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Adaptive dynamic surface control of high-order strict feedback nonlinear systems with parameter estimations

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Appendix A Notations

Throughout this study, the following symbols are used. Trace $[\cdot]$ denotes the trace of the matrix $\cdot$, $\cdot^{\mathrm{T}}$denotes the transponse of the vector or matrix $\cdot$, $\lambda_{\min}[\cdot]$ and $\lambda_{\max}[\cdot]$ respectively denote the minimal and the maximal eigenvalues of the matrix $\cdot$, a marix $P_i > 0$ means that P_i is a positive definite, symmetric matrix, and

$$x^{(0\sim n)} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(n)} \end{bmatrix}, \quad x_{i\sim j}^{(0\sim n)} = \begin{bmatrix} x_i^{(0\sim n)} \\ x_{i+1}^{(0\sim n)} \\ \vdots \\ x_j^{(0\sim n)} \end{bmatrix}, j \geqslant i, \quad x_k^{(n_0\sim n_k)}|_{k=i\sim j} = \begin{bmatrix} x_i^{(n_0\sim n_i)} \\ x_{i+1}^{(n_0\sim n_{i+1})} \\ \vdots \\ x_j^{(0\sim n_j)} \end{bmatrix}, j \geqslant i,$$

$$a_i^{0\sim m_i-1} = \begin{bmatrix} a_{i0} & a_{i1} & \cdots & a_{i(m_i-1)} \end{bmatrix}, \quad A_i(a_i^{0\sim m_i-1}) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_{i0} & -a_{i1} & \cdots & -a_{i(m_i-2)} & -a_{i(m_i-1)} \end{bmatrix},$$

In the control design process, $a_i^{0\sim m_i-1}, i \in \{1, \cdots, n\}$ is chosen such that $A_i(a_i^{0\sim m_i-1})$ is a Hurwitz matrix.

Appendix B Assumptions

To ensure that system (1) is controllable, it is reasonable to give the following assumption.

Assumption 1. For any $\theta_{gi} \in \Omega_{gi}, i \in \{1, 2, \cdots, n\}$, there exists a possibly unknown constant $g_{im} > 0$ such that $\left|g_{i0} + \theta_{gi}^{\mathrm{T}} g_i\right| > g_{im}$.

Remark 1. Assumption 1 is also used in [1]. Assumption 1 implies that the control coefficient $g_{i0} + \theta_{gi}^{\top} g_i, i \in \{1, \cdots, n\}$ is not equal to zero as long as θ_{gi} falls within the set Ω_{gi}. This is a reasonable assumption used to guarantee the controllability of the system to some extent.

The feasible reference signals are supposed to satisfy the assumption below.

Assumption 2. The reference signal y_d and its derivatives $\dot{y}_d, \cdots, y_d^{(m_1)}$ are available, $y_d, \dot{y}_d, \cdots, y_d^{(m_1+1)}$ belong to the set $\Omega_d = \left\{ \begin{bmatrix} y_d & \dot{y}_d & \cdots & y_d^{(m_1+1)} \end{bmatrix}^{\mathrm{T}} \mid y_d^2 + \dot{y}_d^2 + \cdots + \left(y_d^{(m_1+1)}\right)^2 \leqslant d^2 \right\}$, where d is a possibly unknown constant.

For accurate parameter estimation, the following assumption is usually necessary.

Assumption 3. For any $i \in \{1, \cdots, n\}$, there exist $\alpha_i > 0$ and $T_e > 0$ such that $\lambda_{\min}(C_i(T_e)) > \alpha_i$.

Remark 2. As discussed in [6], the above assumption in fact means that the system is assumed to satisfy the so-called sufficient excitation condition rather than stringent persistent excitation condition in [7]. Interested readers can refer to Ref. [6] for more details.

Appendix C The Proof of Lemma 2

Define

$$\xi_i = x_i^{(m_i-1)} - \zeta_i - \omega_i(\theta_i - \theta_i^0), \tag{C1}$$

then one has that

$$\xi_i(0) = 0, \tag{C2}$$

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and

$$\dot{\xi}_i = x_i^{(m_i)} - \dot{\zeta}_i - \dot{\omega}_i(\theta_i - \theta_i^0) = -k_i\xi_i. \tag{C3}$$

Hence, for all $t \geqslant 0$,

$$\xi_i = x_i^{(m_i-1)} - \zeta_i - \omega_i(\theta_i - \theta_i^0) = 0. \tag{C4}$$

Equ.(C4) is equivalent to

$$\omega_i\theta_i = \omega_i\theta_i^0 + x_i^{(m_i-1)} - \zeta_i. \tag{C5}$$

Pre-multiplying both sides by ω_i^{T} yields

$$\omega_i^{\mathrm{T}}\omega_i\theta_i = \omega_i^{\mathrm{T}}\left(\omega_i\theta_i^0 + x_i^{(m_i-1)} - \zeta_i\right). \tag{C6}$$

Taking integration yields

$$\int_0^t \omega_i^{\mathrm{T}}(s)\omega_i(s)\mathrm{d}s\theta_i = \int_0^t \omega_i^{\mathrm{T}}(s)\left(\omega_i(s)\theta_i^0 + x_i^{(m_i-1)}(s) - \zeta_i(s)\right)\mathrm{d}s, \tag{C7}$$

that is,

$$C_i\theta_i = D_i. \tag{C8}$$

According to the definitions of C_{Ti} and D_{Ti}, one has that

$$C_{Ti}\theta_i = D_{Ti}, \tag{C9}$$

and C_{Ti} is an asymmetric matrix, further, according to Assumption 3 in Appendix B, one has $\lambda_{\min}(C_{Ti}) > \alpha_i > 0$. From Equ.(C9), one can conclude that

$$W_i = C_{Ti}\hat{\theta}_i - D_{Ti} = C_{Ti}\hat{\theta}_i - C_{Ti}\theta_i = C_{Ti}\tilde{\theta}_i. \tag{C10}$$

Meanwhile, one has $\|C_{Ti}\| = \sqrt{\lambda_{\max}[C_{Ti}^{\mathrm{T}}C_{Ti}]} = \sqrt{\lambda_{\max}[C_{Ti}^2]} = \lambda_{\max}[C_{Ti}] \leqslant \mathrm{Trace}\,[C_{Ti}] \leqslant \pi_i$. This completes the proof.

Remark 3. In the accurate parameter estimation methods [3–5], Equ.(C8) is used to reconstruct the parameter estimation error. Unfortunately, the regression matrix C_i and the vector D_i may increase persistently (even to infinite values) due to the unbounded integral operation. In the proposed method, by introducing the truncation operation, Equ.(C9) is used for the reconstruction of the parameter estimation error, which has the following two advantages: (1) Since C_{Ti} and D_{Ti} are bounded, the potential persistent growth problem is avoided; (2) the online computation load is reduced. Additionally, the auxiliary systems utilized in [3–5] are also discarded, which makes the parameter estimation error reconstruction mechanism more simpler and further reduces the online computation load.

Appendix D The Definition and Properties of Projection Operator

A set of projection operators $\mathrm{Pr}_{\hat{\theta}_i}(\beta_i) = \left[\left[\mathrm{Pr}_{\hat{\theta}_{i1}}(\beta_{i1})\right] \cdots \left[\mathrm{Pr}_{\hat{\theta}_{i(p_{fi}+p_{gi})}}\left(\beta_{i(p_{fi}+p_{gi})}\right)\right]\right]^{\mathrm{T}}, i = 1,\cdots,n$, are introduced, where $\hat{\theta}_i = \left[\hat{\theta}_{i1},\cdots,\hat{\theta}_{i(p_{fi}+p_{gi})}\right]^{\mathrm{T}}$, $\beta_i = \left[\beta_{i1},\cdots,\beta_{i(p_{fi}+p_{gi})}\right]^{\mathrm{T}}$, $\beta_{ij}, j = 1,\cdots,p_{fi}+p_{gi}$ are continuous functions, and $\left[\mathrm{Pr}_{\hat{\theta}_{ij}}(\beta_{ij})\right], j = 1,\cdots,p_{fi}+p_{gi}$ are defined as

$$\left[\mathrm{Pr}_{\hat{\theta}_{ij}}(\beta_{ij})\right] = \begin{cases} 0, \hat{\theta}_{ij} = \theta_{ij}^{+} \text{ and } \beta_{ij} > 0 \\ 0, \hat{\theta}_{ij} = \theta_{ij}^{-} \text{ and } \beta_{ij} < 0 \\ \beta_{ij}, \text{otherwise} \end{cases}. \tag{D1}$$

The projection operators possess the following properties (See Ref. [8] and references therein).

Property 1. $\tilde{\theta}_i^{\mathrm{T}}\left[\mathrm{Pr}_{\hat{\theta}_i}(\beta_i) - \beta_i\right] \leqslant 0, i = 1,\cdots,n$.

Property 2. If $\dot{\hat{\theta}}_i = \mathrm{Pr}_{\hat{\theta}_i}(\beta_i), i = 1,\cdots,n$, then the condition $\hat{\theta}_i(0) \in \Omega_i = \Omega_{fi} \times \Omega_{gi}$ implies that $\hat{\theta}_i(t) \in \Omega_i$ for all $t \geqslant 0$.

Remark 4. Property 2 means that if the initial value $\hat{\theta}_i(0) \in \Omega_i$, then for all $t \geqslant 0$, $\hat{\theta}_i(t) \in \Omega_i$. Therefore, according to Assumption 1, $g_{i0} + \hat{\theta}_{gi}^{\mathrm{T}}g_i$ are non-zero. This point is very important since the term $\frac{1}{g_{i0}+\hat{\theta}_{gi}^{\mathrm{T}}g_i}$ will be used in the subsequent control algorithm. That is, the introduction of the projection operator can avoid the singularity problem of the control law.

Appendix E The Proof of Theorem 1

For any $i \in \{1,\cdots,n-1\}$, define

$$\begin{aligned} y_{(i+1)1} &= \bar{x}_{(i+1)1} - x_{(i+1)d}, \\ y_{(i+1)2} &= \bar{x}_{(i+1)2} - \bar{x}_{(i+1)1}, \\ &\vdots \\ y_{(i+1)m_{i+1}} &= x_{(i+1)c} - \bar{x}_{(i+1)m_{i+1}-1}, \end{aligned} \tag{E1}$$

then one has that

$$x_{i+1} = x_{(i+1)d} + z_{i+1} + \sum\nolimits_{j=1}^{m_{(i+1)}} y_{(i+1)j}, \tag{E2}$$

and

$$\dot{y}_{(i+1)1} = -\frac{1}{\tau_{(i+1)1}}y_{(i+1)1} - \dot{x}_{(i+1)d},$$

$$\dot{y}_{(i+1)2} = -\frac{1}{\tau_{(i+1)2}} y_{(i+1)2} + \frac{1}{\tau_{(i+1)1}} y_{(i+1)1},$$
$$\vdots$$
$$\dot{y}_{(i+1)m_{i+1}} = -\frac{1}{\tau_{(i+1)m_{i+1}}} y_{(i+1)m_{i+1}} + \frac{1}{\tau_{(i+1)(m_{i+1}-1)}} y_{(i+1)(m_{i+1}-1)}. \tag{E3}$$

From (1), (2) and (E2), one has that

$$\begin{aligned} z_i^{(m_i)} &= f_{i0} + \theta_{fi}^{\mathrm{T}} f_i + \left(g_{i0} + \theta_{gi}^{\mathrm{T}} g_i\right) x_{i+1} - x_{ic}^{(m_i)} \\ &= f_{i0} + \hat{\theta}_{fi}^{\mathrm{T}} f_i + \left(g_{i0} + \hat{\theta}_{gi}^{\mathrm{T}} g_i\right) \left(x_{(i+1)d} + z_{i+1} + \sum\nolimits_{j=1}^{m_{(i+1)}} y_{(i+1)j}\right) - x_{ic}^{(m_i)} - \tilde{\theta}_{fi}^{\mathrm{T}} f_i - \tilde{\theta}_{gi}^{\mathrm{T}} g_i x_{i+1}. \end{aligned} \tag{E4}$$

Note that the estimations of θ_{fi} and θ_{gi} are respectively $\hat{\theta}_{fi}$ and $\hat{\theta}_{gi}$, the estimation errors are respectively $\tilde{\theta}_{fi} = \hat{\theta}_{fi} - \theta_{fi}$ and $\tilde{\theta}_{gi} = \hat{\theta}_{gi} - \theta_{gi}$, $\theta_2 = \begin{bmatrix} \theta_{fi} \\ \theta_{gi} \end{bmatrix}$, $\hat{\theta}_2 = \begin{bmatrix} \hat{\theta}_{fi} \\ \hat{\theta}_{gi} \end{bmatrix}$, $\tilde{\theta}_i = \hat{\theta}_i - \theta_i = \begin{bmatrix} \hat{\theta}_{fi} - \theta_{fi} \\ \hat{\theta}_{gi} - \theta_{gi} \end{bmatrix}$ and $\varphi_i = \begin{bmatrix} f_i \\ g_i x_{i+1} \end{bmatrix}$. Then one can obtain that

$$z_i^{(m_i)} = f_{i0} + \hat{\theta}_{fi}^{\mathrm{T}} f_i + \left(g_{i0} + \hat{\theta}_{gi}^{\mathrm{T}} g_i\right) \left(x_{(i+1)d} + z_{i+1} + \sum\nolimits_{j=1}^{m_{(i+1)}} y_{(i+1)j}\right) - x_{ic}^{(m_i)} - \tilde{\theta}_i^{\mathrm{T}} \varphi_i. \tag{E5}$$

Substituting (3) into (E5), one has

$$z_i^{(m_i)} = -a_i^{0\sim m_i-1} z_i^{(0\sim m_i-1)} + \left(g_{i0} + \hat{\theta}_{gi}^{\mathrm{T}} g_i\right) \left(z_{i+1} + \sum\nolimits_{j=1}^{m_{(i+1)}} y_{(i+1)j}\right) - \tilde{\theta}_i^{\mathrm{T}} \varphi_i. \tag{E6}$$

Hence, define

$$V_{zi} = \left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i z_i^{(0\sim m_i-1)}, \tag{E7}$$

where the matrix $P_i > 0$ is introduced in Lemma 1, then one has that

$$\begin{aligned} \dot{V}_{zi} &= \left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} \left(A_i^{\mathrm{T}} P_i + P_i A_i\right) z_i^{(0\sim m_i-1)} - 2\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_{ilc} \tilde{\theta}_i^{\mathrm{T}} \varphi_i \\ &\quad + 2\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i \begin{bmatrix} 0 \\ g_{i0} + \hat{\theta}_{gi}^{\mathrm{T}} g_i \end{bmatrix} \left(z_{i+1} + \sum\nolimits_{j=1}^{m_{(i+1)}} y_{(i+1)j}\right) \\ &\leqslant -\mu_i \left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i z_i^{(0\sim m_i-1)} - 2\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_{ilc} \tilde{\theta}_i^{\mathrm{T}} \varphi_i \\ &\quad + 2\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i \begin{bmatrix} 0 \\ g_{i0} + \hat{\theta}_{gi}^{\mathrm{T}} g_i \end{bmatrix} \left(z_{i+1} + \sum\nolimits_{j=1}^{m_{(i+1)}} y_{(i+1)j}\right), \end{aligned} \tag{E8}$$

where P_{ilc} is the last column of the matrix P_i, and the parameter $\mu_i > 0$ is introduced in Lemma 1.

From the definition of $\tilde{\theta}_i$, one has that

$$\dot{\tilde{\theta}}_i = \dot{\hat{\theta}}_i. \tag{E9}$$

Hence, define

$$V_{\theta i} = \tilde{\theta}_i^{\mathrm{T}} \tilde{\theta}_i, \tag{E10}$$

then one has that

$$\dot{V}_{\theta i} = 2\tilde{\theta}_i^{\mathrm{T}} \dot{\hat{\theta}}_i. \tag{E11}$$

From (4), one has

$$\begin{aligned} \dot{V}_{zi} + \dot{V}_{\theta i} &\leqslant -\mu_i \left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i z_i^{(0\sim m_i-1)} - 2\gamma_i \tilde{\theta}_i^{\mathrm{T}} C_{Ti} \tilde{\theta}_i \\ &\quad + 2\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i \begin{bmatrix} 0 \\ g_{i0} + \hat{\theta}_{gi}^{\mathrm{T}} g_i \end{bmatrix} \left(z_{i+1} + \sum\nolimits_{j=1}^{m_{(i+1)}} y_{(i+1)j}\right) \\ &\leqslant -\left(\mu_i \left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i z_i^{(0\sim m_i-1)} + 2\gamma_i \tilde{\theta}_i^{\mathrm{T}} C_{Ti} \tilde{\theta}_i\right) \\ &\quad + 2\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i \begin{bmatrix} 0 \\ g_{i0} + \hat{\theta}_{gi}^{\mathrm{T}} g_i \end{bmatrix} \left(z_{i+1} + \sum\nolimits_{j=1}^{m_{(i+1)}} y_{(i+1)j}\right). \end{aligned} \tag{E12}$$

In addition, one has the fact that

$$\begin{aligned} &\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i \begin{bmatrix} 0 \\ g_{i0} + \hat{\theta}_{gi}^{\mathrm{T}} g_i \end{bmatrix} \left(z_{i+1} + \sum\nolimits_{j=1}^{m_{i+1}} y_{(i+1)j}\right) \\ = &\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i^{\frac{1}{2}} P_i^{\frac{1}{2}} \begin{bmatrix} 0 \\ g_{i0} + \hat{\theta}_{gi}^{\mathrm{T}} g_i \end{bmatrix} \left(z_{i+1} + \sum\nolimits_{j=1}^{m_{i+1}} y_{(i+1)j}\right) \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{2}(m_{i+1}+1)\left\|P_i^{\frac{1}{2}} z_i^{(0\sim m_i-1)}\right\|^2+\frac{1}{2}g_{i0}^2\left\|P_i^{\frac{1}{2}}\right\|^2\left(z_{i+1}^2+\sum\nolimits_{j=1}^{m_{i+1}} y_{(i+1)j}^2\right)\\
&\quad+\frac{1}{2}(m_{i+1}+1)\left\|P_i^{\frac{1}{2}} z_i^{(0\sim m_i-1)}\right\|^2+\frac{1}{2}\left(\hat{\theta}_{gi}^{\mathrm{T}} g_i\right)^2\left\|P_i^{\frac{1}{2}}\right\|^2\left(z_{i+1}^2+\sum\nolimits_{j=1}^{m_{i+1}} y_{(i+1)j}^2\right)\\
&\leqslant (m_{i+1}+1)\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i z_i^{(0\sim m_i-1)}\\
&\quad+\frac{1}{2}\lambda_{\max}(P_i)\left(g_{i0}^2+\left(\hat{\theta}_{gi}^{\mathrm{T}} g_i\right)^2\right)\left(\left\|z_{i+1}^{(0\sim m_{i+1}-1)}\right\|^2+\sum\nolimits_{j=1}^{m_{i+1}} y_{(i+1)j}^2\right),
\end{aligned}\tag{E13}
$$

where $P_i^{\frac{1}{2}}$ denotes the positive definite symmetric matrix which satisfies that $P_i^{\frac{1}{2}}P_i^{\frac{1}{2}}=P_i$.

From (6), one has

$$
\begin{aligned}
z_n^{(m_n)}&=f_{n0}+\theta_{fn}^{\mathrm{T}} f_n+\left(g_{n0}+\theta_{gn}^{\mathrm{T}} g_n\right)u-x_{nc}^{(m_n)}\\
&=f_{n0}+\hat{\theta}_{fn}^{\mathrm{T}} f_n+\left(g_{n0}+\hat{\theta}_{gn}^{\mathrm{T}} g_n\right)u-x_{nc}^{(m_n)}-\tilde{\theta}_{fn}^{\mathrm{T}} f_n-\tilde{\theta}_{gn}^{\mathrm{T}} g_n u
\end{aligned}\tag{E14}
$$

Note that the estimations of θ_{fn} and θ_{gn} are respectively $\hat{\theta}_{fn}$ and $\hat{\theta}_{gn}$, the estimation errors are respectively $\tilde{\theta}_{fn}=\hat{\theta}_{fn}-\theta_{fn}$ and $\tilde{\theta}_{gn}=\hat{\theta}_{gn}-\theta_{gn}$, $\theta_n=\begin{bmatrix}\theta_{fn}\\ \theta_{gn}\end{bmatrix}$, $\hat{\theta}_n=\begin{bmatrix}\hat{\theta}_{fn}\\ \hat{\theta}_{gn}\end{bmatrix}$, $\tilde{\theta}_n=\hat{\theta}_n-\theta_n=\begin{bmatrix}\hat{\theta}_{fn}-\theta_{fn}\\ \hat{\theta}_{gn}-\theta_{gn}\end{bmatrix}$ and $\varphi_n=\begin{bmatrix}f_n\\ g_n u\end{bmatrix}$. Then one can obtain that

$$
z_n^{(m_n)}=f_{n0}+\hat{\theta}_{fn}^{\mathrm{T}} f_n+\left(g_{n0}+\hat{\theta}_{gn}^{\mathrm{T}} g_n\right)u-x_{nc}^{(m_n)}-\tilde{\theta}_n^{\mathrm{T}}\varphi_n,\tag{E15}
$$

Substituting (7) into (E15), one has

$$
z_n^{(m_n)}=-a_n^{0\sim m_n-1} z_n^{(0\sim m_n-1)}-\tilde{\theta}_n^{\mathrm{T}}\varphi_n.\tag{E16}
$$

Hence, define

$$
V_{zn}=\left(z_n^{(0\sim m_n-1)}\right)^{\mathrm{T}} P_n z_n^{(0\sim m_n-1)},\tag{E17}
$$

where the matrix P_n is introduced in Lemma 1, then one has that

$$
\dot{V}_{zn}\leqslant-\mu_n\left(z_n^{(0\sim m_n-1)}\right)^{\mathrm{T}} P_n z_n^{(0\sim m_n-1)}-2\left(z_n^{(0\sim m_n-1)}\right)^{\mathrm{T}} P_{nlc}\tilde{\theta}_n^{\mathrm{T}}\varphi_n\tag{E18}
$$

where P_{nlc} is the last column of the matrix P_n, and the parameter $\mu_n>0$ is introduced in Lemma 1.

From the definition of $\tilde{\theta}_n$, one has that

$$
\dot{\tilde{\theta}}_n=\dot{\hat{\theta}}_n.\tag{E19}
$$

Hence, define

$$
V_{\theta n}=\tilde{\theta}_n^{\mathrm{T}}\tilde{\theta}_n,\tag{E20}
$$

then one has that

$$
\dot{V}_{\theta n}=2\tilde{\theta}_n^{\mathrm{T}}\dot{\hat{\theta}}_n,\tag{E21}
$$

From (8), one has

$$
\dot{V}_{zn}+\dot{V}_{\theta n}\leqslant-\mu_n\left(z_n^{(0\sim m_n-1)}\right)^{\mathrm{T}} P_n z_n^{(0\sim m_n-1)}-2\gamma_n\tilde{\theta}_n^{\mathrm{T}} C_{Tn}\tilde{\theta}_n.\tag{E22}
$$

For any $i\in\{2,\cdots,n\}$, define

$$
V_{yi}=\frac{1}{2}\sum\nolimits_{j=1}^{m_i} y_{ij}^2,\tag{E23}
$$

then one has that

$$
\begin{aligned}
\dot{V}_{yi}&=-\frac{1}{\tau_{i1}}y_{i1}^2-y_{i1}\dot{x}_{id}\\
&\quad-\frac{1}{\tau_{i2}}y_{i2}^2+\frac{1}{\tau_{i1}}y_{i1}y_{i2}\\
&\quad\cdots\\
&\quad-\frac{1}{\tau_{i(m_i)}}y_{i(m_i)}^2+\frac{1}{\tau_{i(m_i-1)}}y_{i(m_i-1)}y_{i(m_i)}\\
&\leqslant-\frac{1}{2\tau_{i1}}y_{i1}^2-\sum\nolimits_{j=2}^{m_i-1}\left[\left(\frac{1}{2\tau_{ij}}-\frac{1}{2\tau_{i(j-1)}}\right)y_{ij}^2\right]\\
&\quad-\left(\frac{1}{\tau_{i(m_i)}}-\frac{1}{2\tau_{i(m_i-1)}}\right)y_{im_i}^2-y_{i1}\dot{x}_{id}.
\end{aligned}\tag{E24}
$$

Define

$$
V=\sum\nolimits_{i=1}^{n}\left(V_{zi}+V_{\theta i}\right)+\sum\nolimits_{i=2}^{n} V_{yi}.\tag{E25}
$$

Its time derivative can be calculated from Eqs.(E12,E13,E22,E24) as follows:

$$
\dot{V}\leqslant-\sum\nolimits_{i=1}^{n}\left(\mu_i\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}} P_i z_i^{(0\sim m_i-1)}+2\gamma_i\tilde{\theta}_i^{\mathrm{T}} C_{Ti}\tilde{\theta}_i\right)
$$

$$
\begin{aligned}
&+\sum_{i=1}^{n-1}2(m_{i+1}+1)\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm T}P_iz_i^{(0\sim m_i-1)}\\
&+\sum_{i=1}^{n-1}\lambda_{\max}(P_i)\left(g_{i0}^2+\left(\hat\theta_{gi}^{\mathrm T}g_i\right)^2\right)\left(\left\|z_{i+1}^{(0\sim m_{i+1}-1)}\right\|^2+\sum_{j=1}^{m_{i+1}}y_{(i+1)j}^2\right)\\
&-\sum_{i=2}^{n}\frac{1}{2\tau_{i1}}y_{i1}^2-\sum_{i=2}^{n}\sum_{j=2}^{m_i-1}\left[\left(\frac{1}{2\tau_{ij}}-\frac{1}{2\tau_{i(j-1)}}\right)y_{ij}^2\right]\\
&-\sum_{i=2}^{n}\left(\frac{1}{\tau_{i(m_i)}}-\frac{1}{2\tau_{i(m_i-1)}}\right)y_{im_i}^2-\sum_{i=2}^{n}y_{i1}\dot x_{id}.
\end{aligned}\tag{E26}
$$

It is not difficult to obtain that $\dot x_{id},\forall i\in\{2,\cdots,n\}$ is a function of the variables $z_j^{(0\sim m_j-1)}|_{j=1\sim i-1}$, z_i, y_{21}, $\cdots$, y_{2m_2}, $\cdots$, y_{i1}, $\cdots$, y_{im_i}, $\tilde\theta_1$, $\cdots$, $\tilde\theta_{i-1}$, y_d, $\dot y_d$, $\cdots$, $y_d^{(m_1+1)}$, $C_{T1},\cdots,C_{T(i-1)}$ and the design parameters P_1, $\cdots$, P_{i-1}, γ_1, $\cdots$, γ_{i-1}, $a_1^{0\sim m_1-1}$, $\cdots$, $a_{i-1}^{0\sim m_{i-1}-1}$, τ_{21}, $\cdots$, τ_{2m_2}, $\cdots$, $\tau_{(i-1)1}$, $\cdots$, $\tau_{(i-1)m_{i-1}}$, where $\tau_{11}=0$, $\cdots$, $\tau_{1m_1}=0$. Note that according to Lemma 2, $\|C_{Ti}\|\leqslant\pi_i,\forall i\in\{1,\cdots,n\}$. Hence, there exists a non-negative continuous function η_i, which is dependent on the variables $z_j^{(0\sim m_j-1)}|_{j=1\sim i-1}$, z_i, y_{21}, $\cdots$, y_{2m_2}, $\cdots$, y_{i1}, $\cdots$, y_{im_i}, $\tilde\theta_1$, $\cdots$, $\tilde\theta_{i-1}$, y_d, $\dot y_d$, $\cdots$, $y_d^{(m_1+1)}$ and the design parameters P_1, $\cdots$, P_{i-1}, γ_1, $\cdots$, γ_{i-1}, $a_1^{0\sim m_1-1}$, $\cdots$, $a_{i-1}^{0\sim m_{i-1}-1}$, π_1, $\cdots$, π_{i-1}, τ_{21}, $\cdots$, τ_{2m_2}, $\cdots$, $\tau_{(i-1)1}$, $\cdots$, $\tau_{(i-1)m_{i-1}}$, such that

$$
|\dot x_{id}|\leqslant\eta_i.\tag{E27}
$$

It is also not difficult to obtain that $g_{10}^2+\left(\hat\theta_{g1}^{\mathrm T}g_1\right)^2$ is a non-negative continuous function of the variables $z_1^{(0\sim m_1-1)}$, $\hat\theta_{g1}$ and $y_d^{(0\sim m_1-1)}$, meanwhile $g_{i0}^2+\left(\hat\theta_{gi}^{\mathrm T}g_i\right)^2,\forall i\in\{2,\cdots,n-1\}$, is a non-negative continuous function of the variables $z_j^{(0\sim m_j-1)}|_{j=1\sim i-1}$, $y_{21},\cdots,y_{2m_2}$, $\cdots$, y_{i2}, $\cdots$, y_{im_i}, $\tilde\theta_1$, $\cdots$, $\tilde\theta_{i-1}$, $\hat\theta_{gi}$ and $y_d^{(0\sim m_1)}$, and the design parameters τ_{2m_2}, $\cdots$, τ_{21}, $\cdots$, $\tau_{(i-1)m_{i-1}}$, $\cdots$,$\tau_{(i-1)1}$, τ_{im_i}, $\cdots$, τ_{i2}, $a_1^{0\sim m_1-1}$, $\cdots$, $a_{i-1}^{0\sim m_{i-1}-1}$.

For any $V(0)>0$, the set

$$
\Omega=\left\{z_i^{(0\sim m_i-1)}|_{i=1\sim n},y_{21},\cdots,y_{2m_2},\cdots,y_{n1},\cdots,y_{nm_n},\tilde\theta_1,\cdots,\tilde\theta_n\mid V\leqslant V(0)\right\}\tag{E28}
$$

is a compact set. Note that $\left[\begin{array}{cccc}y_d&\dot y_d&\cdots&y_d^{(m_1+1)}\end{array}\right]^{\mathrm T}\in\Omega_d$. Therefore, $\eta_i,\forall i\in\{2,\cdots,n\}$ has the maximal value on Ω, denoted by $\bar\eta_i$ which among all of the design parameters only depends on P_1, $\cdots$, P_{i-1}, γ_1, $\cdots$, γ_{i-1}, π_1, $\cdots$, π_{i-1}, $a_1^{0\sim m_1-1}$, $\cdots$, $a_{i-1}^{0\sim m_{i-1}-1}$, $\tau_{21},\cdots,\tau_{2m_2},\cdots,\tau_{(i-1)1},\cdots,\tau_{(i-1)m_{i-1}}$. Since $\hat\theta_{gi}\in\Omega_{gi}$, $g_{i0}^2+\left(\hat\theta_{gi}^{\mathrm T}g_i\right)^2,\forall i\in\{1,\cdots,n-1\}$ also has the maximal value on Ω, denoted by $\bar g_i$ which among all of the design parameters only depends on the design parameters τ_{2m_2}, $\cdots$, τ_{21}, $\cdots$, $\tau_{(i-1)m_{(i-1)}}$, $\cdots$,$\tau_{(i-1)1}$, τ_{im_i}, $\cdots$, τ_{i2}, $a_1^{0\sim m_1-1}$, $\cdots$, $a_{i-1}^{0\sim m_{(i-1)}-1}$, where $a_0^{0\sim m_0-1}=0$. Therefore, on the set Ω, one has that

$$
\begin{aligned}
\dot V\leqslant&-\sum_{i=1}^{n}\left(\mu_i\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm T}P_iz_i^{(0\sim m_i-1)}+2\gamma_i\tilde\theta_i^{\mathrm T}C_{Ti}\tilde\theta_i\right)\\
&+\sum_{i=1}^{n-1}2(m_{i+1}+1)\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm T}P_iz_i^{(0\sim m_i-1)}\\
&+\sum_{i=1}^{n-1}\lambda_{\max}(P_i)\bar g_i\left(\left\|z_{i+1}^{(0\sim m_{i+1}-1)}\right\|^2+\sum_{j=1}^{m_{i+1}}y_{(i+1)j}^2\right)\\
&-\sum_{i=2}^{n}\frac{1}{2\tau_{i1}}y_{i1}^2-\sum_{i=2}^{n}\sum_{j=2}^{m_i-1}\left[\left(\frac{1}{2\tau_{ij}}-\frac{1}{2\tau_{i(j-1)}}\right)y_{ij}^2\right]\\
&-\sum_{i=2}^{n}\left(\frac{1}{\tau_{i(m_i)}}-\frac{1}{2\tau_{i(m_i-1)}}\right)y_{im_i}^2+\sum_{i=2}^{n}|y_{i1}|\,\bar\eta_i\\
\leqslant&-(\mu_1-2(m_2+1))\left(z_1^{(0\sim m_1-1)}\right)^{\mathrm T}P_1z_1^{(0\sim m_1-1)}\\
&-\sum_{i=2}^{n-1}(\mu_i-2(m_{i+1}+1))\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm T}P_iz_i^{(0\sim m_i-1)}\\
&-\mu_n\left(z_n^{(0\sim m_n-1)}\right)^{\mathrm T}P_nz_n^{(0\sim m_n-1)}-\sum_{i=1}^{n}2\gamma_i\tilde\theta_i^{\mathrm T}C_{Ti}\tilde\theta_i\\
&+\sum_{i=1}^{n-1}\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_{i+1})}\bar g_i\left(z_{i+1}^{(0\sim m_{i+1}-1)}\right)^{\mathrm T}P_{i+1}z_{i+1}^{(0\sim m_{i+1}-1)}\\
&-\sum_{i=2}^{n}\left(\frac{1}{2\tau_{i1}}-\frac{\bar\eta_i^2}{4\varepsilon}\right)y_{i1}^2+(n-1)\varepsilon-\sum_{i=2}^{n}\sum_{j=2}^{m_i-1}\left[\left(\frac{1}{2\tau_{ij}}-\frac{1}{2\tau_{i(j-1)}}\right)y_{ij}^2\right]\\
&-\sum_{i=2}^{n}\left(\frac{1}{\tau_{i(m_i)}}-\frac{1}{2\tau_{i(m_i-1)}}\right)y_{im_i}^2+\sum_{i=2}^{n}\lambda_{\max}(P_{i-1})\bar g_{i-1}\sum_{j=1}^{m_i}y_{ij}^2\\
\leqslant&-(\mu_1-2(m_2+1))\left(z_1^{(0\sim m_1-1)}\right)^{\mathrm T}P_1z_1^{(0\sim m_1-1)}\\
&-\sum_{i=2}^{n-1}\left(\mu_i-2(m_{i+1}+1)-\frac{\lambda_{\max}(P_{i-1})}{\lambda_{\min}(P_i)}\bar g_{i-1}\right)\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm T}P_iz_i^{(0\sim m_i-1)}\\
&-\left(\mu_n-\frac{\lambda_{\max}(P_{n-1})}{\lambda_{\min}(P_n)}\bar g_{n-1}\right)\left(z_n^{(0\sim m_n-1)}\right)^{\mathrm T}P_nz_n^{(0\sim m_n-1)}-\sum_{i=1}^{n}2\gamma_i\tilde\theta_i^{\mathrm T}C_{Ti}\tilde\theta_i\\
&-\sum_{i=2}^{n}\left(\frac{1}{2\tau_{i1}}-\frac{\bar\eta_i^2}{4\varepsilon}-\lambda_{\max}(P_{i-1})\bar g_{i-1}\right)y_{i1}^2
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=2}^{n}\sum_{j=2}^{m_i-1}\left[\left(\frac{1}{2\tau_{ij}}-\frac{1}{2\tau_{i(j-1)}}-\lambda_{\max}(P_{i-1})\bar{g}_{i-1}\right)y_{ij}^2\right]\\
&-\sum_{i=2}^{n}\left(\frac{1}{\tau_{i(m_i)}}-\frac{1}{2\tau_{i(m_i-1)}}-\lambda_{\max}(P_{i-1})\bar{g}_{i-1}\right)y_{im_i}^2+(n-1)\varepsilon, \qquad \text{(E29)}
\end{aligned}
$$

where $\varepsilon>0$ is a small constant and the equality

$$
\begin{aligned}
&\sum_{i=1}^{n-1}\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_{i+1})}\bar{g}_i\left(z_{i+1}^{(0\sim m_{i+1}-1)}\right)^{\mathrm{T}}P_{i+1}z_{i+1}^{(0\sim m_{i+1}-1)}\\
&=\sum_{i=2}^{n}\frac{\lambda_{\max}(P_{i-1})}{\lambda_{\min}(P_i)}\bar{g}_{i-1}\left(z_i^{(0\sim m_i-1)}\right)^{\mathrm{T}}P_iz_i^{(0\sim m_i-1)} \qquad \text{(E30)}
\end{aligned}
$$

is used in the derivation.

Note that according to Assumption 3,

$$\gamma_i\tilde{\theta}_i^{\mathrm{T}}C_{Ti}\tilde{\theta}_i\geqslant\gamma_i\alpha_i\tilde{\theta}_i^{\mathrm{T}}\tilde{\theta}_i. \qquad \text{(E31)}$$

Hence, if the design parameters satisfy that

$$
\begin{aligned}
&\mu_1-2(m_2+1)\geqslant\kappa,\\
&\mu_i-2(m_{i+1}+1)-\frac{\lambda_{\max}(P_{i-1})}{\lambda_{\min}(P_i)}\bar{g}_{i-1}\geqslant\kappa, i=2,\cdots,n-1,\\
&\mu_n-\frac{\lambda_{\max}(P_{n-1})}{\lambda_{\min}(P_n)}\bar{g}_{n-1}\geqslant\kappa,\\
&2\alpha_i\gamma_i\geqslant\kappa,\\
&\frac{1}{2\tau_{i1}}-\frac{\bar{\eta}_i^2}{4\varepsilon}-\lambda_{\max}(P_{i-1})\bar{g}_{i-1}\geqslant\kappa, i=2,\cdots,n,\\
&\frac{1}{2\tau_{ij}}-\frac{1}{2\tau_{i(j-1)}}-\lambda_{\max}(P_{i-1})\bar{g}_{i-1}\geqslant\kappa, i=2,\cdots,n, j=2,\cdots,m_i-1,\\
&\frac{1}{\tau_{i(m_i)}}-\frac{1}{2\tau_{i(m_i-1)}}-\lambda_{\max}(P_{i-1})\bar{g}_{i-1}\geqslant\kappa, i=2,\cdots,n, \qquad \text{(E32)}
\end{aligned}
$$

where $\kappa>0$ is a constant, then one has that

$$\dot{V}\leqslant-\kappa V+\epsilon, \qquad \text{(E33)}$$

where $\epsilon=(n-1)\varepsilon$. By the comparison principle, one has that

$$V(t)\leqslant\frac{\epsilon}{\kappa}+\left(V(0)-\frac{\epsilon}{\kappa}\right)e^{-\kappa t}. \qquad \text{(E34)}$$

Clearly, if $\kappa\geqslant\frac{\epsilon}{V(0)}$, then one has that

$$V(0)\left(1-e^{-\kappa t}\right)\geqslant\frac{\epsilon}{\kappa}\left(1-e^{-\kappa t}\right), \qquad \text{(E35)}$$

or equivalently,

$$\frac{\epsilon}{\kappa}+\left(V(0)-\frac{\epsilon}{\kappa}\right)e^{-\kappa t}\leqslant V(0). \qquad \text{(E36)}$$

Hence, one has that

$$V(t)\leqslant V(0). \qquad \text{(E37)}$$

This implies that Ω is an invariant set. Additionally, it is easy to know that $V(t)$ ultimately converges to the set

$$\Omega_f=\left\{\left[\left(z_i^{(0\sim m_i-1)}|_{i=1\sim n}\right)^{\mathrm{T}},y_{21},\cdots,y_{2m_2},\cdots,y_{n1},\cdots,y_{nm_n},\tilde{\theta}_1^{\mathrm{T}},\cdots,\tilde{\theta}_n^{\mathrm{T}}\right]^{\mathrm{T}}:V\leqslant\frac{\epsilon}{\kappa}\right\}, \qquad \text{(E38)}$$

of which the size can be made arbitrarily small by tuning the design parameters, since ϵ does not depend on κ. Hence, it is easy to conclude that all the closed-loop signals are semi-global uniformly ultimately bounded, and the ultimate bounds of the tracking error z_1 and the parameter estimation errors $\tilde{\theta}_i, i=1,\cdots,n$, can be adjusted arbitrarily by tuning the design parameters. This completes the proof.

Remark 5. At the ith step, $2\leqslant i\leqslant n$, to obtain $z_i^{(0\sim m_i-1)}$, it seems that the 1st- to (m_i-1)th-order derivatives of the signal x_{ic} are required to be computed. In fact, they can be obtained by some algebraic operations. Let us take $x_{2c}^{(j)}, 1\leqslant j\leqslant m_2-1$ as an example to show this point. From Equ.(5), one has that

$$
\begin{aligned}
\dot{\bar{x}}_{21}&=\frac{1}{\tau_{21}}(x_{2d}-\bar{x}_{21}),\\
\dot{\bar{x}}_{22}&=\frac{1}{\tau_{22}}(\bar{x}_{21}-\bar{x}_{22}),\\
&\vdots
\end{aligned}
$$

$$\dot{\bar{x}}_{2(m_2-1)} = \frac{1}{\tau_{2(m_2-1)}}(\bar{x}_{2(m_2-2)} - \bar{x}_{2(m_2-1)}),$$
$$\dot{x}_{2c} = \frac{1}{\tau_{2m_2}}(\bar{x}_{2(m_2-1)} - x_{2c}).$$

Hence,

$$\begin{aligned} x_{2c}^{(2)} &= \frac{1}{\tau_{2m_2}}(\dot{\bar{x}}_{2m_2-1} - \dot{x}_{2c}) \\ &= \frac{1}{\tau_{2m_2}}\left[\frac{1}{\tau_{2(m_2-1)}}(\bar{x}_{2(m_2-2)} - \bar{x}_{2(m_2-1)}) - \frac{1}{\tau_{2m_2}}(\bar{x}_{2(m_2-1)} - x_{2c})\right]. \end{aligned}$$

Continue this process recursively, it is easy to know that $x_{2c}^{(j)}, 3 \leqslant j \leqslant m_2 - 1$ can also be computed algebraically from the available variables $\bar{x}_{21}, \cdots, \bar{x}_{2(m_2-1)}, x_{2c}$ and the parameters $\frac{1}{\tau_{22}}, \cdots, \frac{1}{\tau_{2m_2}}$.

Remark 6. From the above control design process, one can see that: firstly, if $m_i = 1$ for all $i \in \{1, 2, \cdots, n\}$, the proposed adaptive DSC can be reduced to the traditional one; secondly, if $m_i = 2$ for some $i \in \{1, 2, \cdots, n\}$, one can set $\tau_{i1} = \tau_{i2} = \tau_i$, which is helpful to reduce the number of the design parameters; and thirdly, by regarding every high-order subsystem as a whole instead of converting it into a set of first-order ones, less virtual control laws are required to be derived, which makes the proposed control algorithm more direct and concise.

Remark 7. Different from most of the adaptive backstepping or DSC results, where only the boundedness of the parameter estimation errors is established, the proposed adaptive DSC method could guarantee that the estimations of the unknown parameters converge to their true values with very small errors by utilizing the accurate estimation error reconstruction mechanism. This is very useful to improve the performance of the closed-loop system.

Remark 8. For any $i \in \{1, 2, \cdots, n\}$, μ_i and P_i are introduced in Lemma 1, and the solution of μ_i and P_i can be found in [2]. Once P_i satisfies

$$A_i^{\mathrm{T}}(a_i^{0\sim m_i-1})P_i + P_i A_i(a_i^{0\sim m_i-1}) < -\mu_i P_i,$$

then for any $\sigma > 0$, σP_i also satisfies the above matrix inequality. This implies that $\lambda_{\min}[P_i]$ or $\lambda_{\max}[P_i]$ can be adjusted arbitrarily without affecting the value of μ_i. In theory, the selection of control parameters can be conducted step-by-step in the following order. In the first step, the parameters μ_1, P_1, γ_1 are selected to satisfy $\mu_1 - 2(m_2 + 1) \geqslant \kappa$ and $2\alpha_1\gamma_1 \geqslant \kappa$. In the second step, given that the value of $\bar{\eta}_2$ is related to μ_1, P_1, γ_1, the parameters $\mu_2, P_2, \gamma_2, \tau_{2j}, j = 1, 2, \cdots, m_2$ are selected to satisfy $\mu_2 - 2(m_3 + 1) - \frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_2)}\bar{g}_1 \geqslant \kappa$, $2\alpha_2\gamma_2 \geqslant \kappa$ and

$$\begin{aligned} \frac{1}{2\tau_{i1}} - \frac{\bar{\eta}_i^2}{4\varepsilon} - \lambda_{\max}(P_{i-1})\bar{g}_{i-1} &\geqslant \kappa, i = 2 \\ \frac{1}{2\tau_{ij}} - \frac{1}{2\tau_{i(j-1)}} - \lambda_{\max}(P_{i-1})\bar{g}_{i-1} &\geqslant \kappa, i = 2, j = 2, \cdots, m_i - 1, \\ \frac{1}{\tau_{i(m_i)}} - \frac{1}{2\tau_{i(m_i-1)}} - \lambda_{\max}(P_{i-1})\bar{g}_{i-1} &\geqslant \kappa, i = 2. \end{aligned}$$

Likewise, in the ith ($i \in \{3, 4, \cdots, n\}$) step, given that the value of $\bar{\eta}_i$ is related to μ_1, $\cdots$, μ_{i-1}, P_1, $\cdots$, P_{i-1}, γ_1, $\cdots$,γ_{i-1}, τ_{21}, $\cdots$, τ_{2m_2}, $\cdots$, $\tau_{(i-1)1}$, $\cdots$, $\tau_{(i-1)m_{i-1}}$, the parameters $\mu_i, P_i, \gamma_i, \tau_{ij}, j = 1, 2, \cdots, m_i$ can be selected according to the similar conditions.

Appendix F Numerical Example

Consider the single-link flexible-joint robot system, as shown in Fig.F1, the dynamic model of which can be expressed as follows

$$\begin{aligned} I\ddot{q}_1 + MgL\sin(q_1) + K(q_1 - q_2) &= 0, \\ J\ddot{q}_2 - K(q_1 - q_2) &= u, \end{aligned}$$

where q_1 and q_2 represent the angular positions of the link and the motor, respectively, M is the link mass, $g = 9.8\mathrm{m} \cdot \mathrm{s}^{-2}$ is the gravity constant, L is the location of the mass center, I is the link inertia, K is the torsional spring constant, J is the motor inertia, u denotes the input torque delivered by the motor.

Defining the state variables as $x_1 = q_1$, $x_2 = q_2$ and the output as $y = q_1$, then one can obtain that

$$\begin{aligned} \ddot{x}_1 &= -\theta_{f1}\sin(x_1) - \theta_{g1}(x_1 - x_2), \\ \ddot{x}_2 &= \theta_{f2}(x_1 - x_2) + \theta_{g2}u, \\ y &= x_1, \end{aligned}$$

where $\theta_{f1} = \frac{MgL}{I}$, $\theta_{g1} = \frac{K}{I}$, $\theta_{f2} = \frac{K}{J}$, $\theta_{g2} = \frac{1}{J}$.

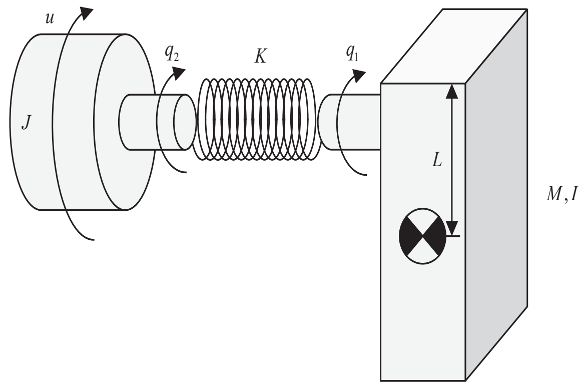

Figure F1 Single-link flexible-joint robot

The nominal values of the system parameters are set as: $I_0 = 0.105\text{kg}\cdot\text{m}^2$, $M_0 = 1.265\text{kg}$, $L_0 = 0.25\text{m}$, $K_0 = 1.073\text{N}\cdot\text{m/rad}$ and $J_0 = 0.06784\text{kg}\cdot\text{m}^2$. Hence, the nominal values of θ_{f1}, θ_{g1}, θ_{f2} and θ_{g2} are $\theta_{f10} = 29.5167$, $\theta_{g10} = 10.2190$, $\theta_{f20} = 17.7794$ and $\theta_{g20} = 14.7059$. The bounds of these unknown parameters are assumed to be $I_{\min} = 0.9I_0$, $I_{\max} = 1.1I_0$, $M_{\min} = 0.9M_0$, $M_{\max} = 1.1M_0$, $L_{\min} = 0.9L_0$, $L_{\max} = 1.1L_0$, $K_{\min} = 0.9K_0$, $K_{\max} = 1.1K_0$, $J_{\min} = 0.9J_0$, $J_{\max} = 1.1J_0$. Hence, one has that $\theta_{f1} \in [21.7350, 39.6835]$, $\theta_{g1} \in [8.3610, 12.4899]$, $\theta_{f2} \in [12.9104, 19.2859]$ and $\theta_{g2} \in [13.3690, 16.3399]$.

By using the proposed adaptive DSC approach, the control algorithm can be designed as follows

$$\begin{cases} z_1 = x_1 - y_d, \dot{z}_1 = \dot{x}_1 - \dot{y}_d, \\ x_{2d} = \frac{1}{\hat{\theta}_{g1}}\left[-\hat{\theta}_{f1}\sin(x_1) + \hat{\theta}_{g1}x_1 - a_1^{0\sim 1}z_1^{(0\sim 1)} + \ddot{x}_{1d}\right], \\ \tau_{21}\dot{\bar{x}}_{21} + \bar{x}_{21} = x_{2d}, \bar{x}_{21}(0) = x_{2d}(0), \\ \tau_{22}\dot{x}_{2c} + x_{2c} = \bar{x}_{21}, x_{2c}(0) = \bar{x}_{21}(0), \\ z_2 = x_2 - x_{2c}, \dot{z}_2 = \dot{x}_2 - \dot{x}_{2c}, \\ u = \frac{1}{\hat{\theta}_{g2}}\left[-\hat{\theta}_{f2}(x_1 - x_2) - a_2^{0\sim 1}z_2^{(0\sim 1)} + \ddot{x}_{2c}\right], \end{cases}$$

with the adaptive laws as follows

$$\begin{cases} \varphi_1 = [-\sin(x_1); -x_1 + x_2], \varphi_2 = [x_1 - x_2; u], \\ \dot{\zeta}_i = -k_i\zeta_i + \left(\theta_i^0\right)^{\mathrm{T}}\varphi_i + k_i\dot{x}_i, \zeta_i(0) = \dot{x}_i(0), i = 1, 2, \\ \dot{\omega}_i = -k_i\omega_i + \varphi_i^{\mathrm{T}}, \omega_i(0) = 0, i = 1, 2, \\ \dot{C}_i = \omega_i^{\mathrm{T}}\omega_i, C_i(0) = 0, i = 1, 2, \\ \dot{D}_i = \omega_i^{\mathrm{T}}\left(\omega_i\theta_i^0 + \dot{x}_i - \zeta_i\right), D_i(0) = 0, i = 1, 2, \\ C_{Ti} = \begin{cases} C_i(t_T), \text{if } C_i(t_T) > 0 \text{ and Trace}\left[C_i(t_T)\right] < \pi_i \\ C_i(t), \text{otherwise} \end{cases}, i = 1, 2, \\ D_{Ti} = \begin{cases} D_i(t_T), \text{if } C_i(t_T) > 0 \text{ and Trace}\left[C_i(t_T)\right] < \pi_i \\ D_i(t), \text{otherwise} \end{cases}, i = 1, 2, \\ W_i = C_{Ti}\hat{\theta}_i - D_{Ti}, i = 1, 2, \\ \dot{\hat{\theta}}_i = \begin{bmatrix} \dot{\hat{\theta}}_{fi} \\ \dot{\hat{\theta}}_{gi} \end{bmatrix} = \mathrm{Pr}_{\hat{\theta}_i}\left(\left(z_i^{(0\sim 1)}\right)^{\mathrm{T}}P_{ilc}\varphi_i - \gamma_i W_i\right), \hat{\theta}_i(0) = \theta_i^0 = \begin{bmatrix} \theta_{fi0} \\ \theta_{gi0} \end{bmatrix}, i = 1, 2, \end{cases}$$

In the simulation, the parameters and the desired trajectory are set as $a_1^{0\sim 1} = [38.21, 12.2]$, $a_2^{0\sim 1} = [400.0001, 40]$, $\tau_{21} = 0.002$, $\tau_{22} = 0.001$, $k_1 = 0.01$, $k_2 = 0.1$, $\pi_1 = 40$, $\pi_2 = 100$, $\gamma_1 = 100$, $\gamma_2 = 4$, $P_1 = \begin{bmatrix} 0.0177 & 0.0001 \\ 0.0001 & 0.0004 \end{bmatrix}$, $P_2 = \begin{bmatrix} 250.6251 & 0.0125 \\ 0.0125 & 0.6253 \end{bmatrix}$, $y_d = 1 + \sin(t) + \sin(2t)$. Meanwhile, the initial conditions are set as

$$\begin{bmatrix} x_1(0) \\ \dot{x}_1(0) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 3.5 \\ 0 \end{bmatrix}, \begin{bmatrix} \zeta_1(0) \\ \zeta_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The actual values of $\theta_i, i = 1, 2$ are set as

$$\theta_1 = \begin{bmatrix} \theta_{f1} \\ \theta_{g1} \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix}, \theta_2 = \begin{bmatrix} \theta_{f2} \\ \theta_{g2} \end{bmatrix} = \begin{bmatrix} 17 \\ 16 \end{bmatrix}.$$

With the above parameter settings, the simulation results are shown in Fig.F2. From Figs.F2(a)-F2(c), one can see that $x_1(t)$ can track the desired trajectory $y_d(t)$ with a small tracking error $z_1(t)$. The curves of $x_2(t)$ and $\dot{x}_2(t)$ are shown in Fig.F2(d). From Figs.F2(e)-F2(h), one can observe that $\hat{\theta}_{f1}(t)$, $\hat{\theta}_{g1}(t)$, $\hat{\theta}_{f2}(t)$ and $\hat{\theta}_{g2}(t)$ converge to the actual values θ_{f1}, θ_{g1}, θ_{f2} and θ_{g2}, respectively. The curve of $u(t)$ is depicted in Fig.F2(i).

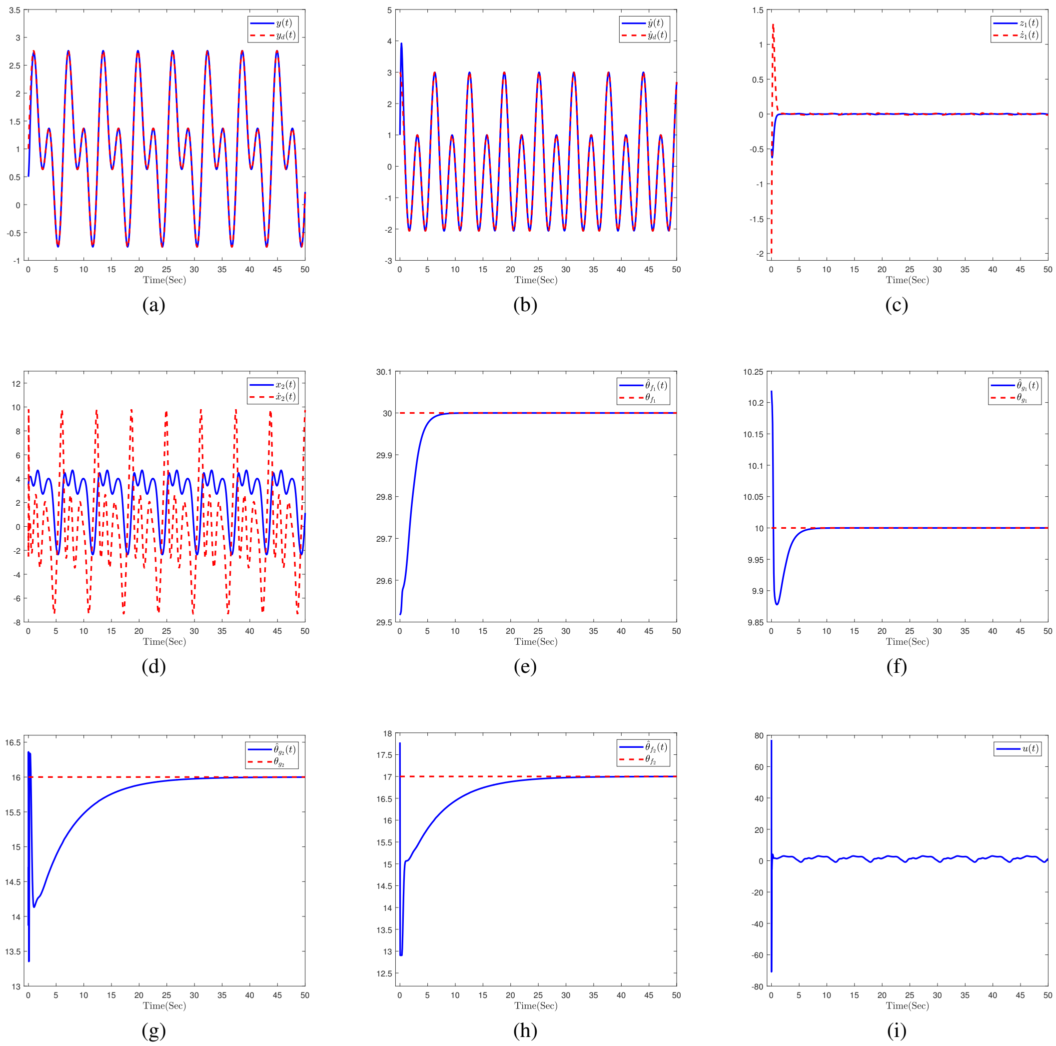

Figure F2 Simulation results: (a): The output $x_1(t)$ and the desired trajectory $y_d(t)$; (b): $\dot{x}_1(t)$ and $\dot{y}_d(t)$; (c): Tracking errors; (d): State variables $x_2(t)$ and $\dot{x}_2(t)$; (e): Adaptive law $\hat{\theta}_{f1}(t)$; (f): Adaptive law $\hat{\theta}_{g1}(t)$; (g): Adaptive law $\hat{\theta}_{f2}(t)$; (h): Adaptive law $\hat{\theta}_{g2}(t)$; (i): Control input $u(t)$.

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