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# Mean square stability of discrete-time linear systems with random impulsive disturbance

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## Appendix A  Proof of Proposition 1

**Lemma 1.** Define a subset $\mathcal{M} := \{x \otimes x \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^{n^2}$. Then, the subspace Span($\mathcal{M}$) is of dimension $\frac{n(n+1)}{2}$ and the base vectors can be chosen as $\mathcal{E}_1 \cup \mathcal{E}_2$, where

$$\mathcal{E}_1 := \{\boldsymbol{e}_i \otimes \boldsymbol{e}_i \mid 1 \leqslant i \leqslant n\},$$

$$\mathcal{E}_2 := \{\boldsymbol{e}_i \otimes \boldsymbol{e}_j + \boldsymbol{e}_j \otimes \boldsymbol{e}_i \mid 1 \leqslant i < j \leqslant n\}.$$

*Proof.* First, we prove that the vectors in $\mathcal{E}_1 \cup \mathcal{E}_2$ are linearly independent. Thus, we assume that

$$\sum_{i=1}^{n} \alpha_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i + \sum_{1 \leqslant j < k \leqslant n} \beta_{j,k} \left[\boldsymbol{e}_j \otimes \boldsymbol{e}_k + \boldsymbol{e}_k \otimes \boldsymbol{e}_j\right] = 0, \tag{A1}$$

where $\alpha_i$, $1 \leqslant i \leqslant n$, and $\beta_{j,k}$, $1 \leqslant j < k \leqslant n$, are constants. We show that all these constants are zero. Notice that

$$[\boldsymbol{e}_i \otimes \boldsymbol{e}_j]^\top [\boldsymbol{e}_{\bar{i}} \otimes \boldsymbol{e}_{\bar{j}}] = \begin{cases} 1, & (i,j) = (\bar{i}, \bar{j}) \\ 0, & (i,j) \neq (\bar{i}, \bar{j}). \end{cases}$$

Multiplying the left-hand side of (A1) by $[\boldsymbol{e}_r \otimes \boldsymbol{e}_r]^\top$, $1 \leqslant r \leqslant n$, leads to $\alpha_r = 0$, $1 \leqslant r \leqslant n$. Similarly, multiplying the left-hand side of (A1) by $[\boldsymbol{e}_r \otimes \boldsymbol{e}_s]^\top - [\boldsymbol{e}_s \otimes \boldsymbol{e}_r]^\top$, $1 \leqslant r < s \leqslant n$, leads to $\beta_{r,s} = 0$, $1 \leqslant r < s \leqslant n$. Thus, the vectors in $\mathcal{E}_1 \cup \mathcal{E}_2$ are linearly independent.

Second, we prove that $\mathcal{E}_1 \cup \mathcal{E}_2 \subset$ Span($\mathcal{M}$). Evidently, $\mathcal{E}_1 \subset \mathcal{M}$. In addition, for any $i$, $j$ with $1 \leqslant i < j \leqslant n$,

$$\boldsymbol{e}_i \otimes \boldsymbol{e}_j + \boldsymbol{e}_j \otimes \boldsymbol{e}_i = (\boldsymbol{e}_i + \boldsymbol{e}_j) \otimes (\boldsymbol{e}_i + \boldsymbol{e}_j) - \boldsymbol{e}_i \otimes \boldsymbol{e}_i - \boldsymbol{e}_j \otimes \boldsymbol{e}_j.$$

Thus, $\mathcal{E}_2 \subset$ Span($\mathcal{M}$).

Finally, it remains to prove that any vector in $\mathcal{M}$ can be represented as a combination of the vectors in $\mathcal{E}_1 \cup \mathcal{E}_2$. For any $x = \sum_{i=1}^{n} x_i \boldsymbol{e}_i \in \mathbb{R}^n$, it holds that

$$x \otimes x = \left(\sum_{i=1}^{n} x_i \boldsymbol{e}_i\right) \otimes \left(\sum_{j=1}^{n} x_j \boldsymbol{e}_j\right) = \sum_{i=1}^{n} x_i^2 \boldsymbol{e}_i \otimes \boldsymbol{e}_i + \sum_{1 \leqslant i < j \leqslant n} x_i x_j \left[\boldsymbol{e}_i \otimes \boldsymbol{e}_j + \boldsymbol{e}_j \otimes \boldsymbol{e}_i\right].$$

This completes the proof. □

**Proof of Proposition 1:** Define

$$\boldsymbol{j}_t^+(x_0) := \max\{j \mid \exists t \in \overline{\mathbb{Z}^+} \text{ s.t. } (t,j) \in \mathcal{H}_{x_0}\} \quad \forall x_0 \in \mathbb{R}^n, \forall t \in \overline{\mathbb{Z}^+}$$

For a solution $x(t, j; x_0)$, we select the sequence of after-jump states at the impulsive instants $\tau_\alpha$ as

$$\zeta_\alpha(x_0) := x(\tau_\alpha, \boldsymbol{j}_{\tau_\alpha}^+; x_0), \quad \alpha \in \overline{\mathbb{Z}^+}.$$

By definition, $\zeta_\alpha$ is subject to the dynamical system

$$\zeta_{\alpha+1} = JA^{\omega_\alpha} \zeta_\alpha \tag{A2}$$

with the initial state $\zeta_0 = JA^{\tau_0} x_0$. Thus,

$$\zeta_{\alpha+1} \zeta_{\alpha+1}^\top = JA^{\omega_\alpha} \zeta_\alpha \zeta_\alpha^\top (JA^{\omega_\alpha})^\top$$

and

$$\Sigma_{\alpha+1} = \mathbb{E}\left(\zeta_{\alpha+1} \zeta_{\alpha+1}^\top\right) = \mathbb{E}\left[JA^{\omega_\alpha} \zeta_\alpha \zeta_\alpha^\top \left(JA^{\omega_\alpha}\right)^\top\right].$$

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Define $\eta_\alpha := \mathrm{Vec}(\Sigma_\alpha)$. Then,

$$\begin{aligned}\eta_{\alpha+1} &= \mathbb{E}\left[\mathrm{Vec}(JA^{\omega_\alpha}\zeta_\alpha\zeta_\alpha^\top(JA^{\omega_\alpha})^\top)\right]\\ &= \mathbb{E}\left[(JA^{\omega_\alpha})\otimes(JA^{\omega_\alpha})\,\mathrm{Vec}(\zeta_\alpha\zeta_\alpha^\top)\right]\\ &= \mathbb{E}\left[(J\otimes J)(A\otimes A)^{\omega_\alpha}\right]\cdot\eta_\alpha\\ &= \mathbb{Q}_\omega\eta_\alpha,\end{aligned}$$

$$\begin{aligned}\eta_0 &= \mathrm{Vec}(\Sigma_0)\\ &= \mathbb{E}\left[\mathrm{Vec}(JA^{\tau_0}x_0x_0^\top(JA^{\tau_0})^\top)\right]\\ &= \mathbb{E}\left[(JA^{\tau_0})\otimes(JA^{\tau_0})\,\mathrm{Vec}(x_0x_0^\top)\right]\\ &= \mathbb{E}\left[(J\otimes J)(A\otimes A)^{\tau_0}\right]\cdot(x_0\otimes x_0)\\ &= \mathbb{Q}_{\tau_0}(x_0\otimes x_0).\end{aligned}$$

Thus,

$$\eta_\alpha = (\mathbb{Q}_\omega)^\alpha\mathbb{Q}_{\tau_0}(x_0\otimes x_0).$$

Consequently, $\lim_{\alpha\to\infty}\Sigma_\alpha = 0 \quad \forall x_0\in\mathbb{R}^n$ if and only if

$$\lim_{\alpha\to\infty}(\mathbb{Q}_\omega)^\alpha\mathbb{Q}_{\tau_0}(x_0\otimes x_0) = 0 \quad \forall x_0\in\mathbb{R}^n. \tag{A3}$$

Here, (A3) is equivalent to

$$\lim_{\alpha\to\infty}\mathbb{Q}_\omega^\alpha\mathbb{Q}_{\tau_0}z = 0 \quad \forall z\in\mathrm{Span}(\mathcal{M}), \tag{A4}$$

where $\mathcal{M}$ is defined in Lemma 1.By Lemma 1, it holds that $\mathrm{Span}(\mathcal{M}) = \mathrm{Span}(\Psi)$. Thus, (A4) is equivalent to $\lim_{\alpha\to\infty}\mathbb{Q}_\omega^\alpha\mathbb{Q}_{\tau_0}\Psi = 0$. □

## Appendix B Proof of Theorem 1

By Proposition 1, the necessity is clearly true because $\zeta_\alpha$ is a subsequence of $x(t)$. We prove the sufficiency in the following. Suppose that (4) holds. We assume that $\tau_{\alpha^*}$ is the any impulse instant. By the definition of $\omega_\alpha$, and for $k$, $t_k$ is the next nonimpulse instant, thus, $\alpha^*$ is the number of impulses that occur within $[0 : t_k - 1]$, it holds that $t_k - \tau_{\alpha^*} \leqslant \omega_\alpha$. Note that

$$x_k(x_0) = A^{t_k-\tau_{\alpha^*}}\zeta_{\alpha^*}(x_0).$$

According to Assumption 2, it holds that $\omega_\alpha \leqslant \boldsymbol{\omega}_m$. We denote

$$\rho_m := \max_{1\leqslant\omega\leqslant\boldsymbol{\omega}_m}\rho\left((A^\omega)^\top A^\omega\right).$$

Then,

$$\begin{aligned}x_k^\top x_k &= \zeta_{\alpha^*}^\top\left(A^{t_k-\tau_{\alpha^*}}\right)^\top A^{t_k-\tau_{\alpha^*}}\zeta_{\alpha^*}\\ &\leqslant \rho\left[\left(A^{t_k-\tau_{\alpha^*}}\right)^\top A^{t_k-\tau_{\alpha^*}}\right]\cdot\zeta_{\alpha^*}^\top\zeta_{\alpha^*}\\ &\leqslant \rho_m\zeta_{\alpha^*}^\top\zeta_{\alpha^*}.\end{aligned} \tag{B1}$$

By the boundedness of $\omega_\alpha$ and $\tau_0$, we have $\alpha^*\to\infty$ as $k\to\infty$. Thus, by Proposition 1,

$$\lim_{k\to\infty}\mathbb{E}x_k^\top x_k \leqslant \rho_m\cdot\lim_{k\to\infty}\mathbb{E}\zeta_{\alpha^*}^\top\zeta_{\alpha^*} = 0.$$

□

## Appendix C Proof of Theorem 2

We only need to prove that (4) is equivalent to (5). We denote by $\mathcal{S}_\omega$ the smallest invariant subspace of the system $\eta_{\alpha+1} = \mathbb{Q}_\omega\eta_\alpha$ such that $\mathrm{Col}(\mathbb{Q}_{\tau_0}\Psi)\subset\mathcal{S}_\omega$, i.e.,

$$\mathcal{S}_\omega := \mathrm{Span}\left\{y\in\mathbb{R}^{n^2} \mid \exists s\in\overline{\mathbb{Z}^+}\ \text{s.t.}\ y\in\mathrm{Col}(\mathbb{Q}_\omega^s\mathbb{Q}_{\tau_0}\Psi)\right\}.$$

According to linear system theory, (4) holds if and only if the dynamics of $\eta_{\alpha+1} = \mathbb{Q}_\omega\eta_\alpha$ restricted on $\mathcal{S}_\omega$ are asymptotically stable. By the Hamilton-Cayley theorem, it is evident that $\mathcal{S}_\omega = \mathrm{Span}(\mathbf{C})$. We can always find a set of column vectors $\boldsymbol{\gamma}_j\in\mathbb{R}^{n^2}$, $j = 1,2,\cdots,n^2-\boldsymbol{r}_c$, such that $\boldsymbol{B}^\top\boldsymbol{\Gamma} = 0$, where $\boldsymbol{\Gamma} = [\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\cdots,\boldsymbol{\gamma}_{n^2-\boldsymbol{r}_c}]$. Note that, because $\mathcal{S}_\omega = \mathrm{Span}(\mathbf{B})$ is an invariant subspace of $\mathbb{Q}_\omega$, it holds that $\mathrm{Col}(\mathbb{Q}_\omega\mathbf{B})\subseteq\mathcal{S}_\omega$. Thus, we have $\boldsymbol{\Gamma}^\top\mathbb{Q}_\omega\mathbf{B} = 0$. Define

$$\begin{bmatrix}\kappa_\alpha\\ \xi_\alpha\end{bmatrix} = \begin{bmatrix}\mathbf{B} & \boldsymbol{\Gamma}\end{bmatrix}^{-1}\eta_\alpha$$

with $\kappa_\alpha \in \mathbb{R}^{r_c}$, $\xi_\alpha \in \mathbb{R}^{n^2-r_c}$. Then,

$$\begin{bmatrix} \kappa_{\alpha+1} \\ \xi_{\alpha+1} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{\Gamma} \end{bmatrix}^{-1} \mathbb{Q}_\omega \begin{bmatrix} \mathbf{B} & \mathbf{\Gamma} \end{bmatrix} \begin{bmatrix} \kappa_\alpha \\ \xi_\alpha \end{bmatrix}.$$

Note that

$$\begin{bmatrix} \mathbf{B}^\top \\ \mathbf{\Gamma}^\top \end{bmatrix} \begin{bmatrix} \mathbf{B} & \mathbf{\Gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{B}^\top\mathbf{B} & 0 \\ 0 & \mathbf{\Gamma}^\top\mathbf{\Gamma} \end{bmatrix}.$$

It holds that

$$\begin{bmatrix} \mathbf{B} & \mathbf{\Gamma} \end{bmatrix}^{-1}_{n^2\times n^2} = \begin{bmatrix} (\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top & 0 \\ 0 & (\mathbf{\Gamma}^\top\mathbf{\Gamma})^{-1}\mathbf{\Gamma}^\top \end{bmatrix}_{n^2\times n^2}.$$

Thus, by using $\mathbf{\Gamma}^\top \mathbb{Q}_\omega \mathbf{B} = 0$,

$$\begin{bmatrix} \mathbf{B} & \mathbf{\Gamma} \end{bmatrix}^{-1} \mathbb{Q}_\omega \begin{bmatrix} \mathbf{B} & \mathbf{\Gamma} \end{bmatrix} = \begin{bmatrix} (\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top\mathbb{Q}_\omega\mathbf{B} & 0 \\ 0 & (\mathbf{\Gamma}^\top\mathbf{\Gamma})^{-1}\mathbf{\Gamma}^\top\mathbb{Q}_\omega\mathbf{\Gamma} \end{bmatrix}.$$

As a result,

$$\begin{bmatrix} \kappa_{\alpha+1} \\ \xi_{\alpha+1} \end{bmatrix} = \begin{bmatrix} (\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top\mathbb{Q}_\omega\mathbf{B} & 0 \\ 0 & (\mathbf{\Gamma}^\top\mathbf{\Gamma})^{-1}\mathbf{\Gamma}^\top\mathbb{Q}_\omega\mathbf{\Gamma} \end{bmatrix} \begin{bmatrix} \kappa_\alpha \\ \xi_\alpha \end{bmatrix}.$$

Thus, the dynamics of $\eta_{\alpha+1} = \mathbb{Q}_\omega \eta_\alpha$ restricted on $\mathcal{S}_\omega$ are

$$\kappa_{\alpha+1} = (\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top\mathbb{Q}_\omega\mathbf{B}\kappa_\alpha. \tag{C1}$$

Hence, (4) holds if and only if the above system is asymptotically stable, that is, the system matrix $(\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top\mathbb{Q}_\omega\mathbf{B}$ is strictly Schur stable. □

## Appendix D Illustrative Example

We consider the discrete-time system with a random impulsive disturbance where

$$A = \begin{bmatrix} 1.1 & 0.3 & 0.3 & 0.3 \\ 0 & 0.8 & 0 & 0 \\ 0.3 & 0.3 & 1.1 & 0.3 \\ -0.3 & -0.3 & -0.3 & 0.5 \end{bmatrix}, J = \begin{bmatrix} 0.4 & -1.1 & -1.1 & -1.1 \\ 0 & 1.5 & 0.5 & 0.5 \\ -0.6 & -1.1 & -0.1 & -1.1 \\ 0.6 & 1.1 & 1.1 & 2.1 \end{bmatrix}.$$

Note that both the underlying linear system without impulsive disturbance and the jumping process are unstable because $\rho(A) = 1.1 > 1$, $\rho(J) = 1.5 > 1$. We show that the impulsive system is mean-square stable if $m$ satisfies a certain condition. First, we considered the case $m = 9$, we calculated matrix $\mathbf{C}$, which is omitted here to save space, and obtained $\text{rank}(\mathbf{C}) = 10$. In this case,

$$\mathbf{H} \approx \begin{bmatrix} 0.4671 & 0.2352 & 0.2352 & 0.2352 & 0.2658 & 0.2658 & 0.2658 & 0.4704 & 0.4704 & 0.4704 \\ 0.0000 & 0.4364 & 0.0485 & 0.0485 & 0.0000 & 0.0000 & 0.0000 & 0.2910 & 0.2910 & 0.0970 \\ 0.2155 & 0.2352 & 0.2928 & 0.2352 & 0.4022 & 0.4598 & 0.4022 & 0.3340 & 0.4704 & 0.3340 \\ 0.2155 & 0.2352 & 0.2352 & 0.5655 & 0.4022 & 0.4022 & 0.3446 & 0.4704 & 0.6068 & 0.6068 \\ 0.0000 & -0.1023 & -0.0341 & -0.0341 & 0.3342 & 0.1114 & 0.1114 & -0.1364 & -0.1364 & -0.0682 \\ 0.2443 & 0.2352 & 0.1670 & 0.2352 & 0.3340 & 0.5568 & 0.3340 & 0.4022 & 0.4704 & 0.4022 \\ -0.2443 & -0.2352 & -0.2352 & -0.3034 & -0.3340 & -0.3340 & -0.1113 & -0.4704 & -0.5386 & -0.5386 \\ -0.0000 & -0.1023 & 0.0629 & -0.0341 & 0.0432 & 0.0144 & 0.0144 & 0.1546 & -0.1364 & 0.0288 \\ 0.0000 & 0.1023 & 0.0341 & 0.1311 & -0.0432 & -0.0144 & -0.0144 & 0.1364 & 0.4273 & 0.1652 \\ -0.2155 & -0.2352 & -0.1670 & -0.3034 & -0.4022 & -0.4310 & -0.3734 & -0.4022 & -0.5386 & -0.2764 \end{bmatrix} \tag{D1}$$

and $\rho(\mathbf{H}) \approx 0.4671 < 1$. The result have been shown before.