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A new analysis approach to the output constraint and its application in high-order nonlinear systems

You WU & Xuejun XIE*

Institute of Automation, Qufu Normal University, Qufu 273165*, China*

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Many practical nonlinear systems need to consider the output constraint due to performance requirements and safety specifications. In the past decade, barrier Lyapunov function (BLF) and nonlinear mapping (NM), which were first defined in [1,2], respectively, have become two representative tools to handle the output constraint. The biggest merit of the NM-based approach is to directly deal with the original output constraint rather than indirectly limit the relevant error signal in the BLF-based approach. However, the current NM-based approach can only prove that the output lies in the constrained open set itself. As discussed in [1], when the output is close to the constrained boundary at some moment, the corresponding control input will grow rapidly to prevent the constraint violation. It is well known that the large control input is often a source of system instability or actuator damage in practical applications.

Based on these facts, two essential problems arise immediately: Is it possible to precisely determine a specific output-constrained subset, which can quantitatively depict how far from the output to the constrained boundary? For any initial point in the output-constrained open set, how to further ensure that the output remains within this subset, and then the theoretical basis can be established for designing the controller to avoid the undesirable large control input?

To solve these problems thoroughly, we introduce some notations in Appendix A and the following concept named barrier-transformed function (BTF).

Definition 1. Consider the non-autonomous system

$$\dot{x} = \phi(t, x),\ x(0) \in \mathbb{R}^n, \tag{1}$$

where $\phi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Suppose that $B \subset \mathbb{R}^n$ is an open region containing the origin and $T : B \to \mathbb{R}$ is $\mathcal{C}^1$ and strictly monotonic. For each solution $x(t)$ of system (1) starting from $x(0) \in B$, if

(1) $T(x(t)) \to \infty$ as $x(t) \to \partial B$, and

(2) $|T(x(t))| \leqslant L$, $\forall t \geqslant 0$ with some $L \in \mathbb{R}^+$,

then $T(x)$ is called the BTF of system (1).

Remark 1. BTF is partially motivated but is essentially different from NM in [2,3]. In fact, NM is developed for only autonomous systems. Since non-autonomous system (1) is only continuous and may not satisfy the Lipschitz condition, the solutions of (1) may be non-unique. Hence, BTF can be regarded as an extension of NM.

Based on Definition 1, for the universal non-autonomous system (1), the following Lemma 1 provides the analytical form of output-constrained compact subset $\bar{\Omega}_y$.

Lemma 1. Given positive constants $\underline{k}$ and $\bar{k}$, let $\Omega_y = \{x_1 \in \mathbb{R} : -\underline{k} < x_1 < \bar{k}\}$ be an open set and $y = x_1$ be the output of system (1) defined on $(t, x) \in \mathbb{R}^+ \times \Omega_y \times \mathbb{R}^{n-1}$. Suppose that there exists a strictly increasing BTF $T_1 : \Omega_y \to \mathbb{R}$ such that Eq. (1) can be transformed to

$$\dot{\xi} = \psi(t, \xi),\ \xi(0) \in \mathbb{R}^n, \tag{2}$$

where $\xi = [\xi_1, \dots, \xi_n]^{\mathrm{T}}$, $\xi_1 = T_1(x_1)$, $\xi_i = x_i$, $i = 2, \dots, n$, $\psi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Then for system (2), let $V : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ positive definite function, and $W : \mathbb{R}^n \to \mathbb{R}$ be a continuous and positive definite function. If

$$\pi_1(\|\xi\|) \leqslant V(\xi) \leqslant \pi_2(\|\xi\|), \tag{3}$$

$$\frac{\partial V(\xi)}{\partial \xi}\psi(t, \xi) \leqslant -W(\xi), \tag{4}$$

where $\pi_1(\cdot)$ and $\pi_2(\cdot)$ are two class $\mathcal{K}_\infty$ functions, then, every solution $x(t)$ of system (1) starting from $x(0) \in \Omega_y \times \mathbb{R}^{n-1}$ is well-defined on $[0, \infty)$ and the output satisfies

$$\begin{aligned} &y(t) \in \bar{\Omega}_y \subsetneq \Omega_y,\ \forall t \geqslant 0, \\ &\bar{\Omega}_y := \{y \in \mathbb{R} :\ T_1^{-1}(-M) \leqslant y \leqslant T_1^{-1}(M), \\ &\qquad M = \pi_1^{-1}(\pi_2(\|\xi(0)\|)) \geqslant 0\}. \end{aligned} \tag{5}$$

Proof. According to the continuity of $\psi(t, \xi)$, the solutions $\xi(t)$ of (2) are well-defined on $[0, t_f)$, where $0 < t_f \leqslant \infty$.

Let the compact set $\bar{\Omega}_\xi = \{\xi \in \mathbb{R}^n : \|\xi\| \leqslant M\}$. $\xi(0) \in \bar{\Omega}_\xi$ can be directly obtained. Next, we prove $\xi(t) \in \bar{\Omega}_\xi$, $\forall t > 0$. Suppose that there exists a $t_1 \in (0, t_f)$ such that $\xi(t_1) \notin \bar{\Omega}_\xi$ for the first time, and then

$$\|\xi(t_1)\| > \pi_1^{-1}(\pi_2(\|\xi(0)\|)). \tag{6}$$

From (4), we know that $0 \leqslant V(\xi(t)) \leqslant V(\xi(0)) < \infty$, $\forall t \in [0, t_1]$, which together with (3) and (6) implies that

* Corresponding author (email: xuejunxie@126.com)

$\pi_2(\|\xi(0)\|) < \pi_1(\|\xi(t_1)\|) \leqslant V(\xi(t_1)) \leqslant V(\xi(0)) \leqslant \pi_2(\|\xi(0)\|)$. It is a contradiction, and then $\xi(t) \in \bar{\Omega}_\xi$, $\forall t \in [0, t_f)$. Since the solutions $\xi(t)$ are bounded on $[0, t_f)$, then $t_f = \infty$ can be easily proved by using the contradiction argument again. Hence, $\xi(t) \in \bar{\Omega}_\xi$, $\forall t \geqslant 0$.

According to Definition 1, the inverse function $x_1 = T_1^{-1}(\xi_1)$ of T_1 exists. Then, $\bar{\Omega}_y$ in (5) can be determined by $\bar{\Omega}_\xi$ and $|\xi_1| \leqslant \|\xi\|$ such that $y(t) \in \bar{\Omega}_y$, $\forall t \geqslant 0$.

Finally, we prove $\bar{\Omega}_y \subsetneq \Omega_y$. When $M = 0$, the conclusion clearly holds. When $M > 0$, since $T_1 : \Omega_y \to \mathbb{R}$ is strictly increasing, we have $T_1^{-1}(-M) < T_1^{-1}(M)$ and $T_1^{-1}(\pm M) \in \Omega_y$, which imply that $[T_1^{-1}(-M), T_1^{-1}(M)] \subsetneq (-\underline{k}, \bar{k})$. Therefore, $\bar{\Omega}_y \subsetneq \Omega_y$.

Remark 2. In Lemma 1, we suppose that BTF $T_1(x_1)$ is strictly increasing. When $T_1(x_1)$ is strictly decreasing, the conclusion $y(t) \in \bar{\Omega}_y \subsetneq \Omega_y$, $\forall t \geqslant 0$ still holds with $\bar{\Omega}_y = \{y \in \mathbb{R} :\ T_1^{-1}(M) \leqslant y \leqslant T_1^{-1}(-M)\}$.

Application in high-order nonlinear systems. As an application of Lemma 1, we investigate the state-feedback stabilization of high-order nonlinear systems

$$\begin{cases} \dot{x}_i = x_{i+1}^{p_i} + f_i(t, \bar{x}_i),\ i = 1, \ldots, n-1, \\ \dot{x}_n = u^{p_n} + f_n(t, x), \\ y = x_1, \end{cases} \tag{7}$$

with an asymmetric output constraint

$$y(t) \in \Omega_y = \{y(t) \in \mathbb{R} : -\underline{k} < y(t) < \bar{k}\},\ \forall t \geqslant 0, \tag{8}$$

under the following weaker growth assumption.

Assumption 1. For $i = 1, \ldots, n$, there exist continuous functions $\bar{f}_i : \mathbb{R}^i \to \mathbb{R}^+$ and constants $\omega \in [0, \infty)$, $\tau \in (-\frac{1}{\sum_{i=1}^n p_0 \cdots p_{i-1}}, 0]$ with $p_0 = 1$ such that

$$|f_i| \leqslant \bar{f}_i(\bar{x}_i) \sum_{j=1}^{i} \left(|x_j|^{\frac{m_i+\tau}{m_j}} + |x_j|^{\frac{r_i+\omega}{r_j}} \right), \tag{9}$$

with constants m_i, r_i being defined by

$$m_1 = r_1 = 1,\ m_{i+1} = \frac{m_i + \tau}{p_i},\ r_{i+1} = \frac{r_i + \omega}{p_i}, \tag{10}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are measurable system states, control input and output, respectively, $p_i \in \mathbb{R}_{\rm odd}^{\geqslant 1}$ are the powers of system (7), $f_i : \mathbb{R}^+ \times \mathbb{R}^i \to \mathbb{R}$ are unknown continuous functions with $f_i(t, 0) = 0$, $i = 1, \ldots, n$, the Lipschitz condition is not necessary for $f_1, \ldots, f_n$, and $\underline{k}$ and $\bar{k}$ are two predetermined positive constants.

The inequality (9) in Assumption 1 includes both low-order and high-order nonlinear terms with respect to system states. Hence, Eq. (9) is weaker than the restrictions with only low-order or only high-order nonlinearities.

The detailed design of $u(t)$ is provided in Appendix C.

We state the main result of constraint and stability, whose proof is placed in Appendix D.

Theorem 1. If Assumption 1 holds for high-order nonlinear system (7), then there exists a continuous state-feedback controller $u(t)$ such that for any initial value $x(0) = [x_1(0), \ldots, x_n(0)]^{\rm T} \in \Omega_y \times \mathbb{R}^{n-1}$,

(1) the solutions $x(t)$ of (7) are well-defined on $[0, \infty)$, and there exists a specific output-constrained compact subset $\bar{\Omega}_y$ defined in (5) such that $y(t) \in \bar{\Omega}_y \subsetneq \Omega_y$, $\forall t \geqslant 0$;

(2) all the closed-loop signals are uniformly bounded on $[0, \infty)$;

(3) the equilibrium point $x = 0$ of the closed-loop system is uniformly asymptotically stable.

Figure 1 clearly depicts the research idea of constraint analysis in this study. When $T_1(x_1)$, $\xi(0)$ and $V(\xi)$ are specified, the location of $y(0)$ in Figure 1(b) can be determined, and the output-constrained compact subset $\bar{\Omega}_y$ in Figure 1(c) certainly exists such that $y(0) \in \bar{\Omega}_y \subsetneq \Omega_y$. Hence, the trajectory of $y(t)$ does not instantly jump from Ω_y to $\bar{\Omega}_y$. For any $t > 0$, $y(t) \in \bar{\Omega}_y \subsetneq \Omega_y$ and $\lim_{t\to\infty} y(t) = 0$ in Figure 1(d) can be further ensured by Theorem 1. Such a new analysis approach establishes the theoretical basis for designing the controller to avoid the undesirable large control input.

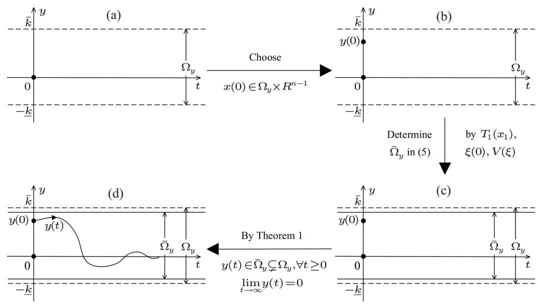

Figure 1 Sketch of constraint analysis. (a) Open set Ω_y; (b) location of $y(0)$; (c) compact subset $\bar{\Omega}_y$; (d) output constraint and asymptotic stability.

To verify the validity of control scheme, we provide the simulation in Appendix E.

Future work. (1) For more general n-dimensional high-order nonlinear systems, can we design an output-feedback controller? (2) We will try to combine some advanced algorithms (see [4–7]) with the proposed analysis approach.

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Supporting information Appendixes A–E. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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