15] proposed a recursive control design for a nonholonomic system in chained form. Similar to [15], Ref. [16] also studied the tracking problem for a class of nonholonomic systems in chained form and achieved the global $\mathcal{K}$-exponential stability of the closed-loop dynamics, where, by using the cascaded method, the design process of the state and output feedback tracking control laws with time-varying gains is much simplified. By combining the cascaded method and the backstepping technique, Ref. [17] solved the tracking problem for a class of Lagrange mechanical systems with nonholonomic constraints. Ref. [18] developed a strict Lyapunov function for a class of force-controlled autonomous vehicles, ensuring uniform convergence of the tracking errors. Moreover, the cooperative control problems for multiple nonholonomic systems were considered in [19, 20].

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The event-triggered method effectively reduces system computation capability and communication bandwidth requirements while maintaining satisfactory control performance. Some results on an event-triggered control strategy for nonholonomic systems have recently been reported [21–28]. In [21], a systematic design framework, including state and output feedback control, was developed to handle the robust event-triggered stabilization problem for a class of nonholonomic systems in chained form subject to disturbances and drift uncertainty nonlinearities. The event-triggered tracking problem for nonholonomic systems was solved in [22–24]. Specifically, Refs. [22,23] adopted an emulation-like approach and model predictive control, respectively. In [24], event-triggered and self-triggered tracking controls were investigated using a modified dynamic feedback linearization technique. Refs. [25–28] focused on the event-triggered cooperative control of multiple mobile robots. In particular, Ref. [25, 26] considered the consensus problem of the hand position centroid, orientation, and moving speed. Later, Refs. [27, 28] further studied the formation tracking control problem featuring the distributed observer approach and relative measurement feedback, respectively. Note that Refs. [22, 23] only achieved practical tracking, i.e., the tracking error could only be driven to a small residual set near the origin. Although asymptotic tracking was successfully obtained in [24], the convergence is not global in the sense that the linear velocity of the mobile robot must be avoided locating in zero.

In this study, we propose a novel event-triggered tracking control approach for a class of nonholonomic systems in the so-called chained form. As shown in [5], given a nonholonomic system, it is often possible to convert it into the chained form by using a coordinate transformation and a control mapping. Indeed, the chained form can not only depict many practical physical systems, such as mobile and hopping robots but also characterize the fundamental difficulties of nonholonomic systems. In contrast to the existing methods, the proposed control approach can achieve global asymptotic tracking; i.e., the tracking error can be driven to the origin globally asymptotically and locally exponentially. The main technical contributions of this paper are two-fold.

• For time-varying reference input, the digital sample-and-hold control inevitably suffers from the so-called ripple phenomenon [29] and thus can only achieve practical tracking. In this study, a novel time-varying event-triggered piecewise continuous control law and a triggering mechanism with a time-varying triggering function are proposed based on the sampled feedback signal. Because the time-varying reference input can be compensated exactly, the proposed control law can achieve global asymptotic tracking for a time-varying reference trajectory.

• A novel integral input-to-state stable (iISS) Lyapunov function for the nonholonomic system is constructed regarding the sampling error as the external input. This iISS-Lyapunov function, together with an exponentially decaying threshold signal, can guarantee the forward completeness of the solution of the whole closed-loop system by viewing its cascaded structure, hence ruling out the Zeno behavior in finite time. Furthermore, as long as the local exponential convergence rate of the threshold signal is set not faster than that of the state of the closed-loop system, the functions in the supply pair of this iISS-Lyapunov function are locally linear, facilitating us to avoid infinitely fast sampling.

The remainder of this paper is organized as follows: Section 2 gives the problem formulation. Section 3 provides the main results. A simulation example is given in Section 4. Section 5 concludes this paper. A preliminary version of this paper was reported in [30]. Compared with [30], all proofs of the main theorem and lemmas herein have been exhibited and reorganized.

Notation. Let $\mathbb{R}_{\geqslant 0} = [0, \infty)$ and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. $\|\cdot\|$ denotes the Euclidean norm, and $|\cdot|$ denotes the sum of the absolute values of all elements. For any rational number a, $\lfloor a \rfloor$ defines the maximum integer not larger than a.

2 Problem statement

Consider a class of nonholonomic systems in the chained form of order three:

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_1 x_3,\ \dot{x}_3 = u_2, \tag{1}$$

where $x = [x_1, x_2, x_3]^{\mathrm{T}} \in \mathbb{R}^3$ is the state and $u = [u_1, u_2]^{\mathrm{T}} \in \mathbb{R}^2$ is the control input. The reference trajectory is assumed to be generated by the following system:

$$\dot{x}_{1r} = u_{1r},\ \dot{x}_{2r} = u_{1r} x_{3r},\ \dot{x}_{3r} = u_{2r}, \tag{2}$$

where $x_r = [x_{1r}, x_{2r}, x_{3r}]^{\mathrm{T}} \in \mathbb{R}^3$ is the desired state and $u_r = [u_{1r}, u_{2r}]^{\mathrm{T}} \in \mathbb{R}^2$ is the desired input. Let the tracking error be $p_x = x - x_r$. The error dynamics is obtained by

$$\dot{p}_{x_1} = u_1 - u_{1r},\ \dot{p}_{x_2} = u_1 p_{x_3} + (u_1 - u_{1r})x_{3r},\ \dot{p}_{x_3} = u_2 - u_{2r}, \tag{3}$$

where $p_x = [p_{x_1}, p_{x_2}, p_{x_3}]^{\mathrm{T}} \in \mathbb{R}^3$. Motivated by [31], we adopt the following coordinate transformation:

$$\begin{bmatrix} x_{1e} \\ x_{2e} \\ x_{3e} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -x_{3r} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{x_1} \\ p_{x_2} \\ p_{x_3} \end{bmatrix} \tag{4}$$

with $x_e = [x_{1e}, x_{2e}, x_{3e}]^{\mathrm{T}} \in \mathbb{R}^3$. Then, system (3) is equivalent to

$$\dot{x}_{1e} = u_1 - u_{1r},\ \dot{x}_{2e} = u_1 x_{3e} - u_{2r} x_{1e},\ \dot{x}_{3e} = u_2 - u_{2r}. \tag{5}$$

In this study, we consider an event-triggered control law of the following form:

$$u_1(t) = \bar{f}_1(t, x_e(t_k)),\ u_2(t) = \bar{f}_2(t, x_e(t_k)),\ t \in [t_k, t_{k+1}),\ k \in \mathbb{S} \subseteq \mathbb{Z}_+, \tag{6}$$

where $\bar{f}_1$ and $\bar{f}_2$ are some nonlinear functions to be specified later. The time sequence $\{t_k\}_{k\in\mathbb{S}}$ represents the sampling time instants with $\mathbb{S} \subseteq \mathbb{Z}_+$ being the corresponding index set. More specifically, the sampling time instants are determined by the following triggering mechanism:

$$t_{k+1} = \inf\{t > t_k :\ \nu(t, x_e(t), x_e(t_k)) \geqslant \eta(t)\}, \tag{7}$$

where ν is a nonlinear function to be determined later, and $\eta(t)$ is a positive threshold signal satisfying

$$\dot{\eta}(t) = -\bar{c}\eta(t),\ \eta(t_0) = \eta_0 > 0 \tag{8}$$

with some positive constant $\bar{c}$ also to be determined later. Then, the event-triggered tracking control problem for systems (1) and (2) is described as follows.

Problem 1. Given systems (1) and (2), find an event-triggered control law of the form (6) incorporated by a triggering mechanism of the form (7), such that, given any $\eta_0 > 0$, for all initial state $x(t_0)$ with any $t_0 \geqslant 0$, the following properties hold: (i) The state $x(t)$ of the closed-loop system composed of (1), (6), and (8) exists over $[t_0, \infty)$, and satisfies $\lim_{t\to\infty} \|x_e(t)\| = 0$. (ii) The inter-event times are lower bounded by a positive number.

Remark 1. From (4),

$$\begin{bmatrix} p_{x_1} \\ p_{x_2} \\ p_{x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x_{3r} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1e} \\ x_{2e} \\ x_{3e} \end{bmatrix}.$$

It can be concluded that $\lim_{t\to\infty} \|x_e(t)\| = 0 \Leftrightarrow \lim_{t\to\infty} \|p_x(t)\| = 0$, whenever x_{3r} is bounded.

Remark 2. The existing event-triggered tracking control for nonholonomic systems (see [22, 23]) can only achieve practical tracking in the sense that the tracking error can only be driven to a small residual set near the origin. In contrast, in this paper, we consider the asymptotic tracking problem by the event-triggered tracking control as described by Problem 1.

3 Main results

Let us specify in detail the synthesis of the event-triggered controller as well as the triggering mechanism.

3.1 Event-triggered control synthesis

Two standing assumptions are given as follows.

Assumption 1. The reference inputs and their derivatives are all bounded signals; i.e., there exist positive constants $b_1, b_2, \bar{b}_1, \bar{b}_2 > 0$, such that $|u_{1r}(t)| \leqslant b_1,\ |u_{2r}(t)| \leqslant b_2,\ |\dot{u}_{1r}(t)| \leqslant \bar{b}_1,\ |\dot{u}_{2r}(t)| \leqslant \bar{b}_2$. Moreover, x_{3r} is bounded.

Assumption 2. There exist constants $T, \mu > 0$, such that, for all $t \geqslant 0$, $\int_t^{t+T} \left(u_{1r}^2(s) + u_{2r}^2(s)\right) \mathrm{d}s \geqslant \mu$.

Remark 3. The boundedness assumption of the reference inputs and their derivatives is common in the literature (see [31] and references therein). In many cases, such an assumption would in turn imply the boundedness of x_{3r} (see the hopping robot in Section 4). Assumption 2 is the standard persistently exciting (PE) condition [31]. Some other variants of the PE condition can be found in [16, 22].

Let the functions $\bar{f}_1$ and $\bar{f}_2$ in (6) satisfy

$$\bar{f}_1(t, x_e(t)) = u_{1r}(t) - k_1 x_{1e}(t) + k_2 u_{2r}(t) x_{2e}(t), \tag{9a}$$
$$\bar{f}_2(t, x_e(t)) = u_{2r}(t) - k_3 x_{3e}(t) - k_2(u_{1r}(t) - k_1 x_{1e}(t) + k_2 u_{2r}(t) x_{2e}(t)) x_{2e}(t), \tag{9b}$$

and

$$\bar{f}_1(t, x_e(t_k)) = u_{1r}(t) - k_1 x_{1e}(t_k) + k_2 u_{2r}(t) x_{2e}(t_k), \tag{10a}$$
$$\bar{f}_2(t, x_e(t_k)) = u_{2r}(t) - k_3 x_{3e}(t_k) - k_2(u_{1r}(t) - k_1 x_{1e}(t_k) + k_2 u_{2r}(t) x_{2e}(t_k)) x_{2e}(t_k), \tag{10b}$$

where k_1, k_2, and k_3 are any positive real numbers. Then, we define

$$\delta_i(t, x_e(t), x_e(t_k)) \triangleq \bar{f}_i(t, x_e(t_k)) - \bar{f}_i(t, x_e(t)),\ i = 1, 2,$$

which can be rewritten as

$$\delta_1(t, x_e(t), x_e(t_k)) = -k_1(x_{1e}(t_k) - x_{1e}(t)) + k_2 u_{2r}(t)(x_{2e}(t_k) - x_{2e}(t)), \tag{11a}$$
$$\begin{aligned}\delta_2(t, x_e(t), x_e(t_k)) = &-k_3(x_{3e}(t_k) - x_{3e}(t)) - k_2 u_{1r}(t)(x_{2e}(t_k) - x_{2e}(t)) \\ &+ k_1 k_2(x_{1e}(t_k) x_{2e}(t_k) - x_{1e}(t) x_{2e}(t)) - k_2^2 u_{2r}(t)\left(x_{2e}^2(t_k) - x_{2e}^2(t)\right).\end{aligned} \tag{11b}$$

Correspondingly, the function ν in the triggering mechanism (7) is defined as

$$\nu(t, x_e(t), x_e(t_k)) = |\delta_1(t, x_e(t), x_e(t_k))| + |\delta_2(t, x_e(t), x_e(t_k))|. \tag{12}$$

Our main theorem is given as follows.

Theorem 1. Given system (5), under Assumptions 1 and 2, there exists $\bar{c} > 0$ such that Problem 1 can be solved by control law (6) together with the triggering mechanism (7), where functions $\bar{f}_1$, $\bar{f}_2$, and ν are specified by (10a), (10b), and (12), respectively.

Remark 4. For the case where $u_r(t)$ is not constant, using purely digital input signal cannot compensate the reference input $u_r(t)$ exactly. In such case, to rule out infinite fast sampling, only practical convergence of the tracking error can be achieved, as in [22, 32]. In contrast, the piecewise continuous control law (6) makes it possible to compensate the time-varying reference input $u_r(t)$ precisely, which thus permits asymptotic convergence of the tracking error.

3.2 Proof of Theorem 1

For simplicity, let $\delta(t) \triangleq [\delta_1(t, x_e(t), x_e(t_k)), \delta_2(t, x_e(t), x_e(t_k))]^{\mathrm{T}}$. Combining error system (5) and controller (6), the dynamics of x_e can be written as

$$\dot{x}_e = A_{0r}(t, x_e) x_e + B(x_e)\delta(t), \tag{13}$$

where

$$A_{0r}(t, x_e) = \begin{bmatrix} -k_1 & k_2 u_{2r}(t) & 0 \\ -u_{2r}(t) & 0 & \bar{f}_1(t, x_e) \\ 0 & -k_2 \bar{f}_1(t, x_e) & -k_3 \end{bmatrix}, \quad B(x_e) = \begin{bmatrix} 1 & 0 \\ x_{3e} & 0 \\ 0 & 1 \end{bmatrix}.$$

Notice that system (13) can be viewed as a perturbed system with a nominal part

$$\dot{x}_e = A_{0r}(t, x_e)x_e \tag{14}$$

and a perturbation $B(x_e)\delta(t)$. Suppose that the solution of the closed-loop system (13) is maximally defined over $t \in [t_0, T_M)$ with $0 \leqslant t_0 < T_M \leqslant \infty$. As in [33], the following three cases may happen:

(a) $\mathbb{S} = \mathbb{Z}_+$ and $\lim_{k\to\infty} t_k < \infty$.

(b) $\mathbb{S} = \mathbb{Z}_+$ and $\lim_{k\to\infty} t_k = \infty$.

(c) $\mathbb{S}$ is a finite set $\{0, 1, 2, \ldots, k^*\}$ with $k^* \in \mathbb{Z}_+$.

Here case (a) means the so-called Zeno behavior that has to be ruled out. Cases (b) and (c) mean the solution of the closed-loop system is forward complete. The behavior of infinitely fast sampling has also to be avoided whenever case (b) happens.

In the remaining of this section, the proof of Theorem 1 is divided into the following four steps.

Step-1: Construct an explicit strict Lyapunov function for the unperturbed nominal system (14) (see Lemma 1 with its proof given in Appendix A);

Step-2: Find an iISS-Lyapunov function for the perturbed system (13) by regarding $\delta(t)$ as the external input (see Lemma 2);

Step-3: Prove the global uniform asymptotic stability of the closed-loop system composed of (13) and triggering mechanism (7) with dynamics of threshold (8) (see Lemma 3);

Step-4: Show that the Zeno behavior and infinitely fast sampling can be avoided by tuning down the convergence rate of the threshold signal (see Lemma 4).

Combining these four steps together, we can conclude that Theorem 1 is true. Hence, the event-based tracking problem for systems (1) and (2) is solved.

Step-1: Chapter 6 in [34] has provided a systematic framework for constructing the time-varying strict Lyapunov function from weak Lyapunov function together with PE signals. Refs. [18, 35] have conducted this process for the wheeled mobile robot with some useful refinements. The construction of the strict Lyapunov function here (see Lemma 1) for the unperturbed nominal system (14) follows the same process as that in [34], meanwhile, jointing with those refinements suggested in [18, 35].

Lemma 1. Consider the nominal system (14). Suppose that Assumptions 1 and 2 hold. Then, there exist linear function $\phi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ and quadratic polynomial function $\rho : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ both with all positive coefficients, such that

$$\begin{aligned} V_2(t, x_e) = {} & \rho(V_1(x_e))V_1(x_e) + \varphi(t)V_1(x_e) + u_{1r}x_{2e}x_{3e} \\ & - u_{2r}x_{1e}x_{2e}\phi\left(V_1(x_e)\right) + \frac{V_1(x_e)}{\min\{1, k_2\}}\left(b_1 + b_2\phi\left(V_1(x_e)\right)\right) \end{aligned} \tag{15}$$

is a uniformly proper positive definite function satisfying

$$V_2(t, x_e) \geqslant \rho\left(V_1(x_e)\right)V_1(x_e) + V_1(x_e), \tag{16}$$

$$V_2(t, x_e) \leqslant \rho\left(V_1(x_e)\right)V_1(x_e) + (1 + T\bar{p})V_1(x_e) + \frac{2}{\min\{1, k_2\}}V_1(x_e)(b_1 + b_2\phi(V_1(x_e))), \tag{17}$$

and

$$\dot{V}_2 \leqslant -\frac{\mu}{T}V_1, \tag{18}$$

where (b_1, b_2) and (μ, T) are given in Assumptions 1 and 2, respectively, k_2 is a positive real number given in (9), $\bar{p}$ denotes the upper bound of $u_{1r}^2(t) + u_{2r}^2(t)$,

$$\varphi(t) = 1 + \frac{2}{T}\int_{t-T}^{t}\int_{s}^{t}\left(u_{1r}^2(m) + u_{2r}^2(m)\right)\mathrm{d}m\mathrm{d}s \text{ and } V_1(x_e) = \frac{1}{2}\left(x_{1e}^2 + k_2x_{2e}^2 + x_{3e}^2\right). \tag{19}$$

Step-2: The strict Lyapunov function V_2 lays a foundation on constructing iISS-Lyapunov function for system (13) (see Lemma 2). From (19), with $m_1 \triangleq (1/2)\min\{1, k_2\}$ and $m_2 \triangleq (1/2)\max\{1, k_2\}$,

$$m_1\|x_e\|^2 \leqslant V_1(x_e) \leqslant m_2\|x_e\|^2. \tag{20}$$

Then from (16) and (17),

$$\|x_e\|^2(\tilde{m}_1\|x_e\|^4 + \tilde{m}_2\|x_e\|^2 + \tilde{m}_3) \leqslant V_2(t,x_e) \leqslant \|x_e\|^2(\bar{m}_1\|x_e\|^4 + \bar{m}_2\|x_e\|^2 + \bar{m}_3), \tag{21}$$

where $\bar{m}_i$ and $\tilde{m}_i, i = 1,2,3$, are appropriate positive real numbers.

Lemma 2. Consider system (13). Suppose that Assumptions 1 and 2 hold. Let

$$W(t,x_e) = \ln(1 + \sqrt{V_2(t,x_e)}), \tag{22}$$

where $V_2(t,x_e)$ is defined in (15). Then, $W(t,x_e)$ is a uniformly proper positive definite function satisfying

$$W(t,x_e) \geqslant \underline{\alpha}(\|x_e\|) \triangleq \ln(1 + \|x_e\|\sqrt{\tilde{m}_1\|x_e\|^4 + \tilde{m}_2\|x_e\|^2 + \tilde{m}_3}), \tag{23}$$

$$W(t,x_e) \leqslant \overline{\alpha}(\|x_e\|) \triangleq \ln(1 + \|x_e\|\sqrt{\bar{m}_1\|x_e\|^4 + \bar{m}_2\|x_e\|^2 + \bar{m}_3}), \tag{24}$$

and

$$\dot{W}(t,x_e) \leqslant -\alpha(\|x_e\|) + c|\delta|, \ \forall \|x_e\| > 0, \tag{25}$$

for some $c > 0$ and some positive definite function

$$\alpha(s) = \frac{\mu m_1 s}{2T\sqrt{\bar{m}_1 s^4 + \bar{m}_2 s^2 + \bar{m}_3}(1 + s\sqrt{\bar{m}_1 s^4 + \bar{m}_2 s^2 + \bar{m}_3})}, \ s \geqslant 0, \tag{26}$$

where m_1, $\bar{m}_i$, and $\tilde{m}_i, i = 1,2,3$ are given the same as those in (20) and (21).

Proof. Since $\ln(1+s)$ is strictly increasing for $s \geqslant 0$, Eqs. (23) and (24) follow (16) and (17), respectively. So $W(t,x_e)$ is also a uniformly proper positive definite function. From (18) and (21), the time derivative of $W(t,x_e)$ along the trajectory of system (13) satisfies

$$\begin{aligned}\dot{W}(t,x_e) &= -\frac{\dot{V}_1(x_e)}{2\sqrt{V_2(t,x_e)}(1 + \sqrt{V_2(t,x_e)})} \\ &\leqslant -\alpha(\|x_e\|) + \frac{\partial V_2}{\partial x_e}\frac{B(x_e)\delta}{2\sqrt{V_2(t,x_e)}(1 + \sqrt{V_2(t,x_e)})}, \ \forall \|x_e\| > 0.\end{aligned} \tag{27}$$

By (27), it remains to show the second term on the right-hand side of (27) is upper bounded by $c|\delta|$ for some $c > 0$. From (15), let $V_2(t,x_e) = \overline{V}_2(t,x_e,V_1)$. Then,

$$\frac{\partial V_2}{\partial x_e}B(x_e)\delta = \frac{\partial \overline{V}_2}{\partial V_1}\frac{\partial V_1}{\partial x_e}B(x_e)\delta + \frac{\partial \overline{V}_2}{\partial x_e}B(x_e)\delta.$$

Again from (15),

$$\frac{\partial \overline{V}_2}{\partial V_1} = \frac{\mathrm{d}\rho(V_1)}{\mathrm{d}V_1}V_1 + \rho(V_1) + \varphi(t) + \frac{b_1 + b_2\phi(V_1)}{\min\{1,k_2\}} + \frac{b_2}{\min\{1,k_2\}}\frac{\mathrm{d}\phi(V_1)}{\mathrm{d}V_1}V_1 - u_{2r}x_{1e}x_{2e}\frac{\mathrm{d}\phi(V_1)}{\mathrm{d}V_1}. \tag{28}$$

Since $\rho(V_1)$ is quadratic and $\phi(V_1)$ is linear, Eq. (28) implies

$$\left|\frac{\partial \overline{V}_2}{\partial V_1}\right| \leqslant \Gamma_1(\sqrt{V_1}), \tag{29}$$

where $\Gamma_1(\sqrt{V_1})$ is a quartic polynomial function of $\sqrt{V_1}$ with positive coefficients. On the other hand, from the definition of $V_1(x_e)$,

$$\begin{aligned}\left|\frac{\partial V_1}{\partial x_e}B(x_e)\delta\right| &\leqslant \left|\frac{\partial V_1}{\partial x_e}B(x_e)\right||\delta| \leqslant |(x_{1e} + k_2x_{2e}x_{3e}, x_{3e})||\delta| \\ &\leqslant (|x_{1e}| + |x_{3e}| + |k_2x_{2e}x_{3e}|)|\delta| \leqslant (\max\{k_2,1\}V_1 + 2\sqrt{V_1})|\delta| \leqslant \Gamma_2(\sqrt{V_1})|\delta|,\end{aligned} \tag{30}$$

where $\Gamma_2(\sqrt{V_1})$ is a quadratic polynomial function of $\sqrt{V_1}$ with positive coefficients. Similarly, from (15),

$$\left|\frac{\partial \overline{V}_2}{\partial x_e}B(x_e)\delta\right| \leqslant \left|\frac{\partial \overline{V}_2}{\partial x_e}B(x_e)\right||\delta| \leqslant \left|\left(-(x_{2e} + x_{1e}x_{3e})u_{2r}\phi(V_1) + u_{1r}x_{3e}^2, u_{1r}x_{2e}\right)\right||\delta|$$

$$\leqslant \left((b_2\phi(V_1)+b_1)|x_{2e}|+b_2|x_{1e}x_{3e}|\phi(V_1)+b_1x_{3e}^2\right)|\delta|$$
$$\leqslant \left((b_2\phi(V_1)+b_1)\sqrt{\frac{2V_1}{k_2}}+b_2\phi(V_1)V_1+2b_1V_1\right)|\delta| \leqslant \Gamma_3(\sqrt{V_1})|\delta|, \quad (31)$$

where $\Gamma_3(\sqrt{V_1})$ is a quartic polynomial function of $\sqrt{V_1}$ with positive coefficients. Combining (29), (30) and (31) together, we know

$$\left|\frac{\partial V_2}{\partial x_e}B(x_e)\delta\right| \leqslant \Gamma_4(\sqrt{V_1})|\delta|, \quad (32)$$

where $\Gamma_4(\sqrt{V_1})$ is a sixth degree polynomial function of $\sqrt{V_1}$ with positive coefficients. On the other hand, by (16), we know

$$2\sqrt{V_2}(1+\sqrt{V_2}) \geqslant P_1(\sqrt{V_1}), \quad (33)$$

where $P_1(\sqrt{V_1})$ is also a sixth degree polynomial function of $\sqrt{V_1}$ with positive coefficients. Consequently, combining (32) and (33), we have

$$\left|\frac{\partial V_2}{\partial x_e}\frac{B(x_e)\delta}{2\sqrt{V_2}(1+\sqrt{V_2})}\right| \leqslant \frac{\Gamma_4(V_1)|\delta|}{2\sqrt{V_2}(1+\sqrt{V_2})} \leqslant \frac{\Gamma_4(\sqrt{V_1})}{P_1(\sqrt{V_1})}|\delta|.$$

Notice that $\frac{\Gamma_4(s)}{P_1(s)}$ is positive and continuous over $s \in [0,\infty)$. Since $\Gamma_4(s)$ and $P_1(s)$ have the same degree, there exists a positive real number $h > 0$, such that $\lim_{s\to\infty}\frac{\Gamma_4(s)}{P_1(s)} = h > 0$, which in turn implies that $\sup_{s\geqslant 0}\frac{\Gamma_4(s)}{P_1(s)} < \infty$. Thus, Eq. (25) holds with $c = \sup_{s\geqslant 0}\frac{\Gamma_4(s)}{P_1(s)}$.

Step-3: We show that, under Assumptions 1 and 2, the triggering times within any finite time interval are finite. Consequently, case (a) can be ruled out, and the state $[x_e^{\mathrm{T}},\eta]^{\mathrm{T}}$ of the closed-loop system exists over $[t_0,\infty)$, i.e., $T_M = \infty$. This together with a closed-loop strict Lyapunov function implies that the origin of the closed-loop system is uniformly globally asymptotically stable (see Lemma 3).

Lemma 3. Under Assumptions 1 and 2, consider the closed-loop system composed of (13) and triggering mechanism (7) with dynamics of threshold (8). Given any $t_0 < T \leqslant T_M$, there exists $\tau_1 > 0$ such that $\inf_{t_k,t_{k+1}\in[t_0,T)}(t_{k+1}-t_k) \geqslant \tau_1$, and hence, the state $[x_e^{\mathrm{T}},\eta]^{\mathrm{T}}$ of the closed-loop system exists over $[t_0,\infty)$. Moreover, the origin of the whole closed-loop system is uniformly globally asymptotically stable.

Proof. Let $\overline{U}(t,x_e,\eta) = W(t,x_e)+U(\eta)$, where $W(t,x_e)$ is defined in (22) and $U(\eta) = ((c+1)/\bar{c})\eta$. Recall the definition of $\delta(t)$. By (7), $|\delta(t)| = |\delta_1(t,x_e(t),x_e(t_k))| + |\delta_2(t,x_e(t),x_e(t_k))| \leqslant \eta(t)$ holds over $t \in [t_k,t_{k+1})\cap[t_0,T_M)$, $k \in \mathbb{S}$. Then, using (25), the derivative of $\overline{U}(t,x_e,\eta)$ along the trajectory of the closed-loop system satisfies

$$\dot{\overline{U}}(t,x_e,\eta) \leqslant -\alpha(\|x_e\|)+c|\delta|-(1+c)\eta \leqslant -\alpha(\|x_e\|)+c\eta-(1+c)\eta$$
$$\leqslant -\alpha(\|x_e\|)-\eta, \quad \forall t\in[t_0,T_M) \text{ and } \forall\|x_e\|>0, \quad (34)$$

which implies $\overline{U}(t,x_e(t),\eta(t)) \leqslant \overline{U}(t_0,x_e(t_0),\eta(t_0))$ for all $t \in [t_0,T_M)$. That means $\|[x_e^{\mathrm{T}}(t),\eta(t)]^{\mathrm{T}}\| \leqslant \Delta_e$ for all $t \in [t_0,T_M)$ and some $\Delta_e > 0$. Then, by (13), there exists $\Delta_f > 0$, such that

$$\|A_{0r}(t,x_e(t))x_e(t)+B(x_e(t))\delta(t)\| \leqslant \Delta_f, \ \forall t\in[t_0,T_M). \quad (35)$$

Meanwhile, it can also be obtained that

$$\left|x_{2e}^2(t_k)-x_{2e}^2(t)\right| \leqslant 2\Delta_e|x_{2e}(t_k)-x_{2e}(t)|, \ \forall t\in[t_k,t_{k+1})\cap[t_0,T_M) \quad (36)$$

and

$$|x_{1e}(t_k)x_{2e}(t_k)-x_{1e}(t)x_{2e}(t)| \leqslant |x_{1e}(t_k)||x_{2e}(t_k)-x_{2e}(t)|+|x_{2e}(t)||x_{1e}(t_k)-x_{1e}(t)|$$
$$\leqslant \Delta_e(|x_{1e}(t_k)-x_{1e}(t)|+|x_{2e}(t_k)-x_{2e}(t)|), \quad \forall t\in[t_k,t_{k+1})\cap[t_0,T_M). \quad (37)$$

Recall the definition of $\delta(t)$ again. Combining (11), (36), and (37),

$$|\delta(t)| \leqslant k_1|x_{1e}(t_k)-x_{1e}(t)|+k_2b_2|x_{2e}(t_k)-x_{2e}(t)|+k_3|x_{3e}(t_k)-x_{3e}(t)|$$

$$+ k_2b_1|x_{2e}(t_k) - x_{2e}(t)| + k_1k_2|x_{1e}(t_k)x_{2e}(t_k) - x_{1e}(t)x_{2e}(t)| + k_2^2b_2|x_{2e}^2(t_k) - x_{2e}^2(t)|$$
$$\leqslant \Delta_s\|x_e(t_k) - x_e(t)\|,\ \forall t \in [t_k, t_{k+1}) \cap [t_0, T_M), \tag{38}$$

where $\Delta_s = \sqrt{3}\max\{k_1(1 + k_2\Delta_e), k_2((b_1 + b_2) + 2k_2b_2\Delta_e + k_1\Delta_e), k_3\}$. Then, from (13), (35), and (38),

$$|\delta(t_{k+1}^-)| \leqslant \Delta_s\|x_e(t_k) - x_e(t_{k+1})\| \leqslant \Delta_s \int_{t_k}^{t_{k+1}} \|A_{0r}(s, x_e(s))x_e(s) + B(x_e(s))\delta(s)\|\mathrm{d}s$$
$$\leqslant \Delta_s\Delta_f(t_{k+1} - t_k).$$

Given any $0 < T \leqslant T_M$, whenever $[t_k, t_{k+1}) \subseteq [0, T)$, by (8), we have $\eta(t_{k+1}) = \mathrm{e}^{-\bar{c}(t_{k+1}-t_k)}\eta(t_k)$. Notice that the triggering mechanism (7) implies that $|\delta(t)| \leqslant \eta(t)$ for all $t \in [t_k, t_{k+1})$ and $|\delta(t_{k+1}^-)| = \eta(t_{k+1}^-)$. Then, $\mathrm{e}^{-\bar{c}(t_{k+1}-t_k)}\eta(t_k) = |\delta(t_{k+1}^-)| \leqslant \Delta_s\Delta_f(t_{k+1}-t_k)$, which in turn implies $(t_{k+1}-t_k)\mathrm{e}^{\bar{c}(t_{k+1}-t_k)} \geqslant \frac{\eta(t_k)}{\Delta_s\Delta_f} \geqslant \frac{\eta(T)}{\Delta_s\Delta_f}$. Let τ_1 be the solution of $\tau_1\mathrm{e}^{\bar{c}\tau_1} = \frac{\eta(T)}{\Delta_s\Delta_f}$. Notice that $s\mathrm{e}^{\bar{c}s}$ is an increasing function in $s \in [0, \infty)$ and $s\mathrm{e}^{\bar{c}s} = 0$ when $s = 0$. Since $\frac{\eta(T)}{\Delta_s\Delta_f} > 0$, we have $\tau_1 > 0$. Then, it follows that $t_{k+1} - t_k \geqslant \tau_1 > 0$.

Now, we claim $T_M = \infty$ by the following two parts. At first, consider $\mathbb{S} = \mathbb{Z}_+$. In this case, we claim that $T_M = \infty$ by contradiction. Assume case (a) occurs, i.e., $\mathbb{S} = \mathbb{Z}_+$ and $T_M < \infty$. From the above facts, the number of the triggering times during the time interval $[0, T_M)$ is finite (no more than $\lfloor \frac{T_M - t_0}{\tau_1} \rfloor + 1$) whenever $T_M < \infty$, which leads to a contradiction of $\mathbb{S} = \mathbb{Z}_+$. Thus, we obtain that case (a) will not happen; i.e., $T_M = \infty$ when $\mathbb{S} = \mathbb{Z}_+$. Next, consider $\mathbb{S}$ is a finite set; i.e., case (c) occurs. In this case, there exists a time t_{k^*}, $0 < t_{k^*} < \infty$, such that the controller is always continuous without sampling after t_{k^*}. Then, by the continuation of the solution, we have $T_M = \infty$. Thus, no mater whether $\mathbb{S} = \mathbb{Z}_+$ or $\mathbb{S}$ is a finite set, we can always obtain that $T_M = \infty$. Finally, from (34), the origin of the whole closed-loop system is uniformly globally asymptotically stable by proposition 4.2 of [36].

Step-4: The previous step indeed has ruled out the Zeno behavior in any finite time, meaning that for any $T^* > 0$, there exists $\tau_1 > 0$ (depending on T^*) such that $\inf_{t_k, t_{k+1} \in [t_0, T^*)}(t_{k+1} - t_k) \geqslant \tau_1$. We then show that there exists a specified $T^* > 0$ such that x_e has a comparable local exponential convergence rate as that of η. This fact guarantees that there exists $\tau_2 > 0$ (depending on T^*) such that $\inf_{t_k, t_{k+1} \in [T^*, \infty)}(t_{k+1} - t_k) \geqslant \tau_2$, which further rules out the behavior of infinitely fast sampling whenever case (b) happens. So the infimum of the inter-event times over all $t \geqslant 0$ is lower bounded by a positive constant $\min\{\tau_1, \tau_2\}$ (Lemma 4).

Lemma 4. Under Assumptions 1 and 2, consider the closed-loop system composed of (13) and event-triggering mechanism (7) with dynamics of threshold (8). Then, the inter-event times are lower bounded by a positive constant.

Proof. When case (c) happens, the number of the triggering times is finite and $T_M = \infty$ by Lemma 3. Hence, the conclusion holds automatically. In what follows, we focus only on case (b), i.e., $\mathbb{S} = \mathbb{Z}_+$.

Given any $0 < \Delta_1 \leqslant 1/\sqrt{\tilde{m}_3}$, for any $0 \leqslant s \leqslant \Delta_1$,

$$\overline{\alpha}_1(s) \triangleq \sqrt{\bar{m}_1\Delta_1^4 + \bar{m}_2\Delta_1^2 + \bar{m}_3}s \geqslant \sqrt{\bar{m}_1s^4 + \bar{m}_2s^2 + \bar{m}_3}s \geqslant \ln(1 + s\sqrt{\bar{m}_1s^4 + \bar{m}_2s^2 + \bar{m}_3}),$$
$$\underline{\alpha}_1(s) \triangleq \left(\sqrt{\tilde{m}_3}/2\right) s \leqslant \ln(1 + \sqrt{\tilde{m}_3}s),$$
$$\alpha_1(s) \triangleq \frac{\mu m_1 s}{2T\sqrt{\bar{m}_1\Delta_1^4 + \bar{m}_2\Delta_1^2 + \bar{m}_3}(1 + \Delta_1\sqrt{\bar{m}_1\Delta_1^4 + \bar{m}_2\Delta_1^2 + \bar{m}_3})} \leqslant \alpha(s),$$

where $\alpha(s)$ is given in (26), and $\tilde{m}_3$ and $\bar{m}_i, i = 1, 2, 3$ are defined in (20) and (21). Recall that $|\delta| \leqslant \eta$. From (23), (24), and (26),

$$\underline{\alpha}_1(\|x_e\|) \leqslant W(t, x_e) \leqslant \overline{\alpha}_1(\|x_e\|), \tag{39}$$
$$\dot{W}(t, x_e) \leqslant -\alpha_1(\|x_e\|) + c\eta,\ \forall 0 < \|x_e\| \leqslant \Delta_1. \tag{40}$$

This further implies

$$\dot{W}(t, x_e) \leqslant -L_1W(t, x_e) + c\eta,\ \forall 0 < W(t, x_e) \leqslant \overline{\alpha}_1(\Delta_1),$$

where $L_1 = \frac{\mu m_1}{2T(\bar{m}_1\Delta_1^4+\bar{m}_2\Delta_1^2+\bar{m}_3)(1+\Delta_1\sqrt{\bar{m}_1\Delta_1^4+\bar{m}_2\Delta_1^2+\bar{m}_3})}$. Then, for any specified $\sigma \in (0,1)$,

$$\dot{W}(t,x_e) \leqslant -(1-\sigma)L_1 W(t,x_e),\ \forall 0 < \frac{c}{\sigma L_1}\eta \leqslant W(t,x_e) \leqslant \overline{\alpha}_1(\Delta_1). \tag{41}$$

Choose $\overline{W}(t,x_e) = (\sigma L_1/c)W(t,x_e)$ and $\bar{c} \leqslant (1-\sigma)L_1$. Then, the following inequality holds:

$$(\sigma L_1/c)\overline{\alpha}_1(\Delta_1) \geqslant \overline{W}(t,x_e) \geqslant \eta > 0 \implies \dot{\overline{W}}(t,x_e) \leqslant -\bar{c}\overline{W}(t,x_e).$$

From Lemma 3, we have known that the origin of the whole closed-loop system is uniformly globally asymptotically stable. Then, given any initial states and any $0 < \Delta_1 \leqslant 1/\sqrt{\tilde{m}_3}$, there exists $T^* \geqslant t_0$ such that $\|x_e(t)\| \leqslant \|[x_e^{\mathrm{T}}(t), \eta(t)]^{\mathrm{T}}\| \leqslant \Delta_1 \leqslant 1/\sqrt{\tilde{m}_3}$ for all $t \geqslant T^*$. Like [33, 37], both $t_k \leqslant T^*$ and $t_k > T^*$ cases are considered to estimate the lower bound of the inter-event times.

Case 1: $t_k \leqslant T^*$. From Lemma 3, there exists $\tau_1 > 0$ such that $\inf_{t_k,t_{k+1}\in[t_0,T^*)}(t_{k+1}-t_k) \geqslant \tau_1$.

Case 2: $t_k > T^*$. We first claim that there exists $P^* \geqslant 1$ such that $\overline{W}(t,x_e(t)) \leqslant P^*\eta(t)$ for all $t \geqslant T^*$. If $\overline{W}(T^*,x_e(T^*)) \leqslant \eta(T^*)$, then $\overline{W}(t,x_e(t)) \leqslant \eta(t)$ for all $t \geqslant T^*$ by the comparison principle. Else if $\overline{W}(T^*,x_e(T^*)) > \eta(T^*)$, noticing $\eta(t) > 0$, let $P^* = \frac{\overline{W}(T^*,x_e(T^*))}{\eta(T^*)} > 1$. Then, $\overline{W}(T^*,x_e(T^*)) = P^*\eta(T^*)$. By (8), $\dot{\overbrace{P^*\eta(t)}} = -\bar{c}\overbrace{P^*\eta(t)}$. Again from the comparison principle, $\overline{W}(t,x_e(t)) \leqslant P^*\eta(t)$ for all $t \geqslant T^*$. Moreover, by definition of $\overline{W}(t,x_e)$ above and (39),

$$\|x_e(t)\| \leqslant L_2\eta(t),\ \forall t \geqslant T^*, \tag{42}$$

where $L_2 = 2cP^*/(\sigma L_1\sqrt{\tilde{m}_3})$. Recall that $\|x_e(t)\| \leqslant \Delta_1$ for all $t \in [T^*,\infty)$. By the same argument as (36) and (37),

$$|x_{2e}^2(t_k) - x_{2e}^2(t)| \leqslant 2\Delta_1|x_{2e}(t_k) - x_{2e}(t)|,\ \forall t \in [t_k,t_{k+1}) \cap [T^*,\infty) \tag{43}$$

and

$$\begin{aligned} |x_{1e}(t_k)x_{2e}(t_k) - x_{1e}(t)x_{2e}(t)| &\leqslant |x_{1e}(t_k)||x_{2e}(t_k) - x_{2e}(t)| + |x_{2e}(t)||x_{1e}(t_k) - x_{1e}(t)| \\ &\leqslant \Delta_1(|x_{1e}(t_k) - x_{1e}(t)| + |x_{2e}(t_k) - x_{2e}(t)|),\ \forall t \in [t_k,t_{k+1}) \cap [T^*,\infty). \end{aligned} \tag{44}$$

Recall the definition of $\delta(t)$ again. Then, it follows from (11), (43), and (44) that

$$\begin{aligned} |\delta(t)| \leqslant\ & k_1|x_{1e}(t_k) - x_{1e}(t)| + k_2b_2|x_{2e}(t_k) - x_{2e}(t)| + k_3|x_{3e}(t_k) - x_{3e}(t)| \\ & + k_2b_1|x_{2e}(t_k) - x_{2e}(t)| + k_1k_2|x_{1e}(t_k)x_{2e}(t_k) - x_{1e}(t)x_{2e}(t)| + k_2^2b_2|x_{2e}^2(t_k) - x_{2e}^2(t)| \\ \leqslant\ & \varsigma_1|x_{1e}(t_k) - x_{1e}(t)| + \varsigma_2|x_{2e}(t_k) - x_{2e}(t)| + \varsigma_3|x_{3e}(t_k) - x_{3e}(t)|,\ \forall t \in [t_k,t_{k+1}) \cap [T^*,\infty), \end{aligned} \tag{45}$$

where $\varsigma_1 \geqslant k_1(1+k_2\Delta_1)$, $\varsigma_2 \geqslant k_2(b_1+b_2+k_1\Delta_1+2k_2b_2\Delta_1)$, $\varsigma_3 \geqslant k_3$. From (13), for $i = 1,2,3$, we have

$$\begin{aligned} &|\dot{x}_{1e}(t)| \leqslant |k_1x_{1e}(t)| + |k_2u_{2r}(t)x_{2e}(t)| + |\delta_1(t,x_e(t),x_e(t_k))|, \\ &|\dot{x}_{2e}(t)| \leqslant |u_{2r}(t)x_{1e}(t)| + |(u_{1r}(t) - k_1x_{1e}(t) + k_2u_{2r}(t)x_{2e}(t))x_{3e}(t)| + |x_{3e}(t)\delta_1(t,x_e(t),x_e(t_k))|, \\ &|\dot{x}_{3e}(t)| \leqslant |k_3x_{3e}(t)| + |k_2(u_{1r}(t) - k_1x_{1e}(t) + k_2u_{2r}(t)x_{2e}(t))x_{2e}(t)| \\ &\qquad\qquad + |\delta_2(t,x_e(t),x_e(t_k))|,\ \forall t \in [t_k,t_{k+1}) \cap [T^*,\infty). \end{aligned}$$

Then, based on the Assumption 1 and above facts, we further obtain

$$|\dot{x}_{1e}(t)| \leqslant (k_1+k_2b_2)\|x_e(t)\| + \eta(t) \leqslant ((k_1+k_2b_2)L_2+1)\eta(t) \triangleq J_1\eta(t), \tag{46a}$$

$$\begin{aligned} |\dot{x}_{2e}(t)| &\leqslant b_2|x_{1e}(t)| + b_1|x_{3e}(t)| + \frac{k_1}{2}\Delta_1\|x_e(t)\| + \frac{k_2b_2}{2}\Delta_1\|x_e(t)\| + \Delta_1\eta(t) \\ &\leqslant \left(\left(\left(\frac{k_1}{2} + \frac{k_2b_2}{2}\right)\Delta_1 + b_1 + b_2\right)L_2 + \Delta_1\right)\eta(t) \triangleq J_2\eta(t), \end{aligned} \tag{46b}$$

$$\begin{aligned} |\dot{x}_{3e}(t)| &\leqslant k_3|x_{3e}(t)| + k_2b_1|x_{2e}(t)| + \frac{k_1k_2}{2}\Delta_1\|x_e(t)\| + k_2^2b_2\Delta_1\|x_e(t)\| + \eta(t) \\ &\leqslant \left(\left(\left(\frac{k_1k_2}{2} + k_2^2b_2\right)\Delta_1 + k_3 + k_2b_1\right)L_2 + 1\right)\eta(t) \triangleq J_3\eta(t),\ \forall t \in [t_k,t_{k+1}) \cap [T^*,\infty). \end{aligned} \tag{46c}$$

Algorithm 1 The design procedure of triggering mechanism

Require: Assumptions 1 and 2 hold. Moreover, store the values of parameters b_1, b_2, $\bar{b}_1$, $\bar{b}_2$, T, and μ.
1: Construct the strict Lyapunov function $V_2(t, x_e)$ according to Lemma 1;
2: Find the values of parameters m_1, $\bar{m}_i$, and $\tilde{m}_i, i = 1, 2, 3$ by (20) and (21);
3: Calculate $(1 - \sigma)L_1$ by the following definition:

$$(1-\sigma)L_1 = \frac{\mu m_1(1-\sigma)}{2T\left(\bar{m}_1\Delta_1^4 + \bar{m}_2\Delta_1^2 + \bar{m}_3\right)\left(1 + \Delta_1\sqrt{\tilde{m}_1\Delta_1^4 + \tilde{m}_2\Delta_1^2 + \tilde{m}_3}\right)},$$

where $\sigma \in (0, 1)$, $0 < \Delta_1 \leqslant \frac{1}{\sqrt{\tilde{m}_3}}$, and others are as above.
4: Obtain the triggering mechanism by letting $0 < \bar{c} \leqslant (1 - \sigma)L_1$.

By (45) and (46),

$$\begin{aligned}\left|\delta(t_{k+1}^-)\right| &\leqslant \varsigma_1|x_{1e}(t_k) - x_{1e}(t_{k+1})| + \varsigma_2|x_{2e}(t_k) - x_{2e}(t_{k+1})| + \varsigma_3|x_{3e}(t_k) - x_{3e}(t_{k+1})| \\ &\leqslant \int_{t_k}^{t_{k+1}} \left(\varsigma_1|\dot{x}_{1e}(s)| + \varsigma_2|\dot{x}_{2e}(s)| + \varsigma_3|\dot{x}_{3e}(s)|\right) \mathrm{d}s \\ &\leqslant \int_{t_k}^{t_{k+1}} \left(\varsigma_1 J_1 + \varsigma_2 J_2 + \varsigma_3 J_3\right) \eta(s)\mathrm{d}s \leqslant J\eta(t_k)(t_{k+1} - t_k),\end{aligned} \tag{47}$$

where $J \geqslant \varsigma_1 J_1 + \varsigma_2 J_2 + \varsigma_3 J_3$. For any $[t_k, t_{k+1}) \in [T^*, \infty)$, by (8), we know $\eta(t_{k+1}) = \mathrm{e}^{-\bar{c}(t_{k+1}-t_k)}\eta(t_k)$. From the triggering mechanism (7), we obtain $|\delta(t)| \leqslant \eta(t)$ for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{S}$, and $|\delta(t_{k+1}^-)| = \eta(t_{k+1}^-)$. Then, it follows from (47) that $\mathrm{e}^{-\bar{c}(t_{k+1}-t_k)}\eta(t_k) = \left|\delta(t_{k+1}^-)\right| \leqslant J\eta(t_k)(t_{k+1} - t_k)$ which implies $(t_{k+1} - t_k)\mathrm{e}^{\bar{c}(t_{k+1}-t_k)} \geqslant 1/J$. Let τ_2 be the solution of $\tau_2\mathrm{e}^{\bar{c}\tau_2} = 1/J > 0$. Since $s\mathrm{e}^{\bar{c}s}$ is an increasing function for $s \in [0, \infty)$ and $s\mathrm{e}^{\bar{c}s} = 0$ when $s = 0$, it gives that $(t_{k+1} - t_k) \geqslant \tau_2 > 0$.

Finally, by letting $\tau = \min\{\tau_1, \tau_2\}$, we can conclude that $\inf_{t_k, t_{k+1} \in [t_0, \infty)}(t_{k+1} - t_k) \geqslant \tau$, meaning that the inter-event times are lower bounded by a positive constant.

Remark 5. An analysis framework based on an iISS time-invariant system with the sampling error as the external input was recently reported in [37], where the key step is to find a suitable iISS-Lyapunov function together with an appropriate threshold signal for the triggering mechanism. Our design here follows the same process of this framework but considers an iISS time-varying system (13). Consequently, different from the time-invariant case in [37], we need to develop a triggering mechanism with a time-varying triggering function (7) and construct a time-varying iISS-Lyapunov function (22) for the time-varying system (13).

Remark 6. As we know, the triggering mechanism needs to ensure the stability of the closed-loop system on one hand, and to avoid Zeno behavior and infinitely fast sampling on the other hand. In this sense, the value of $\bar{c}$ has to be determined from two aspects accordingly. On one hand, from Lemma 3, every $\bar{c} > 0$ is able to guarantee the uniform global asymptotic stability of the closed-loop system with state $[x_e^{\mathrm{T}}, \eta]^{\mathrm{T}}$. Thus, from a stability point of view, $\bar{c}$ is only required to be a positive real number. On the other hand, from Lemma 4, $(1-\sigma)L_1$ characterizes a lower bound of the convergence rate of the state x_e, after a sufficiently large time T^*. By (7) and (8), $\bar{c}$ indeed shows the decay rate of the threshold function, which has to be not faster than that of the state x_e. Thus, to avoid Zeno behavior and the infinitely fast sampling, it suffices to let $\bar{c} \leqslant (1 - \sigma)L_1$. In conclusion, the value of $\bar{c}$ satisfies $0 < \bar{c} \leqslant (1 - \sigma)L_1$. The detailed design procedure of $\bar{c}$ can be seen in Algorithm 1.

Remark 7. In general, the threshold signal $\eta(t)$ in the triggering mechanism (7) can be generated by any system in the following form:

$$\dot{\eta} = -\Omega(\eta),\ \eta(t_0) = \eta_0 > 0 \tag{48}$$

satisfying the following two conditions.

(1) By (34), to rule out the Zeno behavior in finite time, system (48) should ensure the forward completeness of the solutions as well as the global uniform asymptotic stability of the closed-loop system. Suppose that Ω is positive definite and Lipschitz continuous on compacts. Using $W(t, x_e)$ provided in (22), in view of the cascaded structure of the closed-loop system, these requirements could be met if $\Omega(\eta)$ globally dominates $c|\delta|$ up to some positive scalings.

(2) By (42), to avoid infinitely fast sampling, $\eta(t)$ should dominate the state $x_e(t)$ after a sufficient large time T^* so that the local exponential convergence rate of $\eta(t)$ is not faster than that of $x_e(t)$. By

(39) and (40), $W(t,x_e)$ is locally upper and lower bounded by some linear functions of $\|x_e\|$. Moreover, since $\alpha_1(\cdot)$ is linear, it suffices for $\Omega(\eta)$ to be locally linear.

Obviously, simply letting $\Omega(\eta) = -\bar{c}\eta$ with $\bar{c}$ not too large would meet the above two conditions and that explains the design of the threshold signal generator (8).

Remark 8. Besides (22), there are some other types of iISS-Lyapunov functions, such as the one in [35], which is repeated as follows:

$$W(t,x_e) = \ln(1 + V_2(t,x_e)). \tag{49}$$

By applying (49), Eqs. (39) and (40) still hold but with a quadratic function $\alpha_1(\cdot)$. While, in such a scenario, it is impossible to find any locally linear function $\Omega(\cdot)$ so that $\|x_e\|$ is dominated by a linear function of η. Even though it is still able to rule out case (a), it cannot guarantee (42), hence cannot prevent infinitely fast sampling from happening in case (b).

4 Numerical example

Consider the tracking problem for a hopping robot during the flight phase. From [38], the hopping robot's angular momentum is conserved during the flight phase, where the dynamics can be modeled as

$$I\dot{\theta} + m(\ell+d)^2(\dot{\theta}+\dot{\psi}) = 0, \tag{50}$$

where I, m, and d denote the body moment of inertia, leg mass, and distance from the fulcrum of body to leg, respectively, and ψ, ℓ, and θ represent the leg angle, the leg extension, and the body angle, respectively. Let $q = [\psi, \ell, \theta]^{\mathrm{T}}$. Then Eq. (50) can be rewritten as

$$\dot{q} = \lambda_1 v_1 + \lambda_2 v_2, \tag{51}$$

where $\lambda_1 = [0,1,0]^{\mathrm{T}}$, $\lambda_2 = [1, 0, -\frac{m(\ell+d)^2}{I+m(\ell+d)^2}]^{\mathrm{T}}$, and $[v_1, v_2] = [\dot{\ell}, \dot{\psi}]$. With the state transformation

$$x_1 = \ell,\ x_2 = \psi + \frac{I + m(\ell+d)^2}{m(\ell+d)^2}\theta,\ x_3 = -\frac{2I\theta}{m(\ell+d)^3} \tag{52}$$

and input transformation

$$u_1 = v_1,\ u_2 = -\frac{3x_3}{x_1+d}v_1 + \frac{2I}{(I+m(x_1+d)^2)(x_1+d)}v_2, \tag{53}$$

Eq. (51) can be converted to the chained form (1). The corresponding reference system is defined as

$$\dot{q}_r = \lambda_{1r} v_{1r} + \lambda_{2r} v_{2r}. \tag{54}$$

Again with applying the transformations similar to (52) and (53), Eq. (54) is equivalent to (2). Note that the state $[x_1, x_3]^{\mathrm{T}}$ is contained in (53). By means of piecewise integration, it will be replaced by a sampling state and some continuous-time function. The reference trajectory $[x_{1r}, x_{3r}]^{\mathrm{T}}$ is also processed in the same way. For $\forall t \in [t_k, t_{k+1})$, $k \in \mathbb{S}$, we have

$$\begin{aligned}
&x_{1r}(t) = x_{1r}(t_k) + \int_{t_k}^{t} u_{1r}(s)\mathrm{d}s,\ x_{3r}(t) = x_{3r}(t_k) + \int_{t_k}^{t} u_{2r}(s)\mathrm{d}s,\\
&x_1(t) = x_1(t_k) + \int_{t_k}^{t} \Big(u_{1r}(s) - k_1 x_{1e}(t_k) + k_2 u_{2r}(s) x_{2e}(t_k)\Big)\mathrm{d}s,\\
&x_3(t) = x_3(t_k) + \int_{t_k}^{t} \Big(u_{2r}(s) - k_3 x_{3e}(t_k) - k_2\big(u_{1r}(s) - k_1 x_{1e}(t_k) + k_2 u_{2r}(s) x_{2e}(t_k)\big)x_{2e}(t_k)\Big)\mathrm{d}s.
\end{aligned}$$

In the numerical simulation, we set $I = m = 1$ and $d = 2$. The reference inputs are chosen as $v_{1r}(t) = 0$, $v_{2r}(t) = ((1 + (\ell_r + 2)^2)(\ell_r + 2))\cos(t)$. Then, Assumptions 1 and 2 hold. The initial conditions are taken as $[\psi(0), \ell(0), \theta(0)] = [-\pi/7, 0.8, -\pi/6]$, $[\psi_r(0), \ell_r(0), \theta_r(0)] = [-\pi/6, 0.5, 0]$, and $\eta_0 = 0.1$. The control gains are set to $k_1 = k_2 = k_3 = 1$. Direct calculation implies $L_1 = 0.0308$. Let $\sigma = 0.01$.

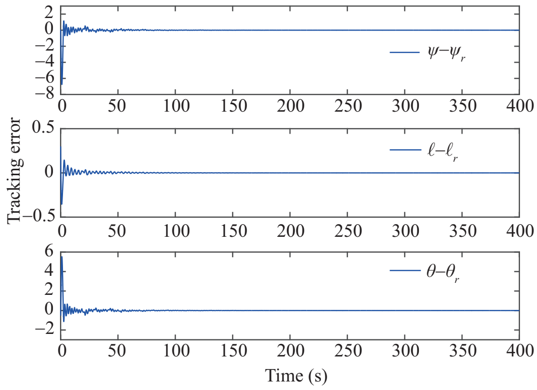

Figure 1 (Color online) Tracking errors.

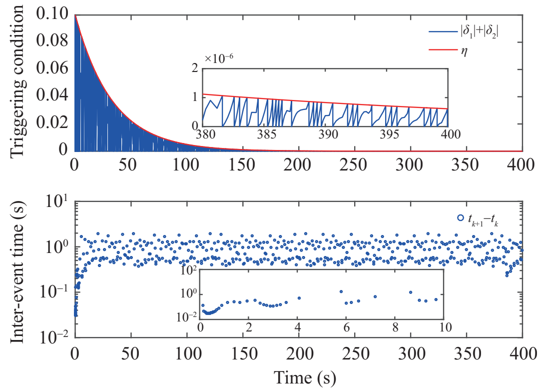

Figure 2 (Color online) Triggering mechanism and inter-event times.

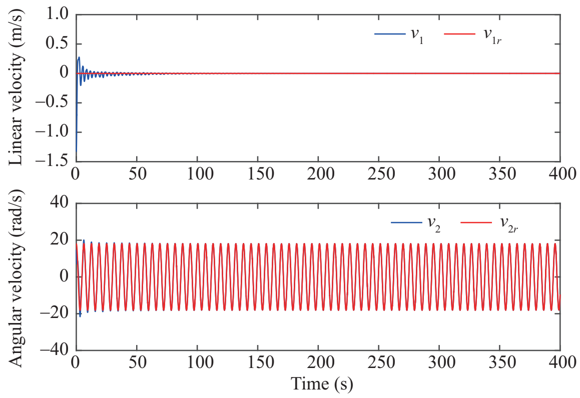

Figure 3 (Color online) Controllers v_1 (m/s) and v_2 (rad/s), and reference inputs v_{1r} (m/s) and v_{2r} (rad/s).

Table 1 Influence of different values of $\bar{c}$ on the triggering performance

$\bar{c}$	The max time (s)	Triggering times	Accuracy of tracking error	Minimum of inter-event time (s)
0.01	600	678	5×10^{-3}	0.0313
	400	463	5×10^{-2}	
0.02	600	703	1×10^{-5}	0.0312
	400	478	1×10^{-3}	
0.03	600	729	1×10^{-7}	0.0311
	400	496	1×10^{-5}	

Then $(1-\sigma)L_1 = 0.0305$. Particularly, we set $\bar{c} = 0.03$ in the numerical simulation. Figure 1 shows the tracking error between the state $[\psi, \ell, \theta]$ and $[\psi_r, \ell_r, \theta_r]$. We can see the state error asymptotically converges to zero. Figure 2 illustrates the triggering mechanism and inter-event times. It can be seen that the error does not exceed the threshold signal and the event is triggered when the equation holds. In particular, according to the numerical simulation, the minimal inter-event time over $t \in [0, 400]$ is 0.0311. Figure 3 indicates the controllers and reference inputs. When time tends to infinity, it is noticed that the controllers converge to the reference inputs.

Let us further discuss the effect of parameter $\bar{c}$ on the triggering performance. From Table 1, we may conclude that the minimum inter-event time gradually decreases from 0.0313 to 0.0311, when $\bar{c}$ increases from 0.01 to 0.03, while the larger $\bar{c}$ will lead to more triggering times but smaller tracking error. These facts are in line with our general understanding. As we know, the allowable value of the controller error will become smaller if the convergence rate of the threshold signal becomes larger. Meanwhile, the number of samples will increase in the same time interval. Of course, the faster the threshold function converges, so does the tracking error.

5 Conclusion

In this study, we developed a novel time-varying event-triggered strategy to solve the asymptotic tracking problem for a class of nonholonomic systems in chained form. By designing a time-varying event-triggered piecewise continuous control law, the time-varying reference input is precisely compensated, which eliminates the ripple phenomenon. To rule out the Zeno behavior, a novel iISS-Lyapunov function for nonholonomic systems is constructed by viewing the sampling error as the external input, which, together with the global exponential decaying threshold signals, leads to a uniformly globally asymptotically stable closed-loop system. Moreover, infinitely fast sampling is avoided by tuning down the convergence rate of the threshold signal. In the future, we may apply the proposed control design to solve the cooperative control problem for multiple nonholonomic systems, where hybrid control [39] or self-trigger control [40] may be adopted. The other interesting direction is to find a universal event-triggered control law whether or not the PE condition (Assumption 2) holds for application to some other practical problems, for example, the parking of a wheeled mobile robot [41].

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 62173092, 62173149).

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Appendix A Proof of Lemma 1

Proof. We first claim that $V_2(t, x_e)$ is a uniformly proper positive definite function. Recall that $V_1(x_e)$ in (19) is a proper positive definite function. From (19),

$$1 \leqslant \varphi(t) \leqslant 1 + T\bar{p} \tag{A1}$$

and

$$\dot{\varphi}(t) = \frac{2}{T}\left(T\left(u_{1r}^2(t) + u_{2r}^2(t)\right) - \int_{t-T}^{t}\left(u_{1r}^2(s) + u_{2r}^2(s)\right)\mathrm{d}s\right) = 2\left(u_{1r}^2(t) + u_{2r}^2(t)\right) - \frac{2}{T}\int_{t-T}^{t}\left(u_{1r}^2(s) + u_{2r}^2(s)\right)\mathrm{d}s. \tag{A2}$$

By Assumption 1 and the mean square inequality,

$$|u_{1r}x_{2e}x_{3e}| \leqslant \frac{b_1}{\min\{1, k_2\}}V_1(x_e), \quad |u_{2r}x_{1e}x_{2e}\phi(V_1)| \leqslant \frac{b_2}{\min\{1, k_2\}}\phi(V_1(x_e))V_1(x_e). \tag{A3}$$

Eqs. (A1)–(A3) together with the definition of $V_1(x_e)$ give (16) and (17). Hence, $V_2(t, x_e)$ is a uniformly proper positive definite function.

Next, we consider the derivative of V_2 along the trajectory of (14). From (19), the derivative of V_1 along the system (14) yields

$$\dot{V}_1(x_e) = -k_1x_{1e}^2 - k_3x_{3e}^2. \tag{A4}$$

Since ρ is a nonnegative polynomial and $V_1(x_e)$ satisfies (A4), we know

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho(V_1)V_1) \leqslant \rho(V_1)(-k_1x_{1e}^2 - k_3x_{3e}^2). \tag{A5}$$

By combining (19) and (A2)–(A4), it gives that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi(t)V_1) \leqslant \left(u_{1r}^2 + u_{2r}^2\right)\left[x_{1e}^2 + k_2x_{2e}^2 + x_{3e}^2\right] - \frac{2}{T}\left[\int_{t-T}^{t}\left(u_{1r}^2(s) + u_{2r}^2(s)\right)\mathrm{d}s\right]V_1. \tag{A6}$$

Notice that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(u_{1r}x_{2e}x_{3e}) = {} & \dot{u}_{1r}x_{2e}x_{3e} + u_{1r}^2x_{3e}^2 - k_1u_{1r}x_{1e}x_{3e}^2 + k_2u_{1r}u_{2r}x_{2e}x_{3e}^2 - u_{1r}u_{2r}x_{1e}x_{3e} \\ & - k_2u_{1r}^2x_{2e}^2 + k_1k_2u_{1r}x_{1e}x_{2e}^2 - k_2^2u_{1r}u_{2r}x_{2e}^3 - k_3u_{1r}x_{2e}x_{3e}.\end{aligned} \tag{A7}$$

Applying the following inequalities ($\epsilon > 0$):

$$\begin{aligned}
&|\dot{u}_{1r}x_{2e}x_{3e}| \leqslant \bar{b}_1\left(\frac{1}{2\epsilon}x_{2e}^2 + \frac{\epsilon}{2}x_{3e}^2\right), && \left|u_{1r}^2x_{3e}^2\right| \leqslant b_1^2x_{3e}^2,\\
&\left|-k_1u_{1r}x_{1e}x_{3e}^2\right| \leqslant \frac{k_1}{2}b_1x_{1e}^2 + k_1b_1V_1x_{3e}^2, && \left|k_2u_{1r}u_{2r}x_{2e}x_{3e}^2\right| \leqslant k_2u_{2r}^2V_1x_{2e}^2 + \frac{k_2b_1^2}{2}x_{3e}^2,\\
&|-u_{1r}u_{2r}x_{1e}x_{3e}| \leqslant \frac{b_1b_2}{2}x_{1e}^2 + \frac{b_1b_2}{2}x_{3e}^2, && \left|k_1k_2u_{1r}x_{1e}x_{2e}^2\right| \leqslant \epsilon k_1b_1V_1x_{1e}^2 + \frac{k_1k_2b_1}{2\epsilon}x_{2e}^2,\\
&\left|-k_2^2u_{1r}u_{2r}x_{2e}^3\right| \leqslant \epsilon k_2u_{2r}^2V_1x_{2e}^2 + \frac{k_2^2b_1^2}{2\epsilon}x_{2e}^2, && |-k_3u_{1r}x_{2e}x_{3e}| \leqslant \frac{k_3b_1}{2\epsilon}x_{2e}^2 + \frac{\epsilon k_3b_1}{2}x_{3e}^2,
\end{aligned}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_{1r}x_{2e}x_{3e}) \leqslant \left(\frac{k_1}{2}b_1 + \frac{b_1b_2}{2} + \epsilon k_1b_1V_1\right)x_{1e}^2 + \left(\frac{\epsilon\bar{b}_1}{2} + b_1^2 + k_1b_1V_1 + \frac{k_2b_1^2}{2} + \frac{b_1b_2}{2} + \frac{\epsilon k_3b_1}{2}\right)x_{3e}^2$$

$$
\begin{aligned}
&+\frac{1}{2\epsilon}\left(\bar{b}_1+k_1k_2b_1+k_2^2b_1^2+k_3b_1\right)x_{2e}^2+\left(k_2V_1+\epsilon k_2V_1\right)u_{2r}^2x_{2e}^2-k_2u_{1r}^2x_{2e}^2\\
&\leqslant \rho_1(V_1)x_{1e}^2+\frac{1}{2\epsilon}(\bar{b}_1+k_1k_2b_1+k_2^2b_1^2+k_3b_1)x_{2e}^2+\rho_2(V_1)u_{2r}^2x_{2e}^2-k_2u_{1r}^2x_{2e}^2+\rho_3(V_1)x_{3e}^2,
\end{aligned}\tag{A8}
$$

where

$$
\rho_1(V_1)=\frac{k_1}{2}b_1+\frac{b_1b_2}{2}+\epsilon k_1b_1V_1,\ \ \rho_2(V_1)=k_2V_1+\epsilon k_2V_1,\ \ \rho_3(V_1)=\frac{\epsilon\bar{b}_1}{2}+b_1^2+k_1b_1V_1+\frac{k_2b_1^2}{2}+\frac{b_1b_2}{2}+\frac{\epsilon k_3b_1}{2}.
$$

Similarly,

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\left(u_{2r}x_{1e}x_{2e}\phi(V_1)\right)=&-\dot{u}_{2r}x_{1e}x_{2e}\phi(V_1)+u_{2r}\frac{\partial\phi(V_1)}{\partial V_1}x_{1e}x_{2e}\left(k_1x_{1e}^2+k_3x_{3e}^2\right)+k_1u_{2r}\phi(V_1)x_{1e}x_{2e}-k_2u_{2r}^2\phi(V_1)x_{2e}^2\\
&+u_{2r}^2\phi(V_1)x_{1e}^2-u_{1r}u_{2r}\phi(V_1)x_{1e}x_{3e}+k_1u_{2r}\phi(V_1)x_{1e}^2x_{3e}-k_2u_{2r}^2\phi(V_1)x_{1e}x_{2e}x_{3e}.
\end{aligned}\tag{A9}
$$

Applying $|-\dot{u}_{2r}x_{1e}x_{2e}\phi(V_1)|\leqslant\bar{b}_2\left(\frac{\epsilon}{2}\phi^2(V_1)x_{1e}^2+\frac{1}{2\epsilon}x_{2e}^2\right)$ and the following inequalities:

$$
\begin{aligned}
&\left|k_1u_{2r}\frac{\partial\phi(V_1)}{\partial V_1}x_{1e}^3x_{2e}\right|\leqslant\frac{k_1b_2}{\min\{1,k_2\}}\frac{\partial\phi(V_1)}{\partial V_1}V_1x_{1e}^2, &&\left|k_3u_{2r}\frac{\partial\phi(V_1)}{\partial V_1}x_{1e}x_{2e}x_{3e}^2\right|\leqslant\frac{k_3b_2}{\min\{1,k_2\}}\frac{\partial\phi(V_1)}{\partial V_1}V_1x_{3e}^2,\\
&|k_1u_{2r}\phi(V_1)x_{1e}x_{2e}|\leqslant\frac{\epsilon k_1b_2}{2}\phi^2(V_1)x_{1e}^2+\frac{k_1b_2}{2\epsilon}x_{2e}^2, &&|-u_{1r}u_{2r}\phi(V_1)x_{1e}x_{3e}|\leqslant\frac{b_1b_2}{2}\phi(V_1)x_{1e}^2+\frac{b_1b_2}{2}\phi(V_1)x_{3e}^2,\\
&\left|k_1u_{2r}\phi(V_1)x_{1e}^2x_{3e}\right|\leqslant k_1b_2\phi(V_1)(V_1x_{3e}^2+\frac{1}{2}x_{1e}^2), &&\left|-k_2u_{2r}^2\phi(V_1)x_{1e}x_{2e}x_{3e}\right|\leqslant\epsilon k_2b_2^2\phi(V_1)V_1x_{3e}^2+\frac{k_2b_2^2}{2\epsilon}x_{2e}^2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\left(u_{2r}x_{1e}x_{2e}\phi(V_1)\right)\leqslant&\left[\left(\frac{\epsilon\bar{b}_2}{2}+\frac{\epsilon k_1b_2}{2}\right)\phi^2(V_1)+\left(\frac{b_1b_2}{2}+\frac{k_1b_2}{2}+b_2^2\right)\phi(V_1)+\frac{k_1b_2}{\min\{1,k_2\}}\frac{\partial\phi(V_1)}{\partial V_1}V_1\right]x_{1e}^2\\
&+\left[\frac{k_3b_2}{\min\{1,k_2\}}\frac{\partial\phi(V_1)}{\partial V_1}V_1+\frac{b_1b_2\phi(V_1)}{2}+k_1b_2\phi(V_1)V_1+\epsilon k_2b_2^2\phi(V_1)V_1\right]x_{3e}^2\\
&+\frac{\left(\bar{b}_2+k_1b_2+k_2b_2^2\right)}{2\epsilon}x_{2e}^2-k_2u_{2r}^2\phi(V_1)x_{2e}^2\\
\leqslant&\ \rho_4(V_1)x_{1e}^2+\rho_5(V_1)x_{3e}^2-k_2u_{2r}^2\phi(V_1)x_{2e}^2+\frac{1}{2\epsilon}\left(\bar{b}_2+k_1b_2+k_2b_2^2\right)x_{2e}^2,
\end{aligned}\tag{A10}
$$

where

$$
\begin{aligned}
\rho_4(V_1)&=\frac{k_1b_2}{\min\{1,k_2\}}\frac{\partial\phi(V_1)}{\partial V_1}V_1+\left(\frac{\epsilon\bar{b}_2}{2}+\frac{\epsilon k_1b_2}{2}\right)\phi^2(V_1)+\left(\frac{b_1b_2}{2}+\frac{k_1b_2}{2}+b_2^2\right)\phi(V_1),\\
\rho_5(V_1)&=\frac{k_3b_2}{\min\{1,k_2\}}\frac{\partial\phi(V_1)}{\partial V_1}V_1+\frac{b_1b_2\phi(V_1)}{2}+k_1b_2\phi(V_1)V_1+\epsilon k_2b_2^2\phi(V_1)V_1.
\end{aligned}
$$

Finally, by using (A5)–(A10), we have

$$
\begin{aligned}
\dot{V}_2\leqslant&-\frac{2\mu}{T}V_1-\left(k_1\rho(V_1)-\rho_1(V_1)-\rho_4(V_1)-(u_{1r}^2+u_{2r}^2)\right)x_{1e}^2-\left[k_3\rho(V_1)-\rho_3(V_1)\right.\\
&\left.-\rho_5(V_1)-(u_{1r}^2+u_{2r}^2)\right]x_{3e}^2+\frac{1}{2\epsilon}\left[\bar{b}_1+\bar{b}_2+k_1k_2b_1+k_2^2b_1^2+k_3b_1+k_1b_2+k_2b_2^2\right]x_{2e}^2\\
&-\left(k_2\phi(V_1)-\rho_2(V_1)-k_2\right)u_{2r}^2x_{2e}^2.
\end{aligned}
$$

Thus, Eq. (18) follows by choosing

$$
\begin{aligned}
\phi(V_1)&=\frac{1}{k_2}\left(\rho_2(V_1)+k_2\right),\\
\rho(V_1)&=\frac{1}{\min\{k_1,k_3\}}\left(\rho_1(V_1)+\rho_3(V_1)+\rho_4(V_1)+\rho_5(V_1)+\bar{p}\right),\\
\epsilon&=\frac{T}{k_2\mu}\left[\bar{b}_1+k_1k_2b_1+k_2^2b_1^2+k_3b_1+\bar{b}_2+k_1b_2+k_2b_2^2\right].
\end{aligned}
$$

That is to say, V_2 is a strict Lyapunov function of the nominal system (14). Finally, notice that $\rho_1(V_1)$, $\rho_2(V_1)$, and $\rho_3(V_1)$ are all linear, which in turn implies that $\phi(V_1)$ is also linear. Hence $\rho_4(V_1)$ and $\rho_5(V_1)$ are quadratic polynomials, so is $\rho(V_1)$.