SCIENCE CHINA
Information Sciences

• **Supplementary File** •

A new analysis approach to output constraint and its application in high-order nonlinear systems

You WU & Xuejun XIE*

Institute of Automation, Qufu Normal University, Qufu 273165, China

Appendix A Notations

$\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^n$ denote the set of all real numbers, the set of all nonnegative real numbers and the real n-dimensional space, respectively. $\mathbb{R}^{\geqslant 1}_{\rm odd} := \{\frac{p}{q} : p$ and q are positive odd integers satisfying $p \geqslant q\}$. $\mathcal{C}^1$ is the set of all functions with continuous partial derivatives. $\mathcal{K}$ is the set of all functions that strictly increasing, continuous and vanishes at the origin. $\mathcal{K}_\infty$ is the set of all functions that are of class $\mathcal{K}$ and unbounded. Given a real vector $x = [x_1, \cdots, x_n]^\top \in \mathbb{R}^n$, $\|x\|$ denotes the 2-norm of x, $\bar{x}_i := [x_1, \cdots, x_i]^\top \in \mathbb{R}^i$, $i = 1, \cdots, n$, obviously $\bar{x}_n = x$. For simplicity, a function $f(x(t))$ is sometimes denoted by $f(x)$, $f(\cdot)$ or f, and f^{-1} denotes the inverse function of f. ∂A denotes the boundary of set A. $\lceil a \rceil^k := \text{sign}(a)|a|^k$, $\forall a \in \mathbb{R}$, $\text{sign}(\cdot)$ denotes the sign function.

Appendix B Some useful lemmas

Lemma 2 [1]. For given positive real numbers m, n, functions $a(x, y)$, $b(x, y) > 0$ and any $x \in \mathbb{R}$, $y \in \mathbb{R}$, there holds $|a(x, y)x^m y^n| \leqslant b(x, y)|x|^{m+n} + \frac{n}{m+n}(\frac{m+n}{m})^{-\frac{m}{n}} b^{-\frac{m}{n}}(x, y)|a(x, y)|^{\frac{m+n}{n}}|y|^{m+n}$.

Lemma 3 [1]. For a given constant $p \geqslant 1$ and any $x \in \mathbb{R}$, $y \in \mathbb{R}$, then $|x + y|^p \leqslant 2^{p-1}|x^p + y^p|$, $(|x| + |y|)^{\frac{1}{p}} \leqslant |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leqslant 2^{1-\frac{1}{p}}(|x| + |y|)^{\frac{1}{p}}$. If $p \in \mathbb{R}^{\geqslant 1}_{\rm odd}$, then $|x - y|^p \leqslant 2^{p-1}|x^p - y^p|$, $|x^{\frac{1}{p}} - y^{\frac{1}{p}}| \leqslant 2^{1-\frac{1}{p}}|x - y|^{\frac{1}{p}}$.

Lemma 4 [2]. For a given continuous function $f(x, y)$ with $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, there exist smooth functions $a(x) \geqslant 0$, $b(y) \geqslant 0$, $c(x) \geqslant 1$, $d(y) \geqslant 1$ such that $|f(x, y)| \leqslant a(x) + b(y)$, $|f(x, y)| \leqslant c(x)d(y)$.

Lemma 5 [3]. If $\frac{m}{n} \in \mathbb{R}^{\geqslant 1}_{\rm odd}$, $n \geqslant 1$, then $|x^{\frac{m}{n}} - y^{\frac{m}{n}}| \leqslant 2^{1-\frac{1}{n}}|\lceil x \rceil^m - \lceil y \rceil^m|^{\frac{1}{n}}$ holds for any $x \in \mathbb{R}$, $y \in \mathbb{R}$.

Lemma 6 [3]. For a given continuous and monotone function $f : [a, b] \to \mathbb{R}$, $b > a$ with $f(a) = 0$, there holds $|\int_a^b f(x)\mathrm{d}x| \leqslant |f(b)|\,|b - a|$.

Lemma 7 [4]. For given real numbers $0 \leqslant \mu_1 \leqslant \cdots \leqslant \mu_n$, $c_j > 0$, $j = 1, \cdots, n$, and any $x \in \mathbb{R}$, there holds $c_1|x|^{\mu_1} + c_n|x|^{\mu_n} \leqslant \sum_{j=1}^n c_j|x|^{\mu_j} \leqslant (\sum_{j=1}^n c_j)(|x|^{\mu_1} + |x|^{\mu_n})$.

Appendix C Constrained control design

We first introduce the transformed states

$$\xi_1 = T_1(x_1) = \frac{x_1}{h_1(x_1)},\ \xi_i = x_i,\ i = 2, \cdots, n, \tag{C1}$$

where $h_1(x_1) = (\underline{k} + x_1)(\bar{k} - x_1)$. It is easy to show that $T_1(x_1)$ is smooth and strictly increasing with respect to $y = x_1$ in Ω_y, $T_1(0) = 0$ and $\xi_1 \to \infty$ if and only if $x_1 \to \partial\Omega_y$, which together with Definition 1 imply that $T_1(x_1)$ is a BTF of system (7). From (7) and (C1), we obtain

$$\begin{cases} \dot{\xi}_1 = H_1(x_1)\xi_2^{p_1} + \tilde{f}_1(t, x_1), \\ \dot{\xi}_i = \xi_{i+1}^{p_i} + \tilde{f}_i(t, \bar{x}_i),\ i = 2, \cdots, n-1, \\ \dot{\xi}_n = u^{p_n} + \tilde{f}_n(t, x), \\ y = h_1(x_1)\xi_1, \end{cases} \tag{C2}$$

where $H_1(x_1) = \frac{\underline{k}\bar{k} + x_1^2}{h_1^2(x_1)} > 0$ in Ω_y, $\tilde{f}_1 = H_1 f_1$, $\tilde{f}_i = f_i$, $i = 2, \cdots, n$.

Next, we give the control design of transformed system (C2). Set $\bar{\xi}_0 = \beta_0 = \alpha_0 = 0$, $\bar{\xi}_k = [\xi_1, \cdots, \xi_k]^\top$, $k = 1, \cdots, n$, $\xi = \bar{\xi}_n$, $\mu = \max_{i=1,\cdots,n+1}\{\frac{r_i}{m_i}\}$, and introduce a coordinate transformation

$$\begin{cases} z_k = \lceil \xi_k \rceil^{\frac{1}{m_k}} - \lceil \alpha_{k-1}(\bar{\xi}_{k-1}) \rceil^{\frac{1}{m_k}}, \\ \alpha_k(\bar{\xi}_k) = -\beta_k^{m_{k+1}}(\bar{\xi}_k)\left\lceil z_k + \lceil z_k \rceil^{\frac{r_{k+1}m_k}{r_k m_{k+1}}} \right\rceil^{m_{k+1}}, \end{cases} \tag{C3}$$

* Corresponding author (email: xuejunxie@126.com)

and integral functions $W_{Lk}:\mathbb{R}^k\to\mathbb{R}$, $W_{Rk}:\mathbb{R}^k\to\mathbb{R}$ as

$$\begin{aligned}W_{Lk}(\bar{\xi}_k)&=\int_{\alpha_{k-1}}^{\xi_k}\left\lceil \lceil s\rfloor^{\frac{1}{m_k}}-\lceil\alpha_{k-1}\rfloor^{\frac{1}{m_k}}\right\rfloor^{2-m_{k+1}p_k}\mathrm{d}s,\\ W_{Rk}(\bar{\xi}_k)&=\int_{\alpha_{k-1}}^{\xi_k}\left\lceil \lceil s\rfloor^{\frac{1}{m_k}}-\lceil\alpha_{k-1}\rfloor^{\frac{1}{m_k}}\right\rfloor^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\mathrm{d}s,\end{aligned}\tag{C4}$$

where $\alpha_k(\cdot)$ and $\beta_k(\cdot)$, $k=1,\cdots,n$, are the virtual controllers and positive smooth functions, respectively. Let $W_k=W_{Lk}+W_{Rk}$. Similar to the proofs of Proposition 1 and Theorem 1 in [3], we infer that $V_i=\sum_{k=1}^{i}W_k$, $i=1,\cdots,n$, are $\mathcal{C}^1$ positive definite and radially unbounded functions, and

$$\frac{\partial W_k}{\partial\xi_k}=\lceil z_k\rfloor^{2-m_{k+1}p_k}+\lceil z_k\rfloor^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}},\tag{C5}$$

$$\begin{aligned}\frac{\partial W_k}{\partial\xi_j}=&-\frac{\partial\lceil\alpha_{k-1}\rfloor^{\frac{1}{m_k}}}{\partial\xi_j}\int_{\alpha_{k-1}}^{\xi_k}\Bigg((2-m_{k+1}p_k)\left|\lceil s\rfloor^{\frac{1}{m_k}}-\lceil\alpha_{k-1}\rfloor^{\frac{1}{m_k}}\right|^{1-m_{k+1}p_k}\\&+\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}\left|\lceil s\rfloor^{\frac{1}{m_k}}-\lceil\alpha_{k-1}\rfloor^{\frac{1}{m_k}}\right|^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}-1}\Bigg)\mathrm{d}s,\ j=1,\cdots,k-1.\end{aligned}\tag{C6}$$

By (C3), we construct the actual controller

$$u=\alpha_n(\xi)=-\beta_n^{m_{n+1}}(\xi)\left\lceil z_n+\lceil z_n\rfloor^{\frac{r_{n+1}m_n}{r_nm_{n+1}}}\right\rfloor^{m_{n+1}}.\tag{C7}$$

The design of $\beta_1,\cdots,\beta_n$ in (C3) and (C7) is proceed in a recursive manner.

Step 1: By $V_1=W_1$, (C2), (C4) and (C5), one gets

$$\begin{aligned}\dot V_1=&\ \frac{\partial W_1}{\partial\xi_1}\dot\xi_1\\=&\ H_1\big(\lceil z_1\rfloor^{2-m_2p_1}+\lceil z_1\rfloor^{2\mu-r_2p_1}\big)\big(\xi_2^{p_1}-\alpha_1^{p_1}\big)+H_1\big(\lceil z_1\rfloor^{2-m_2p_1}+\lceil z_1\rfloor^{2\mu-r_2p_1}\big)\alpha_1^{p_1}\\&+\big(\lceil z_1\rfloor^{2-m_2p_1}+\lceil z_1\rfloor^{2\mu-r_2p_1}\big)\tilde f_1.\end{aligned}\tag{C8}$$

In terms of $\tilde f_1=H_1f_1$, $x_1=h_1\xi_1=h_1z_1$, $1\leqslant\frac{r_2}{m_2}\leqslant\mu$, it follows from Assumption 1 and Lemmas 2,4,7 that

$$\begin{aligned}&\big(\lceil z_1\rfloor^{2-m_2p_1}+\lceil z_1\rfloor^{2\mu-r_2p_1}\big)\tilde f_1\\&\leqslant H_1\bar f_1\big(|z_1|^{2-m_2p_1}+|z_1|^{2\mu-r_2p_1}\big)\big(|h_1z_1|^{1+\tau}+|h_1z_1|^{1+\omega}\big)\\&\leqslant\varpi_1(x_1)\big(|z_1|^{2-m_2p_1}+|z_1|^{2\mu-r_2p_1}\big)\big(|z_1|^{m_2p_1}+|z_1|^{r_2p_1}\big)\\&\leqslant\varphi_1(x_1)\big(z_1^2+z_1^{2\mu}\big),\end{aligned}\tag{C9}$$

where $\varpi_1(\cdot)$ and $\varphi_1(\cdot)$ are known nonnegative smooth functions. By choosing $\beta_1=2^{\frac{1}{m_2p_1}-1}(\frac{\lambda_{11}+\varphi_1}{H_1})^{\frac{1}{m_2p_1}}$ with $\lambda_{11}>0$, substituting $\alpha_1=-\beta_1^{m_2}\lceil z_1+\lceil z_1\rfloor^{\frac{r_2}{m_2}}\rfloor^{m_2}$ and (C9) into (C8), and noting

$$\begin{aligned}&-H_1\beta_1^{m_2p_1}\left\lceil z_1+\lceil z_1\rfloor^{\frac{r_2}{m_2}}\right\rfloor^{m_2p_1}\big(\lceil z_1\rfloor^{2-m_2p_1}+\lceil z_1\rfloor^{2\mu-r_2p_1}\big)\\&\leqslant-H_1\beta_1^{m_2p_1}2^{m_2p_1-1}\big(|z_1|^{m_2p_1}+|z_1|^{r_2p_1}\big)\big(|z_1|^{2-m_2p_1}+|z_1|^{2\mu-r_2p_1}\big)\\&\leqslant-(\lambda_{11}+\varphi_1)\big(z_1^2+z_1^{2\mu}\big),\end{aligned}\tag{C10}$$

from Lemmas 2,5,7, it yields

$$\begin{aligned}\dot V_1\leqslant&-H_1\beta_1^{m_2p_1}\big(\lceil z_1\rfloor^{2-m_2p_1}+\lceil z_1\rfloor^{2\mu-r_2p_1}\big)\left\lceil z_1+\lceil z_1\rfloor^{\frac{r_2}{m_2}}\right\rfloor^{m_2p_1}\\&+\varphi_1\big(z_1^2+z_1^{2\mu}\big)+H_1\big(\lceil z_1\rfloor^{2-m_2p_1}+\lceil z_1\rfloor^{2\mu-r_2p_1}\big)\big(\xi_2^{p_1}-\alpha_1^{p_1}\big)\\\leqslant&-\lambda_{11}\big(z_1^2+z_1^{2\mu}\big)+H_1\big(\lceil z_1\rfloor^{2-m_2p_1}+\lceil z_1\rfloor^{2\mu-r_2p_1}\big)\big(\xi_2^{p_1}-\alpha_1^{p_1}\big).\end{aligned}\tag{C11}$$

Step k ($k=2,\cdots,n$): For sake of consistency, we specify $H_i=h_i=1$, $i=2,\cdots,n$. In Step $k-1$, suppose that there exists a $\mathcal{C}^1$, positive definite and radially unbounded function $V_{k-1}:\mathbb{R}^{k-1}\to\mathbb{R}^+$, smooth functions $\beta_1,\cdots,\beta_{k-1}$ and positive constants $\lambda_{k-1,1},\cdots,\lambda_{k-1,k-1}$ such that

$$\dot V_{k-1}\leqslant-\sum_{j=1}^{k-1}\lambda_{k-1,j}\left(z_j^2+z_j^{\frac{2\mu m_j}{r_j}}\right)+H_{k-1}\left(\lceil z_{k-1}\rfloor^{2-m_kp_{k-1}}+\lceil z_{k-1}\rfloor^{\frac{(2\mu-r_kp_{k-1})m_{k-1}}{r_{k-1}}}\right)\left(\xi_k^{p_{k-1}}-\alpha_{k-1}^{p_{k-1}}\right).\tag{C12}$$

Next, we prove that (C12) still holds at Step k. By $V_k=V_{k-1}+W_k=\sum_{i=1}^{k}W_i$, $H_k=1$, (C2), (C4), (C5) and (C12), one gets

$$\dot V_k\leqslant-\sum_{j=1}^{k-1}\lambda_{k-1,j}\left(z_j^2+z_j^{\frac{2\mu m_j}{r_j}}\right)+H_k\left(\lceil z_k\rfloor^{2-m_{k+1}p_k}+\lceil z_k\rfloor^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right)\left(\xi_{k+1}^{p_k}-\alpha_k^{p_k}\right)$$

$$+\left(\lceil z_k\rceil^{2-m_{k+1}p_k}+\lceil z_k\rceil^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right)\alpha_k^{p_k}+H_{k-1}\left(\lceil z_{k-1}\rceil^{2-m_kp_{k-1}}+\lceil z_{k-1}\rceil^{\frac{(2\mu-r_kp_{k-1})m_{k-1}}{r_{k-1}}}\right)$$
$$\times(\xi_k^{p_{k-1}}-\alpha_{k-1}^{p_{k-1}})+\left(\lceil z_k\rceil^{2-m_{k+1}p_k}+\lceil z_k\rceil^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right)\tilde f_k+\sum_{j=1}^{k-1}\frac{\partial W_k}{\partial \xi_j}(H_j\xi_{j+1}^{p_j}+\tilde f_j), \tag{C13}$$

where $\xi_{n+1}:=u$ when $k=n$. We estimate the last three terms on the right-hand side of (C13). According to (C3) and Lemmas 2,5, it follows that

$$\begin{aligned}
&H_{k-1}\left(\lceil z_{k-1}\rceil^{2-m_kp_{k-1}}+\lceil z_{k-1}\rceil^{\frac{(2\mu-r_kp_{k-1})m_{k-1}}{r_{k-1}}}\right)(\xi_k^{p_{k-1}}-\alpha_{k-1}^{p_{k-1}})\\
&\leqslant 2^{1-m_kp_{k-1}}H_{k-1}|z_k|^{m_kp_{k-1}}\left(|z_{k-1}|^{2-m_kp_{k-1}}+|z_{k-1}|^{\frac{(2\mu-r_kp_{k-1})m_{k-1}}{r_{k-1}}}\right)\\
&\leqslant \lambda_{k,k-1,1}\left(z_{k-1}^2+z_{k-1}^{\frac{2\mu m_{k-1}}{r_{k-1}}}\right)+\varphi_{k1}(\bar x_k)\left(z_k^2+z_k^{\frac{2\mu m_k}{r_k}}\right),
\end{aligned}\tag{C14}$$

where $\lambda_{k,k-1,1}>0$ is a constant to be designed, $\varphi_{k1}(\cdot)$ is a known nonnegative smooth function dependent on $\lambda_{k,k-1,1}$. From (10), we obtain

$$m_i=\frac{1}{p_1\cdots p_{i-1}}+\tau\sum_{l=1}^{i-1}\frac{1}{p_l\cdots p_{i-1}},\ \ r_i=\frac{1}{p_1\cdots p_{i-1}}+\omega\sum_{l=1}^{i-1}\frac{1}{p_l\cdots p_{i-1}}. \tag{C15}$$

By $-\frac{1}{\sum_{i=1}^n p_0\cdots p_{i-1}}<\tau\leqslant 0$ and $\omega\geqslant 0$ in Assumption 1, (C15) implies that for each $i=1,\cdots,n$, $j=1,\cdots,i-1$,

$$\begin{aligned}
r_im_j-r_jm_i\ =&\ \left(\frac{1}{p_1\cdots p_{i-1}}+\omega\sum_{l=1}^{i-1}\frac{1}{p_l\cdots p_{i-1}}\right)\left(\tau\sum_{l=1}^{j-1}\frac{1}{p_l\cdots p_{j-1}}+\frac{1}{p_1\cdots p_{j-1}}\right)\\
&-\left(\frac{1}{p_1\cdots p_{i-1}}+\tau\sum_{l=1}^{i-1}\frac{1}{p_l\cdots p_{i-1}}\right)\left(\frac{1}{p_1\cdots p_{j-1}}+\omega\sum_{l=1}^{j-1}\frac{1}{p_l\cdots p_{j-1}}\right)\\
=&\ \frac{\omega-\tau}{p_1\cdots p_{j-1}}\sum_{l=j}^{i-1}\frac{1}{p_l\cdots p_{i-1}}\geqslant 0,
\end{aligned}\tag{C16}$$

which is meaningful in applying Lemma 7 to enlarge the inequalities in the following proof. In terms of $\tilde f_k=H_kf_k$, $k=1,\cdots,n$, $x_j=h_j\xi_j$, $j=1,\cdots,k$, and (C16), it follows from Assumption 1 and Lemmas 3,4,7 that

$$\begin{aligned}
|\tilde f_k|\ \leqslant&\ H_k\bar f_k\sum_{j=1}^k\left(|h_j\xi_j|^{\frac{m_k+\tau}{m_j}}+|h_j\xi_j|^{\frac{r_k+\omega}{r_j}}\right)\\
\leqslant&\ \varpi_k(\bar x_k)\sum_{j=1}^k\left(|\xi_j|^{\frac{m_k+\tau}{m_j}}+|\xi_j|^{\frac{r_k+\omega}{r_j}}\right)\\
\leqslant&\ \varpi_k(\bar x_k)\sum_{j=1}^k\left(\left|z_j-\beta_{j-1}\left\lceil z_{j-1}+\lceil z_{j-1}\rceil^{\frac{r_jm_{j-1}}{r_{j-1}m_j}}\right\rceil\right|^{m_{k+1}p_k}+\left|z_j-\beta_{j-1}\left\lceil z_{j-1}+\lceil z_{j-1}\rceil^{\frac{r_jm_{j-1}}{r_{j-1}m_j}}\right\rceil\right|^{\frac{r_{k+1}m_jp_k}{r_j}}\right)\\
\leqslant&\ \varpi_k(\bar x_k)\sum_{j=1}^k\left(|z_j|^{m_{k+1}p_k}+|z_j|^{\frac{r_{k+1}m_jp_k}{r_j}}\right),
\end{aligned}\tag{C17}$$

where $\varpi_k(\cdot)$ is a known nonnegative smooth function, which may be changed from place to place. By this fact and Lemmas 2,7, one has

$$\begin{aligned}
&\left(\lceil z_k\rceil^{2-m_{k+1}p_k}+\lceil z_k\rceil^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right)\tilde f_k\\
&\leqslant\varpi_k(\bar x_k)\left(|z_k|^{2-m_{k+1}p_k}+|z_k|^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right)\sum_{j=1}^k\left(|z_j|^{m_{k+1}p_k}+|z_j|^{\frac{r_{k+1}m_jp_k}{r_j}}\right)\\
&\leqslant\sum_{j=1}^{k-2}\lambda_{kj1}\left(z_j^2+z_j^{\frac{2\mu m_j}{r_j}}\right)+\lambda_{k,k-1,2}\left(z_{k-1}^2+z_{k-1}^{\frac{2\mu m_{k-1}}{r_{k-1}}}\right)+\varphi_{k2}(\bar x_k)\left(z_k^2+z_k^{\frac{2\mu m_k}{r_k}}\right),
\end{aligned}\tag{C18}$$

where $\lambda_{kj1}>0$, $j=1,\cdots,k-2$, $\lambda_{k,k-1,2}>0$ are some constants to be designed, $\varphi_{k2}(\cdot)$ is a known nonnegative smooth function dependent on $\lambda_{k,k-1,2}$ and $\lambda_{k11},\cdots,\lambda_{k,k-2,1}$. Finally, we estimate the last term on the right-hand side of (C13). From $H_j=1$, $j=2,\cdots,n$, $\tilde f_k=H_kf_k$, (C3), (C16), (C17) and Lemmas 2,3,4,7, a series of calculations lead to

$$|\xi_j|^{\frac{1}{m_j}-1}\left(|H_j\xi_{j+1}^{p_j}|+|\tilde f_j|\right)\prod_{l=j}^{k-1}\left(1+|z_l|^{\frac{r_{l+1}m_l}{r_lm_{l+1}}-1}\right)$$

$$
\begin{aligned}
&\leqslant \varpi_k(\bar{x}_k)\left(|z_j|^{1-m_j}+|z_{j-1}|^{1-m_j}+|z_{j-1}|^{\frac{r_j m_{j-1}(1-m_j)}{r_{j-1}m_j}}\right)\\
&\quad\times\left(|z_{j+1}|^{m_{j+1}p_j}+\sum_{l=1}^{j}\left(|z_l|^{m_{j+1}p_j}+|z_l|^{\frac{r_{j+1}m_l p_j}{r_l}}\right)\right)\prod_{l=j}^{k-1}\left(1+|z_l|^{\frac{r_{l+1}m_l}{r_l m_{l+1}}-1}\right)\\
&\overset{(1)}{\leqslant} \varpi_k(\bar{x}_k)\sum_{l=1}^{j+1}\left(|z_l|^{1+\tau}+|z_l|^{\frac{(r_{j+1}+\omega m_{j+1})m_l}{r_l m_{j+1}}}\right)\prod_{l=j+1}^{k-1}\left(1+|z_l|^{\frac{r_{l+1}m_l}{r_l m_{l+1}}-1}\right)\\
&\vdots\\
&\overset{(i)}{\leqslant} \varpi_k(\bar{x}_k)\sum_{l=1}^{j+i}\left(|z_l|^{1+\tau}+|z_l|^{\frac{(r_{j+i}+\omega m_{j+i})m_l}{r_l m_{j+i}}}\right)\prod_{l=j+i}^{k-1}\left(1+|z_l|^{\frac{r_{l+1}m_l}{r_l m_{l+1}}-1}\right)\\
&\vdots\\
&\overset{(k-1)}{\leqslant} \varpi_k(\bar{x}_k)\sum_{l=1}^{k}\left(|z_l|^{1+\tau}+|z_l|^{\frac{(r_k+\omega m_k)m_l}{r_l m_k}}\right),
\end{aligned}
\tag{C19}
$$

where $\prod_{l=k}^{k-1}(1+|z_l|^{\frac{r_{l+1}m_l}{r_l m_{l+1}}-1})=1$, and for $i=1,\cdots,k-1$, if $j=k-i$, then (C19) stops at inequality (i). From (C15)-(C16), $-\frac{1}{\sum_{i=1}^{n}p_0\cdots p_{i-1}}<\tau\leqslant 0$, $\omega\geqslant 0$ and $\mu=\max_{i=1,\cdots,n+1}\{\frac{r_i}{m_i}\}=\frac{r_{n+1}}{m_{n+1}}$, it follows that

$$
\begin{aligned}
1+\tau-\frac{r_k+\omega m_k}{\mu m_k}&=\frac{(1+\tau)r_{n+1}m_k-(r_k+\omega m_k)m_{n+1}}{r_{n+1}m_k}\\
&=\frac{\omega-\tau}{p_1\cdots p_{k-1}}\left(\sum_{l=k}^{n}\frac{1}{p_l\cdots p_n}-\frac{1}{p_1\cdots p_n}\right)-\frac{\tau(\omega-\tau)}{p_1\cdots p_n}\sum_{l=1}^{k-1}\frac{1}{p_l\cdots p_{k-1}}\geqslant 0,
\end{aligned}
\tag{C20}
$$

and thus $\frac{r_k+\omega m_k}{2\mu m_k}\leqslant\frac{1+\tau}{2}\leqslant 1$. Hence, we deduce from (C3), (C6), (C19) and Lemmas 2,3,5,6,7 that

$$
\begin{aligned}
&\sum_{j=1}^{k-1}\frac{\partial W_k}{\partial\xi_j}\left(H_j\xi_{j+1}^{p_j}+\tilde{f}_j\right)\\
&=-\int_{\alpha_{k-1}}^{\xi_k}\left((2-m_{k+1}p_k)\left|\lceil s\rceil^{\frac{1}{m_k}}-\lceil\alpha_{k-1}\rceil^{\frac{1}{m_k}}\right|^{1-m_{k+1}p_k}+\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}\right.\\
&\qquad\left.\times\left|\lceil s\rceil^{\frac{1}{m_k}}-\lceil\alpha_{k-1}\rceil^{\frac{1}{m_k}}\right|^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}-1}\right)\mathrm{d}s\sum_{j=1}^{k-1}\frac{\partial\lceil\alpha_{k-1}\rceil^{\frac{1}{m_k}}}{\partial\xi_j}\left(H_j\xi_{j+1}^{p_j}+\tilde{f}_j\right)\\
&\leqslant 2^{1-m_k}\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}\left(|z_k|^{1-\tau}+|z_k|^{\frac{(2\mu-\omega)m_k}{r_k}-1}\right)\sum_{j=1}^{k-1}\left|\frac{\partial\lceil\alpha_{k-1}\rceil^{\frac{1}{m_k}}}{\partial\xi_j}\right|\left(|H_j\xi_{j+1}^{p_j}|+|\tilde{f}_j|\right)\\
&\leqslant\varpi_k(\bar{x}_k)\left(|z_k|^{1-\tau}+|z_k|^{\frac{(2\mu-\omega)m_k}{r_k}-1}\right)\sum_{j=1}^{k-1}|\xi_j|^{\frac{1}{m_j}-1}\left(|H_j\xi_{j+1}^{p_j}|+|\tilde{f}_j|\right)\prod_{l=j}^{k-1}\left(1+|z_l|^{\frac{r_{l+1}m_l}{r_l m_{l+1}}-1}\right)\\
&\leqslant\varpi_k(\bar{x}_k)\left(|z_k|^{1-\tau}+|z_k|^{\frac{(2\mu-\omega)m_k}{r_k}-1}\right)\sum_{l=1}^{k}\left(|z_l|^{1+\tau}+|z_l|^{\frac{(r_k+\omega m_k)m_l}{r_l m_k}}\right)\\
&\leqslant\sum_{j=1}^{k-2}\lambda_{kj2}\left(z_j^2+z_j^{\frac{2\mu m_j}{r_j}}\right)+\lambda_{k,k-1,3}\left(z_{k-1}^2+z_{k-1}^{\frac{2\mu m_{k-1}}{r_{k-1}}}\right)+\varphi_{k3}(\bar{x}_k)\left(z_k^2+z_k^{\frac{2\mu m_k}{r_k}}\right),
\end{aligned}
\tag{C21}
$$

where $\lambda_{kj2}>0$, $j=1,\cdots,k-2$, $\lambda_{k,k-1,3}>0$ are constants to be designed, $\varphi_{k3}(\cdot)$ is a known nonnegative smooth function dependent on $\lambda_{k,k-1,3}$ and $\lambda_{k12},\cdots,\lambda_{k,k-2,2}$. Substituting (C14), (C18) and (C21) into (C13) yields

$$
\begin{aligned}
\dot{V}_k\leqslant&-\sum_{j=1}^{k-1}\lambda_{kj}\left(z_j^2+z_j^{\frac{2\mu m_j}{r_j}}\right)+H_k\left(\lceil z_k\rceil^{2-m_{k+1}p_k}+\lceil z_k\rceil^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right)\left(\xi_{k+1}^{p_k}-\alpha_k^{p_k}\right)\\
&+\left(\lceil z_k\rceil^{2-m_{k+1}p_k}+\lceil z_k\rceil^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right)\alpha_k^{p_k}+\varphi_k\left(z_k^2+z_k^{\frac{2\mu m_k}{r_k}}\right),
\end{aligned}
\tag{C22}
$$

where $\lambda_{kj}=\lambda_{k-1,j}-(\lambda_{kj1}+\lambda_{kj2})>0$ when $j=1,\cdots,k-2$, and $\lambda_{kj}=\lambda_{k-1,j}-(\lambda_{k,k-1,1}+\lambda_{k,k-1,2}+\lambda_{k,k-1,3})>0$ when $j=k-1$, $\varphi_k=\varphi_{k1}+\varphi_{k2}+\varphi_{k3}$. By choosing $\beta_k=2^{\frac{1}{m_{k+1}p_k}-1}(\lambda_{kk}+\varphi_k)^{\frac{1}{m_{k+1}p_k}}$ with $\lambda_{kk}>0$, substituting $\alpha_k=-\beta_k^{m_{k+1}}\lceil z_k+\lceil z_k\rceil^{\frac{r_{k+1}m_k}{r_k m_{k+1}}}\rceil^{m_{k+1}}$ into (C22), and noting

$$
\begin{aligned}
&-\beta_k^{m_{k+1}p_k}\left(\lceil z_k\rceil^{2-m_{k+1}p_k}+\lceil z_k\rceil^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right)\left\lceil z_k+\lceil z_k\rceil^{\frac{r_{k+1}m_k}{r_k m_{k+1}}}\right\rceil^{m_{k+1}p_k}\\
&\leqslant-2^{m_{k+1}p_k-1}\beta_k^{m_{k+1}p_k}\left(|z_k|^{2-m_{k+1}p_k}+|z_k|^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right)\left(|z_k|^{m_{k+1}p_k}+|z_k|^{\frac{r_{k+1}m_k p_k}{r_k}}\right)
\end{aligned}
$$

$$\leqslant -(\lambda_{kk}+\varphi_k)\left(z_k^2+z_k^{\frac{2\mu m_k}{r_k}}\right). \tag{C23}$$

from Lemmas 2,5,7, it follows that

$$\dot{V}_k \leqslant -\sum_{j=1}^{k}\lambda_{kj}\left(z_j^2+z_j^{\frac{2\mu m_j}{r_j}}\right)+H_k(\xi_{k+1}^{p_k}-\alpha_k^{p_k})\left(\lceil z_k\rceil^{2-m_{k+1}p_k}+\lceil z_k\rceil^{\frac{(2\mu-r_{k+1}p_k)m_k}{r_k}}\right). \tag{C24}$$

When $k=n$, we conclude from $\xi_{n+1}=u=\alpha_n$ and (C24) that there exist smooth functions $\beta_1,\cdots,\beta_n$ and the Lyapunov function $V:=V_n=\sum_{k=1}^{n}W_k$ such that

$$\dot{V} \leqslant -\sum_{j=1}^{n}\lambda_{nj}\left(z_j^2+z_j^{\frac{2\mu m_j}{r_j}}\right):=-W(\xi), \tag{C25}$$

where $W(\xi)$ is a continuous and positive definite function.

Appendix D Proof of Theorem 1

1) From (C4), $V=\sum_{k=1}^{n}(W_{Lk}+W_{Rk})$ and Lemma 4.3 in [5], there exist two class $\mathcal{K}_\infty$ functions $\pi_1(\cdot)$ and $\pi_2(\cdot)$ such that

$$\pi_1(\|\xi\|)\leqslant V(\xi)\leqslant \pi_2(\|\xi\|). \tag{D1}$$

Hence, for given $x(0)\in\Omega_y\times\mathbb{R}^{n-1}$, it follows from (C1), (C2), (C25), (D1) and Lemma 1 that $\xi(t)\in\bar{\Omega}_\xi$, the solutions $x(t)$ of system (7) are well-defined on $[0,\infty)$, and

$$y(t)\in\bar{\Omega}_y\subsetneq\Omega_y,\ \forall t\geqslant 0. \tag{D2}$$

Due to the optionality of the initial value, by arbitrarily choosing $x(0)\in\Omega_y\times\mathbb{R}^{n-1}$ and repeating the process from (C3) to (D2), $y(t)\in\bar{\Omega}_y\subsetneq\Omega_y$ for any $t\geqslant 0$ can be rigorously proved. Hence, the design of controller (C7) is valid, and the asymmetric output constraint (8) isn't violated.

2) From 1), we deduce that $\xi(t)$ and $y(t)=x_1(t)$ are uniformly bounded on $[0,\infty)$. Then, the boundedness of $x_2(t),\cdots,x_n(t)$, $z_1(t),\cdots,z_n(t)$, $\alpha_1(t),\cdots,\alpha_{n-1}(t)$ and $u(t)$ can be recursively proved in terms of (C1), (C3), (C7) and the continuity of $\alpha_k(\bar{\xi}_k)$. Hence, all the closed-loop signals are uniformly bounded on $[0,\infty)$.

3) From (C25), (D1) and Theorem A.8 in [6], the equilibrium point $\xi=0$ of ξ-system (C2),(C7) is uniformly asymptotically stable, so is the equilibrium point $x=0$ of the closed-loop system (7),(C7) from $x_1=T_1^{-1}(\xi_1)$ and $x_i=\xi_i$, $i=2,\cdots,n$ in (C1).

Appendix E Simulation

Consider a planar nonlinear system

$$\begin{cases}\dot{x}_1=x_2^{p_1}+f_1,\\ \dot{x}_2=u^{p_2}+f_2,\\ y=x_1,\end{cases} \tag{E1}$$

where $p_1=\frac{11}{9}$, $p_2=1$, $f_1=0$, $f_2=\frac{1}{3}x_2^{\frac{5}{11}}\cos x_1+\frac{\ln(1+|x_2|^{\frac{11}{9}})}{2(1+x_1^2)}$. The output is required to satisfy $y(t)\in\Omega_y=\{y(t)\in\mathbb{R}:-\underline{k}<y(t)<\bar{k}\}$, $\forall t\geqslant 0$ with $\underline{k}=1.5$, $\bar{k}=1.7$. By choosing $\bar{f}_1=0$, $\bar{f}_2=\frac{1}{2}$, $\tau=-\frac{1}{5}\in(-\frac{9}{20},0]$, $\omega=\frac{2}{9}\in[0,\infty)$, $m_1=r_1=1$, we obtain $m_2=\frac{m_1+\tau}{p_1}=\frac{36}{55}$, $m_3=\frac{m_2+\tau}{p_2}=\frac{5}{11}$, $r_2=\frac{r_1+\omega}{p_1}=1$, $r_3=\frac{r_2+\omega}{p_2}=\frac{11}{9}$, $|f_2|\leqslant\frac{1}{2}(|x_2|^{\frac{5}{11}}+|x_2|^{\frac{11}{9}})$. Hence, Assumption 1 holds.

According to Appendix C, by setting $h_1=(\underline{k}+x_1)(\bar{k}-x_1)$, $\xi_1=\frac{x_1}{h_1}$, $\xi_2=x_2$, $H_1=\frac{\underline{k}\bar{k}+x_1^2}{h_1^2}$ and choosing $\mu=\frac{121}{45}$, $z_1=\xi_1$, $V_1=\int_0^{\xi_1}\lceil s\rceil^{\frac{6}{5}}\mathrm{d}s+\int_0^{\xi_1}\lceil s\rceil^{\frac{187}{45}}\mathrm{d}s$, the virtual controller $\alpha_1=-\beta_1^{\frac{36}{55}}\lceil z_1+\lceil z_1\rceil^{\frac{55}{36}}\rceil^{\frac{36}{55}}$, $\beta_1=2^{\frac{1}{4}}(\frac{\lambda_{11}}{H_1})^{\frac{5}{4}}$ leads to $\dot{V}_1\leqslant -\lambda_{11}(z_1^2+z_1^{\frac{242}{45}})+H_1(\lceil z_1\rceil^{\frac{6}{5}}+\lceil z_1\rceil^{\frac{187}{45}})(\xi_2^{\frac{11}{9}}-\alpha_1^{\frac{11}{9}})$. Let $z_2=\lceil\xi_2\rceil^{\frac{55}{36}}-\lceil\alpha_1\rceil^{\frac{55}{36}}$, $V=V_2=V_1+\int_{\alpha_1}^{\xi_2}\lceil\lceil s\rceil^{\frac{55}{36}}-\lceil\alpha_1\rceil^{\frac{55}{36}}\rceil^{\frac{17}{11}}\mathrm{d}s+\int_{\alpha_1}^{\xi_2}\lceil\lceil s\rceil^{\frac{55}{36}}-\lceil\alpha_1\rceil^{\frac{55}{36}}\rceil^{\frac{748}{275}}\mathrm{d}s$, the actual controller

$$u=-\beta_2^{\frac{5}{11}}\lceil z_2+\lceil z_2\rceil^{\frac{44}{25}}\rceil^{\frac{5}{11}},\ \beta_2=2^{\frac{6}{5}}\left(\lambda_{22}+\varphi_2\right)^{\frac{11}{5}} \tag{E2}$$

leads to $\dot{V}\leqslant -\lambda_{21}(z_1^2+z_1^{\frac{242}{45}})-\lambda_{22}(z_2^2+z_2^{\frac{968}{275}})$, where $\varphi_2=2^{\frac{1}{2}}\frac{2}{5}(\frac{5\lambda_{211}}{3})^{-\frac{3}{2}}H_1^{\frac{5}{2}}+2^{\frac{22}{25}}\frac{5}{22}(\frac{242\lambda_{211}}{187})^{-\frac{187}{55}}H_1^{\frac{22}{5}}+\frac{34}{11}(\frac{11\lambda_{212}}{10})^{-\frac{5}{17}}c_{21}^{\frac{22}{17}}+2c_{21}+\frac{6}{5}(\frac{5\lambda_{213}}{8})^{-\frac{2}{3}}c_{22}^{\frac{5}{3}}+\frac{653}{484}c_{22}^{\frac{968}{653}}(\frac{242\lambda_{213}}{315})^{-\frac{315}{653}}+2c_{22}$, $c_{21}=\frac{1}{2}+2(1+\beta_1^{\frac{4}{5}})$, $c_{22}=\frac{187}{45}\frac{271}{63}2^{\frac{19}{55}}\beta_1(1+\beta_1^{\frac{4}{5}})H_1$.

By the analysis of (5) and (D2), for the appropriate initial value $x(0)=[-0.5,0.5]^\top\in\Omega_y\times\mathbb{R}$, we can quantitatively obtain the output-constrained compact subset $\bar{\Omega}_y=\{y(t)\in\mathbb{R}:-0.88\leqslant y(t)\leqslant 0.99\}$ and $y(t)\in\bar{\Omega}_y\subsetneq\Omega_y$, $\forall t\geqslant 0$. Hence, the design of (E2) is valid, and $y(t)$ doesn't approach to the boundary of Ω_y for all time. Since $\lambda_{21}=\lambda_{11}-(\lambda_{211}+\lambda_{212}+\lambda_{213})>0$, we choose design parameters $\lambda_{11}=0.19$, $\lambda_{211}=0.09$, $\lambda_{212}=0.009$, $\lambda_{213}=0.09$, $\lambda_{22}=1$.

Figure E1 demonstrates the effectiveness of this control scheme where Figure E1(d) provides the response of $u(t)$ during the first 0.01s. By Figure E1, it is clear that the closed-loop signals x_1,x_2,u are uniformly bounded, the asymmetric output constraint isn't transgressed, x_1,x_2 converge to zero asymptotically, and the undesirable large u doesn't happen.

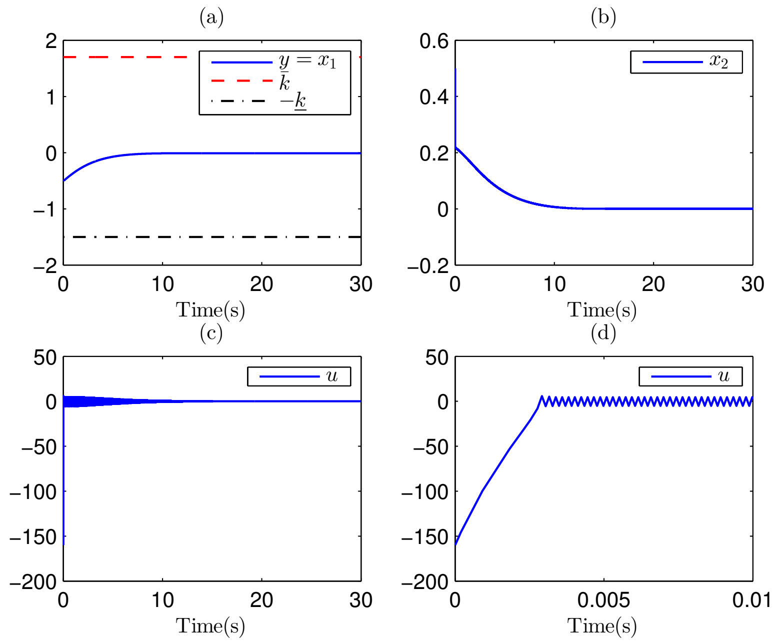

Figure E1 Responses of the closed-loop system (E1)-(E2). (a) Trajectory of $y = x_1$ with constraints $-\underline{k}$ and $\bar{k}$. (b) Trajectory of x_2. (c) Trajectory of u. (d) Trajectory of u during the first 0.01s.

References

1 Qian C J, Lin W. A continuous feedback approach to global strong stabilization of nonlinear systems. IEEE Trans Automat Contr, 2001, 46:1061-1079

2 Lin W, Qian C J. Adaptive control of nonlinearly parameterized systems: the smooth feedback case. IEEE Trans Automat Contr, 2002, 47:1249-1266

3 Sun Z Y, Zhang X H, Xie X J. Continuous global stabilisation of high-order time-delay nonlinear systems. Int J Contr, 2013, 86:994-1007

4 Lei H, Lin W. Reduced-order observer, homogeneous domination and nonsmooth output feedback stabilization of nonlinear systems. Proceedings of the 7th World Congress on Intelligent Control and Automation, Chongqing, China, 2008. 394-399

5 Khalil H K. Nonlinear Systems. New Jersey: Prentice-Hall, 2002

6 Krstić M, Kanellakopoulos I, Kokotovic P V. Nonlinear and Adaptive Control Design. New York: Wiley, 1995