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Mean square stability of discrete-time linear systems with random impulsive disturbances

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Dear editor,

In the real world, many dynamic processes may experience abrupt state [1] or parameter changes. Mathematically, these abrupt changes of state can be modeled as ideal impulsive effects. And also, the moments at which impulses occur might be uncertain because they are triggered by unpredictable factors. Recently, the stability of impulsive differential equations with respect to impulsive effects that occur at uncertain or random moments has been investigated and reported in [2, 3].

Impulsive systems were investigated in the discrete-time framework in the past 20 years [4, 5]. Conventionally, each jump process requires a single step and, thus, does not reflect the instantaneousness of the impulse. Hence, we adopt the hybrid index model [6], which reflects the instantaneousness of the impulses, to describe discrete-time impulsive systems. In addition, we investigate the mean square stability of discrete-time systems with impulses that occur at random instants. The sequence of impulsive instants is assumed to be a random process with a bounded homogeneous independent increment. Thus, the impulsive intervals are independent and identically distributed. Under this assumption, necessary and sufficient conditions for mean square stability of such impulsive systems are obtained.

Model and methodology. The discrete-time linear systems with random impulsive disturbances based on a hybrid-index model is described as

$$\begin{cases} x(t+1,j) = Ax(t,j), \ t \neq \tau_\alpha \text{ or } \gamma(t,j)=1, \\ x(t,j+1) = Jx(t,j), \ t = \tau_\alpha \text{ and } \gamma(t,j)=0, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the states of the system and $x(t+1,j)$ and $x(t,j+1)$ represent the after-step and after-jump states, respectively. τ_α, $\alpha \in \overline{\mathbb{Z}^+}$, is the sequence of random impulsive instants and index j denotes the number of jumps. $\gamma(t,j)$ is an instrumental variable that limits the number of successive jumps at a single instant to 1 with $\gamma(0,0)=0$. The first and the second equations in (1) are referred to as the stepping and the jumping processes, respectively. For any initial state $x_0 \in \mathbb{R}^n$ and a fixed sequence of impulsive instants τ_α, a uniquely determined sequence of hybrid index pairs (t_k, j_k), $k \in \overline{\mathbb{Z}^+}$, exists and satisfies

$$\begin{cases} \left.\begin{aligned} t_{k+1} &= t_k+1, \\ j_{k+1} &= j_k, \end{aligned}\right\}, \ t_k \neq \tau_\alpha \text{ or } \gamma(t_k,j_k)=1, \\ \left.\begin{aligned} t_{k+1} &= t_k, \\ j_{k+1} &= j_k+1, \end{aligned}\right\}, \ t_k = \tau_\alpha \text{ and } \gamma(t_k,j_k)=0, \end{cases} \tag{2}$$

where $(t_0,j_0)=(0,0)$. This subset of hybrid indices is denoted by $\mathcal{H}_{x_0}$, that is

$$\mathcal{H}_{x_0} := \left\{(t_k,j_k) \mid k \in \overline{\mathbb{Z}^+}\right\} \subset \overline{\mathbb{Z}^+} \times \overline{\mathbb{Z}^+}.$$

Evidently, we always have $t_k + j_k = k$, $\forall k \in \overline{\mathbb{Z}^+}$. The solution to (1) with initial state x_0, denoted by $x(t,j;x_0)$, $(t,j) \in \mathcal{H}_{x_0}$, is a sequence of states defined on $\mathcal{H}_{x_0}$ such that $x(0,0;x_0)=x_0$ and satisfies (1). We refer to $\mathcal{H}_{x_0}$ as the hybrid-domain of the solution $x(t,j;x_0)$. In certain instances, let the notation $x(t_k,j_k;x_0)$, $k \in \overline{\mathbb{Z}^+}$, denote the solution when we need to emphasize the order of the hybrid index (t,j) in $\mathcal{H}_{x_0}$.

Assumption 1. The first impulsive instant $\tau_0 \in \overline{\mathbb{Z}^+}$ is a bounded random variable with a probability distribution $\boldsymbol{p}_i^{\tau_0} := \mathbb{P}\{\tau_0 = i\}, i \in \overline{\mathbb{Z}^+}$.

Assumption 2. The sequence of impulsive instants $\tau_\alpha \in \mathbb{Z}^+$, $\alpha \in \overline{\mathbb{Z}^+}$, is a random process with a homogeneous independent increment. Thus the sequence of impulsive intervals defined as $\omega_\alpha := \tau_{\alpha+1} - \tau_\alpha$, $\alpha \in \overline{\mathbb{Z}^+}$, is an independent and identically distributed sequence. In addition, we assume that ω_α is subject to the distribution $\boldsymbol{p}_s^\omega := \mathbb{P}\{\omega_\alpha = s\}$, $s \in \mathbb{Z}^+$, and that $\boldsymbol{\omega}_m := \sup_{\alpha \in \overline{\mathbb{Z}^+}}(\tau_{\alpha+1} - \tau_\alpha) < \infty$.

Definition 1. Impulsive system (1) is mean-square stable if $\lim_{k\to\infty} \mathbb{E}\|x(t_k,j_k;x_0)\|^2 = 0, \forall x_0 \in \mathbb{R}^n$.

Based on the observation that the subsequence of states at impulsive instants, denoted by $\zeta_\alpha(x_0) := x(\tau_\alpha, \boldsymbol{j}^+_{\tau_\alpha}; x_0)$, $\alpha \in \overline{\mathbb{Z}^+}$, constitutes a homogeneous Markov chain, where $\boldsymbol{j}_t^+(x_0) := \max\{j \mid \exists t \in \overline{\mathbb{Z}^+} \text{ s.t. } (t,j) \in \mathcal{H}_{x_0}\}, \forall x_0 \in \mathbb{R}^n, \forall t \in \overline{\mathbb{Z}^+}$, we first derive a necessary and sufficient condition for mean square convergence of $\zeta_\alpha(x_0)$ by using the

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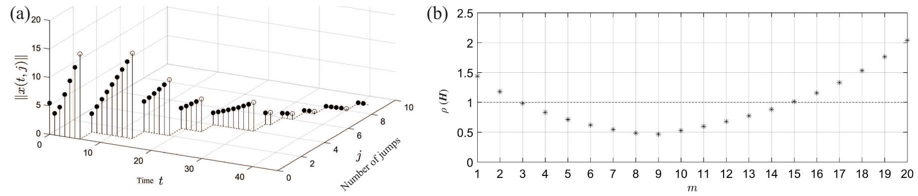

Figure 1 (a) Sample sequence of the solution in the hybrid domain; (b) spectral radius of $\boldsymbol{H}$ over m.

column-stacking technique. This technique was adopted by [7] to assess the mean square stability of networked systems with random packet loss.

Proposition 1. Let $\Sigma_\alpha(x_0) := \mathbb{E}(\zeta_\alpha(x_0)\zeta_\alpha^{\mathrm{T}}(x_0))$. Then, $\lim_{\alpha\to\infty}\Sigma_\alpha(x_0) = 0$, $\forall x_0 \in \mathbb{R}^n$, if and only if $\lim_{\alpha\to\infty}\mathbb{Q}_\omega^\alpha\mathbb{Q}_{\tau_0}\Psi = 0$, where

$$\mathbb{Q}_\omega := (J\otimes J)\mathbb{E}(A\otimes A)^{\omega_\alpha}, \quad \mathbb{Q}_{\tau_0} := (J\otimes J)\mathbb{E}(A\otimes A)^{\tau_0}$$

and Ψ is an $n^2 \times \frac{n(n+1)}{2}$ matrix defined as

$$\begin{aligned}\Psi := \big[\, & \boldsymbol{e}_1\otimes\boldsymbol{e}_1, \boldsymbol{e}_2\otimes\boldsymbol{e}_2,\ldots,\boldsymbol{e}_n\otimes\boldsymbol{e}_n, \boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1,\\ & \boldsymbol{e}_1\otimes\boldsymbol{e}_3+\boldsymbol{e}_3\otimes\boldsymbol{e}_1,\ldots,\boldsymbol{e}_1\otimes\boldsymbol{e}_n+\boldsymbol{e}_n\otimes\boldsymbol{e}_1,\\ & \boldsymbol{e}_2\otimes\boldsymbol{e}_3+\boldsymbol{e}_3\otimes\boldsymbol{e}_2, \boldsymbol{e}_2\otimes\boldsymbol{e}_4+\boldsymbol{e}_4\otimes\boldsymbol{e}_2,\ldots,\\ & \boldsymbol{e}_2\otimes\boldsymbol{e}_n+\boldsymbol{e}_n\otimes\boldsymbol{e}_2,\ldots,\boldsymbol{e}_{n-1}\otimes\boldsymbol{e}_n+\boldsymbol{e}_n\otimes\boldsymbol{e}_{n-1}\,\big],\end{aligned}$$

where $\boldsymbol{e}_j$ represents the jth column of the $n\times n$ identity matrix.

Proof. The proof is shown in Appendix A.

Theorem 1 states that under Assumptions 1 and 2 the obtained condition in Proposition 1 is also necessary and sufficient for the mean square stability of impulsive system (1).

Theorem 1. Impulsive system (1) is mean-square stable if and only if

$$\lim_{\alpha\to\infty}\mathbb{Q}_\omega^\alpha\mathbb{Q}_{\tau_0}\Psi = 0. \tag{3}$$

Proof. The proof is shown in Appendix B.

Corollary 1. Impulsive system (1) is mean-square stable if $\rho(\mathbb{Q}_\omega) < 1$.

It is inconvenient to directly check condition (3) in Theorem 1. However, this condition is equivalent to that all solutions of linear system $z_{t+1} = \mathbb{Q}_\omega z_t$ starting from $\mathrm{Span}(\mathbb{Q}_{\tau_0}\Psi)$ converge to zero. By decomposing the whole state space into stable and unstable subspaces, we have the following necessary and sufficient condition that is much more convenient for checking.

Theorem 2. Suppose that $\boldsymbol{\beta}_j \in \mathbb{R}^{n^2}$, $j = 1,2,\ldots,\boldsymbol{r}_c$, is a maximal linearly independent collection of vectors in $\mathrm{Span}(\boldsymbol{C})$, where $\boldsymbol{r}_c := \mathrm{rank}(\boldsymbol{C})$ and

$$\boldsymbol{C} := \left[\mathbb{Q}_{\tau_0}\Psi, \mathbb{Q}_\omega\mathbb{Q}_{\tau_0}\Psi, \mathbb{Q}_\omega^2\mathbb{Q}_{\tau_0}\Psi,\ldots,\mathbb{Q}_\omega^{n^2-1}\mathbb{Q}_{\tau_0}\Psi\right].$$

Define $\boldsymbol{B} := [\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\ldots,\boldsymbol{\beta}_{\boldsymbol{r}_c}]$. Then, impulsive system (1) is mean-square stable if and only if $\rho((\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B})^{-1}\boldsymbol{B}^{\mathrm{T}}\mathbb{Q}_\omega\boldsymbol{B}) < 1$.

Proof. The proof is shown in Appendix C.

Illustrative example. We consider the impulsive discrete-time system with

$$A = \begin{bmatrix} 1.1 & 0.3 & 0.3 & 0.3\\ 0 & 0.8 & 0 & 0\\ 0.3 & 0.3 & 1.1 & 0.3\\ -0.3 & -0.3 & -0.3 & 0.5 \end{bmatrix},$$

$$J = \begin{bmatrix} 0.4 & -1.1 & -1.1 & -1.1\\ 0 & 1.5 & 0.5 & 0.5\\ -0.6 & -1.1 & -0.1 & -1.1\\ 0.6 & 1.1 & 1.1 & 2.1 \end{bmatrix}.$$

We assume that the initial impulsive instant τ_0 and the impulsive interval ω_α are uniformly distributed in $\{1,2,\ldots,m\}$, with m being a positive integer. Let $\boldsymbol{H} = (\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B})^{-1}\boldsymbol{B}^{\mathrm{T}}\mathbb{Q}_\omega\boldsymbol{B}$ and we calculate the spectral radius of $\boldsymbol{H}$ for different m. The result is presented in Figure 1(b), which shows that, by Theorem 2, the impulsive system is mean-square stable when $3 \leqslant m \leqslant 14$ and the stability would be destroyed if the impulsive disturbance is either too frequent or too slow. The norm of a sample sequence of states $x(t,j;x_0)$ with $x_0 = [1\ 3\ -2\ 4]^{\mathrm{T}}$ and $m = 9$ obtained from the simulation is depicted in Figure 1(a). More details of the example can be found in Appendix D.

Conclusion. This study investigated the mean square stability of discrete-time linear systems with impulsive disturbances at random instants based on a hybrid-index model. Under the provided assumptions, the necessary and sufficient conditions for the mean square stability of the impulsive systems were obtained. Finally, an example was provided and simulations were conducted to show the effectiveness of the results. Our future work may include the extension of our results to the stabilization of the systems.

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Supporting information Appendixes A–D. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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