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• **Supplementary File** •

Event-triggered state estimation for cyber-physical systems with partially observed injection attacks

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Appendix A Notations

$\mathbb{R}$, $\mathbb{R}^n$ $\mathbb{R}^{m\times n}$ are the set of real numbers, the set of $n \times 1$ real vectors and the set of $m \times n$ real matrices respectively. $N(\mu, \Sigma)$ represents a Gaussian distribution with mean μ and covariance matrix Σ. $\Pr(\cdot)$ stands for the probability of a random event. $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ are the sets of $n \times n$ positive semidefinite and positive definite matrices. When $X \in \mathbb{S}^n_+$, we simply write $X \geqslant 0$ ($X > 0$ if $X \in \mathbb{S}^n_{++}$). $\mathbb{E}(\cdot)$ represents the expectation of a random event. O denotes a zero matrix with appropriate dimensions. $\det(\cdot)$ is the determinant of a matrix. $x \propto y$ means x is proportional to y.

Appendix B An example: estimation of traffic densities

Intelligent transport systems for traffic surveillance require some fundamental information including traffic density. Traffic density is defined as the number of vehicles that occupy one unit length of road space per lane. Here we focus on a road segment with n lanes that is a detection zone with an upstream detector and a downstream detector at the entrance and exit of each lane respectively. The two detectors count the vehicles passing through. See [1] for a detailed description of the detectors. $x_{i,k+1} = x_{i,k} + \tilde{d}_{i,k} + u_{i,k} + \omega_{i,k}$ is commonly used in the literature such as [2] to describe the traffic conservation, where $x_{i,k}$ denotes the total number of vehicles in lane i at time step k, and $u_{i,k}$ represents the difference in the numbers of vehicles that enter and leave the upstream and down stream detectors of lane i. The quantity $u_{i,k}$ is directly available from the detectors. However, usually no sensors are installed within a freeway segment. Hence $\tilde{d}_{i,k}$, the vehicles' net gain due to lane-change maneuvers, is not observed in the equation. We note, however, the net gain of lane-changing vehicles aggregated across all the lanes is equal to zero due to traffic conservation (We understand this behavior as an injection attack that destroys estimates), i.e. $\sum_{i=1}^{n} \tilde{d}_{i,k} = 0$. Consequently, the input variables defined as $d_{i,k} = \tilde{d}_{i,k} + u_{i,k}$ are observable at the aggregate level (the segment level).

Appendix C Discussions on problem formulation

To be more specific, at every time step k, the sensor generates an independent and identically distributed (i.i.d.) random variable ξ_k, which is uniformly distributed over [0, 1]. Then the sensor compares ξ_k with a function $\phi(y_k) : \mathbb{R}^m \to [0, 1]$. The sensor transmits if and only if $\xi_k > \phi(y_k)$. Since ξ_k is uniformly distributed, one can interpret $\phi(y_k)$ as the probability of idle and $1 - \phi(y_k)$ as the probability of transmitting for the sensor. The main purpose of the event-triggered mechanism here is to save communication cost. Moreover, when an attacker tries to eavesdrop measurements, an event-triggered mechanism leverages less information.

Even though the event triggered policy form in [5] is more general than the form in this paper, our form needs the lower memory cost and computation cost. At the same time, both the forms use the Gaussianity-preserving property as discussed in [5]. Our result can be extended to the result in [5] trivially.

Appendix D Proof of Theorem 1

To directly estimate the state under injection attacks is a tough problem. To begin with, we need some lemmas which are introduced as follows:

Lemma 1. (1) Given $\Gamma \in \mathbb{R}^{n_1\times n_3}$, $V \in \mathbb{S}^{n_1}_{++}$, $\Psi \in \mathbb{R}^{n_2\times n_3}$, $W \in \mathbb{S}^{n_2}_{++}$. Let x, a, b be vectors with appropriate dimensions. If $V > 0$, $W > 0$, and the matrix $[\Gamma^T, \Psi^T]^T$ has a full column-rank, then the following statement holds:

$$\begin{aligned}&(\Gamma x + a)^T V(\Gamma x + b) + (\Psi x + b)^T W(\Psi x + d)\\ =&[x + (\Gamma^T V\Gamma + \Psi^T W\Psi)^{-1}(\Gamma^T V a + \Psi^T W b)]^T(\Gamma^T V\Gamma + \Psi^T W\Psi)\\ &\times [x + (\Gamma^\top V\Gamma + \Psi^\top W\Psi)^{-1}(\Gamma^\top V a + \Psi^\top W b)] + \star,\end{aligned} \tag{D1}$$

where $\star$ indicates a term uncorrelated with x.

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(2) If $V \in \mathbb{S}_{++}^{n_1}$ and $W \in \mathbb{S}_{++}^{n_2}$, then

$$\begin{aligned} & x^T V x + (\Psi x - b)^T W (\Psi x - b) \\ = & \left[x - (V + \Psi^T W \Psi)^{-1} \Psi^T W b \right]^T (V + \Psi^T W \Psi) \\ & \times [x - (V + \Psi^T W \Psi)^{-1} \Psi^T W b] + b^T \left(\Psi V^{-1} \Psi^T + W^{-1} \right)^{-1} b. \end{aligned} \tag{D2}$$

Proof. The first expression can be verified by direct manipulations and observing that $(\Gamma^T V \Gamma + \Psi^T W \Psi)$ is nonsingular. $(\Gamma^T V \Gamma + \Psi^T W \Psi)$ is nonsingular if $[\Gamma^T, \Psi^T]^T$ has a full column-rank because of the truth that $(\Gamma^T V \Gamma + \Psi^T W \Psi) = [\Gamma^T, \Psi^T] \begin{bmatrix} V & 0 \\ 0 & W \end{bmatrix} [\Gamma^T, \Psi^T]^T$. The second expression can be acquired by direct manipulations with the matrix inversion lemma.

Lemma 2. Let $x \in \mathbb{R}^n$ be a variable with Gaussian distribution and $([\Gamma^T, \tilde{\Gamma}^T])$ be an $n \times n$ invertible partitioned matrix. Assume that $\beta = \tilde{\Gamma} x$ and $\alpha = \Gamma x$, where β follows $N(\mu, \tilde{W})$ and α follows $N(\omega, W)$. Both ω and $W > 0$ are known. Provided hyperparameter vector μ has a noninformative distribution, then

$$f(x) \propto f(\Gamma x). \tag{D3}$$

Proof. Let $f(\mu)$ denote the marginal distribution function of μ. Note that for a given μ, $[\Gamma^T, \tilde{\Gamma}^T]^T x$ follows $N\left(\begin{pmatrix} \omega \\ \mu \end{pmatrix}, \begin{pmatrix} W & \tilde{C} \\ \tilde{C}^T & \tilde{W} \end{pmatrix} \right)$, where $\tilde{C}$ represents the covariance matrix. Then,

$$\begin{aligned} f([\Gamma^T, \tilde{\Gamma}^T]^T x) &= \int f([\Gamma^T, \tilde{\Gamma}^T]^T x | \mu) f(\mu) \, d\mu \\ &\propto \int f([\Gamma^T, \tilde{\Gamma}^T]^T x | \mu) \, d\mu \\ &= f(\Gamma x), \end{aligned} \tag{D4}$$

where the final equality can be deduced by using marginal properties of multivariate Gaussian distributions . Since $f(x) \propto f([\Gamma^T, \tilde{\Gamma}^T]^T x)$, we finally get $f(x) \propto f(\Gamma x)$.

Remark 1. This lemma provides the direction for the design of subsequent filters. In other words, through the Bayesian law, we can transform the filtering problem of the original problem into a partially fully known filtering problem, which will be shown in the proof of Theorem 1.

The following result provides an optimal estimation $\hat{x}_k$ based on $\mathcal{I}_k$. First, let S_k denote $\begin{pmatrix} M_{k-1} \\ N_k C_k \end{pmatrix}$. Under the conditions $Q_k > 0$ and $R_k > 0$, we obtain that $P_{k|k}^{-1}$ is non-singular if S_k has a full column-rank. Since

$$S_k = \begin{pmatrix} D_{k-1} & O \\ O & I \\ N_k C_k G_{k-1} & N_k C_k G_{k-1}^{\perp} \end{pmatrix},$$

the condition that Π_k has a full column-rank is equivalent to S_k has a full column-rank. In the next steps, we will verify Theorem 1 by induction. First, the result is assumed to be true at $k-1$. Second, define $s_k = M_{k-1} x_k$. Noting $M_{k-1} G_{k-1} d_{k-1} = \tilde{r}_{k-1}$, it can be derived that $s_k = M_{k-1} A_{k-1} x_{k-1} + \tilde{r}_{k-1} + M_{k-1} \omega_{k-1}$. Then, by the induction hypothesis, one can obtain

$$s_k \sim N\left(M_{k-1} A_{k-1} \hat{x}_{k-1|k-1} + \tilde{r}_{k-1}, \tilde{P}_{k|k-1} \right), \tag{D5}$$

where

$$\tilde{P}_{k|k-1} = M_{k-1} P_{k|k-1} M_{k-1}^T.$$

Then, consider the case when $\gamma_k = 1$. Due to Bayesian Law, it is true that

$$f(x_k|\mathcal{I}_k) = f(x_k|y_k, \mathcal{I}_{k-1}) \propto f(y_k|x_k, \mathcal{I}_{k-1}) f(x_k|\mathcal{I}_{k-1})$$

Define $\tilde{M}_{k-1} = [F_{0k-1}^T, 0] \Omega_{k-1}^{-1}$and $\tilde{s}_k = \tilde{M}_{k-1} x_k = \tilde{M}_{k-1} A_{k-1} \hat{x}_{k-1|k-1} + \delta_{k-1}^x + \tilde{M}_{k-1} \omega_{k-1}$. Noting that $[M_{k-1}^T, \tilde{M}_{k-1}^T]$ is a $n \times n$ invertible matrix, it follows from the Lemma 2 that $f(x_k|\mathcal{I}_{k-1}) \propto f(s_k|\mathcal{I}_{k-1})$. With similar derivations, one can also obtain $f(y_k|x_k, \mathcal{I}_{k-1}) \propto f(N_k y_k|x_k, \mathcal{I}_{k-1})$. Hence, it can be gotten that

$$f(x_k|\mathcal{I}_k) \propto f(N_k y_k|x_k, \mathcal{I}_{k-1}) f(s_k|\mathcal{I}_{k-1}).$$

Then, it can be derived that

$$\begin{aligned} f(x_k|\mathcal{I}_k) &\propto f(N_k y_k|x_k, \mathcal{I}_{k-1}) f(s_k|\mathcal{I}_{k-1}) \\ &\propto \exp\{-\frac{1}{2}(M_{k-1} x_k - M_{k-1} A_{k-1} \hat{x}_{k-1} - \tilde{r}_{k-1})^T (\tilde{P}_{k|k-1})^{-1} (M_{k-1} x_k - M_{k-1} A_{k-1} \hat{x}_{k-1} - \tilde{r}_{k-1})\} \\ &\quad \times \exp\{-\frac{1}{2}(N_k y_k - N_k C_k x_k - \tilde{q}_k)^T V_k (N_k y_k - N_k C_k x_k - \tilde{q}_k)\}, \end{aligned} \tag{D6}$$

Finally by the item 1 of Lemma 1 and noticing that S_k has a full column-rank, the above equation further implies that the conditional distribution of x_k is a Gaussian distribution with mean

$$\hat{x}_{k|k} = P_{k|k}\{M_{k-1}^T(M_{k-1}P_{k|k-1}M_{k-1}^T)^{-1}[M_{k-1}A_{k-1}\hat{x}_{k-1|k-1} + \tilde{r}_{k-1}] + (N_kC_k)^TV_k(N_ky_k - \tilde{q}_k)\} \tag{D7}$$

and covariance matrix

$$P_{k|k} = [M_{k-1}^T(M_{k-1}P_{k|k-1}M_{k-1}^T)^{-1}M_{k-1}(N_kC_k)^TV_kN_kC_k]^{-1}. \tag{D8}$$

Define $K_k = P_{k|k}(N_kC_k)^TV_k$. The above conditional mean can be rearranged to have the form of (7).

For $\gamma_k = 0$, it follows form the the Bayesian law that

$$\begin{aligned} f(x_k|\mathcal{I}_k) &= f(x_k|\gamma_k = 0, \mathcal{I}_{k-1}) \\ &\propto \Pr(\gamma_k = 0|x_k, \mathcal{I}_{k-1})f(x_k|\mathcal{I}_{k-1}). \end{aligned}$$

By Lemma 2 and noting that the fact $\Pr(\gamma_k = 0)$ is determined by $\mathcal{I}_{k-1}$ and y_k, one can have

$$\begin{aligned} f(x_k|\mathcal{I}_k) &\propto f(s_k|\mathcal{I}_{k-1})\int_{\mathbb{R}^p} \Pr(\gamma_k = 0|x_k, y_k, \mathcal{I}_{k-1})f(y_k|x_k, \mathcal{I}_{k-1})\,dy_k \\ &= f(s_k|\mathcal{I}_{k-1})\int_{\mathbb{R}^p} \Pr(\gamma_k = 0|y_k, \mathcal{I}_{k-1})f(N_ky_k|x_k, \mathcal{I}_{k-1})\,dy_k \\ &= f(s_k|\mathcal{I}_{k-1})\int_{\mathbb{R}^p} \exp\{-\frac{1}{2}y_k^TY_ky_k\}\exp\{-\frac{1}{2}(N_ky_k - N_kC_kx_k - \tilde{q}_k)^TV_k(N_ky_k - N_kC_kx_k - \tilde{q}_k)\}\,dy_k \\ &= f(s_k|\mathcal{I}_{k-1})\int_{\mathbb{R}^p} \exp\{-\frac{1}{2}[(y_k - W_kN_k^TV_k(N_kC_kx_k + \tilde{q}_k)^TW_k^{-1}(y_k - W_kN_k^TV_k(N_kC_kx_k + \tilde{q}_k)]dy_k \\ &\quad\times \exp\{-\frac{1}{2}(N_kC_kx_k + \tilde{q}_k)^T(N_kY_k^{-1}N_k^T + V_k^{-1})^{-1}(N_kC_kx_k + \tilde{q}_k)\} \\ &\propto f(s_k|\mathcal{I}_{k-1})\exp\{-\frac{1}{2}(N_kC_kx_k + \tilde{q}_k)^T(N_kY_k^{-1}N_k^T + V_k^{-1})^{-1}(N_kC_kx_k + \tilde{q}_k)\}, \end{aligned} \tag{D9}$$

where the second to last equality is due to item 2 of Lemma 1. Then, by item 1 of Lemma 1 and noticing that S_k has full column-rank, it is true that

$$\begin{aligned} f(x_k|\mathcal{I}_k) &\propto \exp\{-\frac{1}{2}(M_{k-1}x_k - M_{k-1}A_{k-1}\hat{x}_{k-1} - \tilde{r}_{k-1})^T(\tilde{P}_{k|k-1})^{-1}(M_{k-1}x_k - M_{k-1}A_{k-1}\hat{x}_{k-1} - \tilde{r}_{k-1})\} \\ &\quad\times \exp\{-\frac{1}{2}(N_kC_kx_k + \tilde{q}_k)^T(N_kY_k^{-1}N_k^T + V_k^{-1})^{-1}(N_kC_kx_k + \tilde{q}_k)\} \\ &= \exp\{-\frac{1}{2}(x_k - \hat{x}_{k|k})^TP_{k|k}^{-1}(x_k - \hat{x}_{k|k})\} \end{aligned} \tag{D10}$$

with

$$\hat{x}_{k|k} = P_{k|k}\{M_{k-1}^T(M_{k-1}P_{k|k-1}M_{k-1}^T)^{-1}[M_{k-1}A_{k-1}\hat{x}_{k-1|k-1} + \tilde{r}_{k-1}] - (N_kC_k)^T(V_k - V_kN_kW_kN_k^TV_k)\tilde{q}_k\} \tag{D11}$$

and covariance matrix

$$P_{k|k} = [M_{k-1}^T(M_{k-1}P_{k|k-1}M_{k-1}^T)^{-1}M_{k-1} + (N_kC_k)^T(V_k - V_kN_kW_kN_k^TV_k)N_kC_k]^{-1}. \tag{D12}$$

Define $K_k = P_{k|k}(N_kC_k)^T(V_k - V_kN_kW_kN_k^TV_k)$, the above conditional mean can be directly rearranged to have the form of (7).

In addition, by noting that the results where x_0 follows $(\hat{x}_0, P_{0|0})$ with $\hat{x}_{0|0} = \hat{x}_0$ and $P_{0|0} = P_0$ respectively, the inductive proof is completed.

However, computation of covariance matrix involves the inverses of two matrix. When n is large, the computational cost is high. So we turn to handle the computation issue in the following part.

Remark 2. By the standard result on optimal filtering, it is well known that the mean of the conditional distribution is the MMSE estimation. Notice that $x_k|\mathcal{I}_k$ follows a gaussian distribution, which shows that the conditional mean $\hat{x}_{k|k}$ is the MMSE estimator of x_k.

Lemma 3. ([3], Lemma 2) Let $P > 0$, $R > 0$ and F be a matrix such that $DF = O$. Provided that both the matrix D^T and matrix $[C^T, D^T]^T$ have full column-rank, then it can be shown that

$$\left[D^T\left(DPD^T\right)^{-1}D + C^TR^{-1}C\right]^{-1} = P - PC^TH^{-1}CP + \left[F - PC^TH^{-1}CF\right] \times \left[F^TC^TH^{-1}CF\right]^{-1}\left[F - PC^TH^{-1}CF\right]^T. \tag{D13}$$

$$\left[D^T\left(DPD^T\right)^{-1}D + C^TR^{-1}C\right]^{-1}C^TR^{-1} = PC^TH^{-1} + \left[F - PC^TH^{-1}CF\right] \times \left[F^TC^TH^{-1}CF\right]^{-1}F^TC^TH^{-1}, \tag{D14}$$

where $H = CPC^T + R$.

Notice that K_k and $P_{k|k}$ can be written in the following form:

$$K_k = P_{k|k}(N_kC_k)^T(V_k - (1-\gamma_k)V_kN_kW_kN_k^TV_k) \tag{D15}$$

and

$$P_{k|k} = [M_{k-1}^T(M_{k-1}P_{k|k-1}M_{k-1}^T)^{-1}M_{k-1} + (N_kC_k)^T(V_k - (1-\gamma_k)V_kN_kW_kN_k^TV_k)N_kC_k]^{-1}. \tag{D16}$$

Then by Lemma 3, the following statement is true.

$$\begin{aligned} K_k =& P_{k|k-1}(N_kC_k)^T\Theta_k^{-1} + \left[F_{k-1} - P_{k|k-1}(N_kC_k)^T\Theta_k^{-1}N_kC_kF_{k-1}\right] \\ &\times \left[F_{k-1}^T(N_kC_k)^T\Theta_k^{-1}N_kC_kF_{k-1}\right]^{-1}F_{k-1}^TC_k^T\Theta_k^{-1}, \end{aligned} \tag{D17}$$

$$\begin{aligned} P_{k|k} =& P_{k|k-1} - P_{k|k-1}(N_kC_k)^T\Theta_k^{-1}N_kC_kP_{k|k-1} \\ &+ \left[F_{k-1} - P_{k|k-1}(N_kC_k)^T\Theta_k^{-1}N_kC_kF_{k-1}\right] \\ &\times \left[F_{k-1}^T(N_kC_k)^T\Theta_k^{-1}N_kC_kF_{k-1}\right]^{-1} \\ &\times \left[F_{k-1} - P_{k|k-1}(N_kC_k)^T\Theta_k^{-1}N_kC_kF_{k-1}\right]^T, \end{aligned} \tag{D18}$$

with $\Theta_k = N_kC_kP_{k|k-1}(N_kC_k)^T + (V_k - (1-\gamma_k)V_kN_kW_kN_k^TV_k)^{-1}$, which completes the proof.

Appendix E Stability analysis

In this section, we investigate the stability of the proposed event-triggered estimator. In other words, the purpose here is to evaluate the asymptotic property of the error covariance matrix (11) as $k \to \infty$. As it is shown in [5] that the estimation error matrix $P_{k|k}$ becomes a stochastic process due to the event-triggering signal sequences $\{\gamma_k\}$. Inspired by [5], we define a new Gaussian system model With the help of the new system, we can give a certain condition that guarantee the stability of the origin one:

$$x_{k+1} = A_kx_k + G_kd_k + \omega_k, \tag{E1}$$
$$\breve{y}_k = N_kC_kx_k + \breve{q}_k + \breve{v}_k, \tag{E2}$$

with the same assumptions on x_0, ω_k and d_k as those of Section II. $\breve{v}_k$ is a Gaussian variable with zero mean and covariance matrix $\breve{R}_k = (V_k - (1-\gamma_k)V_kN_kW_kN_k^TV_k)$.

With the model above, the time-based estimation can be written in the following form:

$$\begin{aligned} \hat{x}_{k|k} =& A_{k-1}\hat{x}_{k-1|k-1} + \breve{P}_{k|k}M_{k-1}^T(M_{k-1}\breve{P}_{k|k-1}M_{k-1}^T)^{-1} \\ &\times \tilde{r}_{k-1} + \breve{K}_k(\breve{y}_k - \tilde{q}_k - N_kC_kA_{k-1}\hat{x}_{k-1|k-1}), \end{aligned} \tag{E3}$$

the gain matrix

$$\breve{K}_k = \breve{P}_{k|k}(N_kC_k)^T\breve{R}_k^{-1}, \tag{E4}$$

and the conditional covariance matrix is given by

$$\breve{P}_{k|k} = [M_{k-1}^T(M_{k-1}\breve{P}_{k|k-1}M_{k-1}^T)^{-1}M_{k-1} + (N_kC_k)^T\tilde{R}_kN_kC_k]^{-1}, \tag{E5}$$

$$\breve{P}_{k|k-1} = A_{k-1}\breve{P}_{k-1|k-1}A_{k-1}^T + Q_k. \tag{E6}$$

Theorem 1. The estimation error $e_k = x_k - \hat{x}_{k|k}$ of the filter (E3)-(E5) follows the recursive equation.

$$e_k = \left(A_{k-1} - \breve{K}_kN_kC_kA_{k-1}\right)e_{k-1} + \left[I - \breve{K}_kN_kC_k, -\breve{K}_k\right][\omega_{k-1}, v_k]^T. \tag{E7}$$

Proof. Let $L_{k-1} = \breve{P}_{k|k}M_{k-1}^T\left(M_{k-1}\breve{P}_{k|k-1}M_{k-1}^T\right)^{-1}$, the error dynamics are given by

$$\begin{aligned} e_k =& A_{k-1}x_{k-1} + G_{k-1}d_{k-1} + \omega_{k-1} - A_{k-1}\hat{x}_{k-1|k-1} \\ &- L_{k-1}\tilde{r}_{k-1} - \breve{K}_k\left(\tilde{y}_k - \tilde{q}_k - N_kC_kA_{k-1}\hat{x}_{k-1|k-1}\right) \\ =& \left(A_{k-1} - \breve{K}_kN_kC_kA_{k-1}\right)e_{k-1} + \left(G_{k-1} - \breve{K}_kN_kC_kG_{k-1}\right)d_{k-1} \\ &- L_{k-1}\tilde{r}_{k-1} + \left(I - \breve{K}_kN_kC_k\right)\omega_{k-1} - \breve{K}_k\breve{v}_k. \end{aligned} \tag{E8}$$

Noting that $\tilde{r}_{k-1} = \tilde{M}_{k-1}G_{k-1}d_{k-1}$, it can be obtained that

$$(G_{k-1} - K_kN_kC_kG_{k-1})d_{k-1} - L_{k-1}\tilde{r}_{k-1} = [I - K_kN_kC_k - L_{k-1}M_{k-1}]G_{k-1}d_{k-1}. \tag{E9}$$

Inserting (E4) and (E5) to (E9), we can obtain (E7) by noting that $I - K_kN_kC_k - L_{k-1}M_{k-1} = O$, which completes the proof.

Now we can represent the covariance matrix update equation as

$$\begin{aligned}\breve{P}_{k|k} =&\phi(\breve{K}_k, \breve{P}_{k-1|k-1})\\ =&\left(A_{k-1} - \breve{K}_k N_k C_k A_{k-1}\right) P_{k-1|k-1} \left(A_{k-1} - \breve{K}_k N_k C_k A_{k-1}\right)^T\\ &+\left(I - \breve{K}_k N_k C_k\right) Q_{k-1} \left(I - \breve{K}_k N_k C_k\right)^T + \breve{K}_k R_k \breve{K}_k^T.\end{aligned} \tag{E10}$$

In this way, we can give the condition that guarantee the stability.

Next, we will explain Theorem 2 separately in the following two lemmas.

Lemma 4. If there exists $\tilde{K}_k$ such that$A_k - \tilde{K}_k N_{k+1} C_{k+1} A_k$ are exponentially stable for every k and Π_k has a full column-rank, the covariance matrix satisfying (E10) is asymptotically bounded. Consequently, for the event-triggered estimator in (9)-(9), the estimation error covariance matrix $P_{k|k}$ is asymptotically bounded and the closed-loop matrix of the estimator is exponentially stable for each sample path of $\{\gamma_k\}$.

Proof. Since the pairs $A_k - \breve{K}_k N_{k+1} C_{k+1} A_k$ are exponentially stable, there exists a bounded $\tilde{K}_k$ such that $I - \breve{K}_k N_k C_k$ is bounded. From Lemma 4.2 in [3], the solution P_k to the Lyapunov equation $P_k = \phi(\tilde{K}_k, P_{k-1})$ is bounded as $k \to \infty$. Since the optimality of $\breve{K}_k$, we have $\breve{P}_{k|k} \leqslant P_k$. Moreover, due to that the update covariance matrix $P_{k|k}$ is the same as $\breve{P}_{k|k}$ when the initial condition is the same, we can conclude that the event-triggered MMSE estimator is asymptotically bounded.

Remark 3. It is worth pointing out that from the above derivations, the stable condition does not depend on $\tilde{R}_k$. Furthermore, it can be directly observed that when A_k is stable, the condition is satisfied and hence the estimator is exponentially stable. Specifically, when the system is time-invariant, the condition that (A, NCA) is stabilizable can guarantee the stability of the estimator. To confirm the situation that $A_k - \tilde{K}_k N_{k+1} C_{k+1} A_k$ is exponentially stable, one can refer to [5] for more details

Lemma 5. The expectation of covariance matrix is bounded by a sequence of $\overline{P}_{k|k}$, which means: $\mathbb{E}(P_{k|k}) \leqslant \overline{P}_{k|k}$, where

$$\begin{aligned}\overline{P}_{k|k} = &[M_{k-1}^T (M_{k-1} \overline{P}_{k|k-1} M_{k-1}^T)^{-1} M_{k-1}\\ &+ (N_k C_k)^T (V_k - V_k N_k W_k N_k^T V_k) N_k C_k]^{-1}.\end{aligned} \tag{E11}$$

with

$$\overline{P}_{k|k-1} = A_{k-1} \overline{P}_{k-1|k-1} A_{k-1}^T + Q_{k-1}, \tag{E12}$$

and $\overline{P}_{0|0} = P_{0|0}$.

Proof. Noticing that $P_{k|k} \leqslant \overline{P}_{k|k}$ because of the truth that $V_k - (1-\gamma_k) V_k N_k W_k N_k^T V_k \geqslant V_k - V_k N_k W_k N_k^T V_k$, we complete the proof.

The above discussion investigates the performance of the event-based estimator from the perspective of the average estimation error covariance. Due to the stochastic sequences $\{\gamma_k\}$, it is unlikely to give a tighter upper bound. In the next section, we will show the performance of the proposed estimator by an example.

Appendix F Simulations

In this section, we give a numerical example and illustrate the performance of proposed event-triggered estimator. Consider the following stable linear time-varying system of the form in (1) and (2) with the matrix parameters

$$A_k = \begin{bmatrix} a_{11,k} & a_{12,k} & a_{13,k} \\ a_{21,k} & a_{22,k} & a_{23,k} \\ a_{31,k} & a_{32,k} & a_{33,k} \end{bmatrix},$$

with

$$\begin{aligned}&a_{11,k} = \exp[-h + \sin(kh) - \sin(kh - h)],\\ &a_{12,k} = 0, a_{13,k} = 0,\\ &a_{21,k} = 2\sinh(h/2)\exp[-3h/2 + \sin(kh) - \sin(kh - h)],\\ &a_{22,k} = \exp[-2h + \sin(kh) - \sin(kh - h)], a_{23,k} = 0,\\ &a_{31,k} = 0, a_{32,k} = 0,\\ &a_{33,k} = \exp[-2h + \sin(kh) - \sin(kh - h)],\\ &h = 0.2 \text{ and}\end{aligned}$$

$$Q_k = \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix}, R_k = \begin{bmatrix} 0.1 & 0.03 & 0.05 \\ 0.03 & 0.1 & 0.02 \\ 0.05 & 0.02 & 0.1 \end{bmatrix},$$

$$G_k = \begin{bmatrix} 0.1 & 0 & 0.2 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}^T, H_k = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.3 & 0.6 & 0 \end{bmatrix}^T,$$

$$C_k = \begin{bmatrix} \cos(kh) & \sin(kh) & 1.5 \\ 1 & \sin(2kh) & \cos(2kh) \\ 0 & \sin(3kh) & 2 \end{bmatrix},$$

$$Y_k = \begin{bmatrix} 0.05 & 0.02 & 0 \\ 0.02 & 0.05 & 0.01 \\ 0 & 0.01 & 0.1 \end{bmatrix},$$

$$D_k = \begin{bmatrix} 1 & 0 \end{bmatrix}, E_k = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

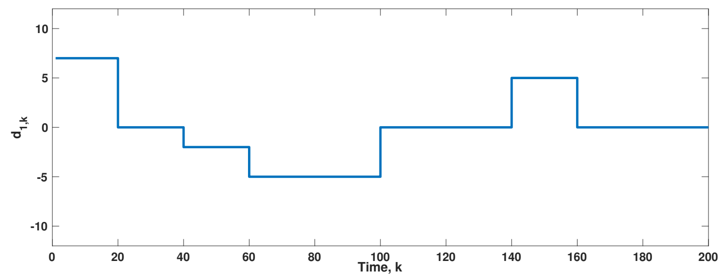

Figure F1 The partially known injection attacks $d_{1,k}$

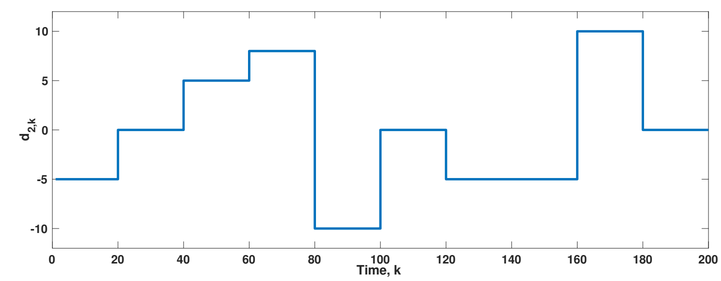

Figure F2 The partially known injection attacks $d_{2,k}$

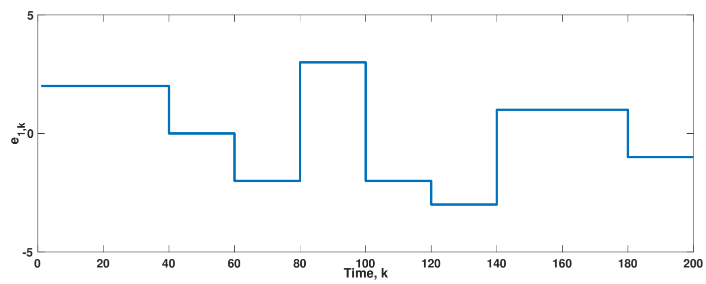

Figure F3 The partially known injection attacks $e_{1,k}$

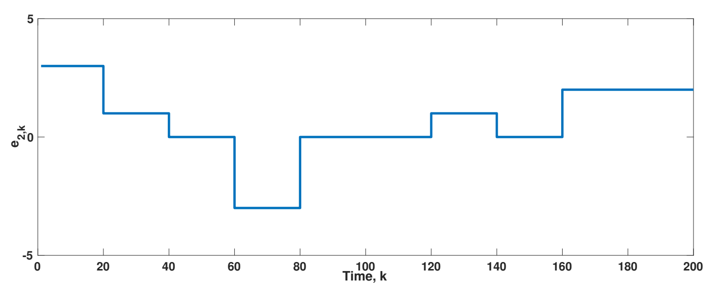

Figure F4 The partially known injection attacks $e_{2,k}$

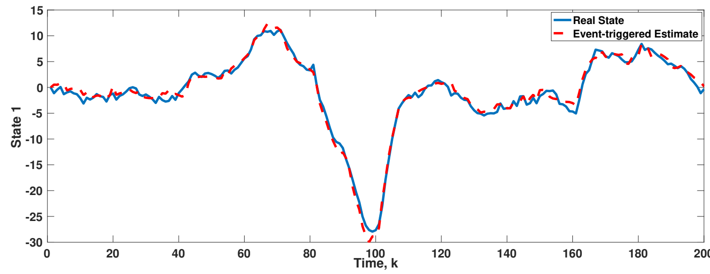

Figure F5 Estimation performance of state 1

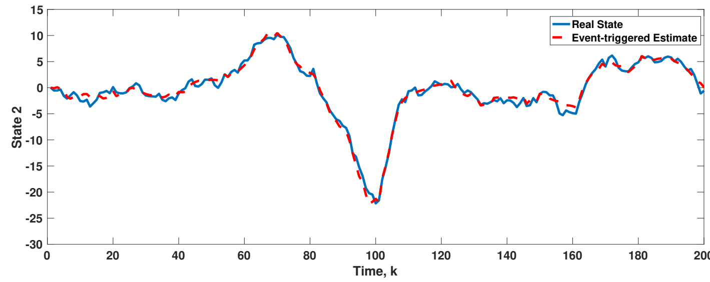

Figure F6 Estimation performance of state 2

d_ks and e_ks are shown in Fig.(F1) - (F4).

The state trajectories are estimated by the event-triggered MMSE estimator in Theorem 1, and the performance is demonstrated in Fig.(F5)-(F7), where the communication rate (communication per time step) is 0.74. The inspection here is that although the communication cost between the sensor and the remote estimator is significantly reduced, event-triggered MMSE estimation can still track the state of the system in the presence of partially observed injection attacks. Also, decreased communication rate contributes to the less information eavesdropped by the attacker. In conclusion, the result shown here indicates that the states can be estimated safely and monitored with satisfactory performance even though the whole system is in an adversarial environment.

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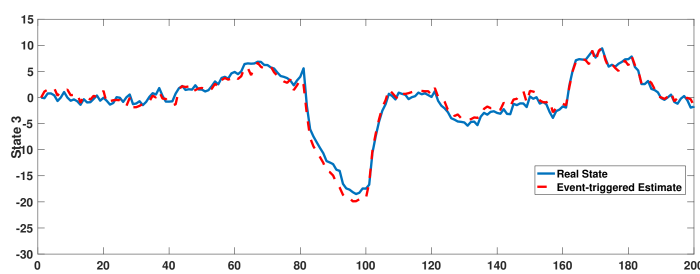

Figure F7 Estimation performance of state 3

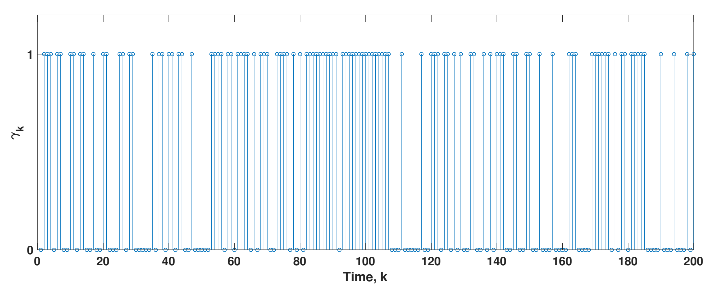

Figure F8 Transmission every time step k