SCIENCE CHINA
Information Sciences

• **Supplementary File** •

Exponential stability analysis of switched positive nonlinear systems with impulsive effects via multiple max-separable Lyapunov function

Dadong TIAN[1], Jianwei XIA[2*], Yuangong SUN[3] & Mei LI[1]

[1]*College of Information Science and Engineering, Shandong Agricultural University, Taian,* 271018*, China;*
[2]*School of Mathematic Sciences, Liaocheng University, Liaocheng,* 252059*, China;*
[1]*School of Mathematical Sciences, University of Jinan, Jinan,* 250022*, China*

Appendix A Preliminaries

Let $\mathbb{R}^n_+ := \{\boldsymbol{x} \in \mathbb{R}^n, x_j \geqslant 0, 1 \leqslant j \leqslant n\}$, where $\mathbb{R}^n$ stands for n-dimensional Euclidean space. $\mathbb{N}_0 = \{0, 1, 2...\}$. I denotes the identity matrix. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, define: $\boldsymbol{x} \gg \boldsymbol{y}$, if $x_j > y_j$ $1 \leqslant j \leqslant n$; $\boldsymbol{x} \geqslant \boldsymbol{y}$, if $x_j \geqslant y_j$, for $1 \leqslant j \leqslant n$; $\boldsymbol{x} > \boldsymbol{y}$, if $\boldsymbol{x} \geqslant \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. $A = (a_{ij})_{n\times n}$ is Metzler matrix if $a_{ij} \geqslant 0 (i \neq j)$. The jth coordinate of vector $\boldsymbol{x}_p$ is denoted by x_{pj}. The weighted l_∞ norm of $\boldsymbol{x} \in \mathbb{R}^n_+$ is given by

$$||\boldsymbol{x}||^{\boldsymbol{v}}_\infty = \max_{1\leqslant j\leqslant n} \frac{|x_j|}{v_j}.$$

For an n-tuple $\boldsymbol{r} = \{r_1, r_2, ..., r_n\}$ and a constant $\lambda > 0$, the dilation map is defined by $\delta^{\boldsymbol{r}}_\lambda$

$$\delta^{\boldsymbol{r}}_\lambda(\boldsymbol{x}) = \begin{bmatrix} \lambda^{r_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda^{r_2} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda^{r_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda^{r_1} x_1 \\ \lambda^{r_2} x_2 \\ \vdots \\ \lambda^{r_n} x_n \end{bmatrix},$$

where $\lambda_j > 0$ $(j = 1, 2, ..., n)$. The dilation map $\delta^{\boldsymbol{r}}_\lambda$ is standard when $\boldsymbol{r} = \{1, 1, ..., 1\}$.

Definition 1. *A* vector field $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is cooperative if the Jacobian matrix $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{a})$ is Metzler matrix for all $\boldsymbol{a} \in \mathbb{R}^n_+ \backslash \{\boldsymbol{0}\}$, where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and continuously differentiable on $\mathbb{R}^n_+ \backslash \{\boldsymbol{0}\}$.

From ([1], Chapter 3, Remark 1.1, p.33), the cooperative vectors satisfy Kamke-condition.

Proposition 1. If $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a cooperative field, then given any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^n_+$ satisfying $\boldsymbol{x} \geqslant \boldsymbol{y}$ and $x_j = y_j$, one has $f_j(\boldsymbol{x}) \geqslant f_j(\boldsymbol{y})$.

Definition 2. *A* vector field $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$ is called order-preserving on $\mathbb{R}^n_+$, if $\boldsymbol{g}(\boldsymbol{x}) \geqslant \boldsymbol{g}(\boldsymbol{y})$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_+$ satisfying $\boldsymbol{x} \geqslant \boldsymbol{y}$.

If for any $t \in [0, +\infty)$, $\boldsymbol{f}_{\sigma(t)}$ is is a cooperative field and $\boldsymbol{g}_{\sigma(t)}$ is order-preserving field, then system (1) is SPNSs. More precisely, in the fist interval $[0, t_1)$, the solution of the first activated subsystem is nonnegative for any nonnegative initial condition. This implies that the initial condition is nonnegative for the second interval $[t_1, t_2)$. Hence, the solution of the system (1) is always nonnegative.

Definition 3. System (1) is exponentially stable (ES), if for nonnegative initial function $\boldsymbol{\theta}(t)$ and any switching signal from a fixed ADT class, there exist two constants $a_0 > 0$ and $b_0 > 0$ such that the solutions of system (1) satisfy

$$||\boldsymbol{x}(t)|| \leqslant a_0 e^{-b_0 t} ||\boldsymbol{\theta}(t)||, \quad t \geqslant 0,$$

where $||.||$ is some norm in $\mathbb{R}^n$.

Definition 4. Denote $N_\sigma(T_1, T_2)$ be the number of switching times of $\sigma(t)$ over the interval $[T_1, T_2]$. If there exist two positive constants τ_a and N_0 such that

$$N_\sigma(T_1, T_2) \leqslant N_0 + \frac{T_2 - T_1}{\tau_a},$$

then τ_a is called an average dwell time (ADT) of $\sigma(t)$.

* Corresponding author (email: njustxjw@126.com)

Appendix B The proof of Lemma 1

Proof: For any $j \in \{1,2,...,n\}$, one can get

$$\begin{aligned}\left(\frac{x_j(t)}{\upsilon_{pj}}\right)^{\frac{r_{max}}{r_j}} &= \left(\frac{x_j(t)}{\upsilon_{qj}}\frac{\upsilon_{qj}}{\upsilon_{pj}}\right)^{\frac{r_{max}}{r_j}} \\ &= \left(\frac{x_j(t)}{\upsilon_{qj}}\right)^{\frac{r_{max}}{r_j}}\left(\frac{\upsilon_{qj}}{\upsilon_{pj}}\right)^{\frac{r_{max}}{r_j}} \\ &\leqslant \alpha^{\frac{r_{max}}{r_j}}\left(\frac{x_j(t)}{\upsilon_{qj}}\right)^{\frac{r_{max}}{r_j}} \\ &\leqslant \alpha^l\left(\frac{x_j(t)}{\upsilon_{qj}}\right)^{\frac{r_{max}}{r_j}} \\ &= \beta\left(\frac{x_j(t)}{\upsilon_{qj}}\right)^{\frac{r_{max}}{r_j}},\end{aligned}$$

then it follows from the definition of the multiple max-separate Lyapunov function that (2) holds.

Appendix C The proof of Theorem 1

Proof: For any $p \in M, j \in \{1,2,...,n\}$, define function $z_{pj}(\lambda) = \lambda + \frac{f_{pj}(\boldsymbol{\upsilon}_p)}{\upsilon_{pj}} + e^{\lambda\tau_{max}}\frac{g_{pj}(\boldsymbol{\upsilon}_p)}{\upsilon_{pj}}$, then it is easily to check that $z_{pj}(0) < 0$ and $z_{pj}(\lambda_{pj}) = 0$. Note that $z_{pj}(\lambda)$ is monotonically increasing in λ, then picking any λ_0 with $0 < \lambda_0 < \min_{p\in M}\min_{1\leqslant j\leqslant n}\lambda_{pj}$, the following inequality holds

$$z_{pj}(\lambda_0) < 0, \quad j \in \{1,2,...,n\}. \tag{C1}$$

Set $\boldsymbol{x}(t) = e^{-\lambda_0 t}\boldsymbol{y}^*(t)$, then it follows from system (1) that

$$\dot{\boldsymbol{y}}^*(t) \leqslant \lambda_0\boldsymbol{y}^*(t) + \boldsymbol{f}_{\sigma(t)}(\boldsymbol{y}^*(t)) + e^{\lambda_0\tau(t)}\boldsymbol{g}_{\sigma(t)}\left(\boldsymbol{y}^*\left(t-\tau(t)\right)\right), t \geqslant 0. \tag{C2}$$

Next, consider the following impulsive switched positive system

$$\begin{cases}\dot{\boldsymbol{y}}(t) = \lambda_0\boldsymbol{y}(t) + \boldsymbol{f}_{\sigma(t)}(\boldsymbol{y}(t)) + e^{\lambda_0\tau(t)}\boldsymbol{g}_{\sigma(t)}\left(\boldsymbol{y}\left(t-\tau(t)\right)\right), & t \geqslant 0, t \neq t_k, \\ \Delta\boldsymbol{y}(t_k) = C_{\sigma(t_k)}\boldsymbol{y}\left((t_k - d(t_k))^-\right), & \\ \boldsymbol{y}(t) = e^{\lambda_0 t}\boldsymbol{\theta}(t), & t \in [-\tau_{max}, 0].\end{cases} \tag{C3}$$

The switching sequence is denoted by $0 = t_0 < t_1 < t_2 < ... < t_k < t_{k+1} < ...$. Then for the first interval $[t_0, t_1)$, let $\sigma(t) = \sigma(t_0)$, $t \in [t_0, t_1)$. Define $||\boldsymbol{\theta}|| = \sup_{-\tau_{max}\leqslant t\leqslant 0} V_{\underline{\boldsymbol{\upsilon}}}(\boldsymbol{\theta}(t))$, where $\underline{\boldsymbol{\upsilon}}$ is defined in Lemma 1. Hence for any $\rho > 1$ one can check

$$V_{\boldsymbol{\upsilon}_{\sigma(t_0)}}(\boldsymbol{y}(0)) = V_{\boldsymbol{\upsilon}_{\sigma(t_0)}}(\boldsymbol{\theta}(0)) \leqslant ||\boldsymbol{\theta}|| < \rho||\boldsymbol{\theta}||.$$

Next, the following inequality will be proved

$$V_{\boldsymbol{\upsilon}_{\sigma(t_0)}}(\boldsymbol{y}(t)) < \rho||\boldsymbol{\theta}||, \quad \forall t \in [t_0, t_1). \tag{C4}$$

By contradiction, assume that (C4) is not always correct. Note that $V_{\boldsymbol{\upsilon}_{\sigma(t_0)}}(\boldsymbol{y}(t))$ is continuous, then there exist a time t^* and at least one index $j_0 \in \{1,2,...,n\}$ such that

$$\left(\frac{y_j(t)}{\upsilon_{\sigma(t_0)j}}\right)^{\frac{r_{max}}{r_j}} < \rho||\boldsymbol{\theta}||, \quad t \leqslant t^*, \quad j \in \{1,2,...,n\}, \tag{C5}$$

$$\left(\frac{y_{j_0}(t^*)}{\upsilon_{\sigma(t_0)j_0}}\right)^{\frac{r_{max}}{r_{j_0}}} = \rho||\boldsymbol{\theta}||, \tag{C6}$$

$$D_-\left(\frac{y_{j_0}(t)}{\upsilon_{\sigma(t_0)j_0}}\right)^{\frac{r_{max}}{r_{j_0}}}|_{t=t^*} \geqslant 0, \tag{C7}$$

where D_- denotes the left derivative.
Case 1: If $t^* - \tau(t^*) \geqslant 0$, then $t^* - \tau(t^*) \in [0, t^*]$. Combining (C5) with (C6) can lead to

$$y_j(t^* - \tau(t^*)) \leqslant (\rho||\boldsymbol{\theta}||)^{\frac{r_j}{r_{max}}}\upsilon_{\sigma(t_0)j}, \quad j \in \{1,2,...,n\}.$$

Since $\rho > 1$, $(\rho||\boldsymbol{\theta}||)^{\frac{1}{r_{max}}} > (||\boldsymbol{\theta}||)^{\frac{1}{r_{max}}}$. Therefore, if $||\boldsymbol{\theta}||^{\frac{1}{r_{max}}} > \lambda^*$, then $(\rho||\boldsymbol{\theta}||)^{\frac{1}{r_{max}}} > \lambda^*$. Furthermore, it follows from Assumption 2 and Assumption 3 that

$$\begin{aligned}\boldsymbol{g}_{\sigma(t_0)}(\boldsymbol{y}(t^* - \tau(t^*)) &\leqslant \boldsymbol{g}_{\sigma(t_0)}\left((\rho||\boldsymbol{\theta}||)^{\frac{r_1}{r_{max}}}\upsilon_{\sigma(t_0)1}, (\rho||\boldsymbol{\theta}||)^{\frac{r_2}{r_{max}}}\upsilon_{\sigma(t_0)2}, ..., (\rho||\boldsymbol{\theta}||)^{\frac{r_n}{r_{max}}}\upsilon_{\sigma(t_0)n}\right) \\ &= \boldsymbol{g}_{\sigma(t_0)}\left(\delta^{\boldsymbol{r}}_{\lambda_{\boldsymbol{\theta}}}\boldsymbol{\upsilon}_{\sigma(t_0)}\right) \\ &\leqslant \delta^{\boldsymbol{r}}_{\lambda_{\boldsymbol{\theta}}}\boldsymbol{g}_{\sigma(t_0)}\left(\boldsymbol{\upsilon}_{\sigma(t_0)}\right),\end{aligned}$$

where $\lambda_{\boldsymbol{\theta}} = (\rho||\boldsymbol{\theta}||)^{\frac{1}{r_{max}}}$. Noting that $f_{\sigma(t_0)}$ is cooperative, this in turn implies

$$\begin{aligned} f_{\sigma(t_0)j_0}(\boldsymbol{y}(t^*)) &\leqslant f_{\sigma(t_0)j_0}\left((\rho||\boldsymbol{\theta}||)^{\frac{r_1}{r_{max}}}\upsilon_{\sigma(t_0)1},(\rho||\boldsymbol{\theta}||)^{\frac{r_2}{r_{max}}}\upsilon_{\sigma(t_0)2},...,(\rho||\boldsymbol{\theta}||)^{\frac{r_n}{r_{max}}}\upsilon_{\sigma(t_0)n}\right)\\ &= f_{\sigma(t_0)j_0}\left(\delta^{\boldsymbol{r}}_{\lambda_{\boldsymbol{\theta}}}\boldsymbol{\upsilon}_{\sigma(t_0)}\right)\\ &\leqslant \lambda_{\boldsymbol{\theta}}^{r_{j_0}} f_{\sigma(t_0)j_0}\left(\boldsymbol{\upsilon}_{\sigma(t_0)}\right). \end{aligned}$$

Then

$$\begin{aligned} &D_-\left(\frac{y_{j_0}(t)}{\upsilon_{\sigma(t_0)j_0}}\right)^{\frac{r_{max}}{r_{j_0}}}|_{t=t^*}\\ &=\frac{r_{max}}{r_{j_0}}\left(\frac{y_{j_0}(t^*)}{\upsilon_{\sigma(t_0)j_0}}\right)^{\frac{r_{max}}{r_{j_0}}-1}\frac{\dot{y}_{j_0}(t^*)}{\upsilon_{\sigma(t_0)j_0}}\\ &=\frac{r_{max}}{r_{j_0}}\left(\frac{y_{j_0}(t^*)}{\upsilon_{\sigma(t_0)j_0}}\right)^{\frac{r_{max}}{r_{j_0}}-1}\frac{1}{\upsilon_{\sigma(t_0)j_0}}\left(\lambda_0 y_{j_0}(t^*)+f_{\sigma(t_0)j_0}(\boldsymbol{y}(t^*))+e^{\lambda_0\tau(t^*)}g_{\sigma(t_0)j_0}\left(\boldsymbol{y}\left(t^*-\tau(t^*)\right)\right)\right)\\ &\leqslant\frac{r_{max}}{r_{j_0}}\lambda_{\boldsymbol{\theta}}^{r_{max}-r_{j_0}}\frac{1}{\boldsymbol{\upsilon}_{\sigma(t_0)j_0}}\left(\lambda_0\lambda_{\boldsymbol{\theta}}^{r_{j_0}}\upsilon_{\sigma(t_0)j_0}+\lambda_{\boldsymbol{\theta}}^{r_{j_0}}f_{\sigma(t_0)j_0}(\boldsymbol{\upsilon}_{\sigma(t_0)})+e^{\lambda_0\tau_{max}}\lambda_{\boldsymbol{\theta}}^{r_{j_0}}g_{\sigma(t_0)j_0}(\boldsymbol{\upsilon}_{\sigma(t_0)})\right)\\ &=\frac{r_{max}}{r_{j_0}}\lambda_{\boldsymbol{\theta}}^{r_{max}-r_{j_0}}\lambda_{\boldsymbol{\theta}}^{r_{j_0}}\left(\lambda_0+\frac{f_{\sigma(t_0)j_0}(\boldsymbol{\upsilon}_{\sigma(t_0)})}{\upsilon_{\sigma(t_0)j_0}}+e^{\lambda_0\tau_{max}}\frac{g_{\sigma(t_0)j_0}(\boldsymbol{\upsilon}_{\sigma(t_0)})}{\upsilon_{\sigma(t_0)j_0}}\right)\\ &=\frac{r_{max}}{r_{j_0}}\lambda_{\boldsymbol{\theta}}^{r_{max}}\left(\lambda_0+\frac{f_{\sigma(t_0)j_0}(\boldsymbol{\upsilon}_{\sigma(t_0)})}{\upsilon_{\sigma(t_0)j_0}}+e^{\lambda_0\tau_{max}}\frac{g_{\sigma(t_0)j_0}(\boldsymbol{\upsilon}_{\sigma(t_0)})}{\upsilon_{\sigma(t_0)j_0}}\right)\\ &<0, \end{aligned}$$

where the last inequality is from (C1). This contradicts (C7).
Case 2: If $t^*-\tau(t^*)<0$, then

$$\boldsymbol{y}(t^*-\tau(t^*))=e^{\lambda_0(t^*-\tau(t^*))}\boldsymbol{\theta}(t^*-\tau(t^*))\leqslant\boldsymbol{\theta}(t^*-\tau(t^*)),$$

which means

$$V_{\boldsymbol{\upsilon}_{\sigma(t_0)}}(\boldsymbol{y}(t^*-\tau(t^*)))<\rho||\boldsymbol{\theta}||.$$

Similar to the analysis in Case 1, $D_-\left(\frac{y_{j_0}(t)}{\upsilon_{\sigma(t_0)j_0}}\right)^{\frac{r_{max}}{r_{j_0}}}|_{t=t^*}<0$ can be obtained, which also yields a contradiction. Thus, (C4) is true for any $\rho>1$. Let ρ tend to 1^+, then

$$V_{\boldsymbol{\upsilon}_{\sigma(t_0)}}(\boldsymbol{y}(t))\leqslant||\boldsymbol{\theta}||,\quad \forall t\in[t_0,t_1).$$

This combines with Lemma 1 leads to

$$V_{\boldsymbol{\upsilon}_{\sigma(t_1)}}(\boldsymbol{y}(t))\leqslant\beta||\boldsymbol{\theta}||,\quad \forall t\in[t_0,t_1).$$

That is, for any $t\in[t_0,t_1)$, $\left(\frac{y_j(t)}{\upsilon_{\sigma(t_1)j}}\right)^{\frac{r_{max}}{r_j}}\leqslant\beta||\boldsymbol{\theta}||$. Let t tend to t_1^-, then

$$y_j(t_1^-)\leqslant(\beta||\boldsymbol{\theta}||)^{\frac{r_j}{r_{max}}}\upsilon_{\sigma(t_1)j}. \tag{C8}$$

Since $-\tau_{max}\leqslant t_1-d(t_1)\leqslant t_1$, the following inequality is always true

$$y_j\left((t_1-d(t_1))^-\right)\leqslant(\beta||\boldsymbol{\theta}||)^{\frac{r_j}{r_{max}}}\upsilon_{\sigma(t_1)j}. \tag{C9}$$

Combining (C8) and (C9) gives

$$\begin{aligned} y_j(t_1) &= y_j(t_1^-)+C_{\sigma(t_1)jj}y_j\left((t_1-d(t_1))^-\right)\\ &\leqslant(\beta||\boldsymbol{\theta}||)^{\frac{r_j}{r_{max}}}(1+C_{\sigma(t_1)jj})\upsilon_{\sigma(t_1)j}\\ &\leqslant(\beta||\boldsymbol{\theta}||)^{\frac{r_j}{r_{max}}}h\upsilon_{\sigma(t_1)j}. \end{aligned}$$

Then it follows that

$$V_{\boldsymbol{\upsilon}_{\sigma(t_1)}}(\boldsymbol{y}(t_1))\leqslant\beta\eta||\boldsymbol{\theta}||.$$

Applying the similar analysis to the one of the first interval, one can verify that

$$V_{\boldsymbol{\upsilon}_{\sigma(t_1)}}(\boldsymbol{y}(t))\leqslant\beta\eta||\boldsymbol{\theta}||,\quad \forall t\in[t_1,t_2)$$

and

$$V_{\boldsymbol{\upsilon}_{\sigma(t_2)}}(\boldsymbol{y}(t_2))\leqslant(\beta\eta)^2||\boldsymbol{\theta}||.$$

By induction, for each $k\in\{1,2,...,n,...\}$, the following inequality can be obtained

$$V_{\boldsymbol{\upsilon}_{\sigma(t_k)}}(\boldsymbol{y}(t))\leqslant(\beta\eta)^k||\boldsymbol{\theta}||,\quad t\in[t_k,t_{k+1}). \tag{C10}$$

It should be pointed out that even if $\lim_{t\to+\infty}t_k-d(t_k)=+\infty$ holds for the impulsive delay, (C10) is always true. The reasons are that $t_k-d(t_k)\leqslant t_k$ and $\beta\eta\geqslant 1$. Furthermore, for any $t\in[0,+\infty)$, assume that $t\in[t_k,t_{k+1})$, then we have

$$V_{\boldsymbol{\upsilon}_{\sigma(t_k)}}(\boldsymbol{y}(t))\leqslant(\beta h)^k||\boldsymbol{\theta}||\leqslant(\beta\eta)^{N_0+\frac{t}{\tau_a}}||\boldsymbol{\theta}||.$$

It follows from (C2) and (C3) that

$$\begin{aligned} V_{\boldsymbol{\upsilon}_{\sigma(t_k)}}(\boldsymbol{x}(t)) &= e^{-\lambda_0 t} V_{\boldsymbol{\upsilon}_{\sigma(t_k)}}(\boldsymbol{y}^*(t)) \\ &\leqslant e^{-\lambda_0 t} V_{\boldsymbol{\upsilon}_{\sigma(t_k)}}(\boldsymbol{y}(t)) \\ &\leqslant (\beta\eta)^{N_0} e^{-\lambda_0 t} (\beta\eta)^{\frac{t}{\tau_a}} ||\boldsymbol{\theta}|| \\ &= (\beta\eta)^{N_0} e^{-(\lambda_0 - \frac{\ln(\beta\eta)}{\tau_a})t} ||\boldsymbol{\theta}||. \end{aligned}$$

Furthermore, from Lemma 1, we can get

$$V_{\overline{\boldsymbol{\upsilon}}}(\boldsymbol{x}(t)) \leqslant V_{\boldsymbol{\upsilon}_{\sigma(t_k)}}(\boldsymbol{x}(t)) \leqslant (\beta h)^{N_0} e^{-(\lambda_0 - \frac{\ln(\beta\eta)}{\tau_a})t} ||\boldsymbol{\theta}||.$$

Since the switching signal satisfies $\tau_a > \frac{\ln(\beta\eta)}{\lambda_0}$, that is $\lambda_0 - \frac{\ln(\beta\eta)}{\tau_a} > 0$, this implies that system (1) is ES.

Appendix D The proof of Theorem 2

Proof: From (5), we have

$$\lambda_0 + \left(a_{jj}^p + \sum_{i=1, i\neq j}^{i=n} \frac{1}{\upsilon_{pj}} \tilde{a}_{ji}^p \upsilon_{pi} \right) + e^{\lambda_0 \tau_{max}} \sum_{i=1}^{i=n} \frac{1}{\upsilon_{pj}} |b_{ji}^p| \upsilon_{pi} < 0, \quad \forall p \in M, \quad j \in \{1, 2, ..., n\}. \tag{D1}$$

For the linear system (4), since $r_1 = r_2 = ... = r_n$, the multiple max-separable Lyapunov function can be written as

$$V_{\boldsymbol{\upsilon_p}}(\boldsymbol{x}) = \max_{1\leqslant j\leqslant n} \left(\frac{|x_j|}{\upsilon_{pj}} \right)^{\frac{r_{max}}{r_j}} = \max_{1\leqslant j\leqslant n} \left(\frac{|x_j|}{\upsilon_{pj}} \right).$$

Similar to the proof of (C4), suppose

$$\frac{|x_j(t)|}{\upsilon_{\sigma(t_0)j}} < \rho||\boldsymbol{\theta}||, \quad t \leqslant t^*, \quad j \in \{1, 2, ..., n\}, \tag{D2}$$

$$\frac{|x_{j_0}(t^*)|}{\upsilon_{\sigma(t_0)j_0}} = \rho||\boldsymbol{\theta}||, \tag{D3}$$

$$D_- \frac{|x_{j_0}(t)|}{\upsilon_{\sigma(t_0)j_0}}|_{t=t^*} \geqslant 0, \tag{D4}$$

Then the left derivative of $|x_{j_0}(t^*)|/\upsilon_{pj_0}$ needs to be calculated, that is

$$\begin{aligned} D_- \frac{|x_{j_0}(t)|}{\upsilon_{pj_0}}|_{t=t^*} &= \frac{sign(x_{j_0}(t^*))\dot{x}_{j_0}(t^*)}{\upsilon_{pj_0}} \\ &= \frac{sign(x_{j_0}(t^*))}{\upsilon_{pj_0}} \left(\lambda_0 x_{j_0}(t^*) + \sum_{i=1}^{i=n} a_{j_0 i}^p x_i(t^*) + e^{\lambda_0 \tau(t^*)} \sum_{i=1}^{i=n} b_{j_0 i}^p x_i(t^*) \right) \\ &\leqslant \frac{1}{\upsilon_{pj_0}} \left(\lambda_0 |x_{j_0}(t^*)| + a_{j_0 j_0}^p |x_{j_0}(t^*)| + \sum_{i=1, i\neq j_0}^{i=n} \tilde{a}_{j_0 i}^p |x_i(t^*)| + e^{\lambda_0 \tau_{max}} \sum_{i=1}^{i=n} |b_{j_0 i}^p||x_i(t^*)| \right). \end{aligned}$$

It follows from (D1) that $D_- \frac{|x_{j_0}(t)|}{\upsilon_{pj_0}}|_{t=t^*} < 0$, which contradicts (D4). Then the rest proof which shows similarities with the one of Theorem 1 is left out here.

References

1 Smith H. Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems. American Mathematical Society: Providence, RI, 1995.